Scott ranks of models of theories of linear orders

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UW Madison Logic Seminar, October 2024

Most of new results in this talk are joint work with David Gonzalez. There is also some joint work with Turbo Ho and Ruiyan Chen.

Infinitary logic

In this talk I will use the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which allows countable conjunctions and disjunctions.

A formula is $\Sigma^{\mathtt{in}}_{\alpha}$ if it has α-many alternations of quantifiers and begins with a disjunction / existential quantifier.

A formula is $\Pi^{\mathtt{in}}_{\alpha}$ if it has α-many alternations of quantifiers and begins with a conjunction / universal quantifier.

Example

There is a $\Pi^{\mathtt{in}}_{2}$ formula which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

The following sentence which says that a vector space is infinite dimensional is Π_3:

$$\underbrace{\bigwedge_{n\in\mathbb{N}} \underbrace{(\exists x_1,\ldots,x_n) \underbrace{\bigwedge_{c_1,\ldots,c_n\in\mathbb{Q}} \underbrace{\left[c_1x_1+\cdots+c_nx_n=0 \rightarrow \left[c_1=c_2=\cdots=c_n=0\right]\right]}_{\Sigma_0}}_{\Pi_1}}_{\Sigma_2}}_{\Pi_3}.$$

Infinitary sentences can characterize countable structures up to isomorphism.

Theorem (Scott)

Let $\mathcal{A}$ be a countable structure. There is an infinitary sentence φ such that for all countable structures $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

We call such a sentence a Scott sentence for $\mathcal{A}$.

Definition (Montalbán)

The Scott rank of $\mathcal{A}$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure and let α a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable without parameters.*
- *Isomorphisms between copies of $\mathcal{A}$ can be computed in a uniformly relatively (boldface) $\mathbf{\Delta}^0_\alpha$ way.*

Definition (Alvir-Greenberg-HT-Turetsky)

The *Scott sentence complexity* of a structure $\mathcal{A}$ is the least complexity of a Scott sentence for $\mathcal{A}$.

This is always one of the complexities Π_α, Σ_α, and $d-\Sigma_\alpha$ (the conjunction of a Σ_α and a Π_α formula).

(Lopez-Escobar, A. Miller, and Alvir-Greenberg-HT-Turetsky: This is the same as the Wadge degree of the isomorphic copies of $\mathcal{A}$.)

Let φ be a Π_α sentence. Think of φ as a theory, defining its class of models

$$\{\mathcal{A} \mid \mathcal{A} \models \varphi\}.$$

Consider all of the models of φ, and their Scott ranks or Scott complexities.

Must there be a model of Scott rank $\approx \alpha$?

More precisely, Montalbán asked at the 2013 BIRS Workshop in Computable Model Theory:

Question

If φ is a Π_2 sentence, must it have a model with a Π_3 Scott sentence and hence Scott rank ≤ 2?

In 2018 I showed that this is very much not true.

Theorem (HT)

For any ordinal α, there is a Π_2 sentence all of whose models have Scott rank $\geq \alpha$.

Recently, Gonzalez and Montalbán proved the ω-Vaught's conjecture for linear orders.

Theorem (Gonzalez, Montalbán)

For every α and every Π_α sentence φ extending the axioms of linear orders, either:

- *There are only countably many models of φ and they all have Scott rank less than $\alpha + \omega$, or*
- *There are uncountably many models of T which are not $\Pi_{\alpha+\omega}$-elementary equivalent with each other.*

Part of this proof gave improved methods for understanding Scott rank in linear orders. So Gonzalez and I started talking about whether we could understand the Scott ranks of the models of a theory of linear orders.

Question

Given a Π_α sentence extending the axioms of linear orders, must it have a model of Scott rank $\approx \alpha$?

The first step was to look at my earlier construction of a Π_2 sentences all of whose models have Scott rank $\geq \alpha$, and see that it is inherently incompatible with linear orders.

I will explain why (vaguely).

The construction uses the back-and-forth relations.

Definition

The *standard asymmetric back-and-forth relations* $\leq_\alpha$, for $\alpha < \omega_1$, are defined by:

- $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas from among the first $|\bar{a}|$-many formulas.
- For $\alpha > 0$, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for each $\beta < \alpha$ and $\bar{d} \in \mathcal{B}$ there is $\bar{c} \in \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

We define $\bar{a} \equiv_\alpha \bar{b}$ if $\bar{a} \leq_\alpha \bar{b}$ and $\bar{b} \leq_\alpha \bar{a}$.

