Enumeration Weihrauch reducibility

Mariya I. Soskova

University of Wisconsin–Madison

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Joint work with Alice Vidrine

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Problems and Weihrauch reduction

Weihrauch reduction is a way of comparing the computational strength of various "problems", represented as partial multifunctions on ω^ω.

We may think of Weihrauch reduction $f \leq_W g$ as a computation of values of f, given the ability to query g as an oracle *exactly once*.

Definition

Let f and g be multifunctions on ω^ω. We say that $f \leq_W g$ if there are Turing operators (Φ, Ψ) such that

1. for every $\alpha \in \operatorname{dom} f$ we have that $\Phi(\alpha) \in \operatorname{dom} g$
2. for every $\alpha \in \operatorname{dom} f$ and $\beta \in g(\Phi(\alpha))$, we have $\Psi(\alpha \oplus \beta) \in f(\alpha)$.

Enumeration reducibility

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: a set of natural numbers A is enumeration reducible to another set B if every enumeration of B uniformly computes an enumeration of A.

Here an enumeration of a set X is a function $e_X \in \omega^\omega$ with $\mathrm{ran}(e_X) = X$.

Definition

$A \leq_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists u)\ \langle n, u \rangle \in W \text{ and } D_u \subseteq B\},$$

where D_u is the uth finite set in a canonical enumeration.

The c.e. set W gives rise to an operator, which we call an *enumeration operator*.

eW-problems and eW-reductions

An eW-problem is a partial multifunction from $\mathcal{P}(\omega)$ to itself.

Definition

Given problems f, g, we say that $f \leq_{eW} g$ if there are enumeration operators Γ, Δ such that

1. for every $A \in \operatorname{dom} f$ we have $\Gamma(A) \in \operatorname{dom} g$,
2. for every $A \in \operatorname{dom} f$ and $X \in g(\Gamma(A))$ we have $\Delta(A \oplus X) \in f(A)$.

In other words, eW-reduction is just Weihrauch reduction where the problems operate on $\mathcal{P}(\omega)$, and enumeration reducibility is used in place of Turing.

Motivation for enumeration reducibility

First, enumeration operators have a robust computational structure, and their use to study problems-as-multifunctions is intrinsically interesting.

Enumeration reducibility gives a notion of computation that works on *positive* information, and was introduced several times by various authors who wanted to extend Turing reducibility to partial functions.

Define $\mathbb{P}$ to be $\mathcal{P}(\mathbb{N})$ equipped with a binary operation (called *application*) given by

$$AB = \{n : \exists m(\langle n, m \rangle \in A \wedge D_m \subseteq B)\}.$$

We read application as left associative (e.g. ABC means $(AB)C$.)

The algebra $\mathbb{P}_\sharp$ is the substructure of $\mathbb{P}$ consisting of the c.e. sets (i.e. enumeration operators).

Dana Scott proved that both these algebras can interpret the untyped lambda calculus.

Motivation from category theory

Second, they are related to an under-studied *realizability topos*.

A topos is a category theoretic model of a kind of intuitionistic set theory. Realizability toposes are built from a model of computation (see van Oosten 2008 for an overview of the area).

Kihara wrote a paper "Lawvere-Tierney topologies for computability theorists". In it he shows a strong relationship between a generalized form of Weihrauch reduction and the lattice of subtoposes of the effective topos. He uses computability to solve open problems about this lattice. For instance, he shows that there exists no minimal LT topology which is strictly above the identity topology on the effective topos.

There is a realizability topos where the underlying model of computation is enumeration reducibility—the topos $\mathsf{RT}(\mathbb{P}, \mathbb{P}_\sharp)$. It is not hard to see that there is a similar relationship between a generalized form of eW-reducibility and LT topologies on $\mathsf{RT}(\mathbb{P}, \mathbb{P}_\sharp)$.

Why not just use Weihrauch reducibility?

Weihrauch reducibility was introduced to study functions on *represented spaces*: pairs (X, δ_X) where $\delta_X : \omega^\omega \to X$ is a partial surjection.

Example

A set of natural numbers A can be represented by any enumeration e_A of A.

