Countable Ordered Groups and Weihrauch Reducibility

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Outline

Reverse mathematics

- Reverse mathematics studies the axiomatic strength needed to prove theorems of ordinary mathematics over a weak base theory.
- It is usually studied using subsystems of second order arithmetic.
- Although we work in the language of arithmetic, other objects such as well-orders and groups can be coded as appropriate subsets of $\mathbb{N}$. We call such coding an ω-presentation or an ω-copy.

Big five

1. RCA_0: $\mathsf{PA}^- + \mathsf{I}\Sigma^0_1 + \Delta^0_1$-$\mathsf{CA}$
2. WKL_0: RCA_0 + some form of weak könig lemma
3. ACA_0: RCA_0 + arithmetical comprehension axiom
4. ATR_0: ACA_0 + arithmetical transfinite recursion scheme
5. Π^1_1-CA_0: RCA_0 + Π^1_1-comprehension axiom

Theorem

The following are equivalent over RCA_0*:*

1. Π^1_1-CA_0
2. *For any sequence of trees* $\langle T_k : k \in \mathbb{N} \rangle$, $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$, *there exists a set* X *such that* $\forall k(k \in X \leftrightarrow T_k$ *has a path*$)$.

Order Type of Countable Ordered Groups

Theorem (Maltsev, 1949)

The order type of a countable ordered group is $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, where α is an ordinal and $\varepsilon = 0$ or 1.

Definition

An ordered group is a pair $(G, \leq_G)$, where G is a group, $\leq_G$ is a linear order on G, and for all $a, b, g \in G$, if $a \leq b$ then $ag \leq bg$ and $ga \leq gb$.

For $\mathbb{Z}^\alpha$ and products of linear orders, we order by the rightmost coordinate on which two elements differ.

Order Type of Countable Ordered Groups

Theorem (Solomon, 2001)

The following are equivalent under RCA_0:

1. Π^1_1-CA_0
2. *If* G *is a countable ordered group, there is a well-order* α *and* $\varepsilon \in \{0, 1\}$ *such that* $\mathbb{Z}^\alpha \mathbb{Q}^\varepsilon$ *is the order type of* G.
3. *If* G *is a countable abelian ordered group, there is a well-order* α *and* $\varepsilon \in \{0, 1\}$ *such that* $\mathbb{Z}^\alpha \mathbb{Q}^\varepsilon$ *is the order type of* G.

Theorems as problems

Statements like the ones in the previous theorem can be written as follows:

$$(\forall x \in X)(\exists y \in Y)[\varphi(x) \to \psi(x, y)].$$

We can naturally translate it to a computational problem, i.e., given an input x such that $\varphi(x)$, the output is y such that $\psi(x, y)$.

For our purposes, we consider problems on Baire space $\mathbb{N}^{\mathbb{N}}$, i.e., relations $f \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, or equivalently partial multi-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.

Remark: for many statements, there could be multiple natural ways to phrase them as a computational problem.

Weihrauch reducibility

Definition

Let f, g be partial multi-valued functions on Baire space. f is Weihrauch reducible to g, denoted $f \leq_W g$ if there are computable Φ, Ψ on Baire space such that:

- given $p \in \mathrm{dom}(f)$, $\Phi(p) \in \mathrm{dom}(g)$, and
- given $q \in g(\Phi(p))$, $\Psi(p, q) \in f(p)$.

Φ, Ψ are called forward functional and backward functional respectively.

$$\begin{array}{ccc} p & \xrightarrow{\Phi} & \Phi(p) \\ \downarrow f & & \downarrow g \\ f(p) & \xleftarrow{\Psi(p,\cdot)} & q \end{array}$$

Algebraic operations

Definition

Among the many operations in Weihrauch degrees, we will use the following:

1. compositional product $f * g$: allows us to "use" g first, then f
2. product $f \times g$: allows us to use both f and g in parallel
3. finite parallelization f^*: allows us to use f finitely many times in parallel
4. parallelization $\widehat{f}$: allows us to use f countably many times in parallel

Big five and Weihrauch reducibility

There are imperfect analogies between the big five and Weihrauch degrees:

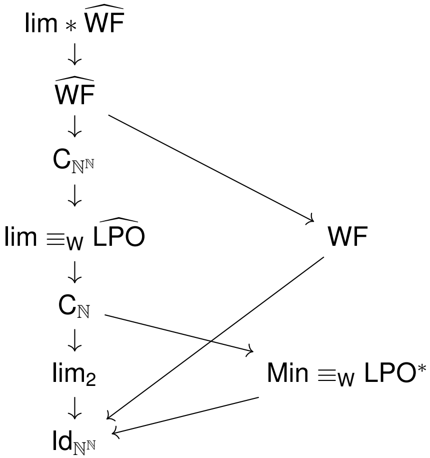
1. RCA_0: $\mathsf{Id}_{\mathbb{N}^\mathbb{N}}$.
2. WKL_0: $\mathsf{C}_{2^\mathbb{N}}$.
3. ACA_0: iterations of lim.
4. ATR_0: many candidates, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, etc.
5. Π^1_1-CA_0: $\widehat{\mathsf{WF}}$.

