The computability of K-theory for operator algebras

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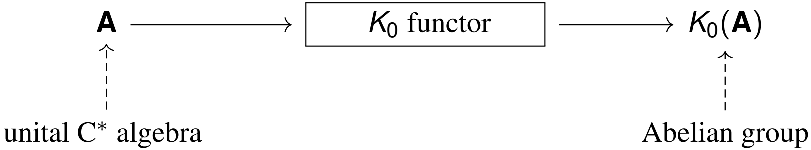
The high-altitude picture

The K_0 functor:



- $K_0(\mathbf{A})$ is countable if $\mathbf{A}$ separable.
- $K_0(\mathbf{A})$ is an invariant, but not a classifier.

Theorem (EGMM 2024+)

If $\mathbf{A}$ is a computably presentable and unital C^*-algebra, then $K_0(\mathbf{A})$ has a c.e. presentation.

Fix a unital C^*-algebra $\mathbf{A}$.

- Vector space over $\mathbb{C}$.
- Has a multiplication with a unit $\mathbf{1_A}$.
  - Left and right distributive.
  - Associative.
  - $\forall \alpha \in \mathbb{C}\ \forall a, b \in \mathbf{A}\ \alpha(ab) = (\alpha a)b = a(\alpha b)$
- Has a submultiplicative norm $\| \ \|$.
  - $\|ab\| \leq \|a\| \, \|b\|$.
- Has isometric adjoint operation $a \mapsto a^*$.
  - $(ab)^* = b^* a^*$
  - $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$
- $\|\mathbf{1_A}\| = 1$.
- $\|aa^*\| = \|a\|^2$ (C^* identity).

Examples:

- $C[0, 1]$ with the supremum norm.
- $M_n(\mathbb{C})$ with the operator norm.

Throughout this talk, $\mathbf{A}^{\#}$ denotes a *presentation* of A.

- Has a sequence of *distinguished points* of $\mathbf{A}$.
- *Requirement*: the distinguished points generate a dense subalgebra of $\mathbf{A}$.

Examples:

- $C[0,1]^{\#}$: special points are just $t \mapsto 1$ and $t \mapsto t$.
- $M_n(\mathbb{C})^{\#}$: standard basis of matrix units.

These presentations are *standard*. We identify $C[0,1]$ and $M_n(\mathbb{C})$ with their standard presentations.

Some more vocab

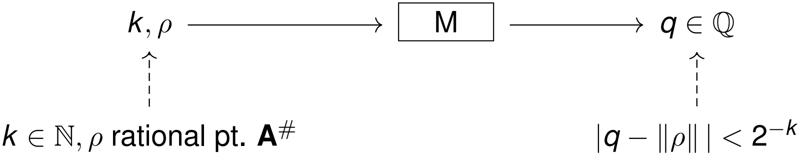
1. *Rational point of* $\mathbf{A}^\#$: $p(s_1, \ldots, s_k)$ where p is a rational $*$-polynomial and each s_j is a special point of $\mathbf{A}^\#$.
2. *Computable point* v *of* $\mathbf{A}^\#$*:* From $k \in \mathbb{N}$ can compute rational point ρ so that $\|\rho - v\| < 2^{-k}$. Code of such a Turing machine is an $\mathbf{A}^\#$ *index* of v.
3. *Computable sequence* $(a_n)_{n\in\mathbb{N}}$ *of* $\mathbf{A}^\#$*:* a_n a computable point of $\mathbf{A}^\#$ uniformly in n.

Remark

The rational points of $\mathbf{A}^\#$ are dense in $\mathbf{A}$.

We assume $\mathbf{A}^{\#}$ is *computable*; that is $\| \ \|$ is computable on the rational points of $\mathbf{A}^{\#}$.

This means there is a Turing machine M that behaves like this:



A code of such a Turing machine is an *index* of $\mathbf{A}^{\#}$.

Remark

The standard presentations of $C[0,1]$ and $M_n(\mathbb{C})$ are computable.

