Between O and Ostaszewski

By

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pour les quatre dames: la mer, la France, ma fille et ma femme

Abstract

An O-space is a regular topology on ω_1 in which every open set is countable or has countable complement. An S-space is a right-separated regular space in which there are no uncountable discrete subspaces. An O-space is always an S-space and an Ostaszewski space is an O-space that is locally compact and countably compact. Eisworth and Roitman have shown that assuming CH there need not be any Ostaszewski spaces and ask if CH implies that there any locally compact O-spaces.

We survey what is known about axioms that follow from $\diamondsuit$ and the existence of various S-spaces. We also answer a question of Juhász about the existence of a $(\mathfrak{c}, \rightarrow)$-HFD without any additional set theoretic assumptions.

Assuming CH, we reduce the existence of a locally compact O-space to that of a first countable O-space. We give a partial answer to Eisworth and Roitman's question by describing a new class of S-spaces: almost left-separated spaces. A space is almost left-separated if it can be non-trivially covered by a $\subseteq^*$ chain of closed discrete sets. Using PID we show that CH can not prove the existence of a normal coherently almost left-separated O-space.

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Chapter 1

Introduction

1.1 Background

We shall mostly be concerned with topological spaces on the set ω_1, the first uncountable ordinal. Topological spaces have various separation properties. A topological space X is T_1 if for every $x \in X$ the set $\{x\}$ is closed. A topological space is *regular* or T_3 if it is T_1 and in addition for every closed K and $x \notin K$ there are disjoint open sets U, V such that $U \cap V = \varnothing$, $x \in U$, $K \subseteq V$.

Almost all of the spaces under consideration will, unless mentioned otherwise, be T_3. Additionally almost all will have various additional topological properties which we shall define subsequently define. The most frequently occurring additional topological properties we shall make use of are

Definition 1.1. *A space X of cardinality κ is* ***right(left)-separated in type*** *κ if there is a well ordering of $X = \{x_\alpha : \alpha < \kappa\}$ such that every initial segment, $\{x_\xi : \xi < \alpha\}$ for $\alpha < \kappa$, is open (closed).*

We shall almost always be concerned with spaces of cardinality $\aleph_1$ that are also right or left-separated in type ω_1. A space will be called **right(left)-separated** if it is right(left)-separated in type ω_1. Right and left-separated spaces are shown in Figures 1 and 2 respectively.

In the language of cardinal functions (see [17]) the height $h(X)$ of a space X is the maximum

Figure 1: A right-separated space

Figure 2: A left-separated space

cardinality of a right-separated subspace. So if X is right-separated then $h(X) = |X|$. The width $z(X)$ of a space X is the maximum cardinality of a left-separated subspace. So if X is left-separated then $z(X) = |X|$. We shall shortly investigate spaces for which $|X| = h(X) > z(X)$ and $|X| = z(X) > h(X)$.

1.2 Set Theoretic and Topological Notation

We shall assume throughout that the reader is familiar with the language of Set Theory. In particular familiarity with the standard notation found in that standard and excellent texts on Set Theory: Kunen's [22] and also Jech's [16]. In particular, all ordinals are Von Neumann ordinals: each of which is equal to the set of its predecessors.

In addition, for convenience, we shall employ the following definitions:

Definition 1.2. *For an ordinal* α*, define* $\mathrm{Lim}(\alpha)$ *to be the set of limit ordinals less than* α*. For any* κ *and any* $\alpha \in \mathrm{Lim}(\kappa)$ *let* $\alpha^+ = \inf(\mathrm{Lim}(\kappa) \setminus \alpha)$*. In the case that* $\alpha = \beta^+$ *for some* $\beta \in \mathrm{Lim}(\kappa)$*, we shall call* α *a* ***successor limit*** *ordinal, and if* $\alpha \in \mathrm{Lim}(\kappa)$ *is not a successor limit ordinal, it will be called a* ***limit of limit ordinals****. In general, define* $\mathrm{Lim}^0(\kappa)$ *to be* $\mathrm{Lim}(\kappa)$ *and*

$$\mathrm{Lim}^{k+1}(\kappa) = \{\alpha \in \mathrm{Lim}^k(\kappa) : \sup(\mathrm{Lim}^k(\kappa) \cap \alpha) = \alpha\}$$

For any subset $C \subseteq \kappa$ *define* $\mathrm{Lim}(C) = \{\xi < \kappa : \sup(C \cap \xi) = \xi\}$.

Definition 1.3. *For two sets* a, b *we write* $a \subseteq^* b$ *if* $a \setminus b$ *is finite and we write* $a =^* b$ *if* $a \subseteq^* b$ *and* $b \subseteq^* a$.

For any $A \in [\kappa]^{\leq \kappa}$, and an ordinal $\alpha < \kappa$ we shall write $A \restriction \alpha$ and A_α, depending on the context, to denote $A \cap \alpha$.

Given a well ordered set $(A, <)$ and $m < \omega$ a subset $K \subseteq [A]^m$ is called **separated** if for every pair of $a, b \in K$ it is the case that $\max_<(a) < \min_<(b)$ or $\max_<(b) < \min_<(a)$.

We shall use $\mathfrak{c}$ to denote the cardinality of the continuum, i.e. $\mathfrak{c} = 2^{\aleph_0}$.

Topological products will always denote the Tychonoff product. In particular, we shall write 2^{ω_1} to denote the ω_1 Tychonoff product of the two point discrete space. We could equivalently write 2^{ω_1} as $\{0, 1\}^{\omega_1}$.

A topological space X is **locally countable** if every $x \in X$ has a countable neighborhood. Similarly the space X is **locally finite** if every $x \in X$ has a finite neighborhood. A topological space X is called **discrete** if $\{x\}$ is open for every $x \in X$. Note that a T_1 space that is locally finite is discrete.

A family of subsets $\{A_\alpha : \alpha < \kappa\}$ is called **locally finite** if for every $x \in X$ there is some open U containing x such that

$$\{\alpha < \kappa : U \cap A_\alpha \neq \varnothing\}$$

is finite. A family of subsets $\{A_\alpha : \alpha < \kappa\}$ is called **discrete** if for every $x \in X$ there is some open U containing x such that

$$\{\alpha < \kappa : U \cap A_\alpha \neq \varnothing\}$$

has cardinality at most one.

A family of open subsets $\mathcal{B}$ of X is called a **base** for X if for every $x \in X$ and every U open with $x \in U$ there is some $V \in \mathcal{B}$ such that $x \in V \subseteq U$. A family of open subsets $\mathcal{B}$ of X is called a **π-base** for X if for every U open there is some $V \in \mathcal{B}$ such that $V \subseteq U$. A family of open subsets $\mathcal{B}$ of X is called a **local base** for X at x if for every U open with $x \in U$ there is some $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

If X is a topological space and $Y \subseteq X$ then Y is called a **subspace** of X if Y is given the topology generated by the open sets $\{Y \cap U : U \text{ is open in } X\}$.

A space X is called **first-countable** if every $x \in X$ has a countable local base. The space X is called **second-countable** if X has a countable base.

If for every $x \in X$ there is a neighborhood of x with compact closure then X is called **locally compact**. A space X is called **countably compact** if X contains no countable infinite closed discrete subspaces.

1.3 *S* and *L*-spaces

In additional to generally assuming all of the topological spaces under consideration are of size $\aleph_1$ and regular, we shall mostly be concerned with a subclass of spaces called S and L-spaces. The standard introduction to this class of spaces is Roitman's [29]. A history of the subject can be found in Juhász's [18].

Definition 1.4. *A space X is an **S-space** if X is T_3, uncountable, right-separated and X contains no uncountable left-separated subspace.*

*A space X is an **L-space** if X is T_3, uncountable, left-separated and X contains no uncountable right-separated subspace.*

In other words, an S-space X has $|X| = h(X) > z(X)$ and an L-space has $|X| = z(X) > h(X)$.

Recall that a space is called **separable** if there is a countable X_0 such that every open $U \subseteq X$ meets X_0 (i.e. X_0 is **dense** in X). A space is called **Lindelöf** if every family $\mathcal{U}$ of open sets such that $X \subseteq \bigcup_{u \in \mathcal{U}} U$ (i.e. $\mathcal{U}$ is a **cover** of X) there is a countable subfamily that covers X. A space is called **hereditarily-P** if every subspace of X has property P.

Traditionally, an S-space (L-space) is defined to be a T_3 space that is hereditarily separable (hereditarily Lindelöf) and not hereditarily Lindelöf (hereditarily separable). This definition is equivalent to Definition 1.4 (see [29] 3.3) if we are allowed to always shrink a space to an uncountable subspace.

It is sometimes necessary to shrink a hereditarily separable (hs), not hereditarily Lindelöf space (hL) to get an S-space in the sense of Definition 1.4. For example, the one point compactification of a locally compact S-space is a compact, hereditarily separable regular space that is not hereditarily Lindelöf. This will not count as an official S-space in light of Definition 1.4 since it is not right separated, but we shall often call such a space an S-space nonetheless. It will typically be clear from the context whether or not any particular space is right separated.

It is straightforward to see that there cannot be any second-countable S-spaces as for any right-separated space it will be the case that $w(X) \geq |X|$. There are however first-countable S-spaces and we shall study various first-countable S-spaces in great detail later.

It was recently shown by Brech and Koszmider in [5] that there is a hereditarily separable, right separated regular space of cardinality $\aleph_2$. Thus, there is an S-space by our definition of cardinality $\aleph_2$. It is still open if there is such a space of cardinality $\aleph_3$. The space constructed in [5] is obtained from a forcing construction. This leaves open the following question:

Question 1.5. *Is there a quotable axiom* $\ast$ *such that* $\ast$ *implies that there is a hereditarily* *separable right separable regular space of cardinality κ for $\kappa \geq \aleph_2$?*

Since right separated spaces have maximal weight, by standard arguments using cardinal functions it is the case that the existence any right-separated, regular, hereditarily separable space refutes CH and so ✱ must be inconsistent with CH.

The two major theorems concerning S and L-spaces are:

Theorem 1.6 (Todorčević)**.** *(PFA) There are no S-spaces.*

Theorem 1.6 is proved in [38] (see 8.9). An alternative proof, using Baumgartner's combinatorial proposition TOP (see [4]), can be found in [29]. The natural question implied by Theorem 1.6 is whether or not there are any L-spaces under PFA. This was answered by Moore in [24] who showed there is an L-space in ZFC.

Theorem 1.7 (Moore)**.** *There is an L-space.*

There are a couple of facts that are essential to many proofs involving S and L-spaces. Since we are always assuming that spaces are T_3 we have:

Proposition 1.8. *A right separated space X is an S-space iff it has no uncountable discrete subspaces.*

Proposition 1.9. *A left-separated space X is an L-space iff it has no uncountable discrete subspaces.*

1.4 S-spaces

Every S-space is hereditarily separable. Thus any subspace Y of an S-space X has a countable dense subset Y_0, i.e. the closure of Y_0 is Y. In many cases the closure of Y_0 may be larger: all of X, or even some Z such that $X \subseteq Z$. We shall now define a series of such S-spaces that have stronger density properties. For a detailed discussion of these types of spaces see Juhász's [20].

Recall that 2^{ω_1} is the space formed by taking the Tychonoff product of the two point discrete space ω_1 times and that therefore the basic clopen neighborhoods are defined by functions in $\mathrm{Fn}(\omega_1, 2)$.

Definition 1.10. *Let $H(\omega_1)$ denote the set of all finite functions from ω_1 into $2 = \{0,1\}$. For $\varepsilon \in H(\omega_1)$ let $[\varepsilon]$ denote*

$$[\varepsilon] = \{f \in 2^{\omega_1} : \varepsilon \subseteq f\}$$

Then $\{[\varepsilon] : \varepsilon \in H(\omega_1)\}$ forms a basis of 2^{ω_1} and each $[\varepsilon]$ is clopen in 2^{ω_1}.

Now we may define subspaces of 2^{ω_1} that have the strongest possible separability properties.

Definition 1.11. *A subspace $X = \{x_\alpha : \alpha < \omega_1\}$ of 2^{ω_1} is an **HFD** if*

(i) for any $Y_0 \in [X]^\omega$ there is an $\alpha < \omega_1$ such that for any $\varepsilon \in H(\omega_1)$ with domain a subset of $\omega_1 \setminus \alpha$ there is some $f \in Y_0$ with $f \in [\varepsilon]$.

(ii) each x_α is such that $x_\alpha(\xi) = 0$ for $\xi < \alpha$ and $x_\alpha(\alpha) = 1$.

*A subspace $X = \{x_\alpha : \alpha < \omega_1\}$ of 2^{ω_1} is an **HFD**$_w$ if*

(i) for any $Y \in [X]^{\omega_1}$ there is a $Y_0 \in [Y]^\omega$ and $\alpha < \omega_1$ such that for any $\varepsilon \in H(\omega_1)$ with domain a subset of $\omega_1 \setminus \alpha$ there is some $f \in Y_0$ with $f \in [\varepsilon]$.

(ii) each x_α is such that $x_\alpha(\xi) = 0$ for $\xi < \alpha$ and $x_\alpha(\alpha) = 1$.

The set Y_0 of Definition 1.11 is called **finally dense**, from which comes the name Hereditarily Finally Dense (HFD) and the weaker form (HFD$_w$, also known as a **weak HFD**).

Immediately from the definition of HFD and HFD_w we may conclude that:

Remark 1.12. *Every HFD is an HFD_w.*

HFD and weak HFDs usually have another interesting property: all of the open sets are countable or co-countable. A regular space with this property is called an ***O*-space**.

Definition 1.13. *An **O-space** is an uncountable T_3 space in which every open set is countable or co-countable.*

There are two names for *O*-spaces in the literature: *O*-space and sub-Ostaszewski space. The American school uses sub-Ostaszewski, e.g. in [11], and the Hungarian school uses *O*-space, e.g. in [20] and [33]. We shall follow the Hungarian school.
Many properties of *O*-spaces and HFDs are examined in detail in [20]. In particular (see [20] 2.24):

Proposition 1.14. *An HFD_w is an O-space.*

An *O*-space will usually be an *S*-space, but this may requiring throwing out some points. Recall that the definition of *S*-space required that the space be right separated and hence locally countable. An *O*-space may not be locally countable, but at most one point may have only uncountable co-countable neighborhoods since the space is T_2. From this we may conclude:

Proposition 1.15. *If there is an O-space, there is a locally countable O-space.*

We can use Proposition 1.15 to prove:

Proposition 1.16. *An O-space is, possibly restricted to an uncountable subspace, an S-space.*

Proof. Let X be an O-space. To prove that X is hereditarily separable and not hereditarily Lindelöf let $Y \subseteq X$ be an uncountable subset. First to see that Y is right separated fix an enumeration of $Y = \{y_\alpha : \alpha < \omega_1\}$. Since X is T_2 then Y is locally countable except possibly at one point y_ξ. Throwing away this point, we may assume that Y is locally countable, and by an induction of length ω_1 we can find an uncountable right separated subspace of Y. Thus Y is not hereditarily Lindelöf and neither is X. Shrinking to this right separated subspace of Y, it suffices to show that there is no uncountable discrete subspace. If there was such an uncountable discrete subspace, then there would be uncountable co-uncountable open set in X which contradicts that X is an O-space. □

There is a dual to 1.15 which demonstrates why the phrase 'possibly restricted to an uncountable subspace' of 1.16 is necessary. If we have a locally countable O-space we can produce an O-space with weight at least $\aleph_1$.

Proposition 1.17. *If there is a locally countable O-space then there is an O-space that is not locally countable.*

Proof. Let X be a locally countable O-space. Define a topology on $X \cup \{\infty\}$ by declaring all sets of the form $Y \cup \{\infty\}$ for Y a co-countable subset of X to be open. Then $X \cup \{\infty\}$ is T_3 and an O-space. □

Thus given any O-space it is possible to construct one that fails to be locally countable, but it must fail to be locally countable by a single point.

Combining the preceding propositions we see that S-spaces, O-spaces, HFDs and weak HFDs can be nicely stratified as shown in Figure 3.

We have now also proved that:

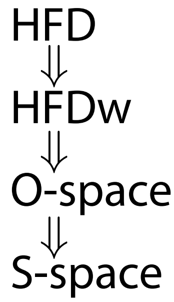

Figure 3: The Hierarchy of S-spaces

Theorem 1.18. *If X is an HFD, HFD$_w$, O-space, then X is, possibly restricted to an uncountable subspace, an S-space.*

1.5 Additional Topological Properties

We shall mostly be concerned with the existence of S-spaces in the hierarchy of Figure 3 which have additional topological properties. For completeness we shall define these properties here.

Definition 1.19. *A space X is **normal** if for every pair of disjoint closed sets in X can be separated by disjoint open neighborhoods.*

We should point out that an O-space that is locally compact is first countable for free, since it is also locally countable. We have already shown that an O-space is, possibly restricted to an uncountable subspace, locally countable.

Definition 1.20. *A space X is an **Ostaszewski** space if X is an O-space, and X is locally compact and countably compact.*

A space X is called **perfectly normal** if every closed set is a G_δ set. Since an Ostaszewski space is countably compact it is consequently perfectly normal. Also, since an Ostaszewski space is right separated (since it's an O-space) it cannot be compact.

There are some standard techniques of constructing right-separated spaces. Typically one constructs such a space of size $\aleph_1$ by inductively constructing a topology τ on ω_1.

Definition 1.21 (Simple Limit Spaces). *Given a topology τ_α on α and τ_β on $\beta > \alpha$ then τ_β is a **conservative extension** of τ_α if*

$$\tau_\beta \cap \mathcal{P}(\alpha) \subseteq \tau_\alpha$$

*For a sequence of topologies $\{\tau_\alpha : \alpha < \omega_1\}$ with the property that $\alpha < \beta$ implies τ_β is a conservative extension of τ_α then let $\sum_{\alpha<\omega_1} \tau_\alpha$ be the topology τ on ω_1 with base $\bigcup_{\alpha<\omega_1} \tau_\alpha$. Such a τ is called the **simple limit space** of the τ_α.*

The standard results we shall use about such simple limit spaces can be found in [29] §2.3. In particular note that any simple limit space is automatically right separated. For this reason simple limit constructions are natural means of constructing *S*-spaces.

Chapter 2

Weakenings of $\diamondsuit$

The primary motivation for this work is several of the open questions concerning S-spaces, in particular the following question originally appearing in [11].

Question 2.1. *(CH) Is there a locally compact O-space?*

In §4 We shall provide a partial solution by means of examining a heretofore unexamined class of S-spaces. In addition, throughout this work, we shall examine several related questions.

We have already seen that it is consistent that there are no S-spaces at all. Thus to construct an S-space at all, much less an S-space with some additional topological properties will require the assumption of some additional axiom. Assuming $\diamondsuit$ it is consistent that there are S-spaces, O-spaces, HFDs and weak HFDS with various other topological properties. If one is interested in constructing any topology on ω_1, the axiom $\diamondsuit$ is a natural place to look since often an S-space is in some way essentially tied to the combinatorial structure of ω_1.

Recall that a $C \subseteq \omega_1$ is called a **club** if C is uncountable and $\mathrm{Lim}(C) \subseteq C$. A set $S \subseteq \omega_1$ is called **stationary** if $S \cap C \neq \varnothing$ for every club C. The axiom $\diamondsuit$ is the the most powerful axiom we shall consider, which follows from Gödel's Axiom of Constructibility ($V = L$).

Definition 2.2. *The axiom $\diamondsuit$ states*

$$\text{There is a sequence } \{A_\alpha : \alpha < \omega_1\} \text{ such that for any } A \in [\omega_1]^{\leq \omega_1} \text{ the set } \{\alpha < \omega_1 : A \cap \alpha = A_\alpha\} \text{ is stationary.} \tag{$\diamondsuit$}$$

The axiom $\diamondsuit$ is independent of the axioms of ZFC, and well studied in the literature. Many of the standard results that follow from $\diamondsuit$ can be found in [22] II.7. One well known consequence of $\diamondsuit$ is CH, the continuum hypothesis, which states that $2^{\aleph_0} = \aleph_1$.

Usually in set theoretic topology $\diamondsuit$ giveth and MA($\aleph_1$) taketh away. Many of the S-spaces with additional topological properties can be built from $\diamondsuit$. Exactly how much of $\diamondsuit$ is needed to build each type of S-space is an interesting question that is the focus of this chapter.

Definition 2.3. *The axiom $\clubsuit$ asserts:*

> *There exists a sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that $x_\alpha \subseteq \alpha$ is unbounded in α of order type ω whereby for any $Y \in [\omega_1]^{\omega_1}$ there is some x_α such that $x_\alpha \subseteq Y$.* **($\clubsuit$)**

As shown in [28] the axiom $\clubsuit$ is $\diamondsuit$ in disguise in the presence of CH.

Proposition 2.4 (Ostaszewski)**.** *CH + $\clubsuit \iff \diamondsuit$.*

In particular $\diamondsuit$ implies $\clubsuit$, but not conversely (see [13]). It turns out that $\diamondsuit$ is more useful for building topologies in the equivalent form $\clubsuit + CH$. This is because often one builds an S-space on ω_1 as a simple limit space and each element x_α of the $\clubsuit$-sequence is used to build the necessary neighborhoods of α from the existing topology on α.