We can represent a (multi)function f on $\mathcal{P}(\omega)$ by a function $\hat{f}$ on ω^ω that maps enumerations of the set A to the set of enumerations of sets in $f(A)$.

We can say that f is reducible to g if $\hat{f} \leq_W \hat{g}$.

Indeed, if $f \leq_{eW} g$ then $\hat{f} \leq_W \hat{g}$.

Problem: The converse need not be true! Different enumerations of A may be mapped to enumerations of different instances of g, which in turn produce different solutions to $f(A)$.

Why not just use Weihrauch reducibility?

Let's turn this into an actual example.

Example

Consider the function f that maps a set A to (the graph of) some total function T such that $A \leq_e T$.

Consider the identity function id that maps a set A to itself.

Then $f \not\leq_{eW} id$ even though $\hat{f} \leq_W \hat{id}$.

- $\hat{f} \leq_W \hat{id}$ because if e_A is an enumeration of A then it computes (an enumeration of the graph of) a total function T such that $A \leq_e T$, namely $e_A = T$;
- There are sets A that are not enumeration equivalent to (the graph of) any total function, e.g. generic or sufficiently random sets.
- If A is such then for any enumeration operator Γ we have that $\Gamma(A) \oplus A \leq_e A$. So $\Lambda(\Gamma(A) \oplus A)$ cannot be a solution to $f(A)$.

Basic results about eW-reduction

The eW degrees form a lattice

By identifying problems that are reducible to each other we get the eW degrees.

Proposition

The eW degrees form a distributive lattice with join and meet as in the Weihrauch degrees.

The least upper bound of f and g is $f \cup g$ with instances $\{0\} \times \operatorname{dom} f \cup \{1\} \times \operatorname{dom} g$ and $(f \cup g)(0, X) = \{0\} \times f(X)$ and $(f \cup g)(1, Y) = \{0\} \times g(Y)$.

The greatest lower bound of f and g is $f \cap g$ with instances $\operatorname{dom} f \times \operatorname{dom} g$ and $(f \cap g)(X, Y) = \{0\} \times f(X) \cup \{1\} \times g(Y)$.

The eW degrees extend the Weihrauch degrees

Turing reducibility can be captured using $\leq_e$: if $\alpha, \beta \in \omega^\omega$ then $\alpha \leq_T \beta$ if and only if $G_\alpha \leq_e G_\beta$.

We call sets (such as G_α) that can enumerate their complements *total*. Thus $\mathcal{D}_T$ lives inside $\mathcal{D}_e$ as the *total degrees*.

This extends to the Weihrauch setting:

Proposition

There is an embedding of the Weihrauch degrees into the eW degrees.

- We map a function f on ω^ω to a function $\tilde{f}$ on $\mathcal{P}(\omega)$ with instances all G_α where $\alpha \in \operatorname{dom} f$ and setting $G_\beta \in \tilde{f}(G_\alpha)$ for every $\beta \in f(\alpha)$.
- It is straightforward to check that $f \leq_W g$ if and only if $\tilde{f} \leq_{eW} \tilde{g}$.

The eW degrees extend the Weihrauch degrees

Proposition

This mapping is not surjective.

- If X is 1-generic and $G_\alpha \leq_e X$ for some $\alpha \in \omega^\omega$ then α is computable.
- Let $g :\subseteq \mathbb{P} \rightrightarrows \mathbb{P}$ have domain consisting of a single 1-generic set X.
- Suppose that there is a Weihrauch problem f with $g \equiv_{eW} \tilde{f}$.
- From $g \leq_{eW} \tilde{f}$ we see that the domain of $\tilde{f}$ must have a computable element C.
- But now a reduction $\tilde{f} \leq_{eW} g$ must send C to X, even though any set e-below C is c.e. which is incompatible with 1-genericity.

Sample problems

Choice problems

The *choice problem* for a represented topological space X, C_X, maps a nonempty closed subset A of X to a member of A. We can represent such problems differently in the Weihrauch setting and in the enumeration Weihrauch setting.

