The covering reflection principle

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This is joint work in progress with myself, Nai Chung Hou, Andreas Leitz, and Farmer Schlutzenberg.

The topic originated in Hou's question [Hou23] on MathOverflow and our various answers to it [Ham23; Sch23; Lie23], in which the solution emerged gradually, ultimately converging to the current collaboration [Ham+].

The covering reflection principle

We consider a model-theoretic covering reflection principle.

Main idea

Every large structure is covered by elementary images of a suitable fixed small structure.

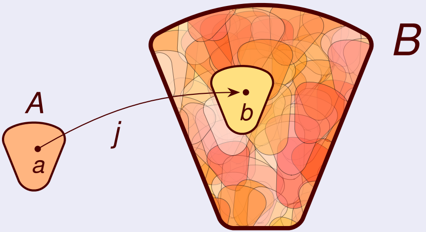
"Looks like model theory...

...but it has a set-theoretic core."

The covering reflection principle

Covering reflection principle CRP_δ

Holds for a cardinal δ, if for every first-order structure B in a countable language, there is substructure A, size less than δ, such that B is covered by the elementary images of A in B.



That is, every element $b \in B$ is in the range of some elementary embedding $j : A \to B$.

Instances of covering reflection

Model theory is full of instances of covering reflection.

- Every uncountable dense linear order, for example, is covered by elementary images of a fixed countable dense linear order.
- Similarly, every model of an $\aleph_0$-categorical theory is covered by elementary images of the unique countable model.
- In fact, same is true for κ-categorical theories in uncountable powers κ—they are covered by elementary images of a fixed countable structure.

Models of κ-categorical theories

Theorem

If a countable theory T is κ-categorical for some infinite κ, then T has covering reflection with respect to countable models.

Furthermore, it is strongly uniform—there is a countable $A \models T$ covering every uncountable $B \models T$ by its elementary images.

Proof.

$\aleph_0$-categorical is easy case—cover by countable elementary substructures.

κ-categorical for uncountable κ. By Morley, all uncountable $B \models T$ are saturated. Morley also proved T is $\aleph_0$-stable, so there is a countable saturated model. It covers. □

Covering reflection cardinal δ

But covering reflection is about covering all models, not just models of a particular theory.

Covering reflection CRP_δ

Every model B in a countable language is covered by elementary images of a fixed model A of size less than δ.

Question

Is there any such δ? Does covering reflection occur? How large is the smallest cardinal exhibiting covering reflection? Is the covering reflection principle consistent? What is the consistency strength?

Easy observations

Covering reflection CRP_δ

Every model *B* is covered by elementary images of some model *A* size $< \delta$.

Closed upward

If covering reflection holds for δ, then also for any larger $\delta' > \delta$.

So our focus might be placed on the smallest δ for which covering reflection holds.

Must be uncountable

Observation

Covering reflection fails for $\delta = \aleph_0$, since the small model *A* would have to be finite, but no infinite model *B* has finite elementary substructures.

So δ must be uncountable. $\omega_1 \leq \delta$.

Bigger than continuum

Observation

If covering reflection holds for δ, then δ is strictly above the continuum $\mathfrak{c}$.

To see this, consider ordered real field $\langle\mathbb{R},+,\cdot,0,1,<\rangle$.

Elementary subfield $A\prec\mathbb{R}$ is determined by the cuts in $\mathbb{Q}$ it fills.

It has no other elementary images. So $\mathbb{R}$ is not covered by any proper subfield.

Thus, $\mathfrak{c}<\delta$.

How big must δ be? Is there any δ at all with covering reflection?

Natural variations are equivalent

Several natural variations of covering reflection are equivalent.

Proposition

Covering reflection is equivalently formulated for finite languages only.

Proof.

Given B size at least δ, expand with pairing function, constant 0, successor S to create distinct definable elements $S0, SS0, \ldots$. We can use these as indices with pairing function to pick out slices of a universal relation. This interprets the countable signature in a finite language. □

Natural variations equivalent

Proposition

Covering reflection is equivalently formulated with mere embeddings instead of elementary embeddings.

Proof.

We can simply Skolemize the language, so that submodels in the expanded language are elementary in the original language. □

Bounded size

Theorem

Covering reflection for δ is equivalently formulated only for structures B of size at most $2^{<\delta}$.

Proof.

Consider any model B in a countable language $\mathcal{L}$.

Let S be all $\mathcal{L}$-structures A with domain bounded in δ. Note S has size at most $2^{<\delta}$.