Definition 2.5. *The axiom $\dagger$ asserts*

> *There is a sequence $\{x_\alpha : \alpha \in \omega_1\} \subseteq [\omega_1]^\omega$ such that for any $Y \in [\omega_1]^{\omega_1}$ there is some x_α such that $x_\alpha \subseteq Y$.* ($\dagger$)

It is clear from the definitions that

Proposition 2.6. $\clubsuit \implies \dagger$ *and CH* $\implies \dagger$.

and so $\stick$ may be thought of as a weakening of both CH and $\clubsuit$. It is rather straightforward to show that MA($\aleph_1$) refutes $\stick$ and hence $\clubsuit$ (just force with $\mathrm{Fn}(\omega_1, 2)$, see [13] Fact 1.3). Another weakening of CH that we shall consider is the cardinal inequality $2^{\aleph_0} < 2^{\aleph_1}$.
It is clear that:

Proposition 2.7. *CH* $\implies 2^{\aleph_0} < 2^{\aleph_1}$.

To define a further weakening of CH requires defining an order on ω^ω, the set of functions from ω to ω. Given two functions $f, g \in \omega^\omega$ we say that $f <^* g$ if there exists $m \in \omega$ such that for every $n > m$ $f(n) < g(n)$.

Definition 2.8. *A set $A \subseteq \omega^\omega$ is $<^*$-**unbounded** if there is no $g \in \omega^\omega$ such that $f <^* g$ for all $f \in A$. The cardinal $\mathfrak{b}$ is*

$$\mathfrak{b} = \min\{|A| : A \textit{ is } <^*\textit{-unbounded}\}$$

The cardinal $\mathfrak{b}$ is one of many cardinal invariants of the continuum. Of these invariants, most is known about the classes of S-spaces that can be built from $\mathfrak{b}$ (see also [20] 6.8). From the definition of $\mathfrak{b}$ it is clear that:

Proposition 2.9. *CH* $\implies \mathfrak{b} = \omega_1$.

We have already seen that CH implies $\stick$. It was an open question for a while if $\stick$ implies $\mathfrak{b} = \omega_1$. In [6] it was recently shown that $\stick$ implies $\mathfrak{b} = \omega_1$.

Theorem 2.10 (Brendle)**.** $\stick$ *implies* $\mathfrak{b} = \omega_1$.

Perhaps the most well know consequence of $\diamondsuit$ in addition to CH is the existence of a particular tree on $2^{<\omega_1}$, a Suslin tree.

Definition 2.11. *A **Suslin tree** $T \subseteq 2^{<\omega_1}$ is a tree with height ω_1, no uncountable levels and no uncountable antichains.*

Such a tree exists assuming $\diamondsuit$, but not if $\mathrm{MA}(\aleph_1)$ holds. The existence of a Suslin tree is an axiom sufficient to build certain types of spaces, so let (ST) denote "there is a Suslin tree."

For every combination of the S-spaces discussed above: HFD, HFD_w, O-space and S-space, and various natural additional topological properties, and the axioms CH, (ST), ⫯, $\mathfrak{b} = \omega_1$, $\diamondsuit$, $\clubsuit$ and $2^{\aleph_0} < 2^{\aleph_1}$ gives rise to a natural existence question. That is, does the space exist assuming the axiom? We leave it as an exercise to the reader to count how many of the questions are not trivially resolved. The summary of what is known about these questions is shown in Figure 4.

It is worth noting that (ST) is independent of CH (see [23] Theorem 9). Also (ST) is independent of $\clubsuit$ (see [8] and [9]). Thus asking which spaces exist assuming (ST) is not the same as asking which spaces exist assuming $\clubsuit$ and CH. In addition it is known that $\clubsuit$ is independent of CH (see [31] and [13]). So $\clubsuit$, CH and (ST) may all give rise to distinct classes of S-spaces.

Recall that the **cofinality** $\mathrm{cf}(\kappa)$ of cardinal κ is the least ordinal λ such that there is a function $f : \lambda \to \kappa$ unbounded in κ. Successor cardinals κ have $\mathrm{cf}(\kappa) = \kappa$ and so:

Proposition 2.12. *CH* $\implies \mathrm{cf}(\mathfrak{c}) = \omega_1$

Thus we may view $\mathrm{cf}(\mathfrak{c}) = \omega_1$ as a weakening of CH.

2.1 Spaces under $\diamondsuit$

Assuming $\diamondsuit$ essentially every S-space with every additional desired topological property can be built. An Ostaszewski space is an O-space that is locally compact and countable compact.

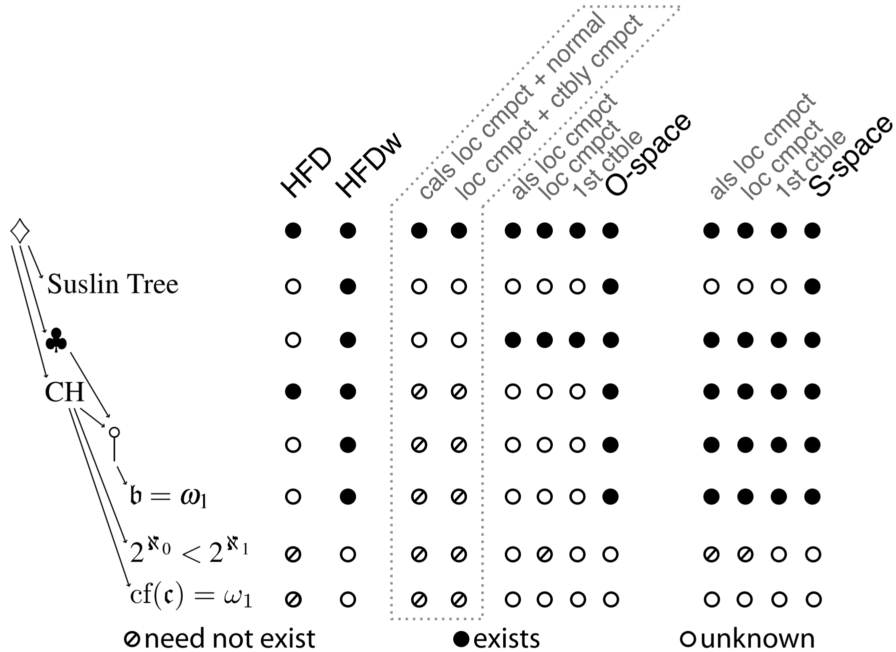

Figure 4: Existence and Non-existence of Various Spaces Under Weakenings of ◊

There is such a space assuming ◊ as shown by Ostaszewski in [28].

Theorem 2.13 (Ostaszewski)**.** *(◊) There is an Ostaszewski space.*

Additionally, since an Ostaszewski space is countably compact it is consequently perfectly normal. Also, since an Ostaszewski space is right separated (since an O-space) it cannot be compact. A space X is called *perfectly normal* if X is normal and every closed set is a G_δ. An Ostaszewski space is perfectly normal since for every pair of disjoint closed sets one of them must be countable, hence compact.

Theorem 2.14 (Ostaszewski)**.** *(◊) There is a locally compact, perfectly normal, countably compact O-space.*

Ostaszewski's construction was the first to use $\clubsuit + CH$. The various properties of Ostaszewski's space come from CH or from ♣. It is straightforward to build spaces with a subset of the properties of an Ostaszewski space using $\clubsuit$ alone or CH alone as we shall see in the following sections.

Since $\diamondsuit$ implies ⫯, CH, $\clubsuit$, $2^{\aleph_0} < 2^{\aleph_1}$, (ST), and $\mathfrak{b} = \omega_1$ any space build from any of those weaker axioms can be built directly from $\diamondsuit$ but the converse does not always hold.

In particular, assuming $\diamondsuit$, since there is an Ostaszewski space, there is a also a locally compact, countably compact *S*-space, and hence a first countable locally compact, countably compact *S*-space. Since $\diamondsuit$ implies CH and CH implies, as we shall soon see, there is an HFD we get that there is every one of the spaces with strong separability properties indicated in Figure 3 from $\diamondsuit$.

2.2 Spaces from a Suslin Tree

Assuming $\diamondsuit$ there is a Suslin tree. It is easy to force a model in which there are no Suslin trees by forcing with the inverse $(T, \geq)$ order of the order $(T, \leq)$ of the Suslin tree itself. Since $(T, \geq)$ is ccc when $(T, \leq)$ is Suslin, there are no Suslin trees under MA$(\aleph_1)$. A tree T is **special** if $T = \bigcup_{i<\omega} T_i$ where each T_i is an antichain of T. If one assumes MA$(\aleph_1)$ then all Aronszajn trees are special and hence there are no Suslin trees.

If one views the branches of a Suslin tree as points, a Suslin tree is an *L*-space. Thus, if there is a Suslin tree there is an *L*-space. Perhaps more surprising, as first discovered by M.E. Rudin in [30], if there is a Suslin tree there is an *S*-space.

Theorem 2.15 (M.E. Rudin)**.** *(ST) There is a normal S-space.*

Assuming (ST) there is not only a normal *S*-space, but also a weak HFD as shown by Todorčević in (see [38], Chapter 5).

Theorem 2.16 (Todorčević)**.** *(ST) There is a weak HFD.*

It is also possible to construct a first countable and locally compact S-space from (ST) (see [18] 1.16) using the techniques of [19] which refines an S-space into another (usually locally compact, first countable) S-space using CH. The construction of a locally compact S-space on a Suslin tree, as has been reported to me by F. Tall, requires the use of CH. Thus this leaves open whether CH is necessary for such a construction or whether CH can be avoided.

Question 2.17. *(ST) Is there a locally compact (or first-countable) S-space?*

2.3 Spaces from CH

Every space of Figure 3 exists assuming CH. Since an HFD gives rise to a weak HFD, O-space and an S-space, if there is an HFD there is every kind of space of 3.
As shown by Hajnal and Juhász in [14]:

Theorem 2.18 (Hajnal, Juhász)**.** *(CH) There is an HFD.*

Since we have already seen that by definition any HFD is an HFD_w and that any HFD_w is an O-space (see Proposition 1.14) to answer Question 2.1 it will suffice to construct a first countable HFD_w. Every HFD_w X is right-separated by definition (see Definition 1.11) and hence locally countable. Thus every HFD_w has countable pseudo-character. However, no HFD_w can be first countable.

Lemma 2.19. *If X is an HFD_w then X is not first countable.*

Proof. Fix $X = \{x_\alpha : \alpha < \omega_1\}$ an HFD_w. Then since $X \subseteq 2^{\omega_1}$, we may fix a base $\mathcal{B} = \{[\varepsilon] : \varepsilon \in H(\omega_1)\}$ for X. Fix a local base $\mathcal{B}_\alpha \subseteq \mathcal{B}$ of x_α for each $\alpha < \omega_1$. We shall prove that each $\mathcal{B}_\alpha$

is not countable for any uncountable subset of $\alpha < \omega_1$. Suppose towards a contradiction that there is some uncountable $E \subseteq \omega_1$ such that $\mathcal{B}_\alpha$ is countable for every $\alpha \in E$. For each $\alpha \in E$ fix a neighborhood $\delta_\alpha \in H(\omega_1)$ such that $x_\alpha \in [\delta_\alpha]$ and such that $\varepsilon_\alpha \in \mathcal{B}_\alpha$ with $\mathrm{dom}(\varepsilon_\alpha) < \mathrm{dom}(\delta_\alpha)$ and

$$[\varepsilon_\alpha] \subseteq [\delta_\alpha] \text{ for all } \alpha \in E$$

By a standard Δ-system argument there is an uncountable $E' \subseteq E$ such that

(i) There is $\delta'_\alpha \subseteq \delta_\alpha$ and $\varepsilon'_\alpha \subseteq \varepsilon_\alpha$ for every $\alpha \in E'$ such that $\{\mathrm{dom}(\delta'_\alpha) : \alpha \in E'\}$ and $\{\mathrm{dom}(\varepsilon'_\alpha) : \alpha \in E'\}$ are separated.

(ii) For every $\alpha < \beta$

$$\mathrm{dom}(\varepsilon'_\alpha) < \mathrm{dom}(\delta'_\alpha) < \mathrm{dom}(\varepsilon'_\beta) < \mathrm{dom}(\delta'_\beta)$$

(iii) It is the case that:

$$[\varepsilon'_\alpha] \subseteq [\delta'_\alpha] \text{ for every } \alpha \in E' \tag{2.1}$$

Since X is an HFD_w there is some $E_0 \subseteq E'$ that is finally dense past $\eta < \omega_1$, i.e. if $\varepsilon \in H(w_1 \setminus \eta)$ then $[\varepsilon] \cap E_0$ is non empty. Fix some $\alpha \in E'$ such that $\mathrm{dom}(\varepsilon'_\alpha) \subseteq \omega_1 \setminus \eta$. Then

$$E_0 \cap [\varepsilon'_\alpha] \setminus [\delta'_\alpha] \neq \varnothing$$

which contradicts (2.1). This E' witnesses that X is not an HFD_w. □

In [19] Juhász, Kunen, and M.E. Rudin gave an example of a first countable, locally compact S-space from CH known as the *Kunen Line* which is a refinement the topology on an set of reals of size $\aleph_1$. The topology on $\mathbb{R}$ is metric and hence second countable. Thus it is hereditarily separable and so there is a refinement that can be made into an S-space provided the refinement is right separated and closed sets are not shrunk by more than a countable set.

Theorem 2.20 (Juhász, Kunen, and M.E. Rudin). *(CH) There is a locally compact S-space.*

There are some interesting techniques using CH that refine a given first countable *S*-space into another *S*-space with additional topological properties. The standard CH techniques turns a first countable *O*-space into a first countable, locally compact *O*-space. We shall explore these techniques later in order to prove (see Theorem 3.8 on page 32):

Theorem 2.21. *(CH) Assume there is a first countable O-space. Then there is a first countable, locally compact O-space.*

We have already seen that under $\diamondsuit$ there is an Ostaszewski space. The original construction of an Ostaszewski space, and most subsequent constructions, use $\clubsuit$ + CH which is equivalent to $\diamondsuit$. An open question for some time, was whether or not an Ostaszewski space existed under CH alone. Nyikos offered a cash award (a total of $50 for a non existence solution, which he tells me has been paid) for a solution to this problem in [27] and it was solved by Eisworth and Roitman in [11]; who showed that CH was consistent with the non-existence of any Ostaszewski space.

Theorem 2.22 (Eisworth, Roitman). *There is a model of CH in which there are no Ostaszewski spaces.*

This leaves open the following question the investigation of which is the major motivation of this work:

Question 2.23. *(CH) Is there a locally compact (hence 1st countable) O-space?*

Using Theorem 2.21 we may re-state Question 2.23 as

Question 2.24. *(CH) Is there a first countable O-space?*

The answer to this question is, at least slightly, elusive for the good reason. Let us say that a *classical CH construction* is a construction of some object of size $\aleph_1$ from an enumeration of $[\omega_1]^\omega$ in order type ω_1. Almost every classically constructed topological object cannot be destroyed by any totally proper forcing as any enumeration of $[\omega_1]^\omega$ stays an enumeration in the extension, and the collection of open sets in the ground model forms a base for a topology in the extension. In [11] the model of CH in which there are no Ostaszewski spaces is constructed by iterating totally proper forcings where each totally proper forcing destroys a locally compact *O*-space. Thus, if there is a positive answer to 2.24, it will not be built by a classical CH construction.

2.4 Spaces from $\clubsuit$

The most fruitful of all weakenings of $\Diamond$ is the axiom $\clubsuit$, originally used by Ostaszewski. Since this principle was formulated with the intention of building a right separated topology on ω_1 it is not surprising that it is so useful in doing exactly that.

In the standard Ostaszewski construction, $\clubsuit$ is used to make the topology into a locally compact *O*-space and CH is used to capture all potential witnesses to the failure of the space to be countably compact.

Theorem 2.25 (Ostaszewski)**.** *($\clubsuit$) There is a locally compact O-space.*

There are several weakenings of $\clubsuit$. We have already mentioned ⫯ which follows from $\clubsuit$ but also from CH. Another weakening of $\clubsuit$ is the following:

Definition 2.26. *The axiom ✤ asserts:*

There exists a sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that $x_\alpha \subseteq \alpha$ is of order type ω whereby for any club $Y \subseteq \omega_1$ there is some x_α such that $x_\alpha \subseteq Y$. (✤)

The axiom ☘ is studied frequently enough that it is called numerous different names in the literature including: $\clubsuit$ for clubs, club guessing on ω_1 and ω-club guessing for ω_1.

It should be clear that:

Proposition 2.27. $\clubsuit \implies$ ☘.

It is the case that in any ccc extension any club contains a club in the ground model. Thus ☘ is ccc indestructible and so in particular ☘ is consistent with MA($\aleph_1$). Thus it is not the case that ☘ $\implies \clubsuit$ and furthermore it is clear that ☘ is consistent with $\neg$CH.

One may wonder if since $\clubsuit$ implies there is a locally compact O-space whether or not ☘ implies there is a locally compact O-space. Since ☘ is consistent with MA($\aleph_1$) and MA($\aleph_1$) implies there are no locally compact S-spaces then one cannot even construct a locally compact S-space from ☘ alone. In order to build a locally compact O-space from ☘, the natural axiom to add to ☘ is CH. However, since it is the case that ☘ is indestructible with respect to ω-proper forcing (see [32] XVII 3.13) and since the forcing used to establish Theorem 2.22 is ω-proper we know that:

Theorem 2.28. *It is consistent with ☘ + CH that there are no Ostaszewski spaces.*

This leaves open the following question:

Question 2.29. *(☘ + CH) Is there a locally compact (first countable) O-space?*

There is a weakening of ☘ that may be of some interest to construct S-spaces. (see [25])

Definition 2.30. *The axiom $\mho$ is defined as:*

> *There exists a sequence $\{f_\alpha : \alpha \in \omega_1\}$ such that f_α is a continuous function from α to ω (whereby α has the order topology and ω the discrete topology) such that whenever $C \subseteq \omega_1$ is club there is a $\delta \in C$ such that f_δ takes all values on $C \cap \delta$.* ($\mho$)

It is easy to verify that $\clubsuit$ implies $\mho$. Furthermore, $\mho$ is impervious to ω-proper forcings, is consistent with MA($\aleph_1$) and the preceding comments about $\clubsuit$ and locally compact S-space apply to $\mho$. This leaves the following question:

Question 2.31. *($\mho$ + CH) Is there a locally compact (first countable) O-space?*

2.5 Spaces from ⫯

The existence of an HFD$_w$ from ⫯ is shown in [20] by means of another combinatorial principle which Juhász calls $V(\omega_1)$.

Theorem 2.32 (Juhász)**.** *(⫯) There is a weak HFD.*

It is worth noting that ⫯ is in fact equivalent to $V(\omega_1)$, even though it is only mentioned that ⫯ $\implies V(\omega_1)$ in [20]. Since there is a weak HFD from ⫯, there is also an O-space and an S-space.

In [6] it was recently shown that ⫯ implies $\mathfrak{b} = \omega_1$. Thus any spaces which can be built from $\mathfrak{b} = \omega_1$ can be built from ⫯.

2.6 Spaces from $\mathfrak{b} = \omega_1$

Let $\{f_\alpha : \alpha < \kappa\}$ be a sequence of functions $f_\alpha : \omega \to \omega$. This sequence is called **unbounded** if there is no $g : \omega \to \omega$ such that $f_\alpha <^* g$ for all $\alpha < \kappa$. The cardinal $\mathfrak{b}$ is defined to be the least κ such that $\{f_\alpha : \alpha < \kappa\}$ is unbounded. It is clear that CH implies $\mathfrak{b} = \omega_1$.

The construction of various S-spaces from $\mathfrak{b} = \omega_1$ is given in [38]. First, there a weak HFD from $\mathfrak{b} = \omega_1$ as shown in §1 of [38]:

Theorem 2.33 (Todorčević)**.** *($\mathfrak{b} = \omega_1$) There is a weak HFD.*

Using CH on can refine a subspace of the real topology of size $\aleph_1$ into a locally compact *S*-space. Similarly it is possible to refine Baire space into a locally compact *S*-space using $\mathfrak{b} = \omega_1$. Thus there is a locally compact *S*-space from $\mathfrak{b} = \omega_1$. (See Theorem 2.5 of [38])

Theorem 2.34 (Todorčević)**.** *($\mathfrak{b} = \omega_1$) There is a locally compact S-space.*

It will be shown in Theorem 3.8 using CH it is possible to refine a first countable *O*-space into a locally compact, first countable *O*-space. This leaves open the following question in light of Theorem 2.34 (see also §3.4):

Question 2.35. *($\mathfrak{b} = \omega_1$) If there is a first countable O-space, is there a locally compact first countable O-space?*

There are numerous other cardinals $\mathfrak{f}$ such that CH implies $\mathfrak{f} = \omega_1$. For each such cardinal $\mathfrak{f}$ one may ask:

Question 2.36. *For which cardinals $\mathfrak{f}$ is it the case that $\mathfrak{f} = \omega_1$ implies there is a locally compact S-space or weak HFD?*

2.7 Spaces from $2^{\aleph_0} < 2^{\aleph_1}$

It remains to mention an important recent result about locally compact *S*-spaces and under $2^{\aleph_0} < 2^{\aleph_1}$. The nonexistence of locally compact *S*-spaces under $2^{\aleph_0} < 2^{\aleph_1}$ is settled by the following Theorem of [10]:

Theorem 2.37 (Eisworth, Nyikos, Shelah)**.** *There is a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no locally compact S-spaces.*

In building a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no locally compact S-spaces, Eisworth, Nyikos and Shelah make use of a very interesting property of S-spaces which puts them under the influence of (PID) which we shall investigate in great detail in the next section.

