A Preparation Theorem for Weierstrass Systems

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Abstract

It is shown that Lion and Rolin's preparation theorem for globally subanalytic functions holds for the collection of definable functions in any expansion of the real ordered field by a Weierstrass system.

Contents

Chapter 1

Introduction

Any real analytic function $f : U \to \mathbb{R}$ on an open neighborhood U of $[-1,1]^n$ defines a corresponding **restricted analytic function** $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^n, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the structure $\mathbb{R}_{\text{an}}$, the expansion of the real ordered field by all restricted analytic functions, and denote its language by $\mathcal{L}_{\text{an}}$.

The definable sets of $\mathbb{R}_{\text{an}}$ are known in geometry as the **globally subanalytic sets**, and have been extensively studied by both geometers and model theorists alike. Gabrielov [6] showed that the complement of a (globally) subanalytic set is (globally) subanalytic, from which it follows that $\mathbb{R}_{\text{an}}$ is model complete. Denef and Van den Dries [4] strengthened this result by using Weierstrass preparation and a generalization of Tarski's theorem to show that $\langle \mathbb{R}_{\text{an}}, / \rangle$, the expansion of $\mathbb{R}_{\text{an}}$ by division, admits quantifier elimination (to be acurate, only a local version of this was shown in [4], from which this result follows). Van den Dries, Macintrye and Marker [20] then showed that if not only division, but also all nth roots are added to the language, to obtain the structure we shall denote by $\mathbb{R}'_{\text{an}}$ with language $\mathcal{L}'_{\text{an}}$, then the theory is universally axiomatizable. Coupling this with the quantifier elimination shows by a simple model theoretic argument that all the definable functions are piecewise given by terms. Lion and Rolin [11] later gave a purely geometric proof of this in their preparation theorem for $\mathbb{R}'_{\text{an}}$, which states that given an $\mathcal{L}'_{\text{an}}$-term $f(x, y)$, where x ranges over $\mathbb{R}^n$ and y over $\mathbb{R}$, $\mathbb{R}^{n+1}$ can be covered by finitely many quantifier-free definable sets of a certain form such that on each of these sets,

$$f(x, y) = a(x)|y - \theta(x)|^q u(x, y),$$

for some $q \in \mathbb{Q}$ and $\mathcal{L}'_{\mathrm{an}}$-terms $a(x)$, $\theta(x)$ and $u(x, y)$, where $u(x, y)$ is a unit. When f is written in this form, we say that f is "prepared." Since a corollary of the preparation theorem is that all definable functions are piecewise given by terms, it follows that all globally subanalytic functions can be prepared. This preparation theorem is our object of study, so we begin by surveying related results.

The preparation theorem was used in [11] to give geometric proofs of other theorems as well which were originally discovered by model theoretic techniques. They considered the structure $\langle \mathbb{R}_{\mathrm{an}}, \exp, \log \rangle$, the expansion of $\mathbb{R}_{\mathrm{an}}$ by the exponential and logarithmic functions, and also the structure $\mathbb{R}^K_{\mathrm{an}} := \langle \mathbb{R}_{\mathrm{an}}, x^r \rangle_{r \in K}$, the expansion of $\mathbb{R}_{\mathrm{an}}$ by all power functions $x \mapsto x^r$ for r in a field $K \subseteq \mathbb{R}$, and showed that the definable functions of both structures are piecewise given by terms (first proven in [20] and C. Miller [15], respectively). Later in [12], the preparation theorem was used to show that volume integrals of subanalytic functions are in fact subanalytic functions themselves, a surprising result not previously know to model theorists.

Speissegger and Van den Dries [23] used the "valuation property" from [24] to show that any polynomially bounded o-minimal expansion $\mathcal{M}$ of the real field has a certain kind of preparation theorem, from which they deduced a preparation theorem for $\langle \mathcal{M}, \exp, \log \rangle$ such as was done in [11] for $\langle \mathbb{R}_{\mathrm{an}}, \exp, \log \rangle$.

In the spirit of o-minimality at its purest, all the sets and functions involved in the preparation theorem of [23] are simply required to be definable in $\mathcal{M}$, so the issue of the quantifier complexity of the formulas needed to define these sets and functions is not addressed. In contrast, quantifier complexity is of central importance in Lion and Rolin's work, but they always consider expansions of the structure $\mathbb{R}_{\mathrm{an}}$, so their language is quite large.

There has been some remarkable progress dealing with quantifier complexity issues in reducts of $\mathbb{R}_{\mathrm{an}}$, particularly in expansions of the real field by restricted Pfaffian functions. A **Pfaffian chain** is a finite list of analytic functions $f_1, \ldots, f_m : U \to \mathbb{R}$ on some open set $U \subseteq \mathbb{R}^n$ such that for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ there are polynomials $p_{ij} \in \mathbb{R}[y_1, \ldots, y_i]$ such that

$$\frac{\partial f_i}{\partial x_j}(x) = p_{ij}(f_1(x), \ldots, f_i(x))$$

on U. Take note of the triangular nature of this system of differential equations. If we relax the definition and simply require that each $p_{ij} \in \mathbb{R}[y_1, \ldots, y_m]$, then $f_1, \ldots, f_m$ is called a **Noetherian chain**.

Wilkie [25] showed that when $[-1,1]^n \subseteq U$, one obtains a model complete structure by expanding the real field by the set of restricted analytic functions $\{\widetilde{f}_1, \ldots, \widetilde{f}_m\}$ corresponding to a Pfaffian chain $f_1, \ldots, f_m : U \to \mathbb{R}$, along with the set $\{c_1, \ldots, c_k\}$ of coefficients occuring in the polynomials p_{ij}. Later, Gabrielov [7] showed that the expansion of the real field by any subalgebra of restricted analytic functions closed under differentiation is model complete. Since each of the derivatives $\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}$ of the functions from a Noetherian chain $f_1, \ldots, f_m$ are given by integral polynomials in $\{f_1, \ldots, f_m, c_1, \ldots, c_k\}$, this generalizes Wilkie's result by showing that given a Noetherian chain $f_1, \ldots, f_m$, the expansion of the real field by $\{\widetilde{f}_1, \ldots, \widetilde{f}_m, c_1, \ldots, c_k\}$ is model complete.

One of the reasons behind the effort to achieve bounds on quantifier complexity in retracts of $\mathbb{R}_{\mathrm{an}}$ is an interest in effectivity questions. Wilkie [25] went on to show that the real exponential field is o-minimal and model complete, and then Wilkie and Macintrye [13] used this work to show that if Schanuel's conjecture is true, then the real exponential field is decidable. Gabrielov and Vorobjov [8] showed that in the case of Pfaffian functions, the cylindrical decomposition theorem from [7] is given by an algorithm for a real number machine which uses an oracle for deciding whether a Pfaffian system of equalities and inequalities has a solution.

All of these effectivity results deal with model complete o-minimal expansions of the real field. Other than for the real ordered field itself [18], I am not currently aware of any progress on effectivity and decidability issues for retracts of $\mathbb{R}'_{\mathrm{an}}$ which have quantifier elimination. But Lion and Rolin's proof of the preparation theorem for $\mathbb{R}'_{\mathrm{an}}$ is a rather explicit geometric construction, and so there arises a natural question: Is there an effective version of their preparation theorem? A positive answer to this question would give an effective quantifier elimination procedure and may shed some new light on determining whether or not the theory of the real field with restricted exponentiation $e^x\big|_{[-1,1]}$ is decidable, which would imply that the theory of the real exponential field is decidable [13].

But an effective preparation theorem could not possibly be about the structure $\mathbb{R}'_{\mathrm{an}}$ itself, since the language $\mathcal{L}'_{\mathrm{an}}$ is clearly not computable. This theorem would have to be about some reduct of $\mathbb{R}'_{\mathrm{an}}$, which we shall call $\mathbb{R}'_{\mathcal{R}}$

with language $\mathcal{L}'_{\mathcal{R}}$, obtained by expanding the real ordered field by division, all nth roots and the functions from $\mathcal{R}$, where $\mathcal{R}$ is a collection of restricted analytic functions in which we have some effective way of representing both the functions in $\mathcal{R}$ and all the geometric operations on these functions needed in the proof of the preparation theorem. To be able to represent the functions in $\mathcal{R}$ they need to be uniquely determined by some finite amount of information. For instance, maybe they could be represented by some polynomial algebraic equations or some polynomial differential equations, along with initial conditions, to which they are the unique solutions. Also, $\mathcal{R}$ clearly has to be countable if we are to have a uniform way of effectively representing all the functions of $\mathcal{R}$.

With this motivation in mind, we consider a Weierstrass system $\mathcal{R}$, of which the system of algebraic restricted analytic functions and the system of differentially algebraic restricted analytic functions are examples.

Main Theorem. *If $\mathcal{R}$ is a Weierstrass system, then every $\mathcal{L}'_{\mathcal{R}}$-term is prepared.*

We shall complete the statement of the Main Theorem in Chapter 2 by precisely defining the structure $\mathbb{R}'_{\mathcal{R}}$ and its language $\mathcal{L}'_{\mathcal{R}}$, defining what it means for a function to be prepared, and defining the notion of a Weierstrass system. Chapter 2 also shows how the preparation theorem implies that definable functions are piecewise given by terms, and how one can obtain countable examples of Weierstrass systems of algebraic and differentially algebraic restricted analytic functions. Chapters 3 and 4 constitute the proof of the Main Theorem. We postpone outlining these chapters until the end of Chapter 2 after all the relevant terminology has been introduced.