S has copy of every $\mathcal{L}$-structure of size less than δ.

If covering reflection fails for B, each $A \in S$ fails to cover some $x_A \in B$. Find $\bar{B} \prec B$ containing every x_A, size at most $2^{<\delta}$. So $\bar{B}$ also fails covering reflection. □

Note that $2^{<\delta} = \delta$ is quite common, including every infinite cardinal under GCH.

Covering reflection is Π^1_1

Corollary

The covering reflection principle for δ is Π^1_1-expressible in $\langle V_\delta, \in\rangle$.

Proof.

One can refer to all structures B of size at most $2^{<\delta}$ with a second-order quantifier over V_δ, since ${}^{<\delta}2 \subseteq V_\delta$.

To assert that a given B is covered by embedding images of a given small structure A is first-order expressible in V_δ.

So the covering reflection principle has complexity Π^1_1 over V_δ. □

A hint: not very large?

Corollary

The least δ for which covering reflection holds is not weakly compact.

Proof.

Weakly compact cardinals are Π^1_1-indescribable, and so if they exhibit covering reflection, then there must be a smaller cardinal also exhibiting covering reflection. So the least cardinal δ with covering reflection cannot be weakly compact. $\square$

Another upper bound on size

Corollary

The first δ with covering reflection is less than the first Σ_2-correct cardinal. In particular, it is less than the first strong cardinal.

Proof.

Since Π^1_1 assertions over V_δ are Π_1 in the language of set theory, the existence of a cardinal δ with the covering reflection principle is a Σ_2 assertion. So if there is one, there will be one below the first Σ_2-correct cardinal. In particular, since every strong cardinal is Σ_2-correct, the first cardinal δ with covering reflection will be less than the first strong cardinal. $\square$

But. . . this is also true of rank-to-rank cardinals, huge cardinals, and more.

A natural weakening

A natural weakening of the covering reflection principle.

Definition

The *covering subreflection principle* (CSRP_δ) holds for δ if for every structure B in a countable language there is a structure A of size less than δ, such that B is covered by the elementary images of the elementary submodels of A.

That is, for every $b \in B$ there is $\bar{A} \prec A$ and elementary embedding $j : \bar{A} \to B$ with $b \in \text{ran}(j)$.

Covering subreflection is settled

Theorem

Covering subreflection holds for δ if and only if $\delta > 2^{\aleph_0}$.

Proof.

($\rightarrow$) The real field $\langle \mathbb{R}, +, \cdot, < \rangle$ cannot be covered by substructures of a structure of size less than continuum.

($\leftarrow$) Consider $\delta > 2^{\aleph_0}$ and B in countable language $\mathcal{L}$.

For each $b \in B$, pick countable $B_b \prec B$ with $b \in B_b$.

Choose family $\{ B_b \mid b \in I \}$ realizing every isomorphism type arising, with I size at most continuum.

Let $A \prec B$ have $B_b \subseteq A$ for all $b \in I$, size at most continuum.

The elementary substructures $B_b \prec A$ for $b \in I$ cover B, as desired. □

Remarkable strength of covering reflection

Despite the earlier hints of weakness, I would like now to establish the remarkable large-cardinal strength of the covering reflection principle.

We shall gradually reveal increasingly strong large cardinal lower bounds to the strength of covering reflection.

Extracting strength

Suppose that covering reflection holds with cardinal δ.

Consider the set-theoretic structure $B = \langle V_{\delta+1}, \in\rangle$.

By covering reflection, there is a small structure A whose elementary images in B cover B.

So A must look like a small version of $V_{\delta+1}$.

Note that A must be well founded. Without loss, A is transitive.

Since $B = V_{\delta+1}$ has a largest ordinal δ, it follows that A also has a largest ordinal δ_0, with $j(\delta_0) = \delta$. Perhaps A is something like V_{δ_0+1}.

It follows that j must have a critical point, $\mathrm{cp}(j) = \kappa < j(\kappa)$.

Extracting strength

We have assumed $B = V_{\delta+1}$ is covered by elementary images $j : A \to B$ of the transitive set A.

Let $\kappa = \mathrm{cp}(j)$ be smallest possible critical point arising $j : A \to B$.

Every $X \subseteq \kappa$ is in B, and so there is some $x \in A$ and $j : A \to B$ with $j(x) = X$.

Since x and $j(x) = X$ must agree up to κ, this implies $X \in A$.

So $P(\kappa) \subseteq A$.

Extracting strength—one measurable.