Example

Consider C_{2^ω}

- In the Weihrauch setting we represent a closed set F via an enumeration of a set $U \subseteq 2^{<\omega}$ with $F = 2^\omega \setminus [U]^\prec \neq \emptyset$. A member of F can be represented by itself.
- In the enumeration Weihrauch case we represent F by U. A member of F can be represented by the set of its initial segments.

We can compare both of these problems in the eW setting by considering $\widetilde{C_{2^\omega}}$ and C_{2^ω}.

C_- and $\widetilde{C_-}$

The eW versions of choice problems tend to fall strictly above their Weihrauch counterparts.

Proposition

$\widetilde{C_{2^\omega}} <_{eW} C_{2^\omega}$.

- The reduction is straightforward.
- Suppose $C_{2^\omega} \leq_{eW} \widetilde{C_{2^\omega}}$ via the enumeration operators Γ and Λ.
- A set U is called *codable* if the closed set $F = 2^\omega \setminus [U]^\prec$ is nonempty and every member of F can enumerate U.
- Miller and co-authors proved that there are non-total codable sets U and that the set of total sets below U is a Scott set S.
- Since all instances of $\widetilde{C_{2^\omega}}$ are total sets $\Gamma(U) = \alpha$ is (the graph of) some total function in S and so $\widetilde{C_{2^\omega}}(\alpha)$ has a solution β in S.
- But then $\Lambda(U \oplus \beta) \leq_e U$ and cannot be a member of the closed set coded by U.

$\mathsf{C}_{\mathbb{N}}$ & $\mathsf{UC}_{\mathbb{N}}$

The first separation that develops in the eW setting concerns $\mathsf{C}_{\mathbb{N}}$ and its restriction to singletons, $\mathsf{UC}_{\mathbb{N}}$.

Fact. $\widetilde{\mathsf{C}_{\mathbb{N}}} \equiv_{eW} \widetilde{\mathsf{UC}_{\mathbb{N}}}$.

Proposition

$\mathsf{UC}_{\mathbb{N}} <_{eW} \mathsf{C}_{\mathbb{N}}$.

- The reduction is immediate from the fact that unique choice is just a restriction of closed choice.
- If $\mathsf{C}_{\mathbb{N}} \leq_{eW} \mathsf{UC}_{\mathbb{N}}$ witnessed by (Γ, Λ) then $\Gamma(\emptyset) = \mathbb{N} \setminus \{k\}$ is the complement of a singleton.
- For any other $A \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$, $\Gamma(A) \supseteq \Gamma(\emptyset)$ and is the complement of a singleton so $\Gamma(A) = \Gamma(\emptyset)$.
- But then $\Lambda(\emptyset \oplus \{k\}) = \{n\}$ is a subset of any $\Lambda(A \oplus \{k\})$ including those A that contain n.

$\mathsf{C}_{2^{\mathbb{N}}}$ and WKL

WKL takes as input an infinite binary tree T and produces a path in $[T]$.

Fact. $\widetilde{\mathsf{C}_{2^{\mathbb{N}}}} \equiv_{eW} \widetilde{\mathsf{WKL}}$.

Proposition

$\mathsf{C}_{2^{\mathbb{N}}} \;|_{eW}\; \mathsf{WKL}$

Suppose in each case below that (Γ, Δ) witnesses the specified reduction.

1. ($\mathsf{C}_{2^{\mathbb{N}}} \not\leq_{eW} \mathsf{WKL}$). We have that $\emptyset$ is an instance of $\mathsf{C}_{2^{\mathbb{N}}}$. So $\Gamma(\emptyset)$ is a c.e. tree that is a subtree of $\Gamma(A)$ for any other instance A. $\Gamma(\emptyset)$ has Δ^0_3 solutions which won't help sufficiently complicated A to enumerate a member.
2. ($\mathsf{WKL} \not\leq_{eW} \mathsf{C}_{2^{\mathbb{N}}}$). Consider the full tree $T = 2^{<\omega}$, and the closed set represented by $\Gamma(T)$. If S is any other tree then $\Gamma(S) \subseteq \Gamma(T)$ and so solutions to $\Gamma(T)$ are solutions to $\Gamma(S)$.