Computable maps between presentations

We need to define what mean by a computable map from $\mathbf{A}^{\#}$ to a presentation $\mathbf{B}^{\#}$. The only maps we care about are $*$-homomorphisms. Hence, we may (and do) take the following as a definition.

Proposition ('Folklore')

Suppose $\mathbf{B}^{\#}$ is a presentation of a C^* algebra $\mathbf{B}$, and suppose f is a $*$-homomorphism of $\mathbf{A}$ into $\mathbf{B}$. Then, f is a computable map from $\mathbf{A}^{\#}$ to $\mathbf{B}^{\#}$ if and only if f is computable on the rational points of $\mathbf{A}^{\#}$.

That is, from a (code of a) rational point ρ of $\mathbf{A}^{\#}$ and $k \in \mathbb{N}$ it is possible to compute a rational point ρ' of $\mathbf{B}^{\#}$ so that $\|\rho' - f(\rho)\|_{\mathbf{B}} < 2^{-k}$.

Fact

If $\phi : \mathbf{A} \to \mathbf{B}$ is a $*$-homomorphism of C^*-algebras, then ϕ is 1-Lipschitz. Hence, if ϕ is a $*$-isomorphism, then it is an isometry.

Finding projections

Recall $p \in \mathbf{A}$ is a *projection* if $p^2 = p = p^*$.

Proposition (EGMM 2024+)

There is a Π^0_1 set $R \subseteq \mathbb{N}$ so that for all $e \in \mathbb{N}$ and $p \in \mathbf{A}$, if e is an $\mathbf{A}^\#$-index of p, then $p \in R$ iff p is a projection.

Proof sketch.

Via e, can enumerate all rational open balls that contain p. We then use the following fact: if $a \in \mathbf{A}$ is self-adjoint and $\|a^2 - a\| < \epsilon$, then there is a projection p' so that $\|p' - a\| < 2\epsilon$. We use this fact to enumerate all rational open balls that contain a projection. R says that for every rational $r > 0$ p is within r of a projection. □

Proof is uniform: an index of R can be computed from an index of $\mathbf{A}^\#$.

Amplifications

Notation

$M_n(\mathbf{A}) =$ set of all $n \times n$ matrices over $\mathbf{A}$.

Fact

There is a C^**-norm* $\| \; \|_*$ *on* $M_n(\mathbf{A})$*.*

Notation

$M_n(\mathbf{A})^\#$ is the presentation of $M_n(\mathbf{A})$ induced by $\mathbf{A}^\#$. That is, the distinguished points of $M_n(\mathbf{A})^\#$ are the matrices whose components are all distinguished points of $\mathbf{A}^\#$. It follows that the rational points of $M_n(\mathbf{A})^\#$ are the matrices whose entries are all rational points of $\mathbf{A}^\#$.

Remark

That $M_n(\mathbf{A})^\#$ *is* a presentation is implied by the following well-known inequality.

$$\max_{r,s} \|a_{r,s}\| \leq \|(a_{r,s})_{r,s}\|_* \leq \sum_{r,s} \|a_{r,s}\| .$$

Theorem (EGMM 2024+)

$M_n(\mathbf{A})^\#$ is computable uniformly in n.

Proof sketch (very sketchy).

By a result of Goldbring, $\mathbf{A}^\#$ induces a computable presentation $(M_n(\mathbb{C}) \otimes \mathbf{A})^\#$. (This is the tricky part.) There is a simple $*$-isomorphism ψ from $M_n(\mathbb{C}) \otimes \mathbf{A}$ onto $M_n(\mathbf{A})$. In fact, ψ maps rational points to rational points. This transfers the computability of the norm. □

Notation

Let $\mathbf{P}_n(\mathbf{A}) =$ the set of projections in $M_n(\mathbf{A})$, and let $\mathbf{P}_{<\omega}(\mathbf{A}) = \bigcup_n \mathbf{P}_n(\mathbf{A})$.

Murray-von Neumann Equivalence

Definition

Suppose $P \in \mathbf{P}_m(\mathbf{A})$ and $P' \in \mathbf{P}_n(\mathbf{A})$. Write $P \sim_{\text{mvn}} P'$ if there exists $V \in M_{m,n}(\mathbf{A})$ so that $P = VV^*$ and $P' = V^*V$.