Theorem (Karp)

$\mathcal{A} \leq_\alpha \mathcal{B}$ *if and only if every* Σ_α *sentence true of* $\mathcal{B}$ *is true of* $\mathcal{A}$
if and only if every Π_α *sentence true of* $\mathcal{A}$ *is true of* $\mathcal{B}$.

So if $\mathcal{A}$ has a Π_α Scott sentence, then

$$\mathcal{A} \leq_\alpha \mathcal{B} \Longrightarrow \mathcal{A} \cong \mathcal{B}.$$

There are other notions of Scott rank using the back-and-forth relations, e.g., the Scott rank of $\mathcal{A}$ is the least α such that

$$\mathcal{A} \equiv_\alpha \mathcal{B} \Longrightarrow \mathcal{A} \cong \mathcal{B}.$$

These are all coarsely equivalent.

Given α, we want a Π_2 sentence all of whose models have Scott rank $\geq \alpha$.

The language will have relations $\sim_\beta$ for $\beta \leq \alpha$ and unary relations R_n.

The Π_2 sentence says that the $\sim_\beta$ satisfy the definition of the back-and-forth relations, using the atomic type in the R_n as the base case: $\bar{a} \sim_0 \bar{b}$ if and only if each a_i satisfies the same relations R_n as b_i.

We also say that there are elements a, b satisfying $a \sim_\beta b$ for all $\beta < \alpha$, but $a \nsim_\alpha b$.

Then we show that the $\sim_\beta$ really are the back-and-forth relations on the structure, even though they did not reference themselves in the base case.

This type of construction cannot build a linear order.

Linear orders have the following back-and-forth property:

Lemma

Given linear orders $\mathcal{A}$ and $\mathcal{B}$, and $a_1 < \cdots < a_n$ in $\mathcal{A}$ and $b_1 < \cdots < b_n$ in $\mathcal{B}$:

$$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}) \iff \text{for } i = 0, \ldots, n,\ (a_i, a_{i+1})^{\mathcal{A}} \leq_\alpha (b_i, b_{i+1})^{\mathcal{B}}.$$

This implies that if $a < b < c$, there is no relationship between a and c that does not come from a relationship between a and b, and a relationship between b and c.

Theorem (Gonzalez, HT)

Let φ be a Π_α theory in the language of linear orders extending the linear order axioms.

Then φ has a model with a $\Pi_{\alpha+4}$ Scott sentence and hence Scott rank $\alpha + 3$.

Theorem (Gonzalez, HT, Ho)

Let λ be a limit ordinal. There is a Π_λ sentence φ such that all models of φ have Scott complexity $\Sigma_{\lambda+1}$.

This is already a counterexample to Montalbán's question at BIRS.

Theorem (Gonzalez, HT)

Let λ be a limit ordinal. There is a Π_λ sentence φ such that all models of φ have Scott complexity $\Pi_{\lambda+2}$.

Theorem (Gonzalez, HT)

For any ordinal α, there is a $\Pi_{\alpha+4}$ sentence φ such that all models of φ have Scott complexity $\Pi_{\alpha+6}$.

Thus there is a gap of size 2 between our upper bounds and our lower bounds.

I am going to talk about the proof a bit to describe what we need to know to decrease this gap.

Given a Π_α sentence φ, we want to show that it has a model $\mathcal{A} \models \varphi$ with a $\Pi_{\alpha+4}$ Scott sentence.

We build $\mathcal{A}$ using a Henkin construction, using formulas of bounded complexity.

There are two key facts about linear orders that we use. The first was already introduced:

Lemma

Given linear orders $\mathcal{A}$ and $\mathcal{B}$, and $a_1 < \cdots < a_n$ in $\mathcal{A}$ and $b_1 < \cdots < b_n$ in $\mathcal{B}$:

$$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}) \iff \text{for } i = 0, \ldots, n,\ (a_i, a_{i+1})^{\mathcal{A}} \leq_\alpha (b_i, b_{i+1})^{\mathcal{B}}.$$

We build $\mathcal{A}$ to have the property that if $(a_1, a_2) \equiv_{\alpha+1} (b_1, b_2)$ then $(a_1, a_2) \cong (b_1, b_2)$.

This implies that $\mathcal{A}$ has a $\Pi_{\alpha+4}$ Scott sentence.