In fact, for very similar reasons, we even have $\mathsf{WKL} \;|_{eW}\; \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$!

A zoo in the making

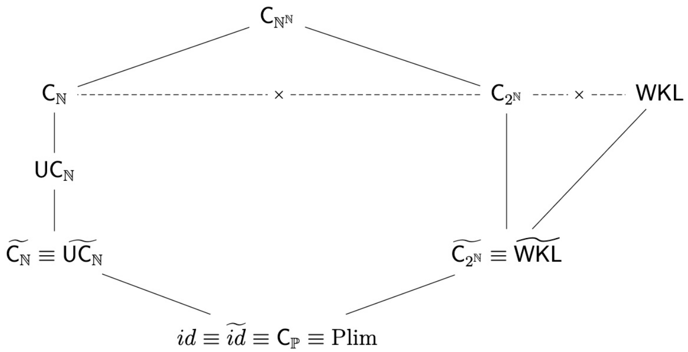

Figure: Reducibilities among choice problems and König's lemmas

The problem id

Definition

The problem id is the identity function on $\mathbb{P}$.

- $id \leq_{eW} f$ if and only if f has a c.e. instance.
- $f \leq_{eW} id$ if and only if there is an enumeration operator Γ such that for all $A \in \operatorname{dom} f$, $\Gamma(A) \in f(A)$.
- So if $f, g \leq_{eW} id$ then $f \leq_{eW} g$ if an only if there is an enumeration operator Γ so that for every element $A \in \operatorname{dom} f$ we have $\Gamma(A) \in \operatorname{dom} g$.

The Dyment lattice

The Dyment lattice was introduced by Dyment (later Skvortsova) in 1976 and studied by her and Andrea Sorbi. It is the enumeration analog of the Medvedev lattice.

Definition

Let $\mathcal{A}$, $\mathcal{B}$ be set of sets of natural numbers. We say that $\mathcal{A} \leq_D \mathcal{B}$ (*Dyment reducible to*) if there is an enumeration operator Γ so that for every element $B \in \mathcal{B}$ we have $\Gamma(B) \in \mathcal{A}$.

So the interval $[\emptyset, id]$ in the eW degrees is isomorphic to the reverse ordering on the Dyment lattice.

(Echoing that the interval $[\emptyset, id]$ in the W degrees is isomorphic to the reverse ordering on the Medvedev lattice.)

The structure of the eW degrees

The W / eW relationship

Question

Are the Weihrauch and eW degrees non-isomorphic? Is there a first order difference between them?

The answer to this turns out to be affirmative.

1. The Dyment lattice is not elementary equivalent to the Medvedev lattice.
2. id is definable in the the Weihrauch degrees and in the eW degrees by the same formula.

Definability in the Dyment lattice

In the Medvedev lattice we have a definable copy of the Turing degrees: $\deg_T(A)$ is mapped to $\deg_M(\{A\})$. The degree of $\{(\Psi, B) : \ \Psi^B = A \;\&\; B >_T A\}$ is a *strong maximal cover* of $\deg_M(\{A\})$.

Theorem (Dyment)

A Medvedev degree is Turing if and only if it has a strong maximal cover.

In the Dyment lattice we have a copy of the enumeration degrees $\mathcal{D}_e$: $\deg_e(A)$ is mapped to $\deg_D(\{A\})$.

Theorem (Dyment)

1. There are sets A and B such that $A \leq_e B$ but $\{A\} \not\leq_D \{A, B\}$.
2. A Dyment degree is enumeration if and only if it has a strong maximal cover.

The e-degrees are downwards dense, but the Turing degrees are not.

Definability of id

Theorem (Lempp, Miller, Pauly, Soskova, and Valenti 2023)

In the Weihrauch degrees id is the greatest degree that is a strong minimal cover.

We will see that the same definition works in the eW setting.

Lemma 0. id is a strong minimal cover of id restricted to non-c.e sets.

Lemma 1. If f is a problem whose eW degree is a strong minimal cover then it contains a problem with singleton domain.

Lemma 2. If f has singleton domain and is not below id then f is not a strong minimal cover.