We conclude the introduction with a problem of Gabrielov's. When $\mathcal{R}$ is a chain of restricted Pfaffian functions, the definable sets of $\mathbb{R}_{\mathcal{R}}$ are called "sub-Pfaffian" [7]. Gabrielov has asked whether the sub-Pfaffian sets have any kind of preparation theorem in the sense of Lion and Rolin. The Main Theorem provides a partial positive answer to this: sub-Pfaffian functions can be prepared within the larger system of differentially algebraic functions. If the sub-Pfaffian functions have a Weierstrass preparation theorem, the Main Theorem would show that the preparation can be done within the system sub-Pfaffian functions, but I am not aware of this being known.

Chapter 2

Preparation and Weierstrass Systems

This chapter defines the terminology used in the statement of the Main Theorem and demonstrates how this theorem can be applied. To motivate this terminology, we begin with an example.

2.1 Example: preparing the general quadratic

Consider the general quadratic,

$$f(x, y) := x_2 y^2 + x_1 y + x_0,$$

where $x = (x_0, x_1, x_2)$, and fix the language $\{<, +, -, \cdot, /, \sqrt{\ }, 0, 1\}$, where we interpret $y/0 := 0$ and $\sqrt{y} := 0$ if $y < 0$.

Claim. We can cover $\mathbb{R}^4$ with quantifier free definable sets $C_1, \ldots, C_9$ such that for each $C \in \{C_1, \ldots, C_9\}$,

$$f(x, y) = a(x)(y - \theta(x))^d u(x, y) \tag{2.1}$$

on C for some $d \in \{0, 1, 2\}$ and terms $a(x)$, $\theta(x)$ and $u(x, y)$, where $\epsilon < u(x, y) < M$ on C for some $0 < \epsilon < M$.

Even though both division and the square root operation have been extended by 0 off their natural domains, this is simply a convention to make them total functions and is of no consequence. We shall neither divide by 0 nor take the square root of a negative number in our calculations.

Let $C_1 := \{(x,y) : x_2 = x_1 = 0\}$. Then on C_1,

$$f(x,y) = x_0.$$

Let $C_2 := \{(x,y) : x_2 = 0, x_1 \neq 0\}$. Then on C_2,

$$f(x,y) = x_1 \left(y + \frac{x_0}{x_1} \right).$$

Let $\widetilde{C}_3 := \{(x,y) : x_2 \neq 0, x_1^2 - 4x_0x_2 < 0\}$. Completing the square gives

$$f(x,y) = x_2 \left(\left(y + \frac{x_1}{2x_2} \right)^2 + \frac{4x_0x_2 - x_1^2}{4x_2^2} \right),$$

so $f(x,y) \neq 0$ on $\widetilde{C}_3$. Consider the two factorizations:

$$f(x,y) = x_2 \left(y + \frac{x_1}{2x_2} \right)^2 \left(1 + \frac{(4x_0x_2 - x_1^2)/(4x_2^2)}{(y + x_1/(2x_2))^2} \right), \tag{2.2}$$

$$f(x,y) = \left(\frac{4x_0x_2 - x_1^2}{4x_2} \right) \left(\frac{(y + x_1/(2x_2))^2}{(4x_0x_2 - x_1^2)/(4x_2^2)} + 1 \right), \tag{2.3}$$

and let

$$C_3 := \left\{ (x,y) \in \widetilde{C}_3 : \frac{(4x_0x_2 - x_1^2)/(4x_2^2)}{(y + x_1/(2x_2))^2} < 2 \right\},$$

$$C_4 := \left\{ (x,y) \in \widetilde{C}_3 : \frac{(y + x_1/(2x_2))^2}{(4x_0x_2 - x_1^2)/(4x_2^2)} < 2 \right\}.$$

Note that $\widetilde{C}_3 = C_3 \cup C_4$ and that (2.2) is of the form (2.1) on C_3 and (2.3) is of the form (2.1) on C_4.

Let $C_5 := \{(x,y) : x_2 \neq 0, x_1^2 - 4x_0x_2 = 0\}$. Then on C_5,

$$f(x,y) = x_2 \left(y + \frac{x_1}{2x_2} \right)^2.$$

Let $\widetilde{C}_6 := \{(x,y) : x_2 \neq 0, x_1^2 - 4x_0x_2 > 0\}$. Then on $\widetilde{C}_6$

$$f(x,y) = x_2(y - \theta_1(x))(y - \theta_2(x)),$$

where $\theta_1(x) := \frac{-x_1+\sqrt{x_1^2-4x_0x_2}}{2x_2}$ and $\theta_2(x) := \frac{-x_1-\sqrt{x_1^2-4x_0x_2}}{2x_2}$. Note that $y - \theta_1(x)$ can be written in the following two ways:

$$
\begin{aligned}
y-\theta_1(x) &= (y-\theta_2(x))\left(1+\frac{\theta_2(x)-\theta_1(x)}{y-\theta_2(x)}\right),\\
y-\theta_1(x) &= (\theta_2(x)-\theta_1(x))\left(1+\frac{y-\theta_2(x)}{\theta_2(x)-\theta_1(x)}\right).
\end{aligned}
$$

So if we only consider the points $(x,y) \in \widetilde{C}_6$ in which $y \neq \theta_2(x)$ and $\frac{\theta_2(x)-\theta_1(x)}{y-\theta_2(x)}$ stays bounded away from -1 and also is bounded in absolute value, then

$$
f(x,y) = x_2(y-\theta_2(x))^2\left(1+\frac{\theta_2(x)-\theta_1(x)}{y-\theta_2(x)}\right)
$$

is of the form (2.1). Similarly, if we only consider the points $(x,y) \in \widetilde{C}_6$ in which $\frac{y-\theta_2(x)}{\theta_2(x)-\theta_1(x)}$ stays bounded away from -1 and also is bounded in absolute value, then

$$
f(x,y) = x_2(\theta_2(x)-\theta_1(x))\cdot(y-\theta_2(x))\cdot\left(1+\frac{y-\theta_2(x)}{\theta_2(x)-\theta_1(x)}\right)
$$

is also of the form (2.1). A similar technique can be applied to $y-\theta_2(x)$ as was done with $y-\theta_1(x)$, so all we need to do is show that these various cases cover $\widetilde{C}_6$. A simple calculation shows that the following sets cover $\widetilde{C}_6$

$$
\begin{aligned}
C_6 &:= \left\{(x,y)\in\widetilde{C}_6 : \left|\frac{y-\theta_1(x)}{\theta_1(x)-\theta_2(x)}\right| < \frac{3}{4}\right\},\\
C_7 &:= \left\{(x,y)\in\widetilde{C}_6 : \left|\frac{y-\theta_2(x)}{\theta_1(x)-\theta_2(x)}\right| < \frac{3}{4}\right\},\\
C_8 &:= \left\{(x,y)\in\widetilde{C}_6 : \frac{y-\theta_2(x)}{\theta_1(x)-\theta_2(x)} > \frac{3}{2}\right\},\\
C_9 &:= \left\{(x,y)\in\widetilde{C}_6 : \frac{y-\theta_1(x)}{\theta_1(x)-\theta_2(x)} < -\frac{3}{2}\right\},
\end{aligned}
$$

and that f is of the following forms on $C_6,\ldots,C_9$, respectively, each of which are as in (2.1):

$$\begin{aligned}
f(x,y) &= x_2(\theta_1(x)-\theta_2(x))\cdot(y-\theta_1(x))\cdot\left(1+\frac{y-\theta_1(x)}{\theta_1(x)-\theta_2(x)}\right),\\
f(x,y) &= x_2(\theta_2(x)-\theta_1(x))\cdot(y-\theta_2(x))\cdot\left(1+\frac{y-\theta_2(x)}{\theta_2(x)-\theta_1(x)}\right),\\
f(x,y) &= x_2\cdot(y-\theta_2(x))^2\cdot\left(1+\frac{\theta_2(x)-\theta_1(x)}{y-\theta_2(x)}\right),\\
f(x,y) &= x_2\cdot(y-\theta_1(x))^2\cdot\left(1+\frac{\theta_1(x)-\theta_2(x)}{y-\theta_1(x)}\right).
\end{aligned}$$

In the next section we shall see that each of the sets C_i are a finite union of sets called “cylinders,” and sometimes we will informally use the phrase “subcylindering” to refer to the technique of proof-by-cases employed here in which we cover a cylinder C with finitely many cylinders $C_1,\ldots,C_k\subseteq C$ so that the function f has more uniform properties on each C_i than it did on C. In Lemma 3.29 we shall see again the subcylindering techniques used in this example.

2.2 Preparation and Quantifier Elimination

Throughout this paper we fix a sequence of variables $\overline{x}:=(x_1,x_2,x_3,\ldots)$. If $n\in\mathbb{N}$ is given we write $x:=(x_1,\ldots,x_n)$ and $y:=x_{n+1}$, and let $\Pi:\mathbb{C}^{n+1}\to\mathbb{C}^n$ denote the projection map $(x,y)\mapsto x$.

For $r:=(r_1,\ldots,r_n)\in(0,\infty)^n$ let

$$\mathcal{B}_r:=\{x\in\mathbb{C}^n:|x_i|\leq r_i \text{ for } i=1,\ldots,n\}$$

and $B_r:=\mathcal{B}_r\cap\mathbb{R}^n$. For $a\in\mathbb{C}^n$ let $\mathcal{B}_r(a):=a+\mathcal{B}_r$, and for $a\in\mathbb{R}^n$ let $B_r(a):=a+B_r$. For $K\subseteq\mathbb{R}$ let $K_+:=K\cap(0,\infty)$.

Partially order $\mathbb{R}^n$ as follows: for $r=(r_1,\ldots,r_n)$ and $s=(s_1,\ldots,s_n)$ in $\mathbb{R}^n$,

$$r\leq s \text{ iff } r_1\leq s_1,\ldots,r_n\leq s_n.$$

Also, write $r<s$ iff $r_1<s_1,\ldots,r_n<s_n$.

