Some Computability-theoretic Aspects of Partition Regularity over Algebraic Structures

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This talk will analyze some very interesting computability-theoretic and reverse mathematical phenomena which arise in a long line of work in Ramsey theory of algebraic structures.
For our analysis the interplay between combinatorics and algebra will be central.

Early Ramsey Theory

Theorem (Schur)

For every finite coloring of the positive integers, there is a monochromatic solution to the equation $x + y = z$*.*

Theorem (Van Der Waerden)

For every finite coloring of the positive integers, there are arbitrarily long monochromatic arithmetic progressions.

Rado's Thesis

Can Schur's theorem and Van Der Waerden's theorem be generalized to get a characterization of all linear systems that have monochromatic solutions under any finite coloring of the positive integers?

If we call these systems partition regular, then Schur's theorem can be restated in the following way:

Theorem (Schur, restated)

Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ *and* $b = 0$. *Then the system* $Ax = b$ *is partition regular.*

Partition Regularity over $\mathbb{Z}$

Definition

Let A be an $m \times n$ matrix with entries in $\mathbb{Z}$ and $b \in \mathbb{Z}^m, b \neq 0$. We say that the system of linear equations $Ax = b$ is **partition regular over** $\mathbb{Z}$ if for every finite coloring of the integers, the system has a monochromatic solution.

Examples

$x + y = 2$ is partition regular over $\mathbb{Z}$.
$x + y = 3$ is not partition regular over $\mathbb{Z}$.

Theorem (Rado, 1933)

Let A be an $m \times n$ matrix with entries in $\mathbb{Z}$ and $b \in \mathbb{Z}^m, b \neq 0$. The system $Ax = b$ is partition regular over $\mathbb{Z}$ if and only if it has a constant solution.

Rado gave a complete characterization of partition regular systems for all $b \in \mathbb{Z}^m$, but we will be interested in the case $b \neq 0$.

The case when $b \neq 0$ is sometimes also called *inhomogeneous partition regularity*.

Partition Regularity over a Commutative Ring R

What happens if we replace $\mathbb{Z}$ with any other commutative ring? Can we get a similar characterization?

Definition

Let R be a commutative ring, A an $m \times n$ matrix with entries in R and $b \in R^m$, $b \neq 0$. We say that the system of linear equations $Ax = b$ is **partition regular over** R if for every finite coloring of R, the system has a monochromatic solution.

Bergelson, Deuber, Hindman and Leffman (1994) showed that the same characterization holds for a special class of integral domains.

Byszewski and Krawczyk (2020) extended the result to all integral domains.

These papers used an indirect approach to get to these results.

Theorem (Leader, Russell, 2021)

Let R be a commutative ring, A an $m \times n$ matrix with entries in R and $b \in R^m, b \neq 0$. The system $Ax = b$ is partition regular over R if and only if it has a constant solution.

Leader, Russell - direct approach, thus giving a new proof to Rado's theorem.

A Common Theme

Theorem (Straus, 1975)

Let $(G, +, 0)$ *be an abelian group and* $b \in G, b \neq 0$. *Let* $n \in \mathbb{Z}, n \geq 1$. *There is a finite* k*-coloring* c *of* G *s.t. the equation*

$$(x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n) = b \qquad (1)$$

has no solution with $c(x_i) = c(y_i)$ *for every* $i \in \{1, ..., n\}$, *where*

$$k = \begin{cases} 2n & \text{if } \operatorname{ord}(b) = \infty \text{ or } 2 | \operatorname{ord}(b) \\ \left\lceil \frac{2np}{p-1} \right\rceil & \text{if } \operatorname{ord}(b) \text{ is odd and } p \text{ is the largest prime divisor of } \operatorname{ord}(b) \end{cases}.$$

Call these solutions *pairwise monochromatic* solutions.

Theorem (Straus, 1975)

Let $(G, +, 0)$ be an abelian group, $b \in G, b \neq 0$ and $n \in \mathbb{Z}, n \geq 1$. Let $f_1, ..., f_n$ be arbitrary mappings from G to G, with $m \leq n$ of them being distinct. There is a finite k-coloring c of G s.t. the equation

$$(f_1(x_1) - f_1(y_1)) + (f_2(x_2) - f_2(y_2)) + \cdots + (f_n(x_n) - f_n(y_n)) = b \tag{2}$$

has no solution with $c(x_i) = c(y_i)$ for every $i \in \{1, ..., n\}$, where

$$k = \begin{cases} (2n)^m & \text{if } \operatorname{ord}(b) = \infty \text{ or } 2 | \operatorname{ord}(b) \\ \left\lceil \frac{2np}{p-1} \right\rceil^m & \text{if } \operatorname{ord}(b) \text{ is odd and } p \text{ is the largest prime divisor of } \operatorname{ord}(b) \end{cases}.$$

Computability-theoretic Questions

Does Straus' Theorem hold computably i.e. is it true that for every computable abelian group, $n \in \mathbb{N}, n \geq 1$ and every $b \in G, b \neq 0$, there is a computable k-coloring that satisfies the theorem? If not, what is the best computability-theoretic bound for this problem?

Definition

A group $(G, +)$ is called a **computable group** if G is a computable set and the function $(x, y) \mapsto x + y$ is computable.