Fact

$\sim_{mvn}$ is an equivalence relation on $P_{<\omega}(\mathbf{A})$.

Theorem (EGMM 2024+)

There is a Σ^0_1 relation $Q \subseteq \mathbb{N}^2$ so that for all $P_0, P_1 \in \mathbf{P}_n(\mathbf{A})$ and $e_0, e_1 \in \mathbb{N}$, if e_j is an $M_n(\mathbf{A})^\#$ index of P_j, then $Q(e_0, e_1)$ iff $P_0 \sim_{mvn} P_1$.

The $\mathcal{D}$ functor

Notation

When $P, P' \in \mathbf{P}_{<\omega}(\mathbf{A})$, let

$$P \oplus P' = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P' \end{pmatrix}$$

Fact

$\sim_{mvn}$ is a congruence relation on $(P_{<\omega}(\mathbf{A}), \oplus)$.

Notation

Set $\mathcal{D}(\mathbf{A}) = (P_{<\omega}(\mathbf{A}), \oplus)/ \sim_{\text{mvn}}$.

Fact

$\mathcal{D}$ is a functor from the category of unital C^* algebras to the category of Abelian semigroups.

Fact

If $P, Q \in \mathbf{P}_n(\mathbf{A})$, and if $\|P - Q\|_* < 1$, then $P \sim_{mvn} Q$.

Remark

Since **A** is separable, it follows that $\mathcal{D}(\mathbf{A})$ is a countable Abelian semigroup.

Detour into computable algebra: Presentations of semigroups

Throughout rest of this talk S is a semigroup, and $X = \{x_0, x_1, \ldots\}$ is a set of indeterminates.

$FS[X]$ = the free semigroup generated by X.

Presentation of S: $S^\# = (S, \nu)$ where ν is an epimorphism of $FS[X]$ onto S.

$S^\#$ is *computable* (*c.e.*) if $\ker(\nu) = \{(w, w') : \nu(w) = \nu(w')\}$ is computable (c.e.).

If $\nu(w) = a$, then w is an $S^\#$*-notation* for a.

Presenting $\mathcal{D}(\mathbf{A})$

Remark

Suppose $\mathcal{D}(\mathbf{A})^\#$ is a presentation of $\mathcal{D}(\mathbf{A})$. Then, $\mathcal{D}(\mathbf{A})^\#$ assigns each $w \in FS[X]$ to an equivalence class $[P]_{\sim \text{mvn}}$. But, even if $\mathcal{D}(\mathbf{A})^\#$ is computable, we may not be able to compute a representative of $[P]_{\sim \text{mvn}}$. This leads to the following definition.

Definition

$\mathcal{D}(\mathbf{A})^\#$ is *supported* by $\mathbf{A}^\#$ if from $w \in FS[X]$ we can compute n and an $M_n(\mathbf{A})^\#$ index of a $P \in \mathbf{P}_n(\mathbf{A})$ so that w is a $\mathcal{D}(\mathbf{A})^\#$-notation for $[P]_{\sim \text{mvn}}$.

The functor $\mathcal{D}^c$

Theorem (EGMM 2024+)

There is a unique (up to computable isomorphism) c.e. presentation $\mathcal{D}(\mathbf{A})^\#$ that is supported by $\mathbf{A}^\#$.

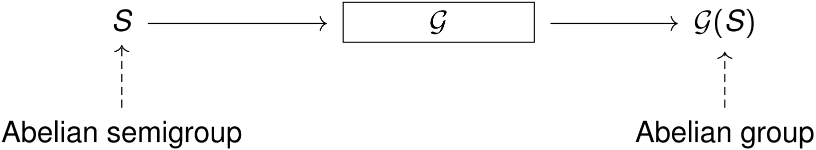
Notation

$\mathcal{D}^c(\mathbf{A}^\#) =$ this presentation.

Theorem (EGMM 2024+)

$\mathcal{D}^c$ is a computable functor from the category of computable presentations of C^* algebras to the category of c.e. presentations of semigroups.

