Universal Sets for Projections

Jacob B. Fiedler (joint with Don Stull)

University of Wisconsin-Madison

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Outline

1. Projections in geometric measure theory
2. Effective dimension
3. Some proof ideas

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Problems in GMT

The basic question: How large are sets?

- Usually, by size we mean measure or some notion of dimension (e.g. Hausdorff, packing, box-counting, Assouad...).
- More precisely, what can we say about sets that we know have a certain geometric property, or that are obtained by some natural geometric operation on a set with known size?

Covers and packings

Let $E \subset \mathbb{R}^n$ and $\{B_i\}_{i \in \mathbb{N}}$ be a collection of open balls in $\mathbb{R}^n$.

We call $\{B_i\}_{i \in \mathbb{N}}$ a δ-cover for E if

- $E \subseteq \bigcup_{i=1}^{\infty} B_i$
- $\text{diam}(B_i) \leq \delta$

We call $\{B_i\}_{i \in \mathbb{N}}$ a δ-packing for E if

- The balls are pairwise disjoint
- The balls have centers in E
- $\text{diam}(B_i) \leq \delta$

Hausdorff and packing dimension

Hausdorff dimension

$$\mathcal{H}^s_\delta(E) = \inf_{\delta-\text{covers}} \{\sum_{i=1}^{\infty} \text{diam}(B_i)^s\}$$

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$$

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\}$$

Packing dimension

$$\bar{\mathcal{P}}^s_\delta(E) = \sup_{\delta-\text{packings}} \{\sum_{i=1}^{\infty} \text{diam}(B_i)^s\}$$

$$\bar{\mathcal{P}}^s(E) = \lim_{\delta \to 0^+} \bar{\mathcal{P}}^s_\delta(E)$$

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} \bar{\mathcal{P}}^s(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\}$$

$$\dim_P(E) = \inf\{s : \mathcal{P}^s(E) = 0\}$$

Regularity

"Regularity" refers to a set looking the same (or at least the same size) at different scales.

- E is **weakly regular** if $\dim_P(E) = \dim_H(E)$
- E is α-**AD regular** if there exists some C such that

$$C^{-1}r^\alpha \leq \mathcal{H}^\alpha(E \cap B(x,r)) \leq Cr^\alpha, \quad x \in A, 0 < r < \text{diam}(A)$$

AD-regularity can be thought of as a generalization of self-similarity. Weak regularity is in turn a generalization of AD regularity.

Size of projections

We work in $\mathbb{R}^2$ and consider projections onto lines. In particular, for $e \in S^1$, let $p_e E = \{x \cdot e : x \in E\}$

Question: How does the Hausdorff dimension of a set change under projections?

We have the upper bound

$$\dim_H(p_e E) \leq \min\{1, \dim_H(E)\}$$

Theorem (Marstrand, 1954)

Let $E \subseteq \mathbb{R}^2$ be Borel. For Lebesgue almost every $e \in S^1$,

$$\dim_H(p_e E) = \min\{1, \dim_H(E)\}$$

Universal sets

Call $e \in S^1$ **maximal** for E if

$$\dim_H(p_e E) = \min\{1, \dim_H(E)\}.$$

Suppose $\mathcal{C}$ is some class of subsets of $\mathbb{R}^2$, i.e. the Borel sets, the weakly regular sets, the AD regular sets...

Call a set of directions $D \subseteq S^1$ such that every $E \in \mathcal{C}$ has a maximal direction in D a $\mathcal{C}$-**universal** set.

Are there small universal sets?

Small universal sets

All the results apply to sets in $\mathbb{R}^2$

Theorem (F. and Stull, 2024)

The class of sets with optimal oracles has a Lebesgue measure zero universal set

Theorem (F. and Stull, 2024)

For any $\varepsilon > 0$, the class of weakly regular sets has a Hausdorff dimension ε universal set.

Theorem (F. and Stull, 2024)

The class of AD regular sets has a Hausdorff dimension 0 universal set.