So we have $j : A \to B = V_{\delta+1}$ with $\kappa = \mathrm{cp}(j)$ and $P(\kappa) \subseteq A$.

This implies that κ is a measurable cardinal!

We can define the induced normal measure $X \in \mu \leftrightarrow \kappa \in j(X)$ for a fixed $j : A \to B$ with critical point κ.

Conclusion

If covering reflection holds for δ, then there is a measurable cardinal $\kappa < \delta$.

Perhaps the earlier result that δ itself is not weakly compact was a distraction.

Extracting strength—more than a measurable

Let's go for more.

Take $\kappa_0 = \kappa$, using the κ just defined.

Let κ_1 be the smallest critical point of some $j : A \to B$ with $\kappa_0 \in \operatorname{ran}(j)$.

Since κ_0 was smallest possible critical point, we have $\kappa_0 \leq \kappa_1$, but since κ_1 is not in the range of j, it must be that $\kappa_0 < \kappa_1$.

Furthermore, we get $P(\kappa_1) \subseteq A$ just as we did with κ_0.

Namely, if $X \subseteq \kappa_1$, there is $j : A \to B$ with $\{ \kappa_0, X \} \in \operatorname{ran}(j)$. So both κ_0 and X are in the range of j. So the critical point of j is at least κ_1, and if $X = j(x)$, then x and $j(x) = X$ agree up to κ_1, which implies $X \in A$.

So $P(\kappa_1) \subseteq A$.

Extracting strength—two measurable cardinals

So we have $j : A \to V_{\delta+1}$ with critical point κ_1 and $P(\kappa_1) \subseteq A$.

This implies κ_1 also is a measurable cardinal, with induced normal measure

$$X \in \mu \leftrightarrow \kappa_1 \in j(X)$$

Conclusion

If covering reflection holds for δ, then there are two measurable cardinals below δ.

Pushing harder—many measurable cardinals

But we can push this much harder.

We can define κ_α in the same way, for $\alpha < \kappa_0$ and more.

Conclusion

If covering reflection holds for δ, there are infinitely many measurable cardinals below δ.

Pushing even harder—1-extendible

For even more strength, let us show κ_0 is far more than merely measurable.

Let κ_0 be the smallest critical point arising via $j : A \to B = V_{\delta+1}$.

Since $P(\kappa_0) \subseteq A$, it follows that $V_{\kappa_0+1} \subseteq A$.

Further, $j(V_{\kappa_0+1}) = V_{j(\kappa_0)+1}$, since $V_{\delta+1}$ is correct about this.

So $j \upharpoonright V_{\kappa_0+1} : V_{\kappa_0+1} \to V_{j(\kappa_0)+1}$ is elementary, with crit point κ_0.

So κ_0 is 1-extendible!

Conclusion

If covering reflection holds for δ, then there is a 1-extendible cardinal below δ.

Extendible cardinals

A cardinal κ is 1-*extendible*, if there is an elementary embedding $j : V_{\kappa+1} \to V_{j(\kappa)+1}$ with critical point κ.

Every 1-extendible cardinal is a limit of measurable cardinals, of very high Mitchell rank.

Thus, we have found many measurable cardinals below δ of high Mitchell rank.

More generally, a cardinal κ is η-*extendible*, if there is an elementary embedding $j : V_{\kappa+\eta} \to V_\theta$ with critical point κ.

The cardinal κ is *extendible*, if η-extendible for all η.

Stronger than supercompact, in the upper realms of the large cardinal hierarchy.

Pushing harder—supercompactness

Reasoning with κ_1 as above shows that $V_{\kappa_1} \subseteq A$.

Therefore the restriction $j \upharpoonright V_{\kappa_1+1} \to V_{j(\kappa_1)+1}$ shows that κ_0 is $(\kappa_1 + 1)$-extendible.

So κ_0 is extendible up to a measurable cardinal, which is a considerable large cardinal hypothesis.

This implies, for example, that κ_0 is a supercompact cardinal in V_{κ_1}.

Conclusion

The consistency strength of covering reflection exceeds a supercompact cardinal.

Pushing still harder—

Let us push still harder.

We defined κ_0 and κ_1, but let us continue the iteration longer.

For each $\beta < \delta$, let κ_β be smallest critical point of some $j : A \to B$ with $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ in $\operatorname{ran}(j)$.

The same kind of reasoning as before shows $P(\kappa_\beta) \subseteq A$ and consequently $V_{\kappa_\beta+1} \subseteq A$ and κ_β is measurable.