It is fairly straightforward to prove that

Theorem 2.38. *There is a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no HFDs.*

We can prove Theorem 2.38 by building a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there exists a special kind of ultrafilter. Recall that a subset $\mathcal{U} \subseteq [\omega]^\omega$ is called an **ultrafilter** if:

1. $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,

2. for every $A \subseteq \omega$ either A or $\omega \setminus A \in \mathcal{U}$,

3. if $A \subseteq B$ and $A \in \mathcal{U}$ then $B \in \mathcal{U}$.

Definition 2.39. *An ultrafilter $\mathcal{U}$ is called a $P_{\omega_2} -$ point if for every sequence $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{U}$ there exists some $A \in \mathcal{U}$ such that $A \subseteq^* A_\alpha$ for all $\alpha < \omega_1$.*

In order to prove Theorem 2.38 it suffices to construct a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there is a P_{ω_2}-point since:

Lemma 2.40. *If there is a P_{ω_2}-point then there are no HFDs.*

Proof. Suppose that $\mathcal{U}$ is a P_{ω_2}-point and that X is an HFD. Let $Y \subseteq X$ be countable and finally dense. Enumerate $Y = \{y_n : n < \omega\}$ and for each $\alpha < \omega_1$ let $i_\alpha \in 2$ be such that $A_\alpha = \{n < \omega : y_n(\alpha) = i_\alpha\} \in \mathcal{U}$. Let A be such that $A \subseteq^* A_\alpha$. There is an uncountable $E \subseteq \omega_1$ and infinite $A' \subseteq A$ such that for all $\alpha \in E$ it is the case that $A_\alpha \cap A = A'$. Then $\{y_n : n \in A'\}$ cannot be finally dense. $\square$

To construct a model of $2^{\aleph_0} < 2^{\aleph_1}$ it suffices to start with a ground model V in which $2^{\aleph_0} < 2^{\aleph_1}$ and to add a a P_{ω_2}-point in ω_2 steps using a finite support iteration.

2.8 Spaces from $\mathrm{cf}(\mathfrak{c}) = \omega_1$

A $(\kappa, \to)$-HFD is a subspace X of 2^κ in which for every $Y \in [X]^\omega$ there is some $\alpha < \kappa$ such that Y is dense in $2^{\kappa \setminus \alpha}$. In the case that $\kappa = \omega_1$ a $(\kappa, \to)$-HFD is an HFD.

The following was asked by Dow and Juhász in [7] and also appears as a question in [20]:

Question 2.41 (Juhász)**.** *Is there a $(\mathfrak{c}, \to)$-HFD?*

This is a ZFC question and is reasonable as there are $\mathfrak{c}$ many requirements to satisfy in $\mathfrak{c}$ steps. Recall that $\mathfrak{r}$, the reaping number, is the least size of a family $\mathcal{A} \subseteq [\omega]^\omega$ such there does not exist a $B \subseteq \omega$ such that for every $A \in \mathcal{A}$ both $A \cap B$ and $A \setminus B$ are infinite. In [20] it is shown that:

Theorem 2.42 (Juhász)**.** *($\mathfrak{r} = \omega_1$) There is a $(\mathfrak{c}, \to)$-HFD.*

If we suppose $\mathrm{cf}(\mathfrak{c}) = \omega_1$ then there is a connection between the existence of a $(\mathfrak{c}, \to)$-HFD and an HFD. The following was remarked to me by Juhász:

Lemma 2.43. *($\mathrm{cf}(\mathfrak{c}) = \omega_1$) If there is a $(\mathfrak{c}, \to)$-HFD then there is an HFD.*

The proof is to simply take the projection of the $(\mathfrak{c}, \to)$-HFD along the graph of any cofinal map. In the preceding section we were able to construct a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there was a P_{ω_2}-point. The same forcing argument can produce a model of $\mathrm{cf}(\mathfrak{c}) = \omega_1$ in which there is a P_{ω_2}-point by starting with a model of $\mathfrak{c} = \aleph_{\omega_1}$. Combining this with Lemma 2.43 we can establish:

Theorem 2.44. *It is consistent that there are no* $(\mathfrak{c}, \rightarrow)$*-HFDs.*

which answers Question 2.41. Thus there cannot be any $(\mathfrak{c}, \rightarrow)$-HFDs unless $\mathfrak{c}$ is regular. In the case that $\mathfrak{c}$ is regular and that $\mathrm{cf}(\mathfrak{c}) > \omega_1$ we can show that there are no $(\mathfrak{c}, \rightarrow)$-HFDs using essentially the same proof as used for Theorem 2.38. Recall that character of a P-point $\mathcal{U}$ is ω_1 in $\beta\omega \setminus \omega$ if $\mathcal{U}$ can be generated from a sequence $\langle A_\xi : \xi < \omega_1 \rangle \subseteq \mathcal{U}$ such that $\xi < \eta$ implies that $A_\xi \subseteq^* A_\eta$. Such a P-point is usually called a simple P-point. Starting with a ground model in which $\mathfrak{c}$ is regular and $\mathrm{cf}(\mathfrak{c}) > \omega_1$ add a P-point $\mathcal{U}$ in ω_1 steps that has character ω_1 in $\beta\omega \setminus \omega$. Then using this $\mathcal{U}$ one can prove:

Lemma 2.45. *If* $\mathrm{cf}(\mathfrak{c}) > \omega_1$*,* $\mathfrak{c}$ *is regular and* $\mathcal{U}$ *has character* ω_1 *in* $\beta\omega \setminus \omega$ *then there is no* $(\mathfrak{c}, \rightarrow)$*-HFDs*

The proof of Lemma 2.45 is mutatis mutandis identical to the proof of Lemma 2.40.

In Lemma 2.43 it was shown that if $\mathrm{cf}(\mathfrak{c}) = \omega_1$ and there is a $(\mathfrak{c}, \rightarrow)$-HFD then there is an HFD. This leaves open the question:

Question 2.46. *(*$\mathrm{cf}(\mathfrak{c}) = \omega_1$*) If there is an HFD is there a* $(\mathfrak{c}, \rightarrow)$*-HFD?*

There is very little known about the existence of various S-spaces from $\mathrm{cf}(\mathfrak{c}) = \omega_1$. In fact, the following is open:

Question 2.47. *(*$\mathrm{cf}(\mathfrak{c}) = \omega_1$*) Is there an S-space?*

Chapter 3

Refinements Using CH

It was first shown by Juhász, Kunen and Rudin in [19] that there is a locally compact S-space assuming CH alone. This space is obtained by an inductive refinement of the topology on $\mathbb{R}$ to obtain a right separated, locally compact space. The trick is to guarantee that the resulting space is also hereditarily separable. Since the topology on $\mathbb{R}$ is second countable it is also hereditarily separable and so it suffices to ensure that no new uncountable discrete spaces are introduced during the refinement.

3.1 S-preserving Refinements

There are standard techniques of refining an existing S-space under CH into another S-space. The means by which to refine one S-space into another is to refine a given topology (ω_1, τ) into another topology (ω_1, ρ) such that for any A, $|\mathrm{cl}_\tau(A) \setminus \mathrm{cl}_\rho(A)| \leq \aleph_0$. Let us call such refinements **S-preserving** refinements. A host of applications of this kind of refinement are discussed in [19], in which the following Theorem is proved:

Theorem 3.1. *If (ω_1, τ) is an S-space and (ω_1, ρ) is an S-preserving refinement then (ω_1, ρ) is an S-space.*

We can improve Theorem 4.43 slightly to produce and almost left-separated locally compact normal S-space by using the following observation: (see [19])

Lemma 3.2 (Juhász, Kunen, Rudin)**.** *Suppose that* (ω_1, τ) *is* T_3 *and hereditarily Lindelöf. If* (ω_1, ρ) *is an S-preserving refinement of* (ω_1, τ) *then* (ω_1, ρ) *is normal.*

Proof. Let H and K be disjoint and closed in (ω_1, ρ). Much like in the standard proof that T_3 and Lindelöf implies normality (see [12] 1.5.15) it will suffice to construct a countable cover of X by open U such that $\mathrm{cl}_\rho(U)$ meets at most one of H or K. Since the refinement is S-preserving, it is the case that

$$|\mathrm{cl}_\tau(H) \cap \mathrm{cl}_\tau(K)| \leq \aleph_0$$

and we may find a neighborhood U of each $x \in \mathrm{cl}_\tau(H) \cap \mathrm{cl}_\tau(K)$ with the property that $\mathrm{cl}_\rho(U)$ meets at most one of H or K. For every $x \in X \setminus (\mathrm{cl}_\tau(H) \cap \mathrm{cl}_\tau(K))$ we may find another such U that is open in (ω_1, τ) and hence (ω_1, ρ). Since (ω_1, τ) is Lindelöf we can find a countable collection covering $X \setminus (\mathrm{cl}_\tau(H) \cap \mathrm{cl}_\tau(K))$. Thus we have obtains a countable collection of U open in (ω_1, ρ) having the necessary property. □

Recall that a space is perfectly normal if every closed set is a G_δ set. It is of some interest if a normal (perfectly normal) space, (ω_1, τ), can be refined into another normal (perfectly normal) space (ω_1, ρ). In the case of S-preserving refinements we can say at least the following:

Lemma 3.3. *Suppose that* (ω_1, τ) *is perfectly normal. If* (ω_1, ρ) *is an S-preserving refinement of* (ω_1, τ) *then* (ω_1, ρ) *is perfectly normal.*

Proof. Let H be closed in (ω_1, ρ). It suffices to show that H is G_δ in (ω_1, ρ). Let $(U_i)_{i<\omega}$ be a family of τ-open sets such that $\mathrm{cl}_\tau(H) = \bigcap U_i$. Since (ω_1, ρ) is an S-preserving refinement of (ω_1, τ) then $\mathrm{cl}_\tau(H) \setminus H$ is countable. Enumerate $\mathrm{cl}_\tau(H) \setminus H$ as $\{x_i : i < \omega\}$ and using regularity let V_i be ρ-open such that $x_i \notin V_i$ and $V_i \supseteq H$. Then $H = \bigcap_{i<\omega} U_i \cap V_i$. □

Suppose that $X = (\omega_1, \tau)$ is an O-space. If X is to be normal we must separate any pair of disjoint closed sets H and K. If both H and K are countable it is fairly straightforward using standard techniques to find disjoint open sets separating H and K. Since X is an O-space it cannot be the case that both H and K are uncountable (hence co-countable) thus one of H or K is countable. So to ensure that X is normal it suffices to separate H and K where we may assume without loss of generality that H is countable.

Lemma 3.4. *If X is an O-space, then X is normal iff for every countable closed H there exists a U open with $U \supseteq H$ such that $\mathrm{cl}(U)$ is countable.*

Proof. First suppose that X is normal. Fix H a countable closed set. Since X is right-separated, every final segment is closed, so fix K closed and uncountable such that $H \cap K = \varnothing$. Let U, V be open and separating H and K respectively. Then U is as needed, as if $\mathrm{cl}(U)$ is uncountable, it is co-countable and $\mathrm{cl}(U)$ would necessarily meet K which is a contradiction.

Now, suppose that for every countable closed H of X there exists a U open such that $U \supseteq H$ with $\mathrm{cl}(U)$ is countable. Fix H and K closed and disjoint such that H is countable and K is co-countable. As above, it suffices to find a cover of X by open sets V such that $\mathrm{cl}(V)$ meets at most one of H or K. Since $\mathrm{cl}(U)$ is countable then $\mathrm{cl}(U) \cap K$ is countable. Let $V' = X \setminus \mathrm{cl}(U)$. Then $\mathrm{cl}(V')$ is disjoint from H. It is routing to cover $\mathrm{cl}(U)$ by open V such that $\mathrm{cl}(V)$ meets at most one of H or K. Thus we have a countable cover of X as needed. □

Now we can apply the previous Lemma in order to show that S-preserving refinements preserve normality in the case of O-spaces.

Corollary 3.5. *Let (ω_1, τ) be a normal O-space and suppose that (ω_1, ρ) is an S-preserving refinement. Then (ω_1, ρ) is normal.*

Proof. It is the case that (ω_1, ρ) is an O-space since the refinement is S-preserving. Thus it suffices to show that any countable closed H can be expanded to an open U with countable closure. Fix such an H. Then since the refinement is S-preserving, it is the case that $\text{cl}_\tau(H)$ is countable. Let U be τ-open such that $\text{cl}_\tau(U)$ is countable. Then $\text{cl}_\rho(U)$ is countable and $H \subseteq U$. Thus (ω_1, ρ) is normal. □

3.2 Refinements of O-spaces

We have already seen how one can refine the topology on $\mathbb{R}$ to obtain a right separated locally compact space. This refinement works equally as well when starting with a first countable O-space. If (X, τ) is an O-space and (X, ρ) is an S-preserving refinement then (X, ρ) will also be an O-space.

We shall shortly show how to produce an S-preserving refinement using CH that refines a first countable O-space with underlying set ω_1 into a first countable locally compact O-space.

Since every O-space is homeomorphic to an O-space with underlying set ω_1, assuming CH *every* first countable O-space will have a locally compact refinement. The following Lemma, originally due to Fodor in this form, is now known in its general form as the Hajnal Free Set Lemma (see [15] Theorem 19.1):

Lemma 3.6. *(Hajnal Free Set Lemma) If $F : \omega_2 \to [\omega_2]^{\leq \omega}$ then there is a subset $A \subseteq \omega_2$ such that $|A| = \aleph_2$ and $\alpha \notin F(\gamma)$ for all $\alpha, \gamma \in F$ with $\alpha < \gamma$.*

We have already seen (see Proposition 1.16) that every O-space is locally countable. Furthermore:

Lemma 3.7. *Every locally countable S-space has cardinality $\aleph_1$.*

Proof. Suppose not. Let X be a locally countable S-space of cardinality at least $\aleph_2$. Without loss of generality, shrinking to a subspace, we may assume that $|X| = \aleph_2$. Enumerate $X = \{x_\alpha : \alpha < \omega_2\}$ and assume, without loss of generality, that $\{x_\alpha : \alpha < \omega_2\}$ is right separated. Let $F : X \to [X]^{\leq\omega}$ be such that $F(x_\alpha)$ is a countable open neighborhood of x_α. Using the Hajnal Free Set Lemma Let $A \subseteq X$ be such that $|A| = \aleph_2$ and such that for x_α, x_β if $\alpha < \beta$ then $x_\alpha \notin F(\beta)$. Then since A is right separated then A is discrete and so X contains an uncountable discrete subspace. This contradicts that X is an S-space. □

Theorem 3.8. *(CH) Every first countable O-space can be refined to a locally compact O-space.*

Proof. Fix X any first countable O-space. We may assume that $|X| = \aleph_1$ by Lemma 3.7. Let $X = (x, \rho)$ be the original topology. We shall construct a sequence τ_α of locally compact, zero dimensional topologies and define $\tau = \sum_{\alpha<\omega_1} \tau_\alpha$ to be the simple limit topology. Further we shall guarantee by induction that τ refines ρ and furthermore for any $A \subseteq X$ that

$$\mathrm{cl}_\rho(A) \setminus \mathrm{cl}_\tau(A) \text{ is countable} \tag{3.1}$$

If we suppose (3.1) for the moment we have

Claim 3.9. (X, τ) *is an O-space.*

Proof. Suppose towards a contradiction that $A \subseteq X$ is such that in τ, A is closed, uncountable and that $X \setminus A$ is also uncountable. Since τ refines ρ, then then $\mathrm{cl}_\rho(A) \supseteq A = \mathrm{cl}_\tau(A)$. By (3.1) it must be the case that $\mathrm{cl}_\rho(A)$ is uncountable and co-uncountable which contracts that (X, ρ) is an O-space. □

Enumerate X as $X = \{x_\alpha : \alpha < \omega_1\}$. Let $X \upharpoonright \gamma = \{x_\xi : \xi < \gamma\}$. Using CH enumerate $[X]^\omega$ as $\{y_\xi : \xi < \omega_1\}$. Let $\mathcal{A}_\alpha$ defined as $\mathcal{A}_\alpha = \{y_\xi : \xi < \alpha, y_\xi \subseteq X \upharpoonright \alpha, x_\alpha \in \mathrm{cl}(\rho)(y_\xi)\}$. Enumerate each $\mathcal{A}_\alpha = \{A_n^\alpha : n < \omega\}$ such that each $y_\xi \in \mathcal{A}_\alpha$ is enumerate ω times.

Suppose that we have already constructed a sequence of topologies $\{\tau_\beta : \beta < \alpha\}$ such that each τ_β is a topology on $X \upharpoonright \beta$ satisfying:

(i) If $\gamma < \beta$ then τ_β is a conservative extension of τ_γ.

(ii) If $x_\gamma \in \mathrm{cl}(y)$ for $y \in \mathcal{A}_\beta$ then $x_\gamma \in \mathrm{cl}_{\tau_{\beta+1}}(y)$.

(iii) Each τ_β is a zero-dimensional locally compact topology on $X \upharpoonright \beta$.

If α is a limit ordinal then there is essentially only one thing to do: define $\tau_\alpha = \sum_{\beta<\alpha} \tau_\beta$. Thus suppose that $\alpha = \beta + 1$ for some β. The relevant topological observation to make is that $(X \upharpoonright \alpha, \rho \upharpoonright \alpha)$ is a first countable, hence second countable space and hence metrizable. (see [12] 4.2.9) Fix δ_α which witnesses that $(X \upharpoonright \alpha, \rho \upharpoonright \alpha)$ is metrizable. Construct a sequence of y_n and ε_n such that

(a) $y_n \in X \upharpoonright \alpha$ such that $y_n \in A_n^\beta$,

(b) $\delta_\alpha(y_n, x_\beta) < 2^{-n}$,

(c) $\varepsilon_n \in \mathbb{R}$,

(d) $x_\beta \notin B_{\varepsilon_n}(y_n)$

Let $u_n \subseteq B_{\varepsilon_n}(y_n)$ be compact open in τ_β. Declare V_k to be

$$V_k = \{x_\beta\} \cup \bigcup_{n>k} u_n$$

and let τ_α be τ_β together with $\rho \upharpoonright \alpha$ and $\{V_k : k < \omega\}$. It is clear that each V_k is compact in τ_α provided that it is closed on τ_α.

Claim 3.10. *V_k is closed in τ_α for each $k < \omega$.*

Proof. Fix $z \in X \restriction \alpha \setminus V_k$. It suffices to produce an open neighborhood U of z such that $U \cap V_k = \varnothing$. Let $\varepsilon = \delta_\alpha(z, x_\beta)$. Then for all but finitely many u_n it is the case that

$$\delta_\alpha(u_n, z) > \frac{\varepsilon}{3}$$

Let m be such that $B_{\frac{\varepsilon}{3}}(z) \cap u_n = \varnothing$ for all $n \geq m$. We may assume without loss of generality that $k < m$. Now since τ_β is zero dimensional there is some open neighborhood V of z such that

$$V \cap \bigcup_{k<n<m} u_n = \varnothing$$

Then $U = B_{\frac{\varepsilon}{3}}(z) \cap V$ works. □

This establishes (iii) above and both (i) and (ii) are immediate from the definition of τ_α. It suffices to establish that (X, τ) satisfies (3.1). Indeed for any uncountable A let α be large enough such that $y_\xi \in \mathcal{A}_\alpha$ is dense in A and $x_\alpha \in A \setminus \mathrm{cl}_\tau(A)$. Then we have that $x_\alpha \in \mathrm{cl}_{\tau_{\alpha+1}}(y_\xi)$ and hence $x_\alpha \in \mathrm{cl}_\tau(y_\xi)$ which contradicts the choice of x_α. □

To construct a normal refinement of an O-space using CH it suffices to ensure that countable closed sets H are separated from every co-countable closed set K such that $H \cap K = \varnothing$. If both H and K are countable then they can be separated since X is regular and H and K can be separated in a countable subspace of X. (see [12] 3.8.2) Spaces for which disjoint closed sets can be separated by open sets whenever at least on of them is countable are called *pseudo-normal*. For O-spaces normality and pseudo-normality coincide. This leaves this following question in light of Theorem 3.8.