More algebra: the Grothendieck Functor



Universality: There exists homomorphism $\gamma_S : S \to \mathcal{G}(S)$ so that:

$$\begin{array}{ccc} S & \xrightarrow{\forall \phi} & \text{Abelian group } H \\ \downarrow{\scriptstyle \gamma_S} & \nearrow{\scriptstyle \psi_\phi} & \\ \mathcal{G}(S) & & \end{array}$$

If S has cancellation property, then γ_S is a monomorphism.

Presenting $\mathcal{G}(S)$

Let $FG[X]$ = free group generated by X.

Group presentations are defined like semigroup presentations except use $FG[X]$ instead of $FS[X]$.

Definition (Computable $S^{\#}$-universality)

γ_S is a computable map from $S^{\#}$ to $\mathcal{G}(S)^{\#}$ and

$$\begin{array}{ccc} S^{\#} & \xrightarrow{\forall\phi} & H^{\#} \\ \Big\downarrow{\scriptstyle \gamma_S} & \nearrow{\scriptstyle \psi_\phi} & \\ \mathcal{G}(S)^{\#} & & \end{array}$$

AND $\phi \mapsto \psi_\phi$ computable.

Proposition (EGMM 2024+)

If $S^\#$ is c.e., then there is a unique (up to computable isomorphism) c.e. presentation of $\mathcal{G}(S)$ that is computably $S^\#$-universal.

Proof sketch.

Follow the classical construction of $\mathcal{G}(S)$. □

Notation

Let $\mathcal{G}^c(S^\#)$ denote this presentation.

Theorem (EGMM 2024+)

$\mathcal{G}^c$ is a computable functor from the category of c.e. presentations of semigroups to the category of c.e. presentations of groups.

And now, K_0!

Definition

$K_0(\mathbf{A}) = \mathcal{GD}(\mathbf{A})$.

Notation

$K_0^c(\mathbf{A}^\#) = \mathcal{G}^c\mathcal{D}^c(\mathbf{A}^\#)$.

Corollary (EGMM 2024+)

K_0^c is a computable functor from the category of computable presentations of C^* algebras to the category of c.e. presentations of groups.

"Application" to AF-algebras

Definition

Suppose $\mathbf{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathbf{A}_n}$ where $\mathbf{1_A} \in \mathbf{A}_n \subseteq \mathbf{A}_{n+1}$. $\mathbf{A}$ is *AF* if each $\mathbf{A}_n$ is a finite-dimensional subalgebra of $\mathbf{A}$.

Fact

If $\mathbf{A}$ *is AF, then* $K_0(\mathbf{A})$ *is torsion-free.*

Theorem (Khisamiev 1986)

If a torsion-free Abelian group has a c.e. presentation, then it has a computable presentation.

Corollary (EGMM 2024+)

If $\mathbf{A}$ *is AF, then* $K_0(\mathbf{A})$ *is computably presentable.*

"Application" to UHF algebras

Definition

Suppose $\mathbf{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathbf{A}_n}$ where $\mathbf{1}_\mathbf{A} \in \mathbf{A}_n \subseteq \mathbf{A}_{n+1}$. $\mathbf{A}$ is *UHF* if for each n there exists k_n so that $\mathbf{A}_n$ is $*$-isomorphic to $M_{k_n}(\mathbb{C})$.

Fact

$k_n | k_{n+1}$.

Definition (Supernatural "number")

If $\mathbf{A}$ is UHF, then for every prime p we let $\epsilon_\mathbf{A}(p) = \sup\{m \in \mathbb{N} \;:\; \exists n \in \mathbb{N}\; p^m | k_n\}$.

Notation

Pr = the set of prime numbers.

Thus, $\epsilon_\mathbf{A} : \mathrm{Pr} \to \mathbb{N} \cup \{\infty\}$.

Notation

When $\epsilon : \mathrm{Pr} \to \mathbb{N} \cup \{\infty\}$, $\mathbb{Q}(\epsilon) =$ the subgroup of $\mathbb{Q}$ generated by $\{\frac{m}{p^k} : m \in \mathbb{Z} \wedge k \in \mathbb{N} \wedge p \in \mathrm{Pr} \wedge k \leq \epsilon(p)\}$.