If $\beta \leq \kappa_\beta$, which is true already for a long way, then all initial segments of $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ are also in the range of the $j : A \to B$ witnessing κ_β. So this embedding is also relevant when defining previous κ_α, and consequently $\kappa_\alpha \leq \kappa_\beta$ for all $\alpha < \beta$.

But since those κ_α are in $\operatorname{ran}(j)$, but κ_β is not, it follows that $\kappa_\alpha < \kappa_\beta$ for all $\alpha < \beta$.

Pushing still harder

In short, the κ_α sequence is strictly increasing for quite a long way, as long as $\beta \leq \kappa_\beta$ remains true.

But it cannot go up forever, since these are all in A.

So it must eventually happen that $\kappa_\gamma < \gamma$ for some γ.

Let $\lambda = \kappa_\gamma$ when this occurs. So λ is critical point of some $j : A \to B$ with $\langle \kappa_\alpha \mid \alpha < \gamma \rangle$ in $\operatorname{ran}(j)$.

This j is relevant for $\alpha < \lambda$, so $\kappa_\alpha < \lambda$. Thus, $\lambda = \sup_{\alpha<\lambda} \kappa_\alpha$.

Each κ_α is λ-extendible by the reasoning we gave earlier.

Conclusion

If covering reflection holds at δ, then there is measurable $\lambda < \delta$ that is a limit of λ-extendible cardinals.

A little more

In fact, in the paper we prove that unboundedly many of the κ_α for $\alpha < \lambda$ are extendible inside V_λ.

Conclusion

If covering reflection holds at δ, then there is measurable cardinal λ below δ such that V_λ has a proper class of extendible cardinals.

Still more

Can still get more.

There are additional measurable cardinals above λ, including κ_β for $\lambda \leq \beta < \gamma$.

So by similar reasoning we get another $\bar\lambda$ higher up that is a limit of $\bar\lambda$-extendible cardinals.

$V_{\bar\lambda}$ will have a proper class of extendible cardinals, with λ as an extendible limit of extendible cardinals inside it.

Conclusion

If covering reflection holds for δ, then it is consistent to have a proper class of extendible cardinals and extendible limits of extendible cardinals, and limits of limits, and so forth.

Upper bound?

OK, so covering reflection is strong, if it is consistent.

But is it consistent?

So far in this talk I have not established consistency from any hypothesis.

Let me do so now.

Huge cardinals

A cardinal κ is *huge*, if it is critical point of elementary $j : V \to M$ with ${}^{j(\kappa)}M \subseteq M$.

This is a large cardinal in the upper realm of large cardinals, above supercompact, extendible, and so forth, but below rank-to-rank.

Theorem

If κ is huge, then the covering reflection principle holds of κ. The least cardinal δ exhibiting covering reflection is therefore strictly less than κ.

Proof of hugeness upper bound

Assume that κ is huge, witnessed by $j : V \to M$.

Suppose covering reflection fails at κ, with structure B of size κ.

So M thinks $j(B)$ is a counterexample to covering reflection for $j(\kappa)$.

By hugeness, M and V have same substructures of $j(B)$ of size $j(\kappa)$, and same embeddings into $j(B)$.

So $j(B)$ is also a counterexample to covering reflection for $j(\kappa)$ in V.

Proof of hugeness upper bound, continued

In particular, in V we think that $j(B)$ is not covered by elementary images of the specific structure B.

So there is $x \in j(B)$ such that

> *x is not in the range of any elementary embedding of B into $j(B)$.*

Applying j, we conclude in M that

> *$j(x)$ is not in the range of any elementary embedding of $j(B)$ into $j(j(B))$*

Now, a delightful trick. $j \upharpoonright j(B)$ is a perfectly good elementary embedding of $j(B)$ into $j(j(B))$.

And it hits $j(x)$. Contradiction.

Exact consistency strength

We settle the consistency strength with a new large cardinal notion.

Theorem

The least cardinal δ with covering reflection is exactly the least anchor cardinal.

A cardinal κ is an *anchor* cardinal if for every $X \subseteq V_\kappa$ there is $\kappa_0 < \kappa_1 < \kappa$ and elementary embedding $j : \langle V_{\kappa_1}, \in, X \cap V_{\kappa_1} \rangle \to \langle V_\kappa, \in, X \rangle$ with $\kappa_0 = \mathrm{cp}(j)$ and $j(\kappa_0) = \kappa_1$.

Related to *links* and *chains* in [SRK78].

Every huge cardinal has a normal measure concentrating on anchor cardinals.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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