For a function $f:\mathbb{C}^n\to\mathbb{C}^m$ which is C^∞ at the origin, $\widehat{f}$ is the Taylor series of f at the origin.

Definition 2.1. Let $K \subseteq \mathbb{R}$ be a field. Suppose that for each $n \in \mathbb{N}$ and $r \in K_+^n$ we have a collection $\mathcal{R}_{n,r}$ of functions $f : \mathcal{B}_r \to \mathbb{C}$. Let $\mathcal{R}_n := \bigcup_{r \in K_+^n} \mathcal{R}_{n,r}$ and $\mathcal{R} := \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$, and suppose that $K = \mathcal{R}_0$. We call $\mathcal{R}$ an **analytic system of functions over** K if for each $n \in \mathbb{N}$ and $r \in K_+^n$,

(i) each $f \in \mathcal{R}_{n,r}$ is holomorphic on $\operatorname{int}(\mathcal{B}_r)$ and $f(B_r) \subseteq \mathbb{R}$;

(ii) $\mathcal{R}_{n,r}$ contains the coordinate projection functions $x_i : \mathcal{B}_r \to \mathbb{C}$, $x \mapsto x_i$, for $i = 1, \ldots, n$;

(iii) $\mathcal{R}_{n,r}$ is a K-algebra, and $\widehat{f}(x) \in K[\![x]\!]$ for each $f \in \mathcal{R}_{n,r}$;

(iv) for each $f \in \mathcal{R}_{n,r}$ there is an $s \in K_+^n$ and a $g \in \mathcal{R}_{n,s}$ such that $s > r$ and $f = g\big|_{\mathcal{B}_r}$;

(v) for each $f \in \mathcal{R}_{n,r}$ and $s \in K_+^n$ such that $s \leq r$, $f\big|_{\mathcal{B}_s} \in \mathcal{R}_{n,s}$;

If in the above definition we replace each instance of $\mathcal{B}_r$ with B_r and replace (i) with

(i′) $\mathcal{R}_{n,r}$ is a collection of functions $f : B_r \to \mathbb{R}$ which are C^∞ on $\operatorname{int}(B_r)$ and such that the Taylor map at the origin $\widehat{\ }: \mathcal{R}_{n,r} \to \mathbb{R}[\![x]\!]$ is injective (**quasianalyticity**),

then $\mathcal{R}$ is called a **quasianalytic system of functions**. Since many of our results apply to both analytic and quasianalytic systems of functions, we shall use the phrase **system of functions** as an abbreviation for either such system.

For $f \in \mathcal{R}_n$ let $r(f) \in K_+^n$ denote the polyradius such that $f \in \mathcal{R}_{n,r(f)}$. For $n \in \mathbb{N}$ and $\epsilon > 0$ we shall write $\mathcal{R}_{n,\epsilon} := \mathcal{R}_{n,(\epsilon,\ldots,\epsilon)}$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a **restricted $\mathcal{R}$-function** if there is a $g \in \mathcal{R}_{n,1}$ such that $f = g$ on $[-1,1]^n$ and $f = 0$ on $\mathbb{R}^n \backslash [-1,1]^n$. The structure $\mathbb{R}_{\mathcal{R}}$ is the expansion of the real ordered field by all restricted $\mathcal{R}$-functions, and $\mathbb{R}'_{\mathcal{R}}$ is the further expansion by division and $\sqrt[n]{\ }$ for $n = 2, 3, 4, \ldots$, where $a/0 := 0$ for all $a \in \mathbb{R}$ and $\sqrt[n]{a} := 0$ for all $a < 0$. The languages of $\mathbb{R}_{\mathcal{R}}$ and $\mathbb{R}'_{\mathcal{R}}$ are $\mathcal{L}_{\mathcal{R}}$ and $\mathcal{L}'_{\mathcal{R}}$, respectively.

Given a language $\mathcal{L}$ and some fixed $\mathcal{L}$-structure $\mathcal{M}$ under consideration, we shall slightly abuse model theoretic terminology and say that an **$\mathcal{L}$-term** is a function obtained by composing functions in the signature of $\mathcal{M}$.

A function $f : M^n \to M$ is **piecewise given by $\mathcal{L}$-terms** if there are finitely many $\mathcal{L}$-terms $t_1(x), \ldots, t_m(x)$ such that for all $a \in M^n$, $f(a) = t_i(a)$ for some $i = 1, \ldots, m$. Note that if f is definable then each of the sets $\{a \in M^n : f(a) = t_i(a)\}$ are definable, so f can be expressed by terms via a definition by cases over definable sets.

Let $\mathcal{L}$ be a language extending the language of the real ordered field. A set $C \subseteq \mathbb{R}^{n+1}$ is an **$\mathcal{L}$-cylinder** if $B := \Pi(C)$ is a quantifier free $\mathcal{L}$-definable set, called the **base** of C, and $C = \{(x, y) \in B \times \mathbb{R} : \psi(x, y)\}$ where $\psi(x, y)$ is either of the form

$$y = t(x), \tag{2.4}$$

or of one of the forms

$$y < s(x), \quad s(x) < y < t(x), \quad \text{or} \quad t(x) < y, \tag{2.5}$$

where $s(x)$ and $t(x)$ are $\mathcal{L}$-terms. We say that C is **thin** if $\psi(x, y)$ is as in (2.4) and that C is **fat** if $\psi(x, y)$ is as in (2.5).

By induction on $n \in \mathbb{N}$, we say that an $\mathcal{L}$-cylinder $C \subseteq \mathbb{R}^{n+1}$ is an **$\mathcal{L}$-term-cell** if $\Pi(C)$ is an $\mathcal{L}$-term-cell.

Example 2.2. Let $\mathcal{L} := \{<, +, -, \cdot, /, \sqrt{\ }, 0, 1\}$ and $x := (x_0, x_1, x_2)$. Each of the sets $C_1, \ldots, C_9$ from the Section 2.1 are finite unions of $\mathcal{L}$-cylinders. For example, if we let $B := \{x \in \mathbb{R}^3 : x_2 \neq 0, x_1^2 - 4x_0x_2 < 0\}$, then

$$\begin{aligned} C_3 \ &= \ \left\{(x, y) \in B \times \mathbb{R} : -\frac{x_1}{2x_2} + \sqrt{\frac{4x_0x_2 - x_1^2}{8x_2^2}} < y\right\} \\ &\quad \bigcup \left\{(x, y) \in B \times \mathbb{R} : y < -\frac{x_1}{2x_2} - \sqrt{\frac{4x_0x_2 - x_1^2}{8x_2^2}}\right\}. \end{aligned}$$

For the rest of this section, fix a system of functions $\mathcal{R}$ over some subfield K of $\mathbb{R}$.

We shall frequently use the following easily verifiable fact: for each $n \in \mathbb{N}$, the collection of all subsets of $\mathbb{R}^n$ which are finite unions of $\mathcal{L}'_{\mathcal{R}}$-cylinders is a boolean algebra.

Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **$\mathcal{R}$-analytic at** $a \in \mathbb{R}^n$ if there is a $b \in K^n$ and an $r \in K^n_+$ such that $f(x+b)\big|_{\mathcal{B}_r} \in \mathcal{R}_{n,r}$ and $a \in \operatorname{int}(B_r(b))$; f is **$\mathcal{R}$-analytic on** $A \subseteq \mathbb{R}^n$ if f is $\mathcal{R}$-analytic at each $a \in A$. If f has constant positive or negative sign on A, we call f a **unit on** A. Note that if A is compact and f is $\mathcal{R}$-analytic on A, the graph of f on A is piecewise given by $\mathcal{L}_{\mathcal{R}}$-terms.

For $A \subseteq \mathbb{R}^{n+1}$, a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is an **$\mathcal{R}$-unit on** A if $A \cap (\mathbb{R}^n \times \{0\}) = \emptyset$ and $u = v \circ \varphi$, where for some $N \in \mathbb{N}$, $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^N$ is given by

$$\varphi(x,y) = \left(a_1(x), \dots, a_{N-2}(x),\, a_{N-1}(x)y,\, \frac{a_N(x)}{y}\right)$$

for some $\mathcal{L}'_{\mathcal{R}}$-terms $a_1(x), \dots, a_N(x)$, $\varphi(A)$ is bounded, and $v : \mathbb{R}^N \to \mathbb{R}$ is an $\mathcal{R}$-analytic unit on $\overline{\varphi(A)}$. If v is positively valued on $\overline{\varphi(A)}$, u is a **positive** $\mathcal{R}$-unit on A.

A function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is **prepared on** A if there are finitely many $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \dots, C_k \subseteq \mathbb{R}^{n+1}$ covering A such that for each $C \in \{C_1, \dots, C_k\}$ either

(i) C is thin, say $C = \{(x,y) \in B \times \mathbb{R} : y = t(x)\}$ for some $\mathcal{L}'_{\mathcal{R}}$-term $t(x)$, and there is an $\mathcal{L}'_{\mathcal{R}}$-term $s(x)$ such that $f(x,t(x)) = s(x)$ for all $x \in B$, or

(ii) C is fat and

$$f(x,y) = a(x)|y-\theta(x)|^q u(x, |y-\theta(x)|^{1/p}) \tag{2.6}$$

on C, where $a(x)$ and $\theta(x)$ are $\mathcal{L}'_{\mathcal{R}}$-terms, $p \in \mathbb{N}_+$, $q \in \mathbb{Q}$, and $u(x,y)$ is a positive $\mathcal{R}$-unit on $\{(x, |y-\theta(x)|^{1/p}) : (x,y) \in C\}$. (Note that $y \neq \theta(x)$ on C in this case.)