Question 3.11. *(CH) If there is a first countable O-space is there a locally compact, normal, first countable O-space?*

3.3 Refinements Using $\stick$

In the preceding sections CH was used to produce S-preserving refinements of various topological spaces X of size $\aleph_1$. In the proofs, CH was used to enumerate $[X]^\omega$ in type ω_1. The proofs that such refinements are S-preserving do not require that $[X]^\omega$ is enumerated entirely, but merely that there is some $\{x_\alpha : \alpha < \omega_1\}$ that is a $\stick$-sequence with respect to $[X]^{\omega_1}$. That is, that for any $Y \in [X]^{\omega_1}$ there is some $x_\alpha \subseteq Y$.

Using this observation, we can strengthen Theorem 3.8 above to:

Theorem 3.12. *($\stick$) Every first countable O-space can be refined to a locally compact O-space.*

3.4 Refinements using $\mathfrak{b} = \omega_1$

In the preceding sections we have constructed S-preserving refinements of first-countable topologies using CH. In this section we shall examine constructing S-preserving refinements of first-countable topologies using $\mathfrak{b} = \omega_1$.

Recall that Baire space is the topology on ω^ω generated by the basic open neighborhoods $[\sigma] = \{x \in \omega^\omega : \sigma \subseteq x\}$ where $\sigma \in \mathrm{Fn}(w, 2)$. Baire space is zero dimensional and second countable. Hence it is first-countable, hereditarily separable, hereditarily Lindelöf and therefore normal. Todorčević has shown in [38] that is is possible to produce an S-preserving refinement of Baire space to produce a locally compact S-space. (see [38] Theorem 2.4)

Theorem 3.13. *($\mathfrak{b} = \omega_1$) There is an S-preserving refinement of Baire space to a locally compact space.*

The topology on ω^ω and the properties of a family $A = \{f_\alpha : \alpha < \omega_1\}$ witnessing that $\mathfrak{b} = \omega_1$ are not unrelated. Thus it is not altogether surprising that an S-preserving refinement can be constructed on Baire space using $\mathfrak{b} = \omega_1$. However what can proved assuming $\mathfrak{b} = \omega_1$ instead of CH is slightly weaker. For under CH we can construct and S-preserving refinement of any first-countable S-space to produce a locally compact one. This leaves open the following question:

Question 3.14. *($\mathfrak{b} = \omega_1$) If X is a a first-countable S-space does X have an S-preserving refinement that is a locally compact space?*

Since Baire space is second-countable it is far from containing any uncountable O-subspace as no second-countable space can contain an O-space as a subspace. Thus, while assuming CH we could show that if there was a first-countable O-space there was a locally compact one, we still have open the following question:

Question 3.15. *($\mathfrak{b} = \omega_1$) Assume there is a first-countable O-space. Is there a locally compact O-space?*

Chapter 4

Almost Left-Separated Spaces

4.1 Introduction

In this chapter we examine a connection between S-spaces and the P-ideal dichotomy. The inspiration for this examination comes from an ideal defined for locally compact S-space defined in [10] in order to prove the following Theorem:

Theorem 4.1 (Eisworth, Nyikos, Shelah)**.** *It is consistent with $2^{\aleph_0} < 2^{\aleph_1}$ that there are no locally compact S-spaces.*

There are in fact a number of ideals that can be naturally defined from a given S-space X. These ideals will have some natural relations to each other, and it will turn out to be very interesting in the case that these ideals turn out to be P-ideals.

Definition 4.2. *An ideal $\mathfrak{I} \subseteq [\omega_1]^\omega$ is called a **P-ideal** if for every sequence I_n $(n < \omega)$ of elements of $\mathfrak{I}$, there is some $J \in \mathfrak{I}$ such that $I_n \subseteq^* J$.*

To define these ideals, fix $X = \{x_\alpha : \alpha < \omega_1\}$ to be an S-space and make the definition:

Definition 4.3. *Let $\mathfrak{I} = \{Y \subseteq X : Y$ is closed and discrete $\}$*

Note that since X is an S-space, then each $Y \in \mathfrak{I}$, being discrete, is necessarily countable. Furthermore note that $\mathfrak{I}$ is indeed an ideal.

Definition 4.4. *Let $\{V_\alpha : \alpha < \omega_1\}$ be a cover of X by clopen sets such that $x_\alpha \in V_\alpha$ for each $\alpha < \omega_1$. Let $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ be defined as*

$$\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}} = \{Y \in [X]^{\leq\omega} : Y \cap V_\alpha \textit{ is finite for each } \alpha < \omega_1\}$$

Note that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is an ideal, and like $\mathcal{I}$, it contains all finite subsets of X. The ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is the aforementioned ideal that occurs in [10]. In [10] it was the case that X was a locally compact S-space and that each of V_α was compact open. In this case $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ has some very interesting applications provided that it is a P-ideal. We shall examine conditions under which $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal shortly.

The P-ideal Dichotomy (PID) is the statement that for any P-ideal $\mathcal{I}$ either

1. there is some $A \subseteq [\omega_1]^{\omega_1}$ such that $[A]^\omega \subseteq \mathcal{I}$ or

2. ω_1 can be partitioned into countably many sets S_i such that each $[S_i]^\omega \cap \mathcal{I} = \varnothing$.

PID is due to Abraham and Todorčević, who showed in [2] that (PID) follows from PFA and is, more immediately relevant, consistent with CH.

For any family of sets $\{V_\alpha : \alpha < \omega_1\}$ in order the ensure that the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is in fact a P-ideal it suffices to assume that $\mathfrak{b} > \omega_1$.

Theorem 4.5 (Eisworth, Nyikos, Shelah)**.** *($\mathfrak{b} > \omega_1$) For any collection of sets $\{V_\alpha : \alpha < \omega_1\}$ with $x_\alpha \in V_\alpha$ for each $\alpha < \omega_1$ it is the case that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal.*

Proof. Fix $\{V_\alpha : \alpha < \omega_1\}$ with $x_\alpha \in V_\alpha$ for each $\alpha < \omega_1$. Let $I_n \in \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ for $n < \omega$. We may assume without loss of generality that the I_n are pairwise disjoint. Enumerate each I_n as $I_n = \{x_n^i : i < \omega\}$. For each $\alpha < \omega_1$ define $f_\alpha : \omega \to \omega$

$$f_\alpha(n) = \inf\{i : V_\alpha \cap I_n \subseteq \{x_n^j : j \leq i\}\}$$

Since $\mathfrak{b} > \omega_1$ there is some $r \in \omega^\omega$ such that $f_\alpha <^* r$ for each $\alpha < \omega_1$. Define $I \in [X]^\omega$ by

$$I = \bigcup_{n<\omega} I_n \setminus \{x_n^i : i \leq r(n)\}$$

By definition of I it is the case that $I_n \subseteq^* I$ for each $n < \omega$. It suffices to show that $I \in \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$. Fix V_α and let $m < \omega$ be such that for all $k \geq m$ that $r(k) > f_\alpha(k)$. Then for $i \in k$ we have that

$$V_\alpha \cap I_n \subseteq \{x_k^i : i \leq f_\alpha(n)\} \subseteq \{x_k^i : i \leq r(n)\}$$

and so $V_\alpha \cap I$ is finite. □

In the case that the family of sets $\{V_\alpha : \alpha < \omega_1\}$ are all open, it is the case that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is contained in $\mathfrak{I}$.

Proposition 4.6. *Suppose that $\{V_\alpha : \alpha < \omega_1\}$ is a collection of open subsets of X. Then $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}} \subseteq \mathfrak{I}$.*

Proof. Fix $\{V_\alpha : \alpha < \omega_1\}$ a collection of open subsets of X. Fix $I \in \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$, it suffices to show that I is closed and discrete. To see that I is closed, for any $x_\alpha \notin I$ there is an open neighborhood V_α with $x_\alpha \in V_\alpha$ such that $V_\alpha \cap I$ is finite. Thus $x \notin \mathrm{cl}(I)$ and hence I is closed. The family $\{V_\alpha : \alpha < \omega_1\}$ witness that I is a locally finite subspace of X and hence is discrete.

□

In the class of locally compact S-spaces, the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ sometimes corresponds to another ideal which we shall now define. Suppose that $\{A_\alpha : \alpha < \omega_1\}$ is a collection of closed discrete subsets of X.

Definition 4.7. *Let $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \{Y \subseteq X : Y \subseteq^* \bigcap_{\alpha\in F} A_\alpha$ for some $F \in [\omega_1]^{<\omega}\}$*

An ideal defined as in $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ above is often called the **ideal generated by** the family $\{A_\alpha : \alpha < \omega_1\}$.

In the next section we shall examine a class of ideals $\mathcal{K}_{\{A_\alpha:\alpha<\omega\}}$ that are generated by particular families of $\{A_\alpha : \alpha < \omega_1\}$. In particular, if the $\{A_\alpha : \alpha < \omega_1\}$ have a natural property the ideal $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ can be made to be a P-ideal.

4.2 Almost Left-Separated Spaces

Recall that a space X is left-separated (see page 2) in type ω_1 if X can be enumerated as

$$X = \{x_\alpha : \alpha < \omega_1\}$$

with the property that $X_\alpha = \{x_\xi : \xi < \alpha\}$ is closed for each $\alpha < \omega_1$. Equivalently, X is left-separated if each final segment $X \setminus X_\alpha$ is open.

Remark 4.8. *If X is left-separated then X contains no uncountable separable space.*

Thus an S-space can be thought of as the antithesis of a left-separated space. Somewhat surprisingly an S-space can be *almost* left-separated if we make the following definition:

Definition 4.9. *An uncountable space X is called **almost left-separated** if X can be enumerated as $X = \{x_\alpha : \alpha < \omega_1\}$ such that there exists a sequence $\{A_\alpha : \alpha < \omega_1\} \subseteq [X]^{\leq\omega}$ such that*

(i) each A_α is closed,

(ii) $A_\alpha \subseteq X_\alpha = \{x_\xi : \xi < \alpha\}$,

(iii) if $\alpha < \beta$ then $A_\alpha \subseteq^* A_\beta$,

(iv) $\{\mathrm{otype}(A_\alpha) : \alpha < \omega_1\}$ *is uncountable and* $X = \bigcup_{\alpha<\omega_1} A_\alpha$.

Of course it is clear from the definition that any left-separated space is almost left-separated by choosing A_α to be X_α.

Remark 4.10. *If X is left-separated then X is almost left-separated.*

If X is almost left-separated as witnessed by the sequence $\{A_\alpha : \alpha < \omega_1\}$, then we shall call $\{A_\alpha : \alpha < \omega_1\}$ an **almost left-separating sequence** for X.

In particular every L-space, since left-separated, is almost left-separated. However, more interestingly, there are S-spaces that are almost left-separated. On the one hand this is surprising, as if the $\subseteq^*$ of Definition (4.9) part (iii) is replaced by $\subseteq$ then no S-space X can possibly satisfy Definition (4.9).

Part (iv) of Definition 4.9 is necessary to avoid some trivialities. If we replace (iv) with

(iv′) $\bigcup_{\alpha<\omega_1} A_\alpha = X$

it is fairly straightforward to construct an S-space X and sequence $\{A_\alpha : \alpha < \omega_1\}$ satisfying (i)-(iii) of Definition 4.9 and (iv′) above. Let us call such an S-space a **trivially almost left-separated** ***S*****-space**.

Proposition 4.11. *Assume there is an S-space, then there is a trivially almost left-separated S space.*

Proof. Fix X an S-space of size $\aleph_1$. Let $\mathscr{T} = \{a_\xi : \xi < \omega_1\}$ be an increasing $\subseteq^*$ tower, i.e. for $\xi < \eta$ we have that $a_\xi \subseteq^* a_\eta$, in $[\omega]^\omega$. Define a topology τ on $X \dot\cup \omega$ such that X is open and ω is discrete in τ. Enumerate $X = \{x_\alpha : \alpha < \omega_1\}$ and define $A_\alpha = a_\alpha \cup \{x_\xi\}$. Then the family of A_α for $\alpha < \omega_1$ witnesses that $\langle X \dot\cup \omega, \tau\rangle$ is a trivially almost left-separated S-space. $\square$

Spaces that are left-separated, have the property that every subspace is also left-separated. Such spaces are called hereditarily left-separated. It is not the case that an almost left-separated space X that is separated by the almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ will be necessarily hereditarily almost left-separated. For a particular $Y \subseteq X$ the sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ may work and it may not for it may be the case that

$$Y \cap A_\alpha \text{ is finite, or of bounded order type}$$

for every $\alpha < \omega_1$. We shall investigate the existence of hereditarily almost left-separated spaces in §4.4.1.

Recall that the ideal $\mathcal{K}_{\{A_\alpha : \alpha < \omega_1\}}$ was defined to be the ideal generated by the sets $\{A_\alpha : \alpha < \omega_1\}$. If X is almost left-separated space with almost left separating sequence $\{A_\alpha : \alpha < \omega_1\}$, then the sequence $\{A_\alpha : \alpha < \omega_1\}$ will generate a P-ideal.

Almost left-separated spaces are most interesting in the case that the space is hereditarily separable for a reason that we now investigate. In the language of [37], a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle \subseteq [\omega_1]^\omega$ is *coherent* if

$$\alpha < \beta \implies E_\beta \cap \alpha =^* E_\alpha$$

Furthermore the sequence is *non-trivial* if there does not exist some $E \in [\omega_1]^{\omega_1}$ such that

$$\forall \alpha < \omega_1 [E \cap \alpha =^* E_\alpha]$$

It is often possible to have the almost left separating sequence $\{A_\alpha : \alpha < \omega_1\}$ be coherent in which case we make the following definition:

Definition 4.12. *An uncountable space X is called* ***coherently almost left-separated*** *if is almost left-separated by the sequence $\{A_\alpha : \alpha < \omega_1\} \subseteq [X]^{\leq \omega}$ and in addition for $\alpha < \beta$, $A_\beta \cap \alpha =^* A_\alpha$.*

It is clear from the definitions that any coherently almost left-separated space is almost left separated. In particular, the sequence of A_α for $\alpha < \omega_1$ form a coherent sequence whenever X is coherently almost left-separated, and provide the underlying space X is hereditarily separable, then this sequence will be nontrivial.

Proposition 4.13. *Assume that X is hereditarily separable and coherently almost left-separated by* $\langle A_\alpha : \alpha < \omega_1 \rangle$. *Then* $\langle A_\alpha : \alpha < \omega_1 \rangle$ *is a non trivial coherent sequence.*

Proof. Fix such an X and almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$. By Definition 4.12 the sequence is coherent. Assume towards a contradiction that it is trivially coherent and fix some $E \in [X]^{\omega_1}$ such that

$$A_\alpha =^* E \cap X_\alpha$$

for all $\alpha < \omega_1$. Then let $F \subseteq E$ be countable and dense in E. Let α be such that $F \subseteq X_\alpha \cap E$. However, then $A_\alpha =^* F$, which is closed, contradicting that F is dense in E. □

We can now connect the notion of almost left-separated space with the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$. Fix $X = \{x_\alpha : \alpha < \omega_1\}$ to be an almost left-separated locally compact S-space with almost left-separating sequence $\{A_\alpha : \alpha < \omega_1\}$. As previously mentioned, the ideal $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is always a P-ideal provided that $\{A_\alpha : \alpha < \omega_1\} \subseteq [X]^\omega$ is an almost left separating sequence. This follows from the fact $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is generated from the sequence $\{A_\alpha : \alpha < \omega_1\}$ and that $A_\alpha \subseteq^* A_\beta$ whenever $\alpha < \beta$. However, the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ need not be a P-ideal, but it is a P-ideal under certain circumstances which we shall now investigate.

There is a connection between the P-ideal $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ for almost left separating sequences $\{A_\alpha : \alpha < \omega_1\}$ and the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$. Given any regular space X, there is always a cover $\{V_\alpha : \alpha < \omega_1\}$ of X by basic open sets which makes the corresponding ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ contain the P-ideal $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$.

Proposition 4.14. *Suppose X is a coherently almost left-separated regular space with basis $\mathcal{B}$. Then there is a cover $\{V_\alpha : \alpha < \omega\} \subseteq \mathcal{B}$ of X such that $A_\alpha \cap V_\beta$ is finite for every $\alpha \neq \beta$.*

Proof. We construct the sequence of V_α by induction. First we shall construct a sequence $\{V_\xi : \xi < \omega_1\}$ with the property that for every $\gamma \leq \beta$, $A_\gamma \cap V_\beta$ is finite. To obtain such a sequence it suffices to choose V_α to be any neighborhood of α in $\mathcal{B}$ with the property that $V_\alpha \cap A_\alpha = \varnothing$. Such a neighborhood exists since X is regular, A_α is closed and $A_\alpha \subseteq \alpha$. Then $V_\alpha \cap A_\beta$ is finite for all $\beta \leq \alpha$.

If suffices to ensure that the cover $\{V_\alpha : \alpha < \omega_1\}$ works. So fix any A_α and V_β. If $\alpha < \beta$ then by construction V_β was chosen so that $V_\beta \cap A_\alpha$ was finite. If $\beta < \alpha$, then we have that $A_\alpha \cap V_\beta \subseteq^* A_\beta \cap V_\beta$ since $A_\alpha \restriction \beta =^* A_\beta$ and so $A_\alpha \cap V_\beta$ is finite. □

Thus we have established that for any X that is a coherently almost left-separated S-space with almost left separating sequence $\{A_\alpha : \alpha < \omega_1\}$ it is the case that we can find some family $\{V_\alpha : \alpha < \omega_1\}$ that covers X such that

$$\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} \subseteq \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}} \tag{4.1}$$

Thus, although it is not the case that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is always a P-ideal, it is the case for a particular class of spaces, e.g. the coherently almost left-separated S-spaces, that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ will contain a P-ideal. The natural question to ask is under what conditions will it be the case that equality holds in (4.1), for in this case it will follow that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal. To answer this question we shall prove the following Theorem in §4.5.2:

Theorem 4.15. *(CH) Assume that $X = (\omega_1, \tau)$ is a first countable O-space with coherent almost left-separating sequence $\{A_\alpha : \alpha < \omega_1\}$. Then there is a refinement (ω_1, ρ) of X and* *cover of X by compact ρ-open sets $\{V_\alpha : \alpha < \omega_1\}$ such that (ω_1, ρ) is a first countable, locally compact, coherently almost left-separated O-space and*

$$\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \mathcal{I}_{\{V_\alpha:\alpha<\omega_1\}}$$

In particular $\mathcal{I}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal.

4.3 Constructions using $\diamondsuit$

Up until now we have seen only trivially almost left-separated *S*-space. The natural question in light of Proposition 4.11 is whether or not there are any *S*-spaces that are almost left-separated at all. Furthermore given the hierarchy of *S*-spaces depicted in Figure 3 on page 10, if there are HFDs, weak HFDs, *O*-spaces and Ostaszewski spaces.

Recall that an Ostaszewski space is an *O*-space that is countably compact, locally compact, and perfectly normal. Such a space exists assuming $\diamondsuit$, but CH is not enough. In §4.4 we shall establish the following Theorem:

Theorem 4.16. *($\diamondsuit$) There is a locally compact, perfectly normal coherently almost left-separated O-space.*

It is natural to wonder if Theorem 4.16 can be strengthened. That is, assuming $\diamondsuit$ is there an almost left-separated Ostaszewski space?

Lemma 4.17. *An almost left-separated S-space is never countably compact.*

Proof. Let X be and *S*-space and suppose that $\langle A_\alpha : \alpha < \omega_1\rangle$ an almost left-separating sequence of X. Suppose that X is countably compact. Since X is countably compact, then each

A_α is compact, and furthermore infinite for a co-bounded subset of ω_1. Hence, without loss of generality, each has some limit point $a_\alpha \in A_\alpha$ which is a limit of the sequence

$$a_\alpha^n \to a_\alpha$$

such that $\{a_\alpha^n : n < \omega\} \subseteq A_\alpha$. Since X is right separated then we may assume that $a_\alpha^n < a_\alpha$ for all $n < \omega$.

Now consider the subspace E of X formed by

$$E = \{a_\alpha : \alpha < \omega_1\}$$

and since X is hereditarily separable let $E_0 \subseteq E$ be countable and dense. Fix $\eta < \omega_1$ such that $E_0 \subseteq X_\eta$. Now since $\langle A_\alpha : \alpha < \omega_1 \rangle$ is almost left-separated then $A_\xi \subseteq^* A_\eta$ for each $\xi < \eta$. In particular

$$\{a_\xi^n : n < \omega\} \subseteq A_\xi \subseteq^* A_\eta$$

whenever $\xi < \eta$. Thus $a_\xi \in A_\eta$ whenever $\xi < \eta$ and so $E_0 \subseteq A_\eta$. This contradicts that E_0 has uncountable closure. $\square$

The natural dual of Lemma 4.17 is not true. That is, given an uncountable countably compact space (such as an Ostaszewski space) there is no uncountable almost left-separated subspace. This is not true, in Theorem 4.20 we shall construct, using ♣+CH, an Ostaszewski space that has an uncountable almost left-separated subspace.

4.4 Spaces Under ♣ and ◇

Ostaszewski originally constructed an Ostaszewski space in [28] by using $\clubsuit + CH$. Those familiar with the standard proof will recall that CH was used to guarantee that the Ostaszewski space is countably compact. In the construction of an almost left-separated space we cannot get the space to be both hereditarily separable and also countably compact by Lemma 4.17. In not attempting to build a countably compact space, we have no immediate need of CH and we shall construct an almost left-separated locally compact O-space assuming $\clubsuit$.