Question

If $\mathcal{A}$ has the property that

$$\mathcal{A} \leq_\alpha \mathcal{B} \implies \mathcal{A} \equiv_\alpha \mathcal{B}$$

then is it true that

$$\mathcal{A} \leq_\alpha \mathcal{B} \implies \mathcal{A} \cong \mathcal{B}.$$

Question

If $\mathcal{A}$ has the property that

$$\mathcal{A} \leq_\alpha \mathcal{B} \implies \mathcal{A} \cong \mathcal{B}$$

then does $\mathcal{A}$ have a Π_α Scott sentence.

The second fact is the key tool that allows us to deal with intervals (a_1, a_2) and (b_1, b_2) that are not disjoint.

Theorem (Tarski and Lindenbaum)

If N and L are order types, and $N \cdot k$ is an initial segment of L for all k, then $N \cdot \omega$ is an initial segment of L, and so $L \cong N + L$.

We might try to apply a similar argument in other classes of structures such as Boolean algebras.

Back-and-forth relations in Boolean algebras break up into relationships between subalgebras (like back-and-forth relations in linear orders break up into relationships between subintervals).

However:

Theorem (Ketonen)

There are Boolean algebras A and B such that for every n there is C such that $A \cong C \oplus B^n$, but $A \ncong A \oplus B$.

Question

Given a Π_α sentence extending the axioms of Boolean algebras, must there be a model of Scott rank $\approx \alpha$?

There is one more aspect of the proof that I want to talk about. To do the Henkin construction, we actually want a more fine-grained version of this lemma:

Lemma

Given linear orders $\mathcal{A}$ and $\mathcal{B}$, and $a_1 < \cdots < a_n$ in $\mathcal{A}$ and $b_1 < \cdots < b_n$ in $\mathcal{B}$:

$$(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}) \iff \text{for } i = 0, \ldots, n,\ (a_i, a_{i+1})^{\mathcal{A}} \leq_\alpha (b_i, b_{i+1})^{\mathcal{B}}.$$

Conjecture

Let $\mathcal{L}$ be a countable linear order and $a_1 < \cdots < a_n$ elements of $\mathcal{L}$. Suppose that $\mathcal{L} \models \varphi(a_1, \ldots, a_n)$ with φ a Π_α formula in the language of linear orders. Then there are Π_α sentences $\theta_0, \ldots, \theta_n$ such that

- *for every $k = 0, \ldots, n$ we have $(a_k, a_{k+1}) \models \theta_k$, and*
- *if $\mathcal{B}$ is any linear order and $b_1 < \cdots < b_n$, if for every $k = 0, \ldots, n$ we have $(b_k, b_{k+1}) \models \theta_k$ then $\mathcal{B} \models \varphi(b_1, \ldots, b_n)$.*

There is one more aspect of the proof that I want to talk about. To do the Henkin construction, we actually want a more fine-grained version of this lemma:

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Given linear orders $\mathcal{A}$ and $\mathcal{B}$, and $a_1 < \cdots < a_n$ in $\mathcal{A}$ and $b_1 < \cdots < b_n$ in $\mathcal{B}$:

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- *if $\mathcal{B}$ is any linear order and $b_1 < \cdots < b_n$, if for every $k = 0, \ldots, n$ we have $(b_k, b_{k+1}) \models \theta_k$ then $\mathcal{B} \models \varphi(b_1, \ldots, b_n)$.*

FALSE

Theorem (Gonzalez, HT)

There is a Π_4 sentence θ expanding the theory of linear orders such that for any consistent Π_4 sentences φ and ψ there are $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \psi$ such that $\mathcal{A} + 1 + \mathcal{B} \not\models \theta$.

Instead, we must use a complexity class $\mathfrak{E}_\alpha/\mathfrak{A}_\alpha$ of formulas that is different from the Σ_α/Π_α hierarchy.

Lemma (Gonzalez, HT)

Let $\mathcal{L}$ be a countable linear order and $a_1 < \cdots < a_n$ elements of $\mathcal{L}$. Suppose that $\mathcal{L} \vDash \varphi(a_1, \ldots, a_n)$ with φ a $\mathfrak{E}_\alpha$ formula in the language of linear orders. Then there are $\mathfrak{E}_\alpha$ sentences $\theta_0, \ldots, \theta_n$ such that

- *for every $k = 0, \ldots, n$ we have $(a_k, a_{k+1}) \vDash \theta_k$, and*
- *if $\mathcal{B}$ is any linear order and $b_1 < \cdots < b_n$, if for every $k = 0, \ldots, n$ we have $(b_k, b_{k+1}) \vDash \theta_k$ then $\mathcal{B} \vDash \varphi(b_1, \ldots, b_n)$.*

Definition

All connectives below are countable.