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Complexity of points in Euclidean space

Fix a universal prefix-free oracle Turing machine U. Given $A \subseteq \mathbb{N}$, the (prefix-free) Kolmogorov complexity of a string σ relative to A is

$$K^A(\sigma) = \min\{|\pi| : U^A(\pi) = \sigma\}$$

We can encode rational vectors $q \in \mathbb{R}^n$ as binary strings, and hence can talk about $K^A(q)$. This in turn allows us to define the complexity of *arbitrary* points in $\mathbb{R}^n$ at any given precision.

$$K^A_r(x) = \min\{K^A(q) : q \in B_{2^{-r}}(x)\}$$

Effective dimension

Definition

The effective Hausdorff dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\dim^A(x) = \liminf_{r \to \infty} \frac{K_r^A(x)}{r}$$

Definition

The effective packing dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\mathrm{Dim}^A(x) = \limsup_{r \to \infty} \frac{K_r^A(x)}{r}$$

The point-to-set principle

Effective dimension is directly related to classical dimension through the following "point-to-set" principle(s):

Theorem (J. Lutz and N. Lutz, 2015)

For all $E \subset \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)$$

and

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)$$

Points in regular sets

Let E be weakly regular. If A is the join of a Hausdorff and a packing oracle for E, for any $\varepsilon > 0$, there is some $x \in E$ such that

$$\mathrm{Dim}^A(x) - \dim^A(x) < \varepsilon$$

If E is AD-regular, we have something stronger: call a *point* $x \in \mathbb{R}^n$ α-AD regular with respect to an oracle $A \subseteq \mathbb{N}$ if there exists some C such that

$$\alpha r - C \log r \leq K_r^A(x) \leq \alpha r + C \log r$$

Proposition

If E is compact and α-AD regular, then there exists an oracle A relative to which $\mathcal{H}^\alpha$-almost every point in E is α-AD regular.

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The reduction

For any E in our class, it suffices to find some e in D with the following property: for every $\varepsilon > 0$, there exists some $x \in E$ such that

$$\dim^B(p_e x) \geq \min\{1, \dim_H(E)\} - \varepsilon$$

where B is a Hausdorff oracle for $p_e E$.

To show this bound holds, we need a few assumptions.

- e has high complexity at certain precisions, and low complexity at other precisions. In particular, e is the result of appropriately adding 0s to a ML random.
- A, the Hausdorff oracle for the set E, does not help in the computation of e
- Oracle access to e does not help in the computation of x

Partitioning

Let a sequence of precisions $1 = r_0, r_1, r_2, \ldots, r_m = r$ be given. Then

$$K_r^A(x) = \sum_{i=1}^{m} \left(K_{r_i}^A(x) - K_{r_{i-1}}^A(x) \right) + K_1^A(x)$$
$$\approx \sum_{i=1}^{m} K_{r_i, r_{i-1}}^A(x)$$

Let $x \in \mathbb{R}^2$ and $a \leq b$. We say that $[a, b]$ is (σ, c)-*teal* if

$$K_{b,s}^A(x \mid x) \leq \sigma(b - s) + c \log b,$$

for all $a \leq s \leq b$. We say that $[a, b]$ is (σ, c)-*yellow* if

$$K_{s,a}^A(x \mid x) \geq \sigma(s - a) - c \log b,$$

for all $a \leq s \leq b$.

Bounds on certain intervals

Lemma

Let $x \in \mathbb{R}^2, e \in \mathcal{S}^1$, $c \in \mathbb{N}$, $\sigma \in \mathbb{Q} \cap (0,1]$, $A \subseteq \mathbb{N}$ and $a < b \in \mathbb{R}_+$. Suppose that b is sufficiently large (depending on e, x, and σ) and $K^A_{s,b}(e \mid x) \geq s - c \log b$, for all $s \leq b - a$. Then the following hold.

1. If $[a, b]$ is (σ, c)-yellow,
$$K^A_{b,b,b,a}(x \mid p_e x, e, x) \leq K^A_{b,a}(x \mid x) - \sigma(b-a) + O_c(\log b)^2.$$
2. If $[a, b]$ is (σ, c)-teal,
$$K^A_{b,b,b,a}(x \mid p_e x, e, x) \leq O_c(\log b)^2.$$

Symmetry of information

Recall, we are interested in lower bounding the quantity $K_r^{A,B,e}(p_e x)$. Using symmetry of information and the assumptions on our points, we have

$$
\begin{aligned}
K_r^A(x \mid e, p_e x) &\geq K_r^{A,e}(x \mid p_e x) - O(\log r) \\
&\geq K_r^{A,B,e}(x \mid p_e x) - O(\log r) \\
&= K_r^{A,B,e}(x, p_e x) - K_r^{A,B,e}(p_e x) - O(\log r) \\
&\geq K_r^{A,B,e}(x) - K_r^{A,B,e}(p_e x) - O(\log r) \\
&\geq K_r^{A}(x) - K_r^{A,B,e}(p_e x) - \text{small error}
\end{aligned}
$$

Then, we can use the upper bound on $K_r^A(x \mid e, p_e x)$ that comes from partitioning.

Thank you!