Theorem 4.18. *($\clubsuit$) There is a locally compact, coherently almost left-separated O-space.*

The O-space is constructed as a simple limit construction (see Definition 1.21) of a topology $\tau = \sum_{\alpha<\beta} \tau_\alpha$ on ω_1 which is a right separated, locally compact, almost left-separated O-space. Since $\clubsuit$ is consistent with $\neg$CH, we will have also established the following interesting Corollary:

Corollary 4.19. *It is consistent with $\neg$CH that there is a locally compact, coherently almost left-separated O-space.*

Assuming $\diamondsuit$, we can prove the following stronger result, which has Theorem 4.18 as an immediate corollary of its proof. Furthermore, we will have established Theorem 4.16 since perfect normality is a hereditary property.

Theorem 4.20. *($\clubsuit$ + CH) There is an Ostaszewski space which has an uncountable coherently almost left-separated subspace.*

Proof. We shall inductively construct a sequence of topologies τ_α on $\alpha < \omega_1$ where

(i) each τ_α is locally compact and zero-dimensional,

(ii) for $\alpha < \beta$, $\tau_\beta \restriction \alpha = \tau_\alpha$

We perform an induction on $\mathrm{Lim}(\omega_1)$. For $\alpha \in \mathrm{Lim}(\omega_1)$ let

$$\alpha^+ = \inf(\mathrm{Lim}(\omega_1) \setminus \alpha)$$

Fix a $\clubsuit$-sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ on ω_1 . Fix an increasing enumeration of each x_α as $x_\alpha = \{\alpha_i : i < \omega\}$. Simultaneously construct a sequence $\{A_\alpha : \alpha \in \omega_1\}$ such that

(iii) $A_\alpha \subseteq \alpha$ is infinite for $\alpha \in \mathrm{Lim}^2(\omega_1)$, and for other if $\alpha = \beta + \omega$ then $A_\alpha \cap [\beta, \beta + \omega) \neq \varnothing$.

(iv) A_α is relatively closed discrete,

(v) for $\alpha < \beta$, $A_\beta \restriction \alpha =^* A_\alpha$,

The requirement that A_α are closed discrete is a necessary one, in light of Lemma 4.17. The almost left-separated subspace will be

$$Y = \bigcup_{\alpha < \omega_1} A_\alpha$$

and the relatively of requirement (iv) is relative to this space Y.

In order to guarantee that the resultant space is an O-space we have the following inductive requirements:

(vii) for any $\alpha < \beta$ the closure of $[\alpha, \alpha + \omega)$ in τ_β is $[\alpha, \beta)$

(viii) the closure of x_α is $[\alpha, \alpha + \omega)$ in τ_{α^+}.

As a bookkeeping measure, to guarantee that the resulting space τ_{ω_1} is countably compact, we fix an enumeration

$$[\omega_1]^\omega = \{B_\alpha : \alpha \in \omega_1\}$$

First, to see that a space meeting (i)-(viii) is indeed a locally compact O-space first note that (ω_1, τ) is locally compact since each τ_α is locally compact. It remains to verify that the space is an O-space. Fix any uncountable subset $Y \in [\omega_1]^{\omega_1}$. Since $\{x_\alpha : \alpha < \omega_1\}$ is a $\clubsuit$-sequence fix $x_\alpha \subseteq Y$. Then by (vii)-(viii) x_α will have co-countable closure. Hence Y has a co-countable closure. Thus (ω_1, τ) is an O-space.

Now it remains to do the actual work of constructing a sequence of τ_α and A_α satisfying (i)-(viii). Suppose that τ_α has been constructed and $\langle A_\xi : \xi < \alpha \rangle$ satisfying (i)-(viii). There are two cases: $\alpha = \beta^+$ for some unique $\beta \in \mathrm{Lim}(\omega_1)$, or α is a limit of ordinals in $\mathrm{Lim}(\omega_1)$.

The easy case: if $\alpha \in \mathrm{Lim}^2(\omega_1)$ (a limit of limits), then let $\tau_\alpha = \sum_{\xi \in \mathrm{Lim}(\alpha)} \tau_\xi$. This guarantees (i)-(ii) and (vii)-(viii). Let $\{\xi_i : i < \omega\} \subseteq \mathrm{Lim}(\alpha)$ be unbounded in α.

Define A_α as

$$A_\alpha = A_{\alpha_0} \cup \bigcup_{i<\omega} A_{\xi_{i+1}} \setminus \xi_i \tag{4.2}$$

We may choose such ξ_i to guarantee A_α satisfies (iii).

Claim 4.21. *A_α is discrete.*

Proof. Since τ_α is zero dimensional then is is regular. Since it is right separated it suffices to find a neighborhood of α_i which is disjoint from $\bigcup_{j<i} A_{\alpha_j}$. Since each is closed in $\tau_{\alpha_{i+1}}$ then there is such a neighborhood and hence A_α is discrete. □

Claim 4.22. *A_α is closed.*

Proof. It suffices to demonstrate that the family

$$\{A_{\alpha_0}\} \cup \{A_{\alpha_{i+1}} \setminus \alpha_i : i < \omega\}$$

is locally finite.(See [12].) Note that this family is pairwise disjoint. Fix any $\xi < \alpha$. Let α_i be least such that $\xi < \alpha_i$. Since τ_α is right separated, there is a neighborhood of ξ which meets only $A_{\alpha_{j+1}} \setminus \alpha_j$ for $j < i$. □

Claim 4.23. *For any $\xi < \alpha$, $A_\alpha \restriction \xi =^* A_\xi$.*

Proof. Fix $\xi < \alpha$. Let α_i be least such that $\xi < \alpha_i$. Note that this suffices, since $A_{\alpha_i} \restriction \xi =^* A_\xi$, to show that $A_\alpha \restriction \alpha_i =^* A_{\alpha_i}$. From (4.2) it is immediate that $A_\alpha \restriction \alpha_i =^* A_{\alpha_i}$. □

The preceding claims are what is needed to demonstrate that A_α satisfies (iii)-(vi). It is immediate that A_α satisfies (vi) and (iii).

The other case: Suppose that $\alpha = \beta^+$, i.e. that $\alpha = \beta + \omega$. To define τ_α is suffices to find a conservative extension of τ_β and is necessary to define a local base at each $\beta + n$ for $n < \omega$. Let $B \subseteq \beta$ be such that $B = B_\xi$ with ξ least such that B_ξ is a closed discrete subspace of τ_β. Without loss of generality we may assume that $B \cap x_\beta = \varnothing$.

Let $x_\beta = \{\beta_i : i < \omega\}$. Fix a sequence of A_{ξ_i} such that $A_{\xi_i} \subseteq \beta_i$ and such that $\{\xi_i : i < \omega\}$ is unbounded in β. Choose compact open neighborhoods K_i for $i < \omega$ such that

(a) $\beta_i \in K_i$,

(b) $A_{\xi_j} \cap K_i = \varnothing$ for $j < i$

and choose compact open neighborhoods F_i for $i < \omega$ such that the collection $\{K_i : i < \omega\} \cup \{F_i : i < \omega\}$ forms a partition of β into compact open sets and furthermore such that

$$B^* = \{i < \omega : F_i \cap B \neq \varnothing\}$$

is infinite.

It remains to define the local base for each $\beta+k$. Let the local base at $\beta+k$ be generated by the collection of $\{V_n(\beta+k) : n<\omega\}$. In the case that $k=0$ let $V_n(\beta+k)$ be

$$V_n(\beta+k)=\{\beta\}\cup\bigcup_{n<i<\omega}K_i$$

Since for each j each V_n eventually misses A_{ξ_j} this will guarantee that β is not in the closure of A_{ξ_j} in τ_α.

For $k>0$, let

$$V_n(\beta+k)=\{\beta+k\}\cup\bigcup_{n<i<\omega}K_i\cup F_i$$

this guarantees that B is no longer closed discrete. Furthermore this choice of V_n guarantees that the closure of any co-bounded subset of β is co-bounded in $\beta+\omega$.

Define $A_{\beta+k}$ and A_α as

$$\{\beta\}\cup A_{\xi_0}\cup\bigcup_{i<\omega}A_{\xi_{i+1}}\setminus\beta_i$$

As proved above, this will be closed in τ_α, and $A_\alpha\cap\xi=^* A_\xi$ for any $\xi<\alpha$.

The fact that $\{\text{otype}(Y\cap A_\alpha):\alpha<\omega_1\}$ is uncountable can be proved by induction on $\eta<\omega_1$ using (iii) and the fact that $\langle A_\alpha:\alpha<\omega_1\rangle$ is coherent. (see Claim 4.34) □

The statement ♣ is actually equivalent to the apparently stronger assertion

> There is a sequence $\{x_\alpha:\alpha\in\text{Lim}(\omega_1)\}$ such that for any uncountable $Y\subseteq\omega_1$ the set $\{\alpha:x_\alpha\subseteq Y\}$ is stationary. (♣$^+$)

The following Lemma (see [13]) will be used to prove Theorem 4.25 below.

Lemma 4.24. *♣ and ♣$^+$ are equivalent.*

Assuming ◊, we have seen that there is an Ostaszewski space and also and almost left-separated locally compact O-space. Usually under ◊ one can get almost anything. In this case we can build a locally compact O-space that avoids all Ostaszewski spaces and all almost left-separated spaces.

Theorem 4.25. *($\clubsuit^+$) There is a locally compact O-space which contains no uncountable almost left-separated subspace and no uncountable countably compact subspace.*

Proof. As above, we shall construct a space (ω_1, τ) in ω_1 steps as a simple limit of topologies τ_α on $\alpha < \omega_1$. Since the resulting space will be an O-space many of the inductive hypotheses will be the same as in Theorem 4.20. Fix club sequences $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ and $\{y_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that $x_\alpha \cap y_\alpha = \varnothing$ for all $\alpha \in \mathrm{Lim}(\omega_1)$. Fix a club $\{N_\alpha : \alpha \in E \subseteq \mathrm{Lim}(\omega_1\}$ of $N_\alpha \prec H(\theta)$, for θ large enough, such that $N_\alpha \cap \omega_1 = \alpha$ for all $\alpha \in E$.
We shall inductively construct a sequence of topologies τ_α on $\alpha < \omega_1$ where

(i) each τ_α is locally compact and zero-dimensional,

(ii) for $\alpha < \beta$, $\tau_\beta \upharpoonright \alpha = \tau_\alpha$

In order to guarantee that the resultant space is an O-space we have the following inductive requirements:

(iii) for any $\alpha < \beta$ the closure of $[\alpha, \alpha + \omega)$ in τ_β is $[\alpha, \beta)$

(iv) the closure of x_α is $[\alpha, \alpha + \omega)$ in τ_{α^+}.

To guarantee that the resulting space contains no uncountable countably compact subspace it suffices to satisfy the requirement:

(v) y_α is closed and discrete in τ_β for $\beta \geq \alpha^+$.

At stage α of the construction there are two cases. First enumerate $\{y_\xi : \xi < \alpha\}$ as $\{y_i : i < \omega\}$ and let $x_\alpha = \{\alpha_i : i < \omega\}$ be an increasing enumeration.

Case 1: $\alpha \notin E$. Construct a partition of α into compact open sets in τ_α such that $\{U_i : i < \omega\}$ is a subset of this partition and

(a) $U_i \cap y_\alpha = \varnothing$

(b) $K_i = \{j < i : \alpha_i \in y_j\}$ and $U_i \cap \bigcup_{m \in i \setminus K_i} y_m = \varnothing$

For any y_j we have that $y_j \cap x_\alpha$ is finite, since $\sup(y_j) < \alpha$. Thus for each j there is some $i > j$ such that $j \notin K_i$ and hence $U_i \cap y_j = \varnothing$.

To define the local base for each $\alpha + k$ let the local base at $\alpha + k$ be generated by the collection of $\{V_n(\alpha + k) : n < \omega\}$. Let $V_n(\alpha + k)$ be

$$V_n(\alpha + k) = \{\alpha + k\} \cup \bigcup_{n < i < \omega} U_i$$

This choice of V_n guarantees that the closure of any co-bounded subset of α is co-bounded in $\alpha + \omega$. Furthermore since $U_i \cap y_j$ for each j and i large enough, then y_j remains closed in $\tau_{\alpha+\omega}$. Also by (a), y_α is closed discrete in $\tau_{\alpha+\omega}$.

Case 2: $\alpha \in E$. For each $i < \omega$ let $A(i)$ be defined as

$$A(i) = \{B \subseteq \alpha_i : B \text{ is closed discrete and } B \text{ is not covered by a finite union of elements of } \{y_i : i < \omega\}\} \tag{4.3}$$

Enumerate $A(i) \cap N_\alpha$ as $\{A^k_{\alpha_i} : k < \omega\}$. Construct a partition of α into compact open neighborhoods such that this partition contains a subset $\{U_i : i < \omega\}$ such that

(a) $U_i \cap y_\alpha = \varnothing$

(b) $K_i = \{j < i : \alpha_i \in y_j\}$ and $U_i \cap \bigcup_{m \in i \setminus K_i} y_m = \varnothing$

(c) $U_i \cap A^m_{\alpha_j} \neq \varnothing$ for $j, m < i$ and in particular

$$U_i \cap \left(A^m_{\alpha_j} \setminus \bigcup_{k<i} U_k \right) \neq \varnothing$$

Such a sequence U_i exists since no $A^k_{\alpha_i}$ is covered by a finite union of $\{y_i : i < \omega\}$.

As before define the local base for each $\alpha + k$ to be generated by the collection of $\{V_n(\alpha + k) : n < \omega\}$. Let $V_n(\alpha + k)$ be

$$V_n(\alpha + k) = \{\alpha + k\} \cup \bigcup_{n<i<\omega} U_i$$

Claim 4.26. *If for some k, $A_\alpha \restriction \alpha_i =^* A^k_{\alpha_i}$ then $\mathrm{cl}_{\tau_{\alpha+\omega}} A_\alpha \supseteq [\alpha, \alpha + \omega)$*

Proof. This follows from the fact that $A^k_{\alpha_i} \cap \bigcup_{i<\omega} U_i$ is infinite by the requirement (c) above. □

It remains to show that the resulting space is a locally compact O-space which contains not uncountable almost left-separated subspace and no uncountable countably compact subspace. Fix $Y \in [\omega_1]^{\omega_1}$. Then Y contains some y_ξ, which is closed and discrete in Y and hence Y is not countably compact.

Suppose towards a contradiction that Y is almost left-separated by the sequence $\langle A_\xi : \xi < \omega_1 \rangle$. Then by the definition of almost left-separated it is the case that the set $\{\mathrm{otype}(A_\xi) : \xi < \omega_1\}$ is uncountable.

Fix N_α such that $Y \in N_\alpha$ and also $x_\alpha \subseteq Y$. By the previous claim using elementarity of N_α some $A_\xi = A^k_{\alpha_i}$ for some $i, k < \omega$. Then by Claim 4.26, some $\alpha > \xi$ must fail to be countable and closed. Hence $\langle A_\xi : \xi < \omega_1 \rangle$ cannot be an almost left-separating sequence. □

By examining the proof it is fairly straightforward to see that we have actually proved something slightly stronger.

Theorem 4.27. *($\clubsuit^+$) There is a locally compact O-space containing no uncountable countably compact subspace, and every uncountable subspace contains no closed discrete subspace of order type more than ω^2.*

4.4.1 Hereditarily Almost Left Separated Spaces

Recall that a space is called **hereditarily-**P if every subspace of X has property P. It is the case that any left-separated space is hereditarily left-separated but this need not hold in the case of almost left-separated spaces. Let X is an almost left-separating space with almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ and let $Y \subseteq X$. It may be the case that there exists some $\eta < \omega_1$ such that

$$\{\text{otype}(Y \cap A_\alpha) : \alpha < \omega_1\} \subseteq \eta$$

even in the case that $\eta < \omega$. In such a case, it won't be that Y is almost left-separated by the almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$. However, it is possible to build a hereditarily almost left-separated locally compact O-space using $\clubsuit$.

In the construction of the almost left-separated O-space in Theorem 4.18 it was the case that the almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ was constructed simultaneously with the topology on ω_1 making $X = (\omega_1, \tau)$ a locally compact almost left-separated O-space. However it is possible, in certain circumstances, to start with an existing coherent sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ and construct a space $X = (\omega_1, \tau)$ making $\langle A_\alpha : \alpha < \omega_1 \rangle$ and almost left-separating sequence for X. Provided that the almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ is not entangled with the $\clubsuit$-sequence it is possible to use the usual Ostaszewski construction of Theorem 4.18 to build a space that is almost left-separated by $\langle A_\alpha : \alpha < \omega_1 \rangle$. We shall need the following definition:

Definition 4.28. *A coherent sequence of sets* $\{A_\alpha : \alpha < \omega_1\}$ ***avoids*** *a sequence* $\{x_\alpha : \alpha \in E\}$ *if it is the case that for all* $\alpha \in E$ *that* $x_\alpha \cap A_\alpha$ *is finite.*

Using a rather straightforward simplification of the proof of Theorem 4.18 one can prove:

Theorem 4.29. *If there is a* $\clubsuit$*-sequence* $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ *and a coherent sequence* $\langle A_\alpha : \alpha < \omega_1\rangle$ *such that* $\langle A_\alpha : \alpha < \omega_1\rangle$ *avoids* $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ *then there is a locally compact, coherently almost left-separated O-space which is almost left-separated by* $\langle A_\alpha : \alpha < \omega_1\rangle$*.*

Thus to produce a hereditarily almost left-separated O-space, it suffices to construct a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ that almost left-separates every uncountable subspace and is such that a construction like used to prove Theorem 4.29 can proceed. Let us make the following definition:

Definition 4.30. *A sequence* $\langle A_\alpha : \alpha < \omega_1\rangle$ *is called* ***hereditarily almost left-separating*** *for* X *if* $\langle A_\alpha : \alpha < \omega_1\rangle$ *if an almost left-separating sequence for* X *and for every* $Y \in [X]^{\omega_1}$ *it is the case that* $\langle A_\alpha \cap Y : \alpha < \omega_1\rangle$ *is an almost left-separating sequence for* Y*.*

The following is an obvious implication of the preceeding definition:

Proposition 4.31. *If* X *has a hereditarily almost left-separating sequence, then* X *is hereditarily almost left-separated.*

In fact an X such that X has a hereditarily almost left-separating sequence is hereditarily almost left-separating in a strong way. The same almost left-separating sequence does all of the work.

Definition 4.32. *A space* X *is called* ***strongly hereditarily almost left-separated*** *if there is a hereditarily almost left-separating sequence for* X*.*

Our immediate goal is to construct a locally compact, strongly hereditarily almost left-separated O-space using Theorem 4.29. In order for this to suceed we need only construct a hereditarily almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ avoiding a ♣-sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$. The existence of the such a pair is proved in the following Lemma.

Lemma 4.33. *(♣) There is a coherent sequence* $\langle A_\alpha : \alpha < \omega_1\rangle$ *and a ♣-sequence* $\{x_\alpha : \alpha < \omega_1\}$ *such that that:*

(i) $\langle A_\alpha : \alpha < \omega_1\rangle$ *avoids* $\{x_\alpha : \alpha < \omega_1\}$,

(ii) $\{\mathrm{otype}(A_\alpha)\cap Y : \alpha < \omega_1\}$ *is uncountable for every uncountable* $Y \subseteq \omega_1$

Proof. Let $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ be a ♣-sequence. We construct a coherent sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ and ♣-sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ simultaneously by induction on $\alpha < \omega_1$. Suppose that we have constructed sequences $\langle A_\xi : \xi < \alpha\rangle$ and $\{x'_\xi : \xi \in \mathrm{Lim}(\alpha)\}$ satisfying:

(a) $A_\gamma \subseteq \gamma$ and $A_\gamma \restriction \beta =^* A_\beta$ for all $\beta < \gamma$,

(b) $A_\beta \cap x'_\beta = \varnothing$ for all $\beta \in \mathrm{Lim}(\alpha)$,

(c) $x'_\beta \subseteq x_\beta$ for all $\beta \in \mathrm{Lim}(\alpha)$

Condition (a) guarantees that the resulting sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ is coherent. Condition (b) guarantees that $\langle A_\alpha : \alpha < \omega_1\rangle$ avoids $\{x'_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$. Condition (c) will guarantee that $\{x'_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ is a ♣-sequence.

Assuming $\alpha = \beta + 1$, then define $A_\alpha = A_\beta$. For limit α, let $x'_\alpha \subseteq x_\alpha$ be infinite and co-infinite and let α_i be an increasing ω-sequence unbounded in α. Define

$$A_\alpha = \left(\bigcup_{i<\omega} A_{\alpha_i} \cup x_\alpha\right) \setminus x'_\alpha$$

It is straightforward to verify that A_α satisfies (a)-(c) above. Thus it suffices to prove the following claim:

Claim 4.34. *For any uncountable $Y \subseteq \omega_1$, the set* $\{\text{otype}(A_\alpha) \cap Y : \alpha < \omega_1\}$ *is uncountable.*

Proof. We prove by induction on $\eta \in \text{Lim}(\omega_1)$ that for every uncountable Y there is some A_α such that $A_\alpha \cap Y$ has order type at least η. In the case that $\eta = \omega$, let α be such that $x_\alpha \subseteq Y$. Then $Y \cap A_\alpha$ will have order type at least ω.