Definition

$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: Given an ill-founded tree in Baire space, find a path through it.

WF: Given a tree in Baire space, tell whether it is well-founded.

A small part of the zoo



Weihrauch problems

We have choices to make when formulating Malstev Theorem into Weihrauch problems.

1. $\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$ and $\mathsf{AOG} \mapsto \alpha\varepsilon\mathsf{f}$: given a countable (abelian) ordered group G, output the ordinal α and $\varepsilon \in \{0, 1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ with an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.
2. $\mathsf{OG} \mapsto \alpha\varepsilon$

As part of our analysis, we also consider the following problems:

3. $\mathsf{OG} \mapsto \varepsilon$
4. $\mathsf{OG} \mapsto \alpha$
5. $\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f}$: given a countable ordered group G with the ordinal α and $\varepsilon \in \{0, 1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, output an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.

Output everthing

Proposition

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \geq_{\mathsf{W}} \widehat{\mathsf{WF}}$

What about the other direction? Suppose we are given a computable ω-copy of the group, the output we need includes an ω-copy of the ordinal α. It is natural to ask: what can α be?

What can α be?

Theorem

If a computable ordered group has order type $\mathbb{Z}^\alpha$, then α is computable.

Proof.

Idea: Build a tree $T \in \mathbb{N}^{<\mathbb{N}}$ by trying to embed $\mathbb{Q}_2 \cap [0,1]$ into the group. This tree has rank $\omega\alpha$. $\square$

Lemma

Given two trees $T_0, T_1 \subseteq \omega^{<\omega}$, if there is a map f from T_1 to T_0 such that $f^{-1}(\sigma)$ has a finite rank as a partial order for any σ, $\mathrm{rk}(f^{-1}(\sigma)) \leq c_l$ for all σ of length l for some constant c_l, and $f(\sigma) \preccurlyeq f(\tau)$ when $\sigma \preccurlyeq \tau$, then $\mathrm{rk}(T_1) \leq^+ \mathrm{rk}(T_0)$. In particular, $\mathrm{rk}(T_1) \leq \mathrm{rk}(T_0)$ when the latter is a limit.

What can α be?

Theorem

If a computable ordered group has order type $\mathbb{Z}^\alpha\mathbb{Q}$, then $\alpha \leq \omega_1^{CK}$.

Proof.

For each positive element g in the group, we build a tree that tries to embed $\mathbb{Q}_2 \cap [0,1]$ into the interval between the identity e and g. Then, $\alpha \leq \omega_1^{CK}$. Otherwise, there is a g mapped to $(0,\ldots,0,1,0,\ldots)$ where 1 is at the ω_1^{CK} position. □

Theorem

There exists a computable countable ordered group with order type $\mathbb{Z}^{\omega_1^{CK}}\mathbb{Q}$.

Proof.

There is a group with order type $\mathbb{Z}^{H}$ where $H \cong \omega_1^{CK}(1+\mathbb{Q})$ is the Harrison linear order. □

Remark: this can also be seen by the Gandy Basis Theorem

One point information

Proposition

$\mathsf{OG} \mapsto \varepsilon \equiv_{\mathsf{W}} \mathsf{WF}$

Proof.

"$\leq_{\mathsf{W}}$" Build a tree T by trying to embed the rationals into the order. Fix a list of rational numbers $\{q_i\}_{i<\omega}$. Define T as follows: any σ is in T if and only if the map from q_i to $\sigma(i) \in G$ for $i < |\sigma|$ preserves the order. Then, T is well-founded if and only if $\varepsilon = 0$.

"$\geq_{\mathsf{W}}$" Follows from the proof of the reverse math result. □

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \leq_{\mathsf{W}} \widehat{\mathsf{WF}}$

Proof.

"$\leq_{\mathsf{W}}$" Assume that the input of $\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$ is computable and $\varepsilon = 1$.

- The forward functional simply makes countably many trees so that the output of $\widehat{\mathsf{WF}}$ is Π^1_1-complete.
- Using this Π^1_1-complete set, the backward functional can identify if two elements are in the same $\mathbb{Z}^\alpha$ copy, and build α via a sequence of approximations.
- It will build the partial order-preserving map according to the current guess of α.
- All the questions the backward functional ask in order to do so are Π^1_1.

□

On the side

If we do not output the isomorphism f, then our Weihrauch problems turn out to be incomparable with fairly weak problems.

Proposition

- $\mathsf{OG} \mapsto \alpha \not\geq_{\mathsf{W}} \mathsf{lim}_2$.
- $\mathsf{OG} \mapsto \alpha\varepsilon \not\geq_{\mathsf{W}} \mathsf{lim}_2 \times \mathsf{lim}_2$.