- $\mathfrak{A}_1 := \Pi_1$
- $\mathfrak{E}_1 := \Sigma_1$
- $\mathfrak{A}_\alpha :=$ closure of $\bigcup_{\beta<\alpha} \overline{\mathfrak{E}}_\beta$ under $\forall$ and $\bigwedge\!\!\!\bigwedge$
- $\mathfrak{E}_\alpha :=$ closure of $\bigcup_{\beta<\alpha} \overline{\mathfrak{A}}_\beta$ under $\exists$ and $\bigvee\!\!\!\bigvee$
- $\overline{\mathfrak{E}}_\alpha :=$ closure of $\mathfrak{E}_\alpha$ under $\bigvee\!\!\!\bigvee, \bigwedge\!\!\!\bigwedge$
- $\overline{\mathfrak{A}}_\alpha :=$ closure of $\mathfrak{A}_\alpha$ under $\bigvee\!\!\!\bigvee, \bigwedge\!\!\!\bigwedge$

We had previously discovered these formulas with Ronnie Chen due to their connections with the back-and-forth relations. (It would be interesting to know if anyone else has seen these before?)

Theorem (Chen, Gonzalez, HT)

Suppose that $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ *for* $\alpha \geq 1$*. Then given a* $\overline{\mathfrak{E}}_\alpha$ *formula* $\varphi(\bar{x})$ *and a* $\overline{\mathfrak{A}}_\alpha$ *formula* $\psi(\bar{x})$*,*

$$\mathcal{B} \models \varphi(\bar{b}) \Longrightarrow \mathcal{A} \models \varphi(\bar{a})$$

and

$$\mathcal{A} \models \psi(\bar{a}) \Longrightarrow \mathcal{B} \models \psi(\bar{b})$$

Theorem (Chen, Gonzalez, HT)

For $\mathcal{A}$ a countable structure, $\bar{a} \in \mathcal{A}$, and $\alpha \geq 1$, there are $\overline{\mathfrak{E}}_\alpha$ formulas $\varphi_{\bar{a},\mathcal{A},\alpha}(\bar{x})$ and $\mathfrak{A}_\alpha$ formulas $\psi_{\bar{a},\mathcal{A},\alpha}(\bar{x})$ such that for $\mathcal{B}$ any structure,

$$\mathcal{B} \models \varphi_{\bar{a},\mathcal{A},\alpha}(\bar{b}) \iff (\mathcal{B},\bar{b}) \leq_\alpha (\mathcal{A},\bar{a})$$

and

$$\mathcal{B} \models \psi_{\bar{a},\mathcal{A},\alpha}(\bar{b}) \iff (\mathcal{B},\bar{b}) \geq_\alpha (\mathcal{A},\bar{a}).$$

We cannot define the back-and-forth relations using the Π/Σ hierarchy without doubling the size of the ordinal.

Theorem (Chen, Gonzalez, HT)

There is a structure $\mathcal{M}$ such that

$$\{\mathcal{N} : \mathcal{N} \geq_2 \mathcal{M}\}$$

is Π^0_4-complete.

Lemma (Gonzalez, HT)

Let $\mathcal{L}$ be a countable linear order and $a_1 < \cdots < a_n$ elements of $\mathcal{L}$. Suppose that $\mathcal{L} \models \varphi(a_1, \dots, a_n)$ with φ a $\mathfrak{E}_\alpha$ formula in the language of linear orders. Then there are $\mathfrak{E}_\alpha$ sentences $\theta_0, \dots, \theta_n$ such that

- *for every $k = 0, \dots, n$ we have $(a_k, a_{k+1}) \models \theta_k$, and*
- *if $\mathcal{B}$ is any linear order and $b_1 < \cdots < b_n$, if for every $k = 0, \dots, n$ we have $(b_k, b_{k+1}) \models \theta_k$ then $\mathcal{B} \models \varphi(b_1, \dots, b_n)$.*

To prove the main theorem, given a Π_α sentence, the Henkin construction works with $\mathfrak{E}_{\alpha+1}$ formulas.

Theorem (Gonzalez, HT)

Let φ be a Π_α theory in the language of linear orders extending the linear order axioms.

Then φ has a model with a $\Pi_{\alpha+4}$ Scott sentence and hence Scott rank $\alpha + 3$.

Thanks!