Suppose that for every uncountable Y and every $\xi \in \text{Lim}(\eta)$ it is the case that there is some A_α such that $A_\alpha \cap Y$ has order type at least ξ.

Case 1: $\eta = \xi + \omega$ for some $\xi < \eta$. In this case, let α be such that $A_\alpha \cap Y$ has order type at least ξ. Let α' be such that $x_{\alpha'} \subseteq E \setminus \alpha$. Then $A_{\alpha'} \cap Y$ has order type $\xi + \omega = \eta$, since $A_\alpha \subseteq^* A_{\alpha'}$.

Case 2: $\eta \in \text{Lim}^2(\omega_1)$. Fix a sequence $(\xi_i)_{i<\omega}$ such that $\sup(\xi_i) = \eta$. By a similar argument as in the previous case, fix $A_{\xi'_{i+1}}$ be such that

$$\text{otype}\left(\left(Y \cap A_{\xi'_{i+1}}\right) \setminus \xi'_i\right) \geq \xi_i$$

Then, let A_α be such that $A_{\xi'_{i+1}} \subseteq^* A_\alpha$ for all $i < \omega$. Then

$$\text{otype}(A_\alpha \cap Y) \geq \sup\left\{\text{otype}(A_{\xi'_{i+1}}) : i < \omega\right\}$$

and so $\text{otype}(A_\alpha \cap Y) \geq \eta$. □

Thus $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a hereditarily almost left-separating sequence which avoids a ♣-sequence $\{x'_\alpha : \alpha \in \text{Lim}(\omega_1)\}$. □

We now have the following Corrollary of Theorem 4.29:

Corollary 4.35. *($\clubsuit$) There is a locally compact, strongly hereditarily coherently almost left-separated O-space.*

4.4.2 Questions

The preceding proofs that construct various almost left-separated spaces have all used $\clubsuit^+$ in a non-trivial way. It is natural to question whether $\clubsuit^+$, i.e. $\clubsuit$, is a necessary component which leads to the following questions:

Question 4.36. *(CH) Is there an almost left-separated locally compact (or even first countable) O-space?*

We now turn our attention to proving Theorem 4.25 using only CH. Since CH is consistent with the non existence of any Ostaszewski spaces, if there is a locally compact (or even first countable) O-space under CH, it will necessarily fail to be countably compact. Thus any locally compact (or even first countable) O-space constructed from CH alone will fail to have any uncountable countably compact subspaces. This does not settle the possibility of constructing, using CH alone an O-space with no uncountable almost left-separated subspaces.

Question 4.37. *(CH) Is there an a locally compact (or even first countable) O-space with no uncountable almost left-separated subspaces?*

If we consider how we may answer the preceding question in the negative it would suffice to build a model of CH in which every locally compact (or even first countable) O-space contains an uncountable almost left subspace. One may provide insight into Question 4.37 by building for any locally compact (or even first countable) O-space X a totally proper order $\mathbb{P}_X$ such that $V^{\mathbb{P}_X}$ models that X contains an uncountable almost left-separated subspace. This will prove that no classical CH construction suffices to provide a positive resolution to Question 4.37. Therefore, a closely related question is:

Question 4.38. *Given any any locally compact (or even first countable) O-space X is there a totally proper $\mathbb{P}_X$ such that $V^{\mathbb{P}_X}$ models that X contains an uncountable almost left-separated subspace?*

4.5 Spaces Under CH

A natural question that comes to mind is if there are any almost left-separated S-spaces under CH alone. In §3.4 we have seen how to produce refinements of topologies that preserve the property of being an S-space and many other properties. Using this techniques allows us to construct various S-spaces under CH, including a space that is a refinement of the topology in the reals.

4.5.1 Almost Left-Separated S-spaces

Note that starting with an almost left-separated space always yields another almost left-separated space, as closed sets will remain closed under any refinement and the $\subseteq^*$ relation is independent of the topology. Thus in order to prove that there exists a locally compact almost left-separated S-space under CH, it suffices to construct and almost left-separated subspace in some convenient space and then using CH refine this space into a locally compact S-space.

The standard topology on the reals is a hereditarily separable, regular space. Therefore it will suffice to find a subspace of cardinality $\aleph_1$ in $\mathbb{R}$ that is almost left-separated. In fact, $\mathbb{R}$ is hereditarily almost left-separated.

Lemma 4.39. *For any $X \subseteq \mathbb{R}$ with $|X| = \aleph_1$ there is an almost left-separated sequence on X.*

Proof. We shall construct sequence $\{A_\alpha : \alpha < \omega_1\}$ by induction on ω_1 such that $\langle A_\alpha : \alpha < \omega_1 \rangle$ is an almost left-separating sequence for X.

Fix a well ordering of $X = \{x_\xi : \xi < \omega_1\}$ in type ω_1. For each $\alpha < \omega_1$ as usual let $X_\alpha = \{x_\alpha : \alpha < \omega\}$. Let $b = \sup(X)$ and fix an increasing sequence $\{q_i : i < \omega\}$ such that $(q_i, q_{i+1}) \cap X$ is uncountable for every $i < \omega$ and such that $\lim_{i<\omega} q_i = b$.

Let $\alpha < \omega_1$ and suppose that:

(i) if $\beta < \alpha$ then $A_\beta \cap (q_i, q_{i+1})$ is finite,

(ii) if $\beta < \gamma < \alpha$ then $A_\beta \subseteq^* A_\gamma$,

(iii) if $\beta + 1 < \alpha$ then $(A_{\beta+1} \cap X) \setminus A_\beta$ is infinite.

Note that (i) implies that each A_β is closed.

If $\alpha = \beta + 1$, then let $x_i \in (q_i, q_{i+1}) \setminus \bigcup_{\xi \leq \beta} A_\xi$. Define

$$A_\alpha = A_\beta \cup \{x_i : i < \omega\}$$

Then A_α satisfies all of the inductive hypothesis.

Suppose $\alpha \in \mathrm{Lim}(\omega_1)$. Fix a sequence $\{\alpha_j : j < \omega\}$ cofinal in α. It suffices to construct a satisfactory A_α such that $A_{\alpha_j} \subseteq^* A_\alpha$ for each $j < \omega$.

Define A_α to be

$$A_\alpha = \bigcup_{j<\omega} A_{\alpha_j} \setminus (q_0, q_j)$$

It suffices to demonstrate that A_α satisfies that inductive hypotheses.

Claim 4.40. *For any $i < \omega$, $(q_i, q_{i+1}) \cap A_\alpha$ is finite.*

Proof. Fix any $i < \omega$. Then $(q_i, q_{i+1}) \cap A_\alpha \subseteq \bigcup_{j<i}(q_i, q_{i+1}) \cap A_{\alpha_j}$ which is finite. □

Thus it must be the case that A_α is closed. Furthermore every $A_{\alpha_j} \subseteq^* A_\alpha$ and A_α satisfies (i)-(iii) above. To demonstrate that $\langle A_\alpha : \alpha < \omega_1 \rangle$ is indeed an almost left-separating sequence it remains only to verify that

Claim 4.41. *The set* $\{\text{otype}(A_\alpha) : \alpha < \omega_1\}$ *is unbounded in* ω_1.

Proof. In a similar manner to the proof of Lemma 4.33 it may be proved by induction on $\eta < \omega_1$ there is some $\alpha < \omega_1$ such that $\text{otype}(A_\alpha \cap X) \geq \eta$. □

Thus X is an almost left-separated space as witnessed by $\langle A_\alpha : \alpha < \omega_1 \rangle$. □

For any X satisfying the hypotheses of the previous Lemma, it is possible to make X almost left-separated. Furthermore X will be hereditarily Lindelöf, hereditarily separable, first countable and perfectly normal. These properties may all be used when refining the space using an S-preserving refinement to produce a refinement with the various other topological properties that were discussed in §3. In fact, by refining this space with the standard S-preserving refinement we will have established:

Theorem 4.42. *(CH) There is a locally compact almost left-separated S-space.*

The preceding Theorem can be proved from $\mathfrak{b} = \omega_1$ alone if we produce an S-preserving refinement of Baire space using an unbounded family $A \subseteq \{\omega\}^\omega$ as in [38] (see §3.4).

Theorem 4.43. *($\mathfrak{b} = \omega_1$) There is a locally compact almost left-separated S-space.*

Since S-preserving refinements of hereditarily Lindelöf spaces produce normal spaces (see Lemma 3.2) we have established the following corollary to Theorem 4.43:

Corollary 4.44. *($\mathfrak{b} = \omega_1$) There is a locally compact normal almost left-separated S-space.*

The space constructed above is almost left-separated, normal, and locally compact. This space is not however coherently almost left-separated. This leaves open the question, can we construct a coherently almost left separated normal S-space using CH alone? It will turn out that this is impossible which we shall prove in §4.5.3.

Since Lemma 4.39 applies to every uncountable subspace X of $\mathbb{R}$ we have established:

Theorem 4.45. *($\mathfrak{b} = \omega_1$) There is a locally compact normal hereditarily almost left-separated S-space.*

The space X satisfying the conclusion of Theorem 4.45 is hereditarily almost left-separated by a separate sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ for each uncountable subspace. Thus, X is not necessarily strongly hereditarily almost left-separated. We shall see in §4.6 that there cannot be a strongly hereditarily almost left-separated S-space under CH alone.

4.5.2 Corresponding Ideals

Now let us turn our attention to proving Theorem 4.15 on page 44. We have already seen how to refine an O-space into another O-space in §3.2. Let X be a first countable O-space with a coherent almost left-separating sequence $\{A_\alpha : \alpha < \omega_1\}$. Since $\{A_\alpha : \alpha < \omega_1\}$ is an almost left-separating sequence it it the case that $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is a P-ideal. We shall construct an S-preserving refinement of the topology on X to ensure that there is a cover of X by a family of compact open $\{V_\alpha : \alpha < \omega_1\}$ such that

$$\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$$

In order to accomplish this goal, it will suffice to guarantee that we kill every potential $Y \in [X]^\omega$ such that Y is a potential witness that $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}} \setminus \mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is not empty, i.e. that

(a) for all $\alpha < \omega_1$, $Y \setminus A_\alpha$ is infinite,

(b) for all V_ξ chosen this far, $Y \cap V_\xi$ is finite.

Any Y satisfying (a), (b) can be killed at stage α of the construction of the refinement by ensuring that V_α meets Y in an infinite set. Now to prove Theorem 4.15, it is fairly straightforward to combine the proofs of Theorem 3.8 and Proposition 4.14 to produce the necessary S-preserving refinement.

Since normality is preserved in S-preserving refinements of O-spaces, (see Corollary 3.5), we have now established:

Theorem 4.46. *(CH) If there is a first countable, normal, coherently almost left-separated O-space X, then there is a locally compact, normal, coherently almost left-separated O-space X with cover by compact open sets $\{V_\alpha : \alpha < \omega_1\}$ such that $\mathfrak{I}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal.*

4.5.3 Applying the P-ideal Dichotomy

Let X be an S-space with enumeration $X = \{x_\alpha : \alpha < \omega_1\}$. Let us call X **almost countably compact** if $X \setminus (X \restriction \alpha)$ is countably compact for some $\alpha < \omega_1$.

Theorem 4.47 (Eisworth, Nyikos, Shelah)**.** *(PID) Suppose that X is a hereditarily separable Property wD space with cover by compact open sets $\{V_\alpha : \alpha < \omega_1\}$ such that $\mathfrak{I}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal. Then X has an uncountable closed countably compact subspace.*

Proof. Using the P-ideal dichotomy, there is an uncountable $Y \subseteq X$ such that either $[Y]^\omega \subseteq \mathfrak{I}_{\{V_\alpha:\alpha<\omega_1\}}$ or $[Y]^\omega \cap \mathfrak{I}_{\{V_\alpha:\alpha<\omega_1\}} = \varnothing$.

First suppose that $[Y]^\omega \subseteq \mathfrak{J}_{\{V_\alpha:\alpha<\omega_1\}}$. Then $Y \cap V_\alpha$ is finite for every V_α. Since V_α is open, Y is an uncountable locally finite subspace of X, which is a contraction since X is hereditarily separable.

Thus it must be the case that $[Y]^\omega \cap \mathfrak{J}_{\{V_\alpha:\alpha<\omega_1\}} = \varnothing$. We shall show that $\mathrm{cl}(Y)$ is countably compact. If not, fix $\{y_n : n < \omega\} \subseteq \mathrm{cl}(Y)$ which is closed and discrete. Since X has Property wD, we can shrink $\{y_n : n < \omega\}$ to an infinite subset such that there is a collection of open sets U_n such that $y_n \in U_n$ and $\{U_n : n < \omega\}$ is a discrete collection of sets, i.e. for any $x \in X$, there is an open V with $x \in V$ and $V \cap U_n \neq \varnothing$ for at most one U_n. Since $y_n \in \mathrm{cl}(Y)$, then there is some $x_n \in U_n \cap Y$ for each $n < \omega$. Fix such a set of $\{x_n : n < \omega\} \subseteq Y$. Then, since $\{x_n : n < \omega\} \in [Y]^\omega$ and $[Y]^\omega \cap \mathfrak{J}_{\{V_\alpha:\alpha<\omega_1\}} = \varnothing$ there is some V_α such that

$$\{x_n : n < \omega\} \cap V_\alpha \text{ is infinite}$$

Since V_α is compact, then $\{x_n : n < \omega\} \cap V_\alpha$ has an accumulation point. Let x be such an accumulation point. Then by the choice of $\{U_n : n < \omega\}$ there is a V such that $x \in V$ can $V \cap U_n \neq \varnothing$ for at most one $n < \omega$. This contradicts that x is an accumulation point of $\{x_n : n < \omega\} \cap V_\alpha$. Hence, $\mathrm{cl}(Y)$ is indeed countably compact. □

Now consider a locally compact O-space X such that there is a cover of X by compact open sets V_α for $\alpha < \omega_1$ such that the corresponding ideal $\mathfrak{J}_{\{V_\alpha:\alpha<\omega_1\}}$ is a P-ideal. If we apply Theorem 4.47 to X we obtain , assuming PID, that X is in fact almost countably compact and hence there is some $\alpha < \omega_1$ such that $X \setminus (X \restriction \alpha)$ is an Ostaszewski space. In particular we have established that:

Theorem 4.48. *(PID) Suppose that there is a locally compact, property wD O-space X with a cover* $\{V_\alpha : \alpha < \omega_1\}$ *by compact open sets such that* $\mathfrak{J}_{\{V_\alpha:\alpha<\omega_1\}}$ *is a P-ideal. Then there is an Ostaszewski space.*

We have proven Theorem 4.46 in the preceeding section. Thus in a model of PID and CH, every normal, locally compact, coherently almost left-separated will be almost-Ostaszewski. On the other hand, by Lemma 4.17 there are no such spaces. Therefore, using a model of CH plus PID, we have established:

Theorem 4.49. *It is consistent with CH that there are no property wD, locally compact, coherently almost left-separated S-spaces by a sequence $\{A_\alpha : \alpha < \omega_1\}$ such that there exists a cover by compact open sets $\{V_\alpha : \alpha < \omega_1\}$ such that $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$.*

In the case of O-spaces, as seen in the preceding section, any normal, first-countable, coherently almost left-separated O-space can be refined into a normal, locally compact O-space such that $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$. We therefore have the following Corollary:

Corollary 4.50. *It is consistent with CH that there are no first countable, normal, coherently almost left-separated O-spaces.*

This is in contrast to what we obtained under $\diamondsuit$ in Theorem 4.16, namely the existence of a perfectly normal, locally compact, coherently almost left separated O-space. In particular, the class of first countable, normal, coherently almost left-separated O-spaces exist under $\diamondsuit$ but need not exist under CH. This is exactly the same behavior as the class of Ostaszewski spaces. Since Theorem 4.49 applies only to property wD, locally compact, coherently almost left-separated O-spaces, and Corollary 4.50 applies only to normal, first countable such spaces this leaves open a few questions.

Recall that in §4.5.1 we constructed using CH an almost left-separated, locally compact S-space using CH alone. Furthermore this example can made normal (see Corollary 4.44) using CH alone. This leaves open the following questions:

Question 4.51. *(CH) Is there a first countable, property D (property wD) coherently almost left-separated S-space (O-space)?*

The almost left-separated S-space constructed in §4.5.1 from CH, was obtained by applying an S-preserving refinement to the topology of an $\aleph_1$ set the reals to produce a space $X = (\omega_1, \rho)$. Since this space is hereditarily-Lindelöf the resulting refinement is unavoidably normal. In order to avoid Theorem 4.49, the ideal $\mathcal{J}_{\{V_\alpha:\alpha<\omega_1\}}$ must fail to be a P-ideal for every cover of X by $\{V_\alpha : \alpha < \omega_1\}$ a collection of compact open sets.

4.6 Hereditarily Almost Left Separated Spaces

Recall that a space X is called strongly hereditarily almost left-separated if there is a single sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ that almost left-separates every uncountable subspace of X. In §4.4.1 it was proved that assuming ♣ there locally compact, strongly hereditarily almost left-separated O-spaces. There cannot be any such spaces constructed from CH. We shall prove that:

Theorem 4.52. *It is consistent with CH that there are no hereditarily separable, strongly hereditarily almost left-separated spaces.*

We shall prove Theorem 4.52 by applying the PID to the ideal generated by hereditarily almost left-separating sequences. Recall that $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is the ideal generated by the sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$. Since $\langle A_\alpha : \alpha < \omega_1 \rangle$ is coherent, then $\mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ is a P-ideal. Thus Theorem 4.52 can be proved by the following Lemma:

Lemma 4.53. *(PID) If X is hereditarily separable with hereditarily almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ then there is an uncountable $Y \subseteq X$ such that $Y \cap A_\alpha$ is finite for every $\alpha < \omega_1$.*

Proof. Let $Y \subseteq [X]$ be uncountable such that $[Y]^{\aleph_0} \subseteq \mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$ or such that $[Y]^{\aleph_0} \cap \mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \varnothing$. If $[Y]^{\aleph_0} \subseteq \mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}}$. In this case, Y is in fact left separating, which contradicts that Y is separable. Thus it must be the case that $[Y]^{\aleph_0} \cap \mathcal{K}_{\{A_\alpha:\alpha<\omega_1\}} = \varnothing$. Then Y is the desired uncountable subset. □

4.7 Summary

By combining the preceeding sections, we can summarize what is known about the existence of almost left-separated *S*-spaces (see Figure 5) and *O*-spaces (see Figure 6).

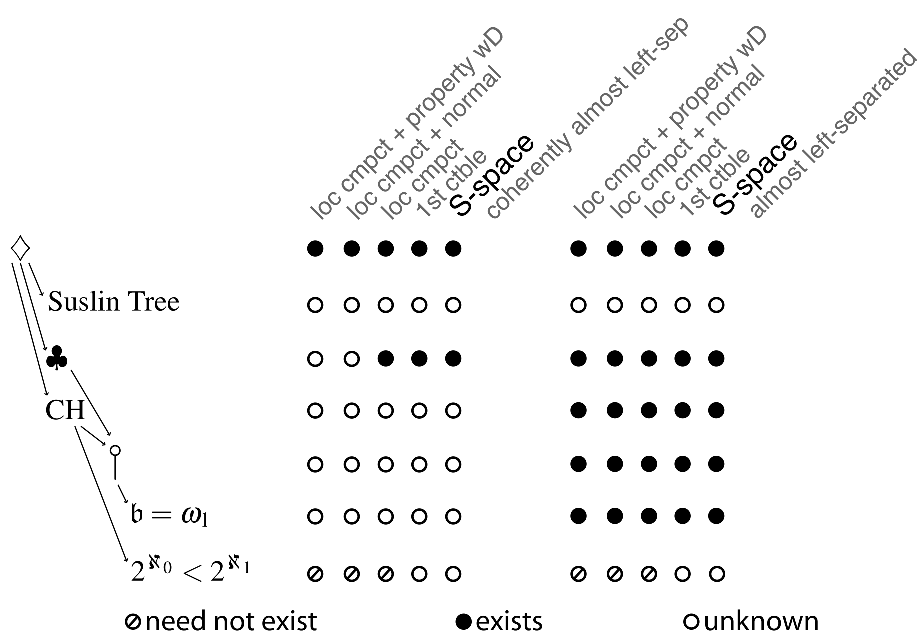

Figure 5: Almost Left Separated *S*-spaces

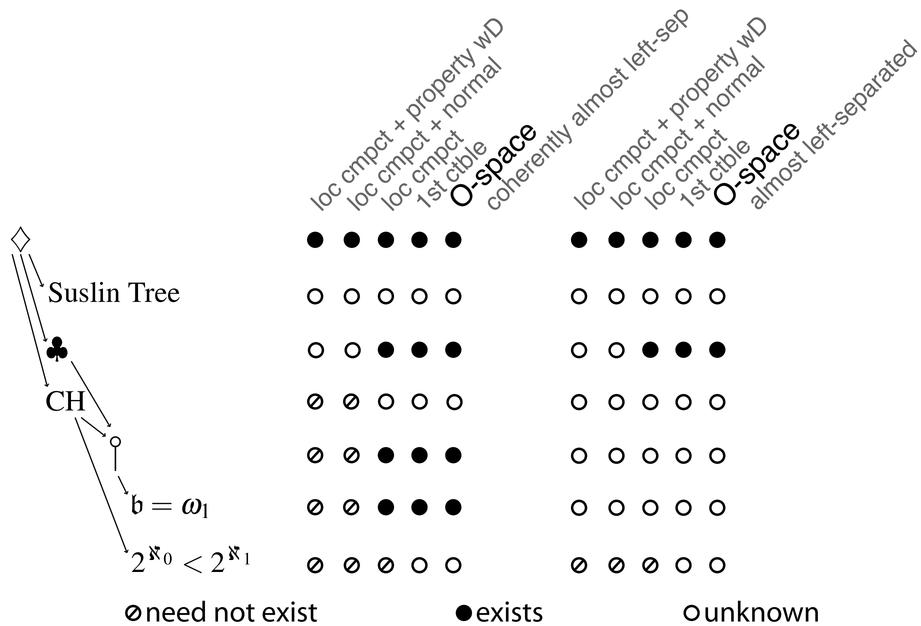

Figure 6: Almost Left Separated *O*-spaces

4.8 Luzin Spaces

We have just seen that a hereditarily almost left-separated space can be made to arise as a subspace of the reals. Recall that a space is called Baire if every countable intersection of open dense sets is dense. Also, the π-weight of a space is the least cardinality of a π-base for the space; a family $\mathcal{B}$ of open sets is a π-base if for any $x \in X$ and u open with $x \in u$ there is some $v \in \mathcal{B}$ with $v \subseteq u$. Assuming CH and using Lemma 4.39 we can construct an hereditarily almost left-separated subspace of $\mathbb{R}$ in which there are no isolated points. Furthermore this space will be Baire and have π-weight no more than ω_1. This is exactly the property needed in order to construct a Luzin space.