If $A = \mathbb{R}^{n+1}$ we say that f is **prepared**.

Remark 2.4. Let us explain some aspects of the above definition which were made for the sake of convenience, not necessity.

(i) In the above definition, when $C \subseteq \mathbb{R}^{n+1}$ is fat we require that $y \neq \theta(x)$ on C since we may need to divide by $y - \theta(x)$ in the expression $|y-\theta(x)|^q$ or in $u(x, |y-\theta(x)|^{1/p})$. This presents no problem since the set $\{(x,y) \in C : y = \theta(x)\}$ is a finite union of thin cylinders and so always may be considered separately.

(ii) Consider a thin cylinder $C \subseteq \mathbb{R}^{n+1}$. If $f(x,y)$ is a term, then condition (i) in Definition 2.3 is automatically satisfied. But we shall also be interested in preparaing all definable functions, so we have incorporated this condition into the definition.

Proposition 2.5. *Suppose that every $\mathcal{L}'_{\mathcal{R}}$-term is prepared. Then*

(i) *$\mathbb{R}'_{\mathcal{R}}$ admits quantifier elimination;*

(ii) *for any $\mathcal{L}'_{\mathcal{R}}$-definable sets $A_1, \ldots, A_m \subseteq \mathbb{R}^n$, $\mathbb{R}^n$ may be partitioned into $\mathcal{L}'_{\mathcal{R}}$-term-cells $C_1, \ldots, C_k$ such that for each $i = 1, \ldots, m$ and $j = 1, \ldots, k$, either $C_j \cap A_i = \emptyset$ or $C_j \subseteq A_i$ (we say that the C_j's* **partition** *the A_i's);*

(iii) *all $\mathcal{L}'_{\mathcal{R}}$-definable functions are piecewise given by $\mathcal{L}'_{\mathcal{R}}$-terms and are prepared.*

Proof. Let $\varphi(x,y)$ be a quantifier-free $\mathcal{L}'_{\mathcal{R}}$-formula. To prove (i), it suffices to show that $\{x \in \mathbb{R}^n : \exists y \varphi(x,y)\}$ is quantifier-free definable. By writing φ in a disjunctive normal form and then distributing the existential quantifier across the disjunction, we may assume that φ is of the form

$$\varphi(x,y) := \bigwedge_{i=1}^{l} f_i(x,y) = 0 \wedge \bigwedge_{j=1}^{m} g_j(x,y) > 0, \tag{2.7}$$

for $\mathcal{L}'_{\mathcal{R}}$-terms f_i and g_j. For notational simplicity, let us suppress the subscripts of the f's and g's. By preparing all the f's and g's and using the fact that the collection of subsets of $\mathbb{R}^{n+1}$ which are finite unions of $\mathcal{L}'_{\mathcal{R}}$-cylinders is a boolean algebra, there are quantifier-free definable sets $B_1, \ldots, B_k$ partitioning $\mathbb{R}^n$ and a finite collection $\mathcal{C} = \bigcup_{i=1}^{k} \mathcal{C}_i$ of $\mathcal{L}'_{\mathcal{R}}$-cylinders partitioning $\mathbb{R}^{n+1}$ such that for each $i = 1, \ldots, k$ and $C \in \mathcal{C}_i$, $\Pi(C) = B_i$ and each f and g is of a prepared form on C. Since on a thin cylinder $C \in \mathcal{C}$ the graphs of the f's and g's are given by terms in x, then on C the statements $f = 0$ and $g > 0$ are quantifier free in x. Similarly, if

$$f(x,y) = a(x)|y - \theta(x)|^q u(x, |y - \theta(x)|^{1/p})$$

on a fat cylinder $C \in \mathcal{C}$, then the sign of f on C is solely determined by the sign of $a(x)$. This is true on C for all the f's and g's, so on C the statements $f = 0$ and $g > 0$ are also quantifier free in x. Therefore, after possibly further partitioning the B_i's we may assume that the signs of all the f's and g's are constant on each of the cylinders of $\mathcal{C}$. It follows that $\{(x,y) \in \mathbb{R}^{n+1} : \varphi(x,y)\} = \cup\mathcal{C}'$ for some $\mathcal{C}' \subseteq \mathcal{C}$, so $\{x \in \mathbb{R}^n : \exists y \varphi(x,y)\} = \bigcup_{i \in I} B_i$ for some $I \subseteq \{1,\ldots,k\}$, proving (i).

We prove (ii) by inductively assuming the result for n and proving it for $n+1$. Let $A_1,\ldots,A_m \subseteq \mathbb{R}^{n+1}$ be $\mathcal{L}'_{\mathcal{R}}$-definable, and hence quantifier free definable by part (i). By partitioning the A_i's and the $\mathbb{R}^{n+1}\backslash A_i$'s into conjunctive components, it suffices to show that for any $A \subseteq \mathbb{R}^{n+1}$ defined by a formula $\varphi(x,y)$ of the form given in (2.7), A is a finite union of $\mathcal{L}'_{\mathcal{R}}$-term-cells. From the above analysis, $A = \cup\mathcal{C}'$ for some $\mathcal{C}' \subseteq \mathcal{C}$. By the induction hypothesis each B_i is a finite union of $\mathcal{L}'_{\mathcal{R}}$-term-cells. But then each $C \in \mathcal{C}'$ is a finite union of $\mathcal{L}'_{\mathcal{R}}$-term-cells, proving (ii).

To prove (iii), let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}'_{\mathcal{R}}$-definable. From (ii) the graph of f is a finite union of $\mathcal{L}'_{\mathcal{R}}$-term-cells. Since f is a function, each of these cells must be thin, so f is piecewise given by terms. Since by assumption each of these terms are prepared, then so is f. □

2.3 Weierstrass systems

We shall need a detailed form of the Weierstrass preparation theorem. Consider a holomorphic function $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^{n+1}$ is a neighborhood of the origin. If $f(0,y) \neq 0$ we say that f is **regular in** y; in this case $f(0) = \frac{\partial f}{\partial y}(0) = \cdots = \frac{\partial^{d-1} f}{\partial y^{d-1}}(0) = 0$ and $\frac{\partial^d f}{\partial y^d}(0) \neq 0$ for some $d \in \mathbb{N}$, called the **order** of f. For $(r,s) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ let $\mathrm{WPT}(f,d,r,s)$ be shorthand for the following statement:

> f is regular in y of order d, and for all $(a,b) \in \mathcal{B}_{(r,s)}$, $f(a,b) \neq 0$ if $|b| = s$, and $f(a,y)$ has exactly d zeros in $\mathrm{int}(\mathcal{B}_s)$ counting multiplicities.

Suppose that $f : U \to \mathbb{C}$ is regular in y of order d. Then $f(0,y)$ has a zero of mulitplicity d at the origin. Let

$$s(f) := \sup\{s > 0 : \{0\} \times \mathcal{B}_s \subseteq U \text{ and } f(0,b) \neq 0 \text{ for all } b \in \mathcal{B}_s\backslash\{0\}\},$$

and note that $s(f) > 0$. By continuity of roots, $s \in (0, s(f))$ iff there is an $r \in (0,\infty)^n$ such that $\mathrm{WPT}(f,d,r,s)$.

Weierstrass Preparation Theorem. *Let $f : U \to \mathbb{C}$ be a holomorphic function, where $U \subseteq \mathbb{C}^{n+1}$ is a neighborhood of the origin, and suppose* $\mathrm{WPT}(f, d, r, s)$. *Then there are unique holomorphic functions $w_0, \ldots, w_{d-1}$:* $\mathrm{int}(\mathcal{B}_r) \to \mathbb{C}$ *and $u : \mathrm{int}(\mathcal{B}_{(r,s)}) \to \mathbb{C}$ such that $w_0(0) = \cdots = w_{d-1}(0) = 0$, u is a unit on* $\mathrm{int}(\mathcal{B}_{(r,s)})$, *and*

$$f(x, y) = (y^d + w_{d-1}(x)y^{d-1} + \cdots + w_0(x))u(x, y)$$

on $\mathrm{int}(\mathcal{B}_{(r,s)})$. *We call $w(x, y) := y^d + w_{d-1}(x)y^{d-1} + \cdots + w_0(x)$ a* **Weierstrass polynomial** *in y.*

Proof. This is a more detailed form of the Weierstrass preparation theorem than is usually stated, but it follows from a well known proof. For instance, see Griffiths and Harris [9, Chapter 0] or Gunning [10, Chapter A, Theorem 4]. □

If $\mathrm{WPT}(f, d, r, s)$ then there is some $(r', s') > (r, s)$ such that $\mathrm{WPT}(f, d, r', s')$. To see this, note that $\{(x, y) \in \mathcal{B}_{(r,s)} : f(x, y) = 0\}$ is a compact subset of $\mathcal{B}_r \times \mathrm{int}(\mathcal{B}_s)$, so by continuity of roots there is an $r' > r$ such that $\mathrm{WPT}(f, d, r', s)$; but then $\mathrm{WPT}(f, d, r', s')$ for any $s' > s$ sufficiently close to s. Therefore, if $\mathrm{WPT}(f, d, r, s)$ and $f = wu$, where w is the Weierstrass polynomial and u is the unit given by Weierstrass preparation on $\mathrm{int}(\mathcal{B}_{(r,s)})$, then w and u extend uniquely to $\mathcal{B}_{(r,s)}$, u is a unit on $\mathcal{B}_{(r,s)}$ and for each $a \in \mathcal{B}_r$, $\{b \in \mathbb{C} : w(a, b) = 0\} \subseteq \mathrm{int}(\mathcal{B}_s)$.