Fact

If $\mathbf{A}$ *is UHF, then* $K_0(\mathbf{A}) \approx \mathbb{Q}(\epsilon_{\mathbf{A}})$.

Definition

$\mathbf{A}^{\#}$ is *computably UHF* if there is a computable sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $(\phi_n)_{n\in\mathbb{N}}$ so that:

1. ϕ_n is a unital $*$-monomorphism of $M_{k_n}(\mathbb{C})$ into $\mathbf{A}$.
2. $\mathrm{ran}(\phi_n) \subseteq \mathrm{ran}(\phi_{n+1})$.
3. $\mathbf{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathrm{ran}(\phi_n)}$.
4. ϕ_n is a computable map of $M_{k_n}(\mathbb{C})$ to $\mathbf{A}^{\#}$.

Let's borrow a definition from computable analysis.

Definition

$\epsilon : \mathrm{Pr} \to \mathbb{N} \cup \{\infty\}$ is *lower semi-computable* if there is a uniformly computable and nondecreasing sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of functions from Pr to $\mathbb{N}$ so that $\epsilon(p) = \lim_n \epsilon_n(p)$ for all $p \in \mathrm{Pr}$.

Theorem (EGMM 2024+)

Suppose **A** *is UHF. TFAE:*

1. **A** *is computably presentable.*
2. $\epsilon_{\mathbf{A}}$ *is lower semi-computable.*
3. **A** *has a computably UHF presentation.*
4. $K_0(\mathbf{A})$ *is computably presentable.*

Proof sketch.

Suppose $\mathbf{A}^{\#}$ is computable. Thus, $K^c(\mathbf{A}^{\#})$ is computable. Search for relations of the form $p^m \cdot a = 1$. $m \leq \epsilon_{\mathbf{A}}(p)$ for each such m. All such values of m will be discovered by this process.

Suppose $\epsilon_{\mathbf{A}}$ is lower semi-computable. We can then build a system $(M_{k_n}(\mathbb{C}), \psi_n)_{n\in\mathbb{N}}$ where $\psi_n : M_{k_n}(\mathbb{C}) \to M_{k_{n+1}}(\mathbb{C})$ is the standard unital $*$-embedding and $\mathbf{A}$ is $*$-isomorphic to the inductive limit of $(M_{k_n}(\mathbb{C}), \psi_n)_{n\in\mathbb{N}}$. By a theorem of Goldbring, this inductive limit has a computable presentation. □

Remark

Proof is not uniform.

Claim (EGMM 2024+)

If **A** *is UHF, then all of its computable presentations are computably UHF.*

Adding order

Definition

1. $\mathbf{A}$ is *finite* if $\mathbf{1_A} = u^*u$ implies $\mathbf{1_A} = uu^*$.
2. $\mathbf{A}$ is *stably finite* if $M_n(\mathbf{A})$ is finite for all n.

Fact

All AF algebras are stably finite.

Notation

$K_0(\mathbf{A})^+ = \mathrm{ran}(\gamma_{\mathcal{D}(\mathbf{A})})$.

Fact

If $\mathbf{A}$ *is stably finite, then* $K_0(\mathbf{A})^+$ *is an order cone; that is:*

- $K_0(\mathbf{A})^+ + K_0(\mathbf{A})^+ \subseteq K_0(\mathbf{A})^+$.
- $K_0(\mathbf{A})^+ - K_0(\mathbf{A})^+ = K_0(\mathbf{A})$.
- $K_0(\mathbf{A})^+ \cap (-K_0(\mathbf{A})^+) = \{0\}$.

Thus, if $\mathbf{A}$ is stably finite, $K_0(\mathbf{A})$ admits a partial order. This *partially* ordered group is denoted $(K_0(\mathbf{A}), K_0(\mathbf{A})^+)$.

Corollary

If $\mathbf{A}$ *is a computably presentable UHF algebra, then* $(K_0(\mathbf{A}), K_0(\mathbf{A})^+)$ *has a computable presentation.*

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