Definition 4.54. *A T_3 space X is a **Luzin space** if X is uncountable, X has no isolated points and every nowhere dense subset of X is countable.*

Recall that a space X is called ccc if there are no uncountable families of open subsets of X that are pairwise disjoint. A Luzin space is hereditarily ccc and also hereditarily Lindelöf. (see [29] §4.3) Thus, to construct an almost left-separated Luzin space, it suffices to find a Luzin space that is an L-space (i.e. non-separable) as any left-separated space is automatically almost left-separated. It remains to see that there is a hereditarily separable almost left-separated Luzin space.

In order to construct such a space we shall use the following Lemma which is proved in [29] §4.3:

Lemma 4.55. *(CH) If X is an uncountable ccc Baire space with π-weight no more than ω_1 and with no isolated points, then X contains an uncountable Luzin subspace.*

Thus, in order to construct a hereditarily separable Luzin almost left-separated space from CH we may proceed as follows. We start with a subset X of $\mathbb{R}$ in which there are no isolated points. Then we construct an almost left-separated sequence for X using Lemma 4.39. Finally, using Lemma 4.55 we construct an uncountable subspace Y of X that is Luzin. Since Y may be almost left-separated, this Luzin space will be almost left-separated and hereditarily separable. The proceeding construction uses Lemma 4.55 the proof of which utilizes a full enumeration of $[\omega_1]^\omega$ in type ω_1. This leaves open the following question:

Question 4.56 ($\mathfrak{b} = \omega_1$)**.** *Is there a hereditarily separable, almost left-separated Luzin space?*

Chapter 5

Coherent *L*-spaces

We have already mentioned the recent result of Moore in [24] in which it is shown that there is an *L*-space without assuming any additional set theoretic axioms. We have also seen that there exist *S*-spaces that have stronger density properties: such as HFD_w and HFD spaces. These notions dualize to *L*-spaces to produce *L*-spaces that have stronger covering properties. We shall investigate a particular class of these spaces here.

The spaces that we shall investigate here are discussed in great length in Juhász's [20] and we shall follow the notation used therein.

Definition 5.1. *Let $D_\kappa(\lambda)$ be the collection of all elements B of $[[\kappa]^{<\omega}]^\lambda$ such that each pair $\sigma, \tau \in B \in D_\kappa(\lambda)$ are disjoint, and $|\sigma| = n$ is constant. For any such B, let $n(B)$ denote the unique n such that $n = |\sigma|$ for every $\sigma \in B$.*

We shall concern ourself primarily with $D_{\omega_1}(\omega_1)$. In this case each of the $B \in D_{\omega_1}(\omega_1)$ can enumerated in type ω_1 such that B will be separated in our usual sense of the word.

Definition 5.2. *For any finite function $\sigma : \omega_1 \to 2$ we shall define*

$$[\sigma] = \{f \in 2^{\omega_1} : \sigma \subseteq f\}$$

For a function $\varepsilon : n \to 2$ and a $C \subseteq B \in D_{\omega_1}(\omega_1)$ let $[\varepsilon * C]$ be

$$[\varepsilon * C] = \bigcup_{c \in C} [\varepsilon * c]$$

where $\varepsilon * c$ is the function with domain c and if $c = \{c_i : i < n\}$ is an increasing enumeration of c then $(\varepsilon * c)(c_i) = \varepsilon(i)$.

Definition 5.3. *A space $X \subseteq 2^{\omega_1}$ is called an HFC$_w$ or weak HFC if for every $B \in D_{\omega_1}(\omega_1)$ there is a $C \in [B]^{\omega}$ such that for every $\varepsilon \in 2^{n(B)}$*

$$|X \setminus [\varepsilon * C]| \leq \aleph_0$$

A space $X \subseteq 2^{\omega_1}$ is called an HFC if for every $C \in D_{\omega}(\omega_1)$ and every $\varepsilon \in 2^{n(C)}$

$$|X \setminus [\varepsilon * C]| \leq \aleph_0$$

The 'HFC' in the definition of HFC spaces stands for Hereditraily Finally Covered. Such spaces are covered, except possibly a countable set, by any nice collection of open neighborhoods in 2^{ω_1}. This is exactly the dual of the notion of density used in the definition of HFD and HFD$_w$. (see Definition 1.11)

It is well know that there are HFCs and weak HFCs under CH. (See [20]) Both HFCs and weak HFCs are L-spaces if they are non-separable. Note that any countable space is hereditarily separable, and hence any weak HFC that is an L-space is necessarily uncountable. We shall henceforth assume that all weak HFCs (and hence also HFCs) are uncountable. We may guarantee that such spaces are in addition non-separable by assuming that $X \subseteq 2^{\omega_1}$ is "canonically" left-separated.

Remark 5.4. *If there is an uncountable HFC$_w$ then there is an L-space.*

Proof. Fix $X = \{x_\alpha : \alpha < \omega_1\} \subseteq 2^{\omega_1}$ a weak HFC. Construct $Y = \{y_\alpha : \alpha < \omega_1\} \subseteq 2^{\omega_1}$ such that

$$y_\alpha(\xi) = \begin{cases} x_\alpha(\xi) & \text{if } \xi < \alpha \\ 1 & \text{if } \xi = \alpha \\ 0 & \text{if } \xi > \alpha \end{cases}$$

The space Y is left-separated, canonically, by the neighborhoods $U_\alpha = \bigcup_{\xi > \alpha}[\langle \xi, 0 \rangle]$.

It remains to show that Y has no uncountable, discrete subspaces. Suppose towards a contradiction that $Y_0 \in [Y]^{\omega_1}$ is discrete and that $Y_0 = \{z_\alpha : \alpha < \omega_1\}$. Let $[\sigma_\alpha]$ be a basic clopen neighborhood with the property that

$$[\sigma_\alpha] \cap Y_0 = \{y_\alpha\}$$

Note that we may assume each $\mathrm{dom}(\sigma_\alpha) < \alpha$ and that the $\{\mathrm{dom}(\sigma_\alpha) : \alpha < \omega_1\}$ are pairwise disjoint. Then X cannot be a weak HFC. $\square$

As mentioned above, there are L-spaces outright, as proved by Moore in [24]. Somewhat surprisingly, the space constructed in [24] is in fact a weak HFC.

Theorem 5.5 (Moore)**.** *There is an HFC$_w$.*

For the rest of this chapter we shall identify each space $X \subseteq 2^{\omega_1}$ with a fixed enumeration $X = \langle x_\alpha : \alpha < \omega_1 \rangle$ of 'partial' functions $\langle x_\alpha : \alpha < \omega_1 \rangle$ where each x_α is viewed as a function with domain α. We shall canonically extend x_α to a total function from ω_1 to 2 by defining $x_\alpha(\alpha) = 1$ and $x_\alpha(\xi) = 0$ for $\xi > \alpha$.

5.1 Coherent HFC$_w$ under $\diamondsuit$, $\clubsuit$ and CH

Recall that a Suslin tree T is a tree on $2^{<\omega_1}$ all of whose levels are countable with no uncountable antichains and no uncountable chains (see Definition 2.11). Assuming $\diamondsuit$ there is a Suslin tree and furthermore there is a Suslin tree $T \subseteq 2^{<\omega_1}$ such that $T = \{x_\alpha \restriction \beta : \beta \leq \alpha < \omega_1\}$ with $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a coherent sequence of functions. Let us call such a Suslin tree a **coherent Suslin tree**. Such Suslin trees are also called *thin* (see [36], Theorem 6.9).

Theorem 5.6 (Jensen)**.** *($\diamondsuit$) There is a coherent Suslin tree.*

The fact that Suslin trees 'are', with the correct topological interpretation, L-spaces is well known. Recall that a Suslin line is a a total order $(X,<)$ such that in the order topology X is ccc and not separable. A Suslin tree with the lexicographic ordering contains a Suslin line (see [36] §6). A Suslin line is an L-space for free since a ccc space is always hereditarily Lindelöf. Thus a Suslin tree with the correct topology can be interpreted as an L-space. However, they are in fact weak HFCs (see [38] Theorem 5.4) and hence:

Theorem 5.7 (Todorčević)**.** *($\diamondsuit$) There is a coherent HFC_w.*

Furthermore, assuming there is a coherent Suslin tree, there is also a coherent Luzin space (see [29] §4.3). However, $\diamondsuit$ is not necessary in order to construct a coherent weak HFC. We shall prove the following:

Theorem 5.8. *($\clubsuit$) There is a coherent HFC_w.*

This result is somewhat surprising given that no analogous result holds for HFCs, even assuming $\diamondsuit$. In fact HFCs fail to be coherent in a rather strong way. Call an open $U \subseteq 2^{\omega_1}$ *nicely split* if

$$U = \bigcup_{n<\omega} [\sigma_n]$$

where $\{\mathrm{dom}(\sigma_n) : n < \omega\}$ are pairwise disjoint.

Proposition 5.9. *If $X = \langle x_\alpha : \alpha < \omega_1 \rangle$ and $T = \{x_\alpha \upharpoonright \beta : \beta \leq \alpha < \omega_1\}$ is Aronszajn then X is not an HFC.*

Proof. Let $X = \langle x_\alpha : \alpha < \omega_1 \rangle$ be such that the tree T is Aronszajn. Fix any U a nicely split open subset of 2^{ω_1} and let $\xi < \omega_1$ be sufficiently large so that ξ bounds $\mathrm{dom}(\sigma_n)$ for all $n < \omega$. Since T is Aronszajn there is some uncountable $D \subseteq \omega_1$ and $x \in 2^\xi$ such that for all $\alpha \in D$ $x_\alpha \upharpoonright \xi = x$. Let $\{\xi_n : n < \omega\} = \{\min(\mathrm{dom}(\sigma_n)) : n < \omega\}$. Let $A \subseteq \omega$ be infinite such that $A = \{n < \omega : x(\xi) = i\}$ for some fixed $i \in 2$. Now let $U' = \bigcup_{n \in A} \langle \xi_n, 1 - i \rangle$. Then U' is also nicely split and $\{x_\alpha : \alpha \in D\} \cap U' = \varnothing$. Thus X fails to HFC. □

In order to prove Theorem 5.8 we need to fist derive a variant of ♣.

Definition 5.10.

There is a sequence $\langle x_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ such that:

1. *$x_\alpha \subseteq Fn(\omega_1, 2)$, $x_\alpha = \langle \sigma_{\alpha,i} : i < \omega \rangle$ with each $|\sigma_{\alpha,i}| = n$,*
2. *for any $\varepsilon \in 2^n$, $\{i : \sigma_{\alpha,i} = \varepsilon * \mathrm{dom}(\sigma_{\alpha,i})\}$ is infinite*
3. *$\{\sup(\mathrm{dom}(\sigma_{\alpha,i})) : i < \omega\}$ is unbounded in α*

$(♣_{\mathrm{Fn}(\omega_1,2)})$

With the property that for any $B \in D_{\omega_1}(\omega_1)$ there is some x_α with $\{\mathrm{dom}(\sigma_{\alpha,i}) : i < \omega\} \subseteq B$.

Lemma 5.11. ♣ $\iff$ $♣_{\mathrm{Fn}(\omega_1,2)}$

Proof. One direction is obvious, thus it suffices to show that ♣ $\implies$ $♣_{\mathrm{Fn}(\omega_1,2)}$. Fix a ♣-sequence $\langle y_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$. Since ♣ implies for any $Z \in [\omega_1]^{\omega_1}$ the set $\{\alpha : x_\alpha \subseteq Z\}$ is stationary, it will suffice to construct a sequence of $\langle x_\alpha : \alpha \in \mathrm{Lim}^2(\omega_1)\rangle$ where $\beta \in \mathrm{Lim}^2(\omega_1)$ iff $\sup(\beta \cap \mathrm{Lim}(\omega_1)) = \beta$.

Let $\mathrm{Fn}(\omega_1, 2)$ be enumerated as $\langle p_\xi : \xi < \omega_1\rangle$ where

$$\mathrm{Fn}([\gamma, \gamma+\omega), 2) = \{p_\xi : \xi \in [\gamma, \gamma+\omega)\}$$

For each $\alpha \in \mathrm{Lim}^2(\omega_1)$ let x_α be defined as follows. Let $y_\alpha = \langle \alpha_i : i < \omega\rangle$. Let $x_\alpha = \langle \sigma_{\alpha,i} : i < \omega\rangle$ where

$$\sigma_{\alpha,i} = p_\xi$$

where ξ is least such that $\xi \in [\alpha_i, \alpha_i+\omega)$ and $\mathrm{dom}(p_\xi) \setminus \alpha_i \neq \varnothing$. Then $\sup(\mathrm{dom}(\sigma_{\alpha,i})) : i < \omega$ is unbounded in α as needed. □

Lemma 5.12. *($\clubsuit_{\mathrm{Fn}(\omega_1,2)}$) There is a coherent HFC$_w$.*

Proof. We construct the space $X = \{x_\alpha : \alpha < \omega\}$ by induction on $\alpha < \omega_1$. The resulting space will be canonically left-separated since for each x_α, it will be the case that $\mathrm{supp}(x_\alpha) \subseteq \alpha + 1$.

Fix a $\clubsuit_{\mathrm{Fn}(\omega_1,2)}$-sequence $\langle y_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\rangle$. Enumerate each y_α as $y_\alpha = \langle \sigma_{\alpha,i} : i < \omega\rangle$.

For a particular $\langle \sigma_{\alpha,i} : i < \omega\rangle$, if for all $\beta < \alpha$

$$\{\mathrm{dom}(\sigma_{\alpha,i}) : i < \omega\} \cap \beta$$

is finite, call $\langle \sigma_{\alpha,i} : i < \omega\rangle$ *nice*. In such cases we may assume without loss of generality that $\{\mathrm{dom}(\sigma_{\alpha,i}) : i < \omega\}$ is separated.

We shall maintain the following inductive hypothesis:

1. if $\beta < \alpha$ is a limit ordinal and y_β is nice, $x_\beta \in \bigcap_{n<\omega}[\sigma_{\beta,n}]$

2. if $\gamma < \beta < \alpha$ then $x_\beta \restriction \gamma =^* x_\gamma$

At stage α is comes time to define x_α.

Case 1: $\alpha = \beta + 1$. Since we shall make X canonically right separated there is only one way to define x_α. Define $x_\alpha = x_\beta \cup \{\langle \alpha, 1 \rangle\}$.

Case 2: $\alpha \in \mathrm{Lim}(\omega_1)$. We consider the following two cases. First suppose that y_α is nice. Fix a sequence of β_i unbounded in α for $i < \omega$ such that $\beta_i < \mathrm{dom}(\sigma_{\alpha,i+1}) < \beta_{i+1}$ for all $i < \omega$. Define x_α such that

$$x_\alpha(\xi) = \begin{cases} \sigma_{\alpha,i}(\xi) & \text{if } \xi \in \mathrm{dom}(\sigma_{\alpha,i}) \\ x_{\beta_i}(\xi) & \text{for } \beta_i > \xi \text{ least, otherwise} \end{cases}$$

Note that x_α maintains the inductive hypothesis.

If y_α is not nice, let β_i for $i < \omega$ be unbounded in α with $\beta_0 = 0$ and define x_α such that $x_\alpha \restriction [\beta_i, \beta_{i+1}) = x_{\beta_{i+1}} \restriction [\beta_i, \beta_{i+1})$.

Now it remains to show that X is a weak HFC. For any $B \in D_{\omega_1}(\omega_1)$, fix y_α such that $\{\mathrm{dom}(\sigma_{\alpha,i} : i < \omega\} \subseteq B$. Note that any such y_α is nice and so $x_\alpha \in \bigcap_{n<\omega}[\sigma_{\alpha,n}]$. For any $\beta > \alpha$ we note that

$$x_\beta \in \bigcup_{n<\omega} [\sigma_{\alpha,n}]$$

since $x_\beta \restriction \alpha =^* x_\alpha$. Since, by the definition of $\clubsuit_{\mathrm{Fn}(\omega_1,2)}$, for any $\varepsilon \in 2^n$ with $n = |\sigma_{\alpha,i}|$ we have that

$$\{i : \sigma_{\alpha,i} = \varepsilon * \mathrm{dom}(\sigma_{\alpha,i})\}$$

is infinite, we are done. □

The preceding two lemmas combined prove Theorem 5.8.

A space X is called a **strong HFC$_w$** if X^n is an HFC$_w$ for every $n < \omega$ (see [20] 3.7). If X is a coherent HFC$_w$ then it is easy to see that $T(X)$ is Aronszajn. It is also the case that (see [24])

Lemma 5.13 (Moore)**.** *If X is an L-space and $T(X)$ is Aronszajn then X^2 is non-Lindelöf.*

From this Lemma we can conclude that:

Theorem 5.14. *There are no coherent strong HFC_w.*

The construction of a coherent HFC_w in Theorem 5.8 used ♣. This leaves the obvious question, is CH sufficient to get a coherent weak HFC? The answer will turn out to be no, as any such spaces are prohibited by PID (the P-ideal dichotomy, see §4.1) and therefore there are no weak HFCs from CH alone.

Definition 5.15. *An ideal $\mathfrak{I} \subseteq [\omega_1]^\omega$ is called a **P-ideal** if for every sequence I_n $(n < \omega)$ of elements of I, there is some $J \in \mathfrak{I}$ such that $I_n \subseteq^* J$.*

The P-ideal Dichotomy (PID) is the statement that for any P-ideal $\mathfrak{I}$ either

1. there is some $A \subseteq [\omega_1]^{\omega_1}$ such that $[A]^\omega \subseteq \mathfrak{I}$ or

2. ω_1 can be partitioned into countably many sets S_i such that each $[S_i]^\omega \cap \mathfrak{I} = \varnothing$.

PID is due to Abraham and Todorčević, who showed in [2] that (PID) follows from PFA and is, more immediately relevant, consistent with CH. Thus to show that there are no coherent weak HFCs, or coherent Luzin spaces under CH, it suffices to show that all such spaces are precluded by (PID).

Theorem 5.16. *(PID) Any coherent $X \subseteq 2^{\omega_1}$ fails to be hereditarily ccc.*

Since any L-space is hereditarily Lindelöf, it is also hereditarily ccc. Thus we may conclude:

Corollary 5.17. *(PID) There are no coherent L-spaces.*

Recall that a Luzin space is an uncountable regular space with no isolated points in which every nowhere dense subspace is countable (see Definition 4.54). Luzin subspaces of 2^{ω_1} exist. So it makes sense to ask if there any coherent Luzin spaces. However, Luzin spaces are also hereditarily ccc and so we may conclude:

Corollary 5.18. *(PID) There are no coherent Luzin spaces.*

To prove Theorem 5.16 we shall identify the sequence of functions $X = \{x_\alpha : \alpha < \omega\}$ that is coherent with the sequence $\{A_\alpha : \alpha < \omega_1\}$ such that $x_\alpha = \chi_{A_\alpha}$ the characteristic function of A_α. Note that by definition of coherent, each x_α has support a subset of α and so X is naturally left-separated. Since $A_\alpha \subseteq \alpha$ we may then apply (PID) to the P-ideal $\mathfrak{I}$ generated by $\{A_\alpha : \alpha < \omega\}$, where $\mathfrak{I}$ is generated by $\{A_\alpha : \alpha < \omega_1\}$ if $\mathfrak{I} = \{A \in [\omega_1]^\omega : A \subseteq^* A_\alpha \text{ for some } \alpha < \omega_1\}$.