Definition 2.6. An analytic system of functions $\mathcal{R} = \bigcup_{n,r} \mathcal{R}_{n,r}$ over K is called a **Weierstrass system** if $\mathcal{R}$ is closed under differentiation, composition and Weierstrass preparation, as defined below:

(i) *differentiation*: for all $n \in \mathbb{N}$, $r \in K_+^n$ and $i = 1, \ldots, n$, if $f \in \mathcal{R}_{n,r}$ then $\frac{\partial f}{\partial x_i} \in \mathcal{R}_{n,r}$ (note that by properties (i) and (iv) of Definition 2.1, $\frac{\partial f}{\partial x_i}$ is indeed well-defined on the boundary of $\mathcal{B}_r$);

(ii) *composition*: for all $m \in \mathbb{N}_+$, $s \in K_+^m$, $n \in \mathbb{N}$ and $r \in K_+^n$, if $f \in \mathcal{R}_{m,s}$ and $g \in \mathcal{R}_{n,r}^m$ are such that $g(\mathcal{B}_r) \subseteq \mathcal{B}_s$, then $f \circ g \in \mathcal{R}_{n,r}$;

(iii) *Weierstrass preparation*: for all $n, d \in \mathbb{N}$, $(r, s) \in K_+^n \times K_+$ and $f \in \mathcal{R}_{n+1,(r,s)}$ such that $\mathrm{WPT}(f, d, r, s)$, if $f = wu$, where w and u are the Weierstrass polynomial and unit given by Weierstrass preparation on $\mathcal{B}_{(r,s)}$, then $w, u \in \mathcal{R}_{n+1,(r,s)}$.

Definition 2.6 completes the introduction of all the terminology used in the statement of the Main Theorem. We now discuss how to find countable examples of Weierstrass systems which are subsystems of two naturally occurring Weierstrass systems over $\mathbb{R}$.

Examples 2.7. Let $\mathcal{O}_{n,r}$ be the collection of all functions $f : \mathcal{B}_r \to \mathbb{C}$ such that $f(B_r) \subseteq \mathbb{R}$ and f extends to a holomorphic function on a neighborhood of $\mathcal{B}_r$. Then $\mathcal{O} := \bigcup_{n,r} \mathcal{O}_{n,r}$ is the largest Weierstrass system, and by definition $\mathbb{R}_{\mathcal{O}} = \mathbb{R}_{\text{an}}$. Here are two more examples.

1. $\mathcal{A} := \{f \in \mathcal{O} : f$ is algebraic over $\mathbb{R}[\overline{x}]\}$ is the smallest Weierstrass system over $\mathbb{R}$ (see Bochnak, Coste and Roy [3, Section 8.2] and Van den Dries [19]);

2. $\mathcal{D} := \{f \in \mathcal{O} : f$ is differentially algebraic$\}$ is a Weierstrass system over $\mathbb{R}$ [19] such that $\mathcal{A} \subset \mathcal{D} \subset \mathcal{O}$, where $\subset$ denotes proper inclusion. This was the motivating example for proving the Main Theorem. Appendix A contains a brief overview of some basic definitions and facts about differentially algebraic power series.

Definition 2.8. Let $\mathcal{S}$ be a system of functions over a subfield L of $\mathbb{R}$. We say that $\mathcal{S}$ is closed under **local composition** if for all $m \in \mathbb{N}_+$, $s \in L_+^m$, $n \in \mathbb{N}$ and $r \in L_+^n$, if $f \in \mathcal{S}_{m,s}$ and $g \in \mathcal{S}_{n,r}^m$ are such that $g(0) = 0$ and $g(\mathcal{B}_r) \subseteq \mathcal{B}_s$ when $\mathcal{S}$ is analytic (and $g(B_r) \subseteq B_s$ when $\mathcal{S}$ is quasianalytic), then $f \circ g \in \mathcal{S}_{n,r}$.

We say that $\mathcal{S}$ is closed under **translation** if for all $n \in \mathbb{N}$, $r \in L_+^n$ and $f \in \mathcal{R}_{n,r}$, if $a \in \text{int}(B_r) \cap L^n$ and $s \in L_+^n$ are such that $B_s(a) \subseteq B_r$, then $f(x+a)\big|_{\mathcal{B}_s} \in \mathcal{R}_{n,s}$.

Note that a system of functions is closed under composition iff it is closed under local composition and translation.

If in the definition of a Weierstrass system, closure under composition is replaced by closure under local composition, then we call the system of functions a **local** Weierstrass system.

Examples 2.9. Let L be any subfield of $\mathbb{R}$.

1. $\mathcal{A}(L) := \{f \in \mathcal{O} : \widehat{f} \in L[\![\overline{x}]\!]$ and f is algebraic over $L[\overline{x}]\}$ is a local Weierstrass system over L.

2. $\mathcal{D}(L) := \{f \in \mathcal{D} : \widehat{f} \in L[\![\overline{x}]\!]\}$ is a local Weierstrass system over L. More generally, if $\mathcal{R}$ is any Weierstrass system over a subfield K of $\mathbb{R}$

and $L \subseteq K$, then $\{f \in \mathcal{R} : \widehat{f} \in L[\![\overline{x}]\!]\}$ is a local Weierstrass system over L.

Lemma 2.10. *For any field* $L \subseteq \mathbb{R}$, $|\mathcal{A}(L)| = |\mathcal{D}(L)| = |L|$.

Proof. Since $L \subseteq \mathcal{A}(L) \subseteq \mathcal{D}(L)$ then $|L| \leq |\mathcal{A}(L)| \leq |\mathcal{D}(L)|$, so it suffices to show that $|\mathcal{D}(L)| \leq |L|$.

We first show that $|\mathcal{D}_1(L)| \leq |L|$. For each $f \in \mathcal{D}_1(L)$ there is a polynomial $p(y_0, \ldots, y_d) \in \mathbb{Q}[y_0, \ldots, y_d]$ such that $p(f(t), f'(t), \ldots, f^{(d)}(t)) = 0$ and $\frac{\partial p}{\partial y_d}(f(t), f'(t), \ldots, f^{(d)}(t)) \neq 0$. Also, there is a $k \in \mathbb{N}$ such that $f(t)$ is the unique solution to the initial value problem

$$p(y, y', \ldots, y^{(n)}) = 0, \tag{2.8}$$

$$y(0) = f(0),\ \ldots,\ y^{(k)}(0) = f^{(k)}(0)$$

(see, for instance, the proof of Lemma 2.3 in Denef and Lipshitz [5]). Since there are only countably many polynomials over $\mathbb{Q}$ and each of these initial conditions are in L, it follows that there are $|L|$ many initial value problems of the form (2.8), so $|\mathcal{D}_1(L)| \leq |L|$.

Now fix $n \in \mathbb{N}$; we show that $\mathcal{D}_n(L) \leq |L|$. We may assume that $|L| < |\mathbb{R}|$, else the result is trivial. But then we may choose a $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ which is algebraically independent over L. For any $f \in \mathbb{R}[\![x]\!]$ and $z = (z_1, \ldots, z_n)$, define $\Delta[f](z) := \{\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) : \alpha \in \mathbb{N}^n\}$ and $f_\lambda(t) := f(\lambda t)$, where $\lambda t := (\lambda_1 t, \ldots, \lambda_n t)$. Let $f \in \mathcal{D}_n(L)$. By definition, the transcendence degree of $\mathbb{Q}(\Delta[f](x))$ over $\mathbb{Q}$ is finite; we shall write $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\Delta[f](x)) < \infty$ to say this. Therefore $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\Delta[f](x), \lambda) < \infty$, so $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\Delta[f](\lambda t), \lambda) < \infty$. Since $\Delta[f_\lambda](t) \subseteq \mathbb{Q}(\Delta[f](\lambda t), \lambda)$, we have $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\Delta[f_\lambda](t)) < \infty$, so by definition $f_\lambda(t) \in \mathcal{D}_1(L(\lambda))$.

Therefore $f \mapsto f_\lambda$ maps $\mathcal{D}_n(L)$ into $\mathcal{D}_1(L(\lambda))$. Since $|\mathcal{D}_1(L(\lambda))| \leq |L(\lambda)| = |L|$, it suffices to show that the map $L[\![x]\!] \to L(\lambda)[\![t]\!]$ given by $f \mapsto f_\lambda$ is injective. Since this map is a ring homomorphism, it suffices to show that its kernel is $\{0\}$. So compute

$$f_\lambda(t) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) \lambda^\alpha t^{|\alpha|} = \sum_{i \in \mathbb{N}} p_i(\lambda) t^i,$$

where

$$p_i(x) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| = i} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) x^\alpha.$$

If $f \in L[\![x]\!]$ is such that $f_\lambda = 0$, then $p_i(\lambda) = 0$ for all $i \in \mathbb{N}$. Since each $p_i(x) \in L[x]$ and λ was chosen to be algebraically independent over L, it follows that $p_i(x) = 0$, so $f(x) = 0$. $\square$

Proposition 2.11. *Suppose that $\mathcal{S}$ is a local Weierstrass system over a subfield L of $\mathbb{R}$. Let $\mathcal{R}_0 := K := \{f(1) : f \in \mathcal{S}_{1,1}\}$, and for each $n \in \mathbb{N}_+$ and $r \in K_+^n$ let*

$$\mathcal{R}_{n,r} := \{f(x,1)\big|_{B_r} : f \in \mathcal{S}_{n+1,(s,1)} \text{ for some } s \in L_+^n \text{ such that } s \geq r\}.$$

Then $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \bigcup_{r \in K_+^n} \mathcal{R}_{n,r}$ is the smallest Weierstrass system containing $\mathcal{S}$.