Definition

$\mathsf{lim}_2 :\subseteq 2^{\mathbb{N}} \to 2$ is the limit operation on $\{0, 1\}$.

We will show this by looking at the first-order parts of these problems.

First-order part

Definition (Dzhfarov, Solomon, & Yokoyama)

A problem P is first-order if the codomain of P is $\mathbb{N}$.
We let $\mathcal{F}$ denote the class of all first-order problems.

For every Weihrauch problem P, there is a first order problem ${}^1\mathsf{P}$ such that ${}^1\mathsf{P} \equiv_{\mathsf{W}} \sup_{\leq_{\mathsf{W}}}\{\mathsf{Q} \in \mathcal{F} : \mathsf{Q} \leq_{\mathsf{W}} \mathsf{P}\}$. It is called the first-order part of P.

First-order part

Definition

$$\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0,1\},\ \mathsf{LPO}(p) = \begin{cases} 0 \text{ if } (\exists k)[p(k) = 0], \\ 1 \text{ otherwise.} \end{cases}$$

Theorem (Brattka, Gherardi, Marcone, & Pauly)

- $\mathsf{LPO}^* \equiv_W \mathsf{Min}$.
- $\mathsf{lim}_2 |_W \mathsf{LPO}^*$.
- $\mathsf{lim}_2, \mathsf{LPO}^* <_W \mathsf{C}_{\mathbb{N}}$.

Proposition

- ${}^1\mathsf{OG} \mapsto \alpha \equiv_W \mathsf{LPO}^*$.
- ${}^1\mathsf{OG} \mapsto \alpha\varepsilon \equiv_W \mathsf{LPO}^* \times \mathsf{WF}$.

Corollary

$\mathsf{OG} \mapsto \alpha \not\geq_W \mathsf{lim}_2$. $\mathsf{OG} \mapsto \alpha\varepsilon \not\geq_W \mathsf{lim}_2 \times \mathsf{lim}_2$.

How much is needed to output α?

Proposition

$\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f} \leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

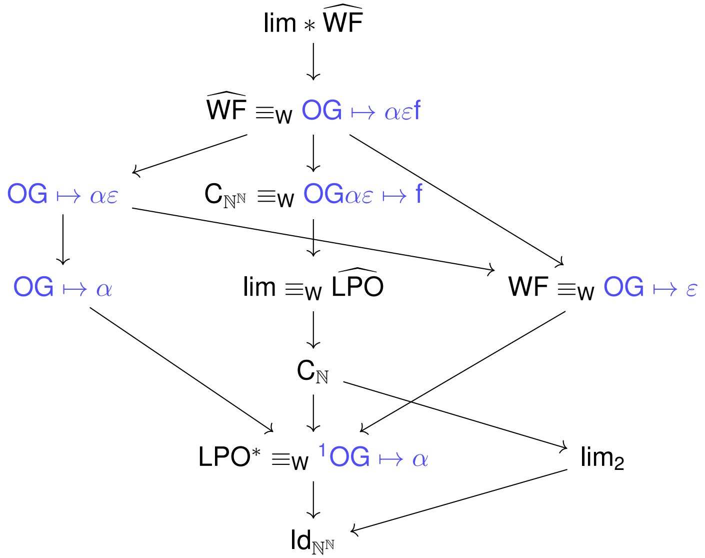
Idea: build a back-and-forth tree whose paths represent isomorphism between $\mathbb{Z}^{\alpha}\mathbb{Q}^{\varepsilon}$ and the group.

Corollary

$\mathsf{OG} \mapsto \alpha\varepsilon \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \mathsf{WF}$

$\mathsf{OG} \mapsto \alpha \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

A slightly bigger part of the zoo



References

- Chris J Ash and Julia Knight, *Computable structures and the hyperarithmetical hierarchy*, Elsevier, 2000.
- Vasco Brattka, Matthew de Brecht, and Arno Pauly, *Closed choice and a uniform low basis theorem*, Ann. Pure Appl. Log. **163** (2010), 986–1008.
- Vasco Brattka, Guido Gherardi, and Alberto Marcone, *The bolzano–weierstrass theorem is the jump of weak kőnig's lemma*, Annals of Pure and Applied Logic **163** (2012), no. 6, 623–655, Computability in Europe 2010.
- Vittorio Cipriani, Alberto Marcone, and Manlio Valenti, *The weihrauch lattice at the level of Π^1_1-CA_0: the cantor-bendixson theorem*, 2022.
- Damir D Dzhafarov, Reed Solomon, and Keita Yokoyama, *On the first-order parts of problems in the weihrauch degrees*, Computability (2023), no. Preprint, 1–13.
- Reed Solomon, Π^1_1-CA_0 *and order types of countable ordered groups*, The Journal of Symbolic Logic **66** (2001), no. 1, 192–206.

Thank You!