Proof. Fix $X \subseteq 2^{\omega_1}$ that is hereditarily ccc and let $\{A_\alpha : \alpha\}$ and $\mathfrak{I}$ be as above. Since $\mathfrak{I}$ is a P-ideal, by (PID) there is either an $S \in [\omega_1]^{\omega_1}$ such that $[S]^\omega \subseteq \mathfrak{I}$ or an $S \in [\omega_1]^{\omega_1}$ such that $[S]^\omega \cap \mathfrak{I} = \varnothing$.

First assume that there is $S \in [\omega_1]^{\omega_1}$ such that $[S]^\omega \subseteq \mathfrak{I}$. Then in particular, $S \cap \alpha \in \mathfrak{I}$ for each $\alpha < \omega_1$. Thus $S \cap \alpha \subseteq^* A_\alpha$ for each $\alpha < \omega_1$. We consider the uncountable subspace of 2^{ω_1} given by S. For each $\alpha \in S$ such that $S \cap \alpha$ is infinite it is the case that $x_\alpha(\xi) = 1$ for $\xi \in (S \cap \alpha) \setminus F_\alpha$ for $F_\alpha \in [\alpha]^{<\omega}$. Shrinking S we may assume that:

(i) $\{F_\alpha : \alpha \in S\}$ form a Δ-system with root F such that $|F_\alpha| = n$ for some fixed $n < \omega$

(ii) for all $\alpha \in S$, $\sup(F) < \inf(F'_\alpha)$ with $F'_\alpha \neq \varnothing$ where $F'_\alpha = F_\alpha \setminus F$

Define for $\alpha \in S$ the function $\varepsilon_\alpha : F'_\alpha \to 2$ to be identically 0. Then for $\alpha, \beta \in S$ and $\alpha < \beta$ we have that $x_\beta \notin [\varepsilon_\alpha]$. Thus $\{x_\alpha : \alpha \in S\}$ can be right-separated by the sequence of open sets $U_\xi = \bigcup_{\xi<\alpha}[\varepsilon_\xi]$. Since X is left-separated, then X is not hereditarily ccc.

Suppose there is some $S \in [\omega_1]^{\omega_1}$ such that $[S]^{\omega} \cap \mathcal{I} = \varnothing$. Then $S \cap A_\alpha$ is finite for every $\alpha < \omega_1$. Thus for $\alpha \in S$, $x_\alpha(\xi) = 0$ for all $\xi \in (S \cap \alpha) \setminus F_\alpha$ for some $F_\alpha \in [\alpha]^{<\omega}$. We may then right separate an uncountable subspace of X as in the previous case. □

5.2 An O-space and HFC$_w$ simultaneously

In Chapter 4 the notion of almost left-separated spaces were introducted. We are now in a position to consider the construction of a rather curious space that is an O-space, but also with the correct point of view, contains an HFC$_w$. That is, an almost left-separated O-space that codes an HFC$_w$ be the almost left separating sequence.

Recall that a sequence of sets $\{A_\xi : \xi < \omega_1\}$ is coherent if for every $\alpha < \beta$ it is the case that

$$A_\beta \cap \alpha =^* A_\alpha$$

and that this sequence is called non-trivially coherent if also there does not exist any uncountable E such that for all $\alpha < \omega_1$ it is the case that $E \cap \alpha =^* A_\alpha$. Throughout this section we shall identify a set with its characteristic function. With that identification in mind, it is clear that if $X = \{x_\alpha : \alpha < \omega_1\}$ is a coherent HFC$_w$, then $\{x_\alpha : \alpha < \omega_1\}$ is a non-trivial coherent sequence.

We have established in §4.4 that under ♣ there is an almost left-separating O-space. Furthermore, it is an artifact of the construction of that space that the almost left-separating sequence had the following additional property:

Definition 5.19. *A coherent sequence of sets $\{A_\alpha : \alpha < \omega_1\}$* ***avoids*** *a sequence $\{x_\alpha : \alpha \in E\}$ if it is the case that for all $\alpha \in E$ that $x_\alpha \cap A_\alpha$ is finite.*

The aforementioned space of §4.4 in fact produces a coherent sequence that necessarily avoids any given $\clubsuit$-sequences $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$.

Proposition 5.20. *($\clubsuit$) There is a locally compact, almost left-seperated O-space such that the almost left-separating sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ avoids any given $\clubsuit$-sequences $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$.*

The natural 'converse' of the preceding proposition is of some interest. That is, given a fixed coherent sequence $\{A_\alpha : \alpha < \omega_1\}$ is there a $\clubsuit$-sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that $\{A_\alpha : \alpha < \omega_1\}$ avoids $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$? It is clear for such a $\clubsuit$-sequence to exist, it cannot be the case that the coherent sequence $\{A_\alpha : \alpha < \omega_1\}$ is trivial.

Question 5.21. *($\diamondsuit$ or $\clubsuit$) Suppose that $\{A_\alpha : \alpha < \omega_1\}$ is a non-trivial coherent sequence. Is there a $\clubsuit$-sequence $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ such that $\{A_\alpha : \alpha < \omega_1\}$ avoids $\{x_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$?*

If the answer to this question is yes, then we may prove the existence of the following peculiar space:

Theorem 5.22. *Let $X = \{x_\alpha : \alpha < \omega_1\}$ be a coherent HFC_w. If there is a $\clubsuit$-sequence avoiding $\{x_\alpha : \alpha < \omega_1\}$ then there is a locally compact, almost left-separated O-space Z such that $\{x_\alpha : \alpha < \omega_1\}$ is an almost left-separating sequence for Z.*

The proof of Theorem 5.22 is, mutatis mutandis, identical to the proof of Theorem 4.18. The space Z constructed in Theorem 5.22 is such that it is an O-space (an S-space), and contains a weak HFC_w whose 'points' are closed discrete subsets of Z.

5.3 Almost left-separated Spaces and MA($\aleph_1$)

In Chapter 4 we have mostly been concerned with proving the existence of various S-space assuming some weakening of $\diamondsuit$ or CH. It is often the case that these spaces fail to exist assuming MA($\aleph_1$). The most immediately relevant of these type of results is due to Szentmiklóssy (see [34]):

Theorem 5.23 (Szentmiklóssy). *(MA($\aleph_1$)) There are no compact S-spaces.*

Since the one point compactification of a locally compact S-space is a compact S-space, as an immediate corollary of Theorem 5.23 we have:

Corollary 5.24. *(MA($\aleph_1$)) There are no locally compact S-spaces*

One may be tempted to try to strengthen Theorem 5.23 to show that there are no S-spaces at all assuming MA($\aleph_1$). However, this is impossible. Szentmiklóssy has shown in [35] that it is possible to have an S-space in a model of MA($\aleph_1$):

Theorem 5.25 (Szentmiklóssy). *It is consistent with MA($\aleph_1$) that there is an S-space.*

The method of proving Theorem 5.25 is to first construct a ccc indestructible S-space. A space is ccc indestructible if it has the additional property that any ccc forcing extension will fail to destroy X. There is only one way to destroy an S-space with a ccc forcing: to force an uncountable discrete subspace by finite approximations. Given an S-space which is immune to this forcing, then one can produce a model of MA($\aleph_1$) by means of a ccc extension in which the space X will remain an S-space. In the case of of Theorem 5.25 the ccc indestructible space is a special type of HFD that is constructed from CH. Extending the work of Szentmiklóssy Abraham and Todorčević in [1] proved that

Theorem 5.26 (Abraham and Todorčević). *It is consistent with MA($\aleph_1$) that there is a first countable S-space.*

It is known that there are no HFDs under MA($\aleph_1$), as they can be destroyed by applying Silver's Lemma (see [29] 6.1.1). Thus the remaining space to consider is an O-space. Since an O-space is an S-space then there cannot be any locally compact O-spaces under MA($\aleph_1$) by Theorem 5.23. However Soukup has shown in [33] that it is consistent with MA($\aleph_1$) that there is a first countable O-space.

Theorem 5.27 (Soukup)**.** *It is consistent with MA($\aleph_1$) that there is a first countable O-space.*

In order to prove Theorem 5.27 requires the construction of a ccc indestructible O-space. We shall soon establish a slight strengthening of Theorem 5.27 and show that:

Theorem 5.28. *It is consistent with MA($\aleph_1$) that there is an almost left-separated first countable O-space.*

The proof of Theorem 5.28 is similar to the proof of Theorem 5.27 and we shall employ the following definitions.

Definition 5.29. *Let K be a set and $m \in \omega$. Let $\mathrm{Fn}_m(\omega_1, K) = \{f \in \mathrm{Fn}(\omega_1, K) : |f| = m\}$. A sequence $\langle s_\alpha : \alpha < \omega_1 \rangle \subseteq \mathrm{Fn}_m(\omega_1, K)$ is called* ***domain disjoint*** *if the set $\{\mathrm{dom}(s_\alpha) : \alpha < \omega_1\}$ is separated. For $s, t \in \mathrm{Fn}_m(\omega_1, K)$ with disjoint domains let $[s,t]$ denote*

$$[s,t] = \{\{\langle \xi, s(\xi) \rangle, \langle \eta, t(\eta) \rangle\} : \xi \in \mathrm{dom}(s) \text{ and } \eta \in \mathrm{dom}(t)\}$$

Definition 5.30. *A* ***graph*** *G is a subset of $[\omega_1 \times K]^2$ for some set K. The graph G is called* m-***solid*** *if for any domain disjoint sequence $\langle s_\alpha : \alpha < \omega_1 \rangle \subseteq \mathrm{Fn}_m(\omega_1, K)$ there is $\alpha < \beta$ such that*

$$[s_\alpha, s_\beta] \subseteq G$$

G is called ***strongly solid*** *if G is m-solid for each $m < \omega$.*

The relevant preservation theorem need to prove Theorem 5.28 is the following:

Theorem 5.31 (Soukup)**.** *Assume $2^{\omega_1} = \omega_2$. If G is a strongly solid graph on $\omega_1 \times K$ where $|K| \leq 2^{\omega_1}$ then for each $m < \omega$ there is a ccc poset $\mathbb{P}$ of size ω_2 such that $V^{\mathbb{P}} \models$ "G is ccc-indestructibly m-solid".*

In order to prove Theorem 5.28 we start with a ground model of $2^{\omega_1} = \omega_2$. We construct a ccc poset $\mathbb{Q}$ of size ω_1 such that forcing with $\mathbb{Q}$ produces a 0-dimensional, first countable almost left-separated space $X = \langle \omega_1, \tau \rangle$. In $V^{\mathbb{Q}}$ we shall define a graph $\hat{G}$ on $\omega_1 \times \omega \times \omega$ such that $V^{\mathbb{Q}} \models \hat{G}$ is strongly solid. Furthermore we shall prove in Lemma 5.33 if $\hat{G}$ is 2-solid then X is an O-space.

Then using the preservation theorem above we get a ccc poset $\mathbb{P}$ such that

$$V^{\mathbb{Q}*\mathbb{P}} \models \hat{G} \text{ is ccc-indestructibly 2-solid}$$

We then force MA($\aleph_1$) using a ccc partial order $\mathbb{R}$ to obtain

$$V^{\mathbb{Q}*\mathbb{P}*\mathbb{R}} \models \text{MA}(\aleph_1) \text{ and } \hat{G} \text{ is 2-solid}$$

Assuming $\clubsuit$, fix $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ a coherent HFC$_w$ and identify each A_α with the set $\{\xi < \alpha : A_\alpha(\xi) = 1\}$. We shall define the forcing order $\mathbb{Q}$ as follows. Each element of $\mathbb{Q}$ will be a finite approximation of a neighborhood base for a finite subset of ω_1. Define $\mathbb{Q}$ be the set of $q = \langle I_q, u_q, n_q, A_q \rangle$ such that

(i) $I_q \in [\omega_1]^{<\omega}$,

(ii) $n_q \in \omega$,

(iii) $u_q : I_q \times n_q \to \mathcal{P}(I_q)$,

(iv) $\alpha \in u_q(\alpha, k)$ for each $\alpha \in I_q$ and each $k < n_q$,

(v) $A_q = \{A_{\xi_0}, \cdots, A_{\xi_{k-1}}\}$ where $I_q < \{\xi_0, \cdots, \xi_{k-1}\}$.

Given $p, q \in \mathbb{Q}$ then $p \leq q$ if and only if

(a) $I_p \supseteq I_q$,

(b) $n_p \geq n_q$,

(c) $u_p(\alpha, j) \setminus I_q \cap A_\eta = \varnothing$ for every $A_\eta \in A_q$, $\alpha \in I_p$, and $j < n_p$,

(d) if $u_q(\alpha, i) \cap u_q(\beta, j) = \varnothing$ then $u_p(\alpha, i) \cap u_p(\beta, j) = \varnothing$ for any $i, j < n_q$, and $\alpha, \beta \in I_q$,

(e) if $u_q(\alpha, i) \subseteq u_q(\beta, j)$ then $u_p(\alpha, i) \cap u_p(\beta, j)$ for any $i, j < n_q$, and $\alpha, \beta \in I_q$.

In $V^{\mathbb{Q}}$ let G be $\mathbb{Q}$-generic over V and define

$$U^G(\alpha, k) = \bigcup_{q \in G} \{u_q(\alpha, k) : \alpha \in I_q, k < n_q\}$$

Let $X = \langle \omega_1, \tau \rangle$ be the topology generated by taking $\{U^G(\alpha, k) : k < \omega\}$ to be a local base at $\alpha \in \omega_1$.

The forcing order $\mathbb{Q}$ was defined to make each A_η in the ground model become locally finite and hence closed and discrete in X. It is straightforward to prove (see [33] Theorem 3.7)

Lemma 5.32. $V^{\mathbb{Q}} \models$ *"X is an almost left-separated first countable, zero dimensional space."*

It is easy to show that ω is dense in X. Now define a graph $\hat{G}$ as follows. Let $K = \omega \times \omega$ and let $I = \{\langle \alpha, k, d \rangle \in \alpha \times K : d \in U^G(\alpha, k)\}$. Define $\hat{G}$ to be

$$\hat{G} = ([\omega_1 \times K]^2 \setminus [I]^2) \cup \big\{\{\langle \alpha_0, k_0, d_0 \rangle, \langle \alpha_0, k_0, d_0 \rangle\} \in [I]^2 : \\ d_0 \neq d_1 \text{ or } \alpha_0 \in U^G(\alpha_1, k_1) \text{ or } \alpha_1 \in U^G(\alpha_0, k_0)\big\}$$

The following is proved in [33] (see Lemma 3.9):

Lemma 5.33. *If $\hat{G}$ is 2-solid then X is an O-space.*

We have not yet shown that $\mathbb{Q}$ is ccc or that $\hat{G}$ is strongly solid in $V^{\mathbb{Q}}$. We can prove both by proving the following Lemma:

Lemma 5.34. *If $n < \omega$, $\{q_\alpha : \alpha < \omega_1\} \subseteq \mathbb{Q}$, $s_\alpha : \alpha < \omega_1 \subseteq \mathrm{Fn}_n(\omega_1, K)$ is dom-disjoint, then there are $\{\alpha, \beta\} \in [\omega_1]^2$ and $q \in \mathbb{Q}$ such that $q \leq q_\alpha, q_\beta$ and $q \Vdash [s_\alpha, s_\beta] \subseteq \hat{G}$.*

Lemma 5.34 is essentially proved in [33] (Lemma 3.8) with a simplified version of $\mathbb{Q}$. The proof given of Lemma 3.8 in [33] is a proof of Lemma 5.34 mutatis mutandis provided that we first make an observation. Fix $\{q_\alpha : \alpha < \omega_1\} \subseteq \mathbb{Q}$. Let $\Delta(A_{q_\alpha})$ be defined as

$$\Delta(A_{q_\alpha}) = \min_{i \neq j}\{\Delta(A_{\alpha_i}, A_{\alpha_j})\}$$

where A_{q_α} is enumerated at $\{A_{\alpha_0}, \cdots\}$ and where

$$\Delta(A_{\alpha_i}, A_{\alpha_j}) = \min\{\xi < \min\{\alpha_i, \alpha_j\} : A_{\alpha_i}(\xi) \neq A_{\alpha_j}(\xi)\}$$

Then shrinking $\{q_\alpha : \alpha < \omega_1\}$ if necessary we may assume that $\{\Delta(A_{q_\alpha}) : \alpha < \omega_1\}$ is uncountable and strictly increasing since $T(A)$ is Aronszajn. Furthermore using that A is an HFC_w we may assume without loss of generality that there is an $\alpha < \omega_1$ such that for all $\beta > \alpha$ it is the case that $A_{\beta_0} \cap I_{q_\alpha} = \varnothing$ and furthermore since $\Delta(A_{q_\beta}) > \alpha$ it is the case that $A_{\beta_i} \cap I_{q_\alpha} = \varnothing$ for every i. Thus condition (c) in the definition of $p \leq q$ is no impediment to compatibility between q_α and q_β.

Chapter 6

Additional Open Questions

We close with some open questions. First and foremost is the question that was in large part motivation for this work. I offer US$50 for a solution to the following question, which brings the total award to US$100 including the US$50 award offered by Nyikos.

Question 6.1. *(CH) Is there a first countable O-space?*

6.1 club O-spaces

An Ostaszewski space was originally constructed by Ostaszewski in [28] using $\clubsuit$ and CH. Recall that $\rotatebox[origin=c]{180}{$\clubsuit$}$ is a weakening of $\clubsuit$ such that the elements $\langle x_\alpha : \alpha < \omega_1 \rangle$ of the $\rotatebox[origin=c]{180}{$\clubsuit$}$-sequence have the property that for any club $C \subseteq \omega_1$, there exists some x_α such that $x_\alpha \subseteq C$. The Ostaszewski construction can be performed using the $\rotatebox[origin=c]{180}{$\clubsuit$}$-sequence and CH. The resulting space $X = (\omega_1, \tau)$ is regular, locally compact and countably compact. Hence X is perfectly normal.

In the original Ostaszewski, the hereditary separability of the Ostaszewski space comes from the property of the $\clubsuit$-sequence to capture all uncountable subsets of ω_1. In particular, the closure of every element x_α of the $\clubsuit$-sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is co-countable and thus so is the closure of every uncountable subset. Hereditary separability is not guaranteed by performing the Ostaszewski construction with a $\rotatebox[origin=c]{180}{$\clubsuit$}$-sequence. This makes natural the following definition:

Definition 6.2. *A space X is called a **club O-space** if $X = (\omega_1, \tau)$ is regular, hereditarily separable, and the closure of every club $C \subseteq \omega_1$ is co-countable.*
*A space X is called a **club Ostaszewski space** if X is a locally compact, countably compact club O-space.*

Question 6.3. *(CH + $\clubsuit$) Is there a club Ostaszewski space?*

Question 6.4. *(CH + $\clubsuit$) Is there a club O-space?*

It is unknown if CH is a necessary hypothesis in the above questions. It is the case however that $\clubsuit$ is indestructible with respect to ω-proper forcings and so $\clubsuit$ is consistent with MA($\aleph_1$). Thus it cannot be the case that $\clubsuit$ can be used to construct a locally compact *S*-space alone. We can however still ask:

Question 6.5. *($\clubsuit$) Is there an S-space?*

6.2 Spaces from ⫯

Since ⫯ implies $\mathfrak{b} = \omega_1$, it is the case that ⫯ implies there is a locally compact *S*-space. (see §2.6) It is also the case that ⫯ implies that there is a HFD_w and hence an *O*-space. This leaves open the following question:

Question 6.6. *(⫯) Is there a locally compact O-space?*

Using Theorem 3.12, it will suffice to produce a 1st-countable *O*-space to answer Question 6.6 positively.

6.3 weak HFDs, O-spaces and S-spaces

We have seen that with a few minor assumptions, all HFDs are weak HFDs, all weak HFDs are O-spaces and all O-spaces are S-spaces. It is unknown if any of these implications can be reversed.

It is known that there are models in which there are weak HFDs and no HFDs. If there is an O-space then it is possible to construct an HFD_w under certain assumptions. In [20] (see 4.27) it was shown that assuming $MA(\aleph_1)$ for countable partial orders that:

Theorem 6.7 (Juhász)**.** *Assuming $MA(\aleph_1)$ for countable partial orders, if there is an O-space, there is an HFD_w.*

This leaves open the following question:

Question 6.8. *If there is an O-space, is there an HFD_w?*

In a model in which $MA(\aleph_1)$ holds, if there is an S-space X, then using Szentmiklóssy's Lemma (see [29] §6.4) there is a natural topology on X which makes X a T_1 space in which every open set is countable or co-countable. It is not clear how to make this topology T_3, even assuming $MA(\aleph_1)$. This leaves open the question:

Question 6.9. *If there is an S-space, is there an O-space?*

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