Proof. Clearly $\mathcal{R}$ contains $\mathcal{S}$, and since Weierstrass systems are closed under composition, $\mathcal{R}$ is contained in any Weierstrass system containing $\mathcal{S}$. To show that $\mathcal{R}$ is a Weierstrass system, the only nontrivial properties that need to be verified is that K is a field and that $\mathcal{R}$ is closed under composition and Weierstrass preparation.

We shall need the following notation: for $n, d \in \mathbb{N}$, $(r,s) \in L_+^n \times L_+$ and $f \in \mathcal{S}_{n+1,(r,s)}$ let

$$\begin{aligned}
I_d[f](x) &:= \sum_{i=0}^{d} \frac{1}{d!}\frac{\partial^i f}{\partial y^i}(x,0);\\
T_d[f](x,y) &:= \sum_{i=d+1}^{\infty} \frac{1}{d!}\frac{\partial^i f}{\partial y^i}(x,0)y^i;\\
|T_d|[f](x,y) &:= \sum_{i=d+1}^{\infty} \frac{1}{d!}\left|\frac{\partial^i f}{\partial y^i}(x,0)y^i\right|;\\
f_d(x,y) &:= I_d[f](x) + T_d[f](x,y).
\end{aligned}$$

Note that $I_d[f] \in \mathcal{S}_{n,r}$ and $T_d[f], |T_d|[f], f_d \in \mathcal{S}_{n+1,(r,s)}$. The following observations will be used:

(i) if $s \geq 1$ then $f(x,1) = f_d(x,1)$;

(ii) $|T_d[f](x,y)| \leq |T_d|[f](x,y)$;

(iii) by property (iv) of Definition 2.1, $\lim_{d\to\infty} |T_d|[f](x,y) = 0$ uniformly on $\mathcal{B}_{(r,s)}$; when $s \geq 1$ it follows that $\lim_{d\to\infty} I_d[f](x) = f(x,1)$ uniformly on $\mathcal{B}_r$.

Since K is clearly a ring, to show that K is a field it suffices to verify that K is closed under multiplicative inverses. So let $a \in K$ be nonzero, and let $f \in \mathcal{S}_{1,1}$ be such that $a = f(1)$. Fix $d \in \mathbb{N}$ such that $I_d[f] \neq 0$ and $|T_d|[f](1) < |I_d[f]|$. For any $y \in \mathcal{B}_1$, since $|T_d[f](y)| \leq |T_d|[f](1)$, $f_d(y) = I_d[f] + T_d[f](y) \neq 0$. So by closure under Weierstrass preparation in $\mathcal{S}$, and hence also under multiplicative inverses, $1/f_d \in \mathcal{S}_{1,1}$. Therefore $1/a = 1/f_d(1) \in K$.

To show that $\mathcal{R}$ is closed under composition, let $f \in \mathcal{R}_{m,s}$ and $g = (g_1, \ldots, g_n) \in \mathcal{R}^m_{n,r}$ be such that $g(\mathcal{B}_r) \subseteq \mathcal{B}_s$, where $s \in K^m_+$ and $r \in K^n_+$. We must show that $f \circ g \in \mathcal{R}_{n,r}$. By properties (iv) and (v) of Definition 2.1, we may slighlty enlarge r and s to assume that $r \in L^n_+$ and $s \in L_+$. By definition of $\mathcal{R}$ and property (iv) of Definition 2.1, there are $s' \in L^m_+$, $F \in \mathcal{S}_{m+1,(s',1)}$ and $G = (G_1, \ldots, G_m) \in \mathcal{S}^m_{n+1,(r,1)}$ such that $s' > s$, $f(x_1, \ldots, x_m) = F(x_1, \ldots, x_m, 1)$ on $\mathcal{B}_s$ and $g(x) = G(x, 1)$ on $\mathcal{B}_r$. Since $\lim_{d\to\infty} |T_d|[G_i](x, 1) = 0$ uniformly on $\mathcal{B}_r$, we may fix $d \in \mathbb{N}$ sufficiently large so that for $i = 1, \ldots, n$,

$$\begin{aligned}
|g_i(x) - G_{i,d}(x, y)| &= |(I_d[G_i](x) + T_d[G_i](x, 1)) - (I_d[G_i](x) + T_d[G_i](x, y))|, \\
&\leq 2|T_d|[G_i](x, 1), \\
&< s'_i - s_i,
\end{aligned}$$

for all $(x, y) \in \mathcal{B}_{(r,1)}$. Hence for all $(x, y) \in \mathcal{B}_{(r,1)}$, $G_d(x, y) \in \mathcal{B}_{s'}$, so also $yG_d(x, y) \in \mathcal{B}_{s'}$. Therefore by closure under local composition in $\mathcal{S}$, $H(x, y) := F(yG_d(x, y), y)$ is in $\mathcal{S}_{n+1,(r,1)}$. Since $g(x) = G(x, 1) = yG_d(x, y)\big|_{y=1}$, $f \circ g(x) = H(x, 1) \in \mathcal{R}_{n,r}$.

To show that $\mathcal{R}$ is closed under Weierstrass preparation, let $f \in \mathcal{R}_{n+1,(r,s)}$ be such that $\mathrm{WPT}(f, d, r, s)$, where $(r, s) \in K^n_+ \times K_+$ and $d \in \mathbb{N}$. Let $f = wu$ on $\mathcal{B}_{(r,s)}$, where w and u are the Weierstrass polynomial and unit given by Weierstrass preparation on $\mathcal{B}_{(r,s)}$. We must show that $w, u \in \mathcal{R}_{n+1,(r,s)}$. By slightly enlarging (r, s), we may assume $(r, s) \in L^n_+ \times L_+$. Fix $F \in \mathcal{S}_{n+2,(r,s,1)}$ such that $f(x, y) = F(x, y, 1)$.

Since $F(0, 0, 1) = \cdots = \frac{\partial^{d-1} F}{\partial y^{d-1}}(0, 0, 1) = 0$, by replacing $F(x, y, z)$ with $F(x, y, z) - \sum_{i=0}^{d-1} \frac{1}{i!} \frac{\partial^i F}{\partial y^i}(0, 0, z) y^i$ we retain the property that $f(x, y) = F(x, y, 1)$ on $\mathcal{B}_{(r,s)}$ and gain the property that $\frac{\partial^i F}{\partial y^i}(0, 0, z) = 0$ for $i = 0, \ldots, d - 1$. By writing $\frac{\partial^i F}{\partial y^i}(x, y, z) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^{i+j} F}{\partial y^i \partial z^j}(x, y, 0) z^j$ we see that $\frac{\partial^{i+j} F}{\partial y^i \partial z^j}(0, 0, 0) = 0$

for all $j \in \mathbb{N}$ and $i = 0, \dots, d-1$. Since

$$\frac{\partial^i F_k}{\partial y^i}(x,y,z) = \sum_{j=0}^{k} \frac{1}{j!} \frac{\partial^{i+j} F}{\partial y^i \partial z^j}(x,y,0) + \sum_{j=k+1}^{\infty} \frac{1}{j!} \frac{\partial^{i+j} F}{\partial y^i \partial z^j}(x,y,0) z^j,$$

it follows that for all $k \in \mathbb{N}$, F_k also satisfies $F_k(x,y,1) = f(x,y)$ and $\frac{\partial^i F_k}{\partial y^i}(0,0,z) = 0$ for $i = 0, \dots, d-1$; in particular, $\frac{\partial^i F_k}{\partial y^i}(0,0,0) = 0$ for all $k \in \mathbb{N}$ and $i = 0. \dots, d-1$. Since for all $i \in \mathbb{N}$ and all $(x,y,z) \in \mathcal{B}_{(r,s,1)}$,

$$\begin{aligned} \left| \frac{\partial^i F_k}{\partial y^i}(x,y,z) - \frac{\partial^i f}{\partial y^i}(x,y) \right| &= \left| \begin{array}{l} \left(\frac{\partial^i I_k[F]}{\partial y^i}(x,y) + \frac{\partial^i T_k[F]}{\partial y^i}(x,y,z) \right) \\ - \left(\frac{\partial^i I_k[F]}{\partial y^i}(x,y) + \frac{\partial^i T_k[F]}{\partial y^i}(x,y,1) \right) \end{array} \right|, \\ &\le 2 \frac{\partial^i |T_k|[F]}{\partial y^i}(x,y,1), \\ &\to 0, \end{aligned}$$

uniformly as $k \to \infty$, by continuity of roots $\mathrm{WPT}(F_k, d, (r,1), s)$ holds for all sufficiently large $k \in \mathbb{N}$; fix such a k. By closure under Weierstrass preparation in $\mathcal{S}$ we have $W, U \in \mathcal{S}_{n+2,(r,s,1)}$, where W and U are the Weierstrass polynomial and unit given by Weierstrass preparing F_k in y on $\mathcal{B}_{(r,s,1)}$. Therefore $w(x,y) = W(x,y,1)$ and $u(x,y) = U(x,y,1)$ are in $\mathcal{R}_{n+1,(r,s)}$. □

Corollary 2.12. *If we let $\mathcal{S}$ be either $\mathcal{A}(\mathbb{Q})$ or $\mathcal{D}(\mathbb{Q})$ and let $\mathcal{R}$ be the smallest Weierstrass system containing $\mathcal{S}$, then $\mathcal{R}$ is countable. In the case that $\mathcal{S} = \mathcal{A}(\mathbb{Q})$, $K := \mathcal{R}_0$ is the field of algebraic reals.*

Proof. Note that for any $\mathcal{S}$ and $\mathcal{R}$ as in Proposition 2.11, $|\mathcal{R}| = |\mathcal{S}|$. Therefore by letting $\mathcal{S}$ be either $\mathcal{A}(\mathbb{Q})$ or $\mathcal{D}(\mathbb{Q})$, Lemma 2.10 shows that $\mathcal{S}$ is a countable local Weierstrass system, so $\mathcal{R}$ is a countable Weierstrass system.