Lemma 1

Lemma 1. If f is a problem whose eW degree is a strong minimal cover then it contains a problem with singleton domain.

- Suppose f is a strong minimal cover of g.
- First we build h with finite domain $\{(n_0, X_0), \ldots (n_k, X_k)\}$ where $X_i \in \operatorname{dom} f$ and $n_i \neq n_j$ and with $h(n_i, X_i) = f(X_i)$.
- At even stages $s = 2e$ we check whether $f \leq_{eW} h_s$ via (Γ_e, Λ_e) and if not we preserve the difference: If $\Gamma_e(X) = (k, Y)$ that is not in the domain of h_s then we promise never to use k again.
- As odd stages $s = 2e + 1$ we extend h_s to h_s^* allowing (n, X) in the domain for all unused and nonforbidden n. Since h^* is equivalent to f it is not reducible to g via (Γ_e, Λ_e). We preserve this difference by adding a problematic instance to the domain of h_s.
- We must stop at some even stage or else we contradict that f is a strong minimal cover of g.

Lemma 1 cont.

Lemma 1. If f is a problem whose eW degree is a strong minimal cover then it contains a problem with singleton domain.

- We have $f \equiv_{eW} h$ where $\operatorname{dom} h = \{Y_0, \ldots Y_k\}$ where $Y_i \cap Y_j = \emptyset$ for $i \neq j$.
- Since $h \restriction \{Y_0\} \cup h \restriction \{Y_1, \ldots Y_k\} = h$ and h is a strong minimal cover one of the two sides must be equivalent to h.
- Inductively we reduce h to a problem with singleton domain.

Lemma 2

Lemma 2. If f has singleton domain $\{A\}$ and is not below id then f is not a strong minimal cover.

- Suppose f is as above and $g <_{eW} f$.
- Build a set D so that $f \cap \chi_D <_{eW} f$ and $f \cap \chi_D \not\leq_{eW} g$.
- Since $f \not\leq_{eW} id$ we know that $f(A) \not\leq_e A$. Since every instance of $f \cap \chi_D$ has a computable solution, we know that $f \not\leq_{eW} f \cap \chi_D$.
- Suppose (Γ, Λ) are a threat to making $f \cap \chi_D \leq_{eW} g$.
- If for some n we have that $\Gamma(A, n) = Y$ is an instance of g and for every X to solution to $g(Y)$ we have that $\Lambda(A, n, X) = \{0\} \times f(A)$. Then $f \leq_{eW} g$ contradicting our choice of g.
- So fix n such that $D(n)$ is not yet determined. If $\Gamma(A, n) = Y$ is an instance of g then for some solution to $g(Y)$, say X we have that $\Lambda(A, n, X) = \{1\} \times \{i\}$. Set $D(n) = 1 - i$.

One more consequence of the definability of id

Lewis-Pye, Nies, and Sorbi and independently Shafer proved that the theory of the Medvedev degrees is computably isomorphic to third order arithmetic.

The definability of the total degrees in the enumeration degrees and the enumeration degrees in the Dyment degrees allow us to transfer this result.

Theorem

The theories of the Dyment degrees and of the eW degrees are each computably isomorphic to third order arithmetic.

The End: Thank you!

Some references:

1. Dyment, E. Z. (1976). Certain properties of the Medvedev lattice. *Mat. Sb. (N.S.)*, 101(143)(3):360-379. 455.
2. Kihara, T. (2023). Lawvere-Tierney topologies for computability theorists. *Trans. Amer. Math. Soc. Ser. B*, 10:48-85.
3. Lempp, S., Miller, J. S., Pauly, A., Soskova, M. I., and Valenti, M. (2023). Minimal covers in the Weihrauch degrees.
4. Lewis, A., Nies, A., Sorbi, A. (2009). The First Order Theories of the Medvedev and Muchnik Lattices. *LNCS*, 5635.
5. Scott, D. (1976). Data types as lattices. *SIAM J. Comput.*, 5(3):522-587. Semantics and correctness of programs.
6. Van Oosten, J. (2008). Realizability: An Introduction to its Categorical Side. *Studies in Logic and the Foundations of Mathematics*, Elsevier 152.