PA Degrees

Turing degrees - measure of complexity of sets in computability theory. Two sets have the same Turing degree if they are "equally hard to compute".

Definition

A set X has **PA degree** if every computable infinite binary tree has an X-computable path.

Proposition

A degree is PA if and only if it is the degree of a complete extension of Peano Arithmetic.

Proposition

A degree is PA if and only if it computes a total $\{0,1\}$*-valued extension of the* $\{0,1\}$*-valued function* $e \mapsto \Phi_e(e)$.

The Computability of Straus' Theorem

Theorem (L., 2024)

There is a computable abelian group $(G, +, 0)$ *and an element* $b \in G, b \neq 0$ *for which every 2-coloring with no monochromatic solutions to*

$$x_1 - y_1 = b$$

has PA degree.

Proposition

For every computable group, there is a 2-coloring of PA degree with no monochromatic solutions to $x_1 - y_1 = b$*.*

Hence, PA degrees are *the best possible bound.*

Our theorem gives the best possible result of this sort:

Proposition

Let $n > 2, n \in \mathbb{Z}$. Then for every computable group, there is a computable n-coloring with no monochromatic solutions to $x_1 - y_1 = b$.

Sketch of the Proof

- Build a computable group $(G, +, 0)$ and a monomorphism $h : G \to \mathbb{Z}^{(\omega)}$ in stages.
- Ensure that for every e, there are x_e and y_e s.t.
  - if $\Phi_e(e) \downarrow = 0$ then $h(y_e) = h(x_e) + kh(b)$ for k even,
  - if $\Phi_e(e) \downarrow = 1$ then $h(y_e) = h(x_e) + kh(b)$ for k odd.
- If c is a 2-coloring of G with no monochromatic solutions to $x - y = b$, then the function

$$g(e) = \begin{cases} 0, & \text{if } c(x_e) = c(y_e) \\ 1, & \text{if } c(x_e) \neq c(y_e) \end{cases}$$

is a $\{0, 1\}-$valued extension of the function $e \mapsto \Phi_e(e)$.

Does combinatorial complexity imply computability-theoretic complexity?

Theorem (L., 2024)

Let $n \in \mathbb{Z}, n \geq 2$. There is a computable group $(G, +, 0)$ and an element $b \in G, b \neq 0$ such that, if c is a $2n$-coloring of G for which the equation

$$(x_1 - y_1) + ... + (x_n - y_n) = b$$

has no pairwise monochromatic solutions and there is an $m > 0$ such that for every $k \in \mathbb{Z}$, all the mkb are colored with the same color, then c has PA degree.

Reverse Mathematics

RCA_0 - usual base system in reverse mathematics (roughly corresponding to computable mathematics)

Lemma (Weak König's Lemma)

Every infinite binary tree has an infinite path.

Weak König's Lemma does not hold computably.

$WKL_0 = RCA_0 +$ Weak König's Lemma

The Reverse Mathematics of Straus' Theorem

Theorem (L., 2024)

Over RCA_0, WKL_0 *is equivalent to the following statement:*

For every abelian group G *and every element* $b \in G, b \neq 0$, *there is a* $k-$*coloring of* G *for which the equation* $x - y = b$ *has no monochromatic solutions, where* $k = 2$ *if* $\mathrm{ord}(b) = \infty$ *or* $\mathrm{ord}(b)$ *is even and* $k = 3$ *if* $\mathrm{ord}(b)$ *is odd.*

Sketch of the Proof

- The backwards direction holds by our previous theorem.
- For the forward direction, consider the graph on m vertices $a_0, a_1, ..., a_{m-1}$, where we say that a_i and a_j are connected iff $|a_i - a_j| = b$.
- If $\text{ord}(b)$ is infinite, this is a graph with no loops.
- If $\text{ord}(b)$ is even/odd, there might be loops on an even/odd number of vertices, respectively.
- In all cases, we can color the graph with k colors so that no two neighboring vertices have the same color.

The Path Ahead

1. The reverse mathematics of the full Straus' theorem.
2. Computability-theoretic analysis of Leader/Russell theorem.
3. The reverse mathematics of the fact that the DNC_k degrees coincide with the PA degrees.
4. What happens in the case $b = 0$?
5. The nonlinear case.

Thank you!

Main Results

Theorem (L., 2024)

There is a computable abelian group $(G, +, 0)$ and an element $b \in G, b \neq 0$ for which every 2-coloring with no monochromatic solutions to

$$x_1 - y_1 = b$$

has PA degree.

Theorem (L., 2024)

Let $n \in \mathbb{Z}, n \geq 2$. There is a computable group $(G, +, 0)$ and an element $b \in G, b \neq 0$ such that, if c is a $2n$-coloring of G for which the equation

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has no pairwise monochromatic solutions and there is an $m > 0$ such that for every $k \in \mathbb{Z}$, all the mkb are colored with the same color, then c has PA degree.

Theorem (L., 2024)

Over RCA_0, WKL_0 is equivalent to the following statement:
For every abelian group G and every element $b \in G, b \neq 0$, there is a $k-$coloring of G for which the equation $x - y = b$ has no monochromatic solutions, where $k = 2$ if $\mathrm{ord}(b) = \infty$ or $\mathrm{ord}(b)$ is even and $k = 3$ if $\mathrm{ord}(b)$ is odd.