Now consider the case that $\mathcal{S} = \mathcal{A}(\mathbb{Q})$. Let $a \in K$, and fix $f \in \mathcal{S}_1$ such that $a = f(1)$. There is a $p(x,y) \in \mathbb{Q}[x,y]$ such that $p(x, f(x)) = 0$, so $p(1,a) = 0$, showing that a is algebraic over $\mathbb{Q}$. In Lemma 4.4 we shall see that K is real closed, so K must in fact be the entire field of algebraic reals. □

The following proposition is used in Section 4.4 to prove the Main Theorem for general Weierstrass systems.

Proposition 2.13. *Let $\mathcal{R}$ be a Weierstrass system over K, and let E be a field such that $K \subseteq E \subseteq \mathbb{R}$. Define*

$$\mathcal{S}_0 := L := \bigcup_{m\in\mathbb{N}} \bigcup_{s\in K_+^m} \{f(a) : f \in \mathcal{R}_{m,s}, a \in E^m \cap B_s\},$$

and for $n \in \mathbb{N}_+$ and $r \in L_+^n$ define

$$\mathcal{S}_{n,r} := \bigcup_{m\in\mathbb{N}} \bigcup_{\substack{(r',s)\in K_+^{n+m}\\ r'\geq r}} \{f(x,a)\big|_{\mathcal{B}_r} : f \in \mathcal{R}_{n+m,(r',s)}, a \in E^m \cap B_s\}.$$

Then $\mathcal{S} := \bigcup_{n\in\mathbb{N}, r\in L_+^n} \mathcal{S}_{n,r}$ is the smallest Weierstrass system containing $E \cup \mathcal{R}$.

Proof. Clearly $\mathcal{S}$ contains $\mathcal{R}$, and since Weierstrass systems are closed under composition, $\mathcal{S}$ is contained in any Weierstrass system containing $E \cup \mathcal{R}$. To show that $\mathcal{S}$ is a Weierstrass system, the only nontrivial properties that need to be checked are that L is a field and $\mathcal{S}$ is closed under composition and Weierstrass preparation.

Claim. Let $f \in \mathcal{S}_{n,r}$, where $r \in L_+^n$, and let $\epsilon > 0$. Let $I \subseteq \mathbb{N}^n$ be a finite set such that $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) = 0$ for all $\alpha \in I$. There is an $F \in \mathcal{R}_{n+k,(r',s)}$, where $(r', s) \in K_+^n \times K_+^k$ and $r' > r$, and also an $a \in E^k \cap \mathcal{B}_s$ such that $f(x) = F(x,a)\big|_{\mathcal{B}_r}$, $|F(x,a) - F(x,z))| < \epsilon$ for all $(x,z) \in \mathcal{B}_{(r',s)}$, and $\frac{\partial^{|\alpha|} F}{\partial x^\alpha}(0,z) = 0$ for all $\alpha \in I$.

To show the claim, fix $F \in \mathcal{R}^m_{n+k,(r',s)}$ such that $r' \geq r$ and $f(x) = F(x,a)$ for some $a \in E^m \cap \mathcal{B}_s$. By property (iv) of Definition 2.1 we may assume that $r' > r$ and that $a \in \mathrm{int}(\mathcal{B}_s)$. By replacing $F(x,z)$ with $F(x,z) - \sum_{\alpha\in I} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} F}{\partial x^\alpha}(0,z)x^\alpha$, we retain the property that $f(x) = F(x,a)$ on $\mathcal{B}_r$ and gain the property that $\frac{\partial^{|\alpha|} F}{\partial x^\alpha}(0,z) = 0$ for all $\alpha \in I$. Since $\mathcal{B}_{r'}$ is compact, we may choose $b \in K^m \cap \mathcal{B}_s$ and $t \in K_+^m$ such that $a \in \mathcal{B}_t(b) \subseteq \mathcal{B}_s$ and $|F(x,z) - F(x,a)| < \epsilon$ on $\mathcal{B}_{(r',t)}(0,b)$. Then $G(x,z) := F(x,z+b)\big|_{\mathcal{B}_{(r',t)}} \in \mathcal{R}_{n+m,(r',t)}$, $a - b \in E^m \cap \mathcal{B}_t$, $f(x) = G(x,a-b)$, $|G(x,z) - G(x,a-b)| < \epsilon$ on $\mathcal{B}_{(r',t)}$, and $\frac{\partial^{|\alpha|} G}{\partial x^\alpha}(0,z) = 0$ for all $\alpha \in I$, proving the claim.

Since L is clearly a ring, to show that L is a field we fix a nonzero $a \in L$ and show that $1/a \in L$. Since $a \neq 0$, by the claim we may fix $f \in \mathcal{R}_{n,r}$ and

$b \in E^n \cap B_r$ be such that $a = f(b)$ and $f(x) \neq 0$ on $\mathcal{B}_r$. Then by closure under Weierstrass preparation in $\mathcal{R}$, so also under mulitplicative inverses, $1/f(x) \in \mathcal{R}_{n,r}$, so $1/a = 1/f(b) \in L$.

To show that $\mathcal{S}$ is closed under composition, let $f \in \mathcal{S}_{m,s}$ and $g \in \mathcal{S}^m_{n,r}$ be such that $g(\mathcal{B}_r) \subseteq \mathcal{B}_s$, where $s \in L^m_+$ and $r \in L^n_+$. By adding dummy variables (just to simplify notation), by the claim we may fix $s' \in K^m_+$, $r' \in K^n_+$, $t \in K^k_+$, $F \in \mathcal{R}_{m+k,(s',t)}$, $G \in \mathcal{R}^m_{n+k,(r',t)}$ and $a \in E^k \cap B_t$ such that $s' > s$, $r' \geq r$, $f(x_1, \ldots, x_m) = F(x_1, \ldots, x_m, a)\big|_{\mathcal{B}_s}$, $g(x) = G(x,a)\big|_{\mathcal{B}_r}$ and $G(\mathcal{B}_{(r,t)}) \subseteq \operatorname{int}(\mathcal{B}_{s'})$. By possibly shrinking $r' \geq r$ (if $r' \neq r$), we may obtain $G(\mathcal{B}_{(r',t)}) \subseteq \mathcal{B}_{s'}$. Then $H(x,z) := F(G(x,z),z) \in \mathcal{R}_{n+k,(r',t)}$, so $f \circ g(x) = H(x,a)\big|_{\mathcal{B}_r} \in \mathcal{S}_{n,r}$.

Finally, to show that $\mathcal{S}$ is closed under Weierstrass preparation let $f \in \mathcal{S}_{n+1,(r,s)}$, where $(r,s) \in L^n_+ \times L_+$, and assume $\mathrm{WPT}(f,d,r,s)$. Let $f(x,y) = w(x,y)u(x,y)$, where w and u are the Weierstrass polynomial and unit given by Weierstrass preparation in y on $\mathcal{B}_{(r,s)}$. By the claim, for any $\epsilon > 0$ we may fix $F \in \mathcal{R}_{n+1+m,(r',s',t)}$, where $(r',s',t) \in K^n_+ \times K_+ \times K^m_+$ and $(r',s') \geq (r,s)$, and also $a \in E \cap \mathcal{B}_t$ such that $f(x,y) = F(x,y,a)$ on $\mathcal{B}_{(r,s)}$, $|F(x,y,z) - F(x,y,a)| < \epsilon$ on $\mathcal{B}_{(r',s',t)}$ and $\frac{\partial^i F}{\partial y^i}(0,0,z) = 0$ for $i = 0, \ldots, d-1$. So by continuity of roots we may assume $\mathrm{WPT}(F,d,(r',t),s')$. Let $F(x,y,z) = W(x,y,z)U(x,y,z)$, where W and U are the Weierstrass polynomial and unit given by Weierstrass preparation in y on $\mathcal{B}_{(r',s',t)}$. By closure under Weierstrass preparation in $\mathcal{R}$, $W, U \in \mathcal{R}_{n+1+m}$, so $w(x,y) = W(x,y,a)\big|_{\mathcal{B}_{(r,s)}}$ and $u(x,y) = U(x,y,a)\big|_{\mathcal{B}_{(r,s)}}$ are in $\mathcal{S}_{n+1,(r,s)}$. □

2.4 Outline of the proof

In Chapter 3 we prove the Main Theorem for the special case that $\mathcal{R}$ is a Weierstrass system over $\mathbb{R}$. Most of the work involves proving certain singularity resolution theorems which we refer to as "normalization theorems." Such theorems have the following general form: given a certain function $f(x)$ on a certain set $A \subseteq \mathbb{R}^n$ (the assumptions upon f and A depend on the particular theorem), there are finitely many coordinate transformations $\mu_1, \ldots, \mu_m \in \mathcal{R}^n_n$ of a certain form such that $A \subseteq \bigcup_{i=1}^m \mu_i(B_{r(\mu_i)})$ and $f \circ \mu_i(x)$ is "normal" on $B_{r(\mu_i)}$, meaning that $f \circ \mu_i(x) = x^{\alpha_i} u_i(x)$ for some $\alpha_i \in \mathbb{N}^n$ and unit u_i on $B_{r(\mu_i)}$. These μ_i will be formed by composing very special coordinate transformations which we shall call "admissible transformations."

The general technique we employ to prepare a function f is to first normalize f in a careful manner and then unwind the coordinate transformations μ_i to show that f is prepared in the original coordinates.

In Chapter 3 the assumption that $K = \mathbb{R}$ is needed solely because these normalization theorems are local geometric constructions, and so a function $f \in \mathcal{R}_{n,r}$ must be allowed to be translated by arbitray points of $\text{int}(\mathcal{B}_r)$ and still remain in $\mathcal{R}$, which is only true when $K = \mathbb{R}$. So one way to prove the Main Theorem for a general Weierstrass system $\mathcal{R}$ over K would be to prove these normalization theorems in a slightly more global fashion so as to show that the translations can always be taken to be by points of $K^n \cap \text{int}(\mathcal{B}_r)$, so that we never leave the system $\mathcal{R}$. This can be done, but the only proofs I found seems excessively complicated, so we shall adopt a simpler strategy involving a little bit of model theory.

In Chapter 4 we prove a normalization theorem, Theorem 4.2, for functions from a system $\mathcal{R}$ over K, but only by cheating; namely, we will have to expand our notion of what is considered to be an "admissible transformation" by also including linear coordinate transformations of the form $(x, y) \mapsto (x + \lambda y, y)$ for $\lambda \in K^n$. An unfortunate consequence of this will be that unwinding the coordinate transformations given by Theorem 4.2 will not prepare the normalized function back in the original coordinates. But this will not matter because we will be able to use Theorem 4.2 to show that $K_{\mathcal{R}}$ and $\mathbb{R}_{\mathcal{R}}$ have the same theory, where $K_{\mathcal{R}}$ is the submodel of $\mathbb{R}_{\mathcal{R}}$ with universe K. Coupling this fact with Proposition 2.13 will enable us to give a very simple model theoretic proof of the Main Theorem for general Weierstrass systems.

It should be pointed out that in Sections 3.1 and 4.2 it is assumed that the reader is familiar with both the proofs and the results of Rolin, Speissegger and Wilkie [17].

Chapter 3

Proof of the preparation theorem for Weierstrass systems over the reals

This Chapter proves the Main Theorem for Weierstrass systems over $\mathbb{R}$. It is organized as follows.

Section 3.1 proves a version of the formal normalization theorem from [17, Section 2]. It differs from [17] only in that it does not use linear coordinate transformations of the form $(x, y) \mapsto (x + \lambda y, y)$, $\lambda \in \mathbb{R}^n$, to make functions $f(x, y)$ regular in y.

Section 3.2 shows how the formal normalization theorem of Section 3.1 can be interpreted as a local normalization theorem for functions from certain quasianalytic classes, and then shows in Proposition 3.18 that this gives a preparation theorem for such functions. The axiomatic setting of Section 3.2 is much weaker than that of a Weierstrass system.

Section 3.3 is the heart of the chapter, where we show in Proposition 3.27 how to prepare functions of the form $f(x, g(x)/y, y)$ if f and g are from a Weierstrass system. This is a consequence of our main technical result, Proposition 3.24, where we show how to normalize such functions. Lion and Rolin [11] originally proved Proposition 3.27 via a "splitting argument." Here we use Weierstrass preparation instead, and our argument is modeled after Parusiński's proof of [16, Theorem 7.5].

Finally, Section 3.4 uses Proposition 3.27 to complete the proof of the Main Theorem for Weierstrass systems over $\mathbb{R}$. Its proof uses Weierstrass preparation only in that it relies on this proposition. The ideas of this section come directly from Lion and Rolin's original argument [11], as articulated by Rolin in a series of lectures he gave at the University of Wisconsin-Madison in the fall of 2000.

3.1 A Formal Normalization Theorem

Note that for $\alpha, \beta \in \mathbb{N}^n$, $\alpha \leq \beta$ iff x^α divides x^β.

Definition 3.1. A series $f(x) \in \mathbb{R}[\![x]\!]$ is **normal** if $f(x) = x^\alpha u(x)$ for some $\alpha \in \mathbb{N}^n$ and unit $u(x) \in \mathbb{R}[\![x]\!]$. A set of series $\{f_1(x), \ldots, f_m(x)\} \subseteq \mathbb{R}[\![x]\!]$ is **normal** if each $f_i(x)$ is normal, say $f_i(x) = x^{\alpha_i} u_i(x)$ with $\alpha_i \in \mathbb{N}^n$ and $u_i(x)$ a unit, and if $\{\alpha_1, \ldots, \alpha_m\}$ is linearly ordered. (Note: Having $\alpha_i = \alpha_j$ for $i \neq j$ is permissible.)

Lemma 3.2. *Let $f_1, \ldots, f_m \in \mathbb{R}[\![x]\!]\backslash\{0\}$.*

(i) *$f_1 \cdots f_m$ is normal iff f_i is normal for all $i = 1, \ldots, m$.*

(ii) *If f_i and $f_i - f_j$ are normal for all $i, j = 1, \ldots, m$ such that $f_i \neq f_j$, then the set $\{f_1, \ldots, f_m\}$ is normal.*

Proof. See Bierstone and Milman [2, Lemma 4.7]. □

Let $\mathbb{R}[\![\overline{x}]\!] := \bigcup_{n \in \mathbb{N}} \mathbb{R}[\![x_1, \ldots, x_n]\!]$, the ring of formal power series in $\overline{x}$ with real coefficients. For $\mathcal{F} \subseteq \mathbb{R}[\![\overline{x}]\!]$ and $n \in \mathbb{N}$, let $\mathcal{F}_n := \mathcal{F} \cap \mathbb{R}[\![x_1, \ldots, x_n]\!]$.

For the rest of this section fix a ring $\mathcal{F}$ such that $\mathbb{R}[\overline{x}] \subseteq \mathcal{F} \subseteq \mathbb{R}[\![\overline{x}]\!]$ and which is closed under the following operations:

(i) *differentiation*: if $f \in \mathcal{F}$, then $\frac{\partial f}{\partial x_i} \in \mathcal{F}$ for all $i \in \mathbb{N}_+$;

(ii) *formal composition*: for all $m \in \mathbb{N}_+$ and $n \in \mathbb{N}$, if $f \in \mathcal{F}_m$, and $g \in \mathcal{F}_n^m$ is such that $g(0) = 0$, then $f \circ g \in \mathcal{F}_n$;

(iii) *monomial factorization*: for all $n \in \mathbb{N}$, if $f \in \mathcal{F}_{n+1}$ is such that $f(x, y) = y\, g(x, y)$ for some $g(x, y) \in \mathbb{R}[\![x, y]\!]$, then $g \in \mathcal{F}_{n+1}$;

(iv) *implicit functions*: for all $n \in \mathbb{N}$, if $f \in \mathcal{F}_{n+1}$ is such that $f(0) = 0$ and $\frac{\partial f}{\partial y}(0) \neq 0$, there is a $g \in \mathcal{F}_n$ such that $g(0) = 0$ and $f(x, g(x)) = 0$.

Given a series $f(x) \in \mathcal{F}_n$, we shall construct a certain set T of homomorphisms of $\mathcal{F}_n$ such that for each $\mu \in T$, $\mu f(x)$ is normal. In Sections 3.2 and 3.3 $\mathcal{F}_n$ will play the role of the collection of Taylor series about the origin of a system of functions under consideration, and the homomorphisms of T will correspond to charts of a sequence of coordinate transformations.

Definition 3.3. By induction on $n \in \mathbb{N}$, define a **formal admissible transformation in** (x, y) to be either a formal admissible transformation in x, considered to be a homomorphism from $\mathcal{F}_{n+1}$ into $\mathcal{F}_{n+1}$, or one of the following three types of homomorphisms from $\mathcal{F}_{n+1}$ into $\mathcal{F}_{n+1}$:

(i) **functional translation**: for $\theta \in \mathcal{F}_n$ such that $\theta(0) = 0$,

$$t_\theta(x, y) := (x, y + \theta(x));$$

(ii) **power substitution**: for $m \in \mathbb{N}_+$, $1 \le i \le n$ and $\sigma \in \{1, -1\}$,

$$p^m_{i,\sigma}(x, y) := (x_1, \ldots, \sigma(\sigma x_i)^m, \ldots, x_n, y);$$

(iii) **blow-up substitution**: for $\lambda \in \mathbb{R} \cup \{\infty\}$ and $1 \le i \le n$,

$$b^{i,n+1}_\lambda(x, y) := \begin{cases} (x_1, \ldots, x_n, x_i(y + \lambda)), & \text{if } \lambda \in \mathbb{R}, \\ (x_1, \ldots, x_i y, \ldots, x_n, y), & \text{if } \lambda = \infty. \end{cases}$$

Given a formal admissible transformation μ, define the **family of** μ as the set $\{t_\theta\}$ if $\mu = t_\theta$, the set $\{p^m_{i,1}, p^m_{i,-1}\}$ if $\mu = p^m_{i,\sigma}$ for some $\sigma \in \{1, -1\}$, and the set $\{b^{i,n+1}_\lambda : \lambda \in \mathbb{R} \cup \{\infty\}\}$ if $\mu = b^{i,n+1}_\lambda$ for some $\lambda \in \mathbb{R} \cup \{\infty\}$. In Sections 3.2 and 3.3 a family of admissible transformations will correspond to a single geometric operation whose charts are given by the individual members of the family.

We call $\mu = \langle \mu_1, \ldots, \mu_m \rangle$ a **formal transformation sequence in** x if each μ_i is a formal admissible transformation in x. For $f \in \mathcal{F}_n$ define $\mu f := \mu_m \cdots \mu_1 f$, and note that the closure properties of $\mathcal{F}$ imply that $\mu f \in \mathcal{F}_n$.

Given a set T of transformation sequences, define the **height** of T by $\operatorname{ht}(T) := \sup\{m \in \mathbb{N} : \langle \mu_1, \ldots, \mu_m \rangle \in T\} \in \mathbb{N} \cup \{\infty\}$. We will be interested in sets T of transformation sequences in x such that $\operatorname{ht}(T) < \infty$ and for each $\mu = \langle \mu_1, \ldots, \mu_m \rangle \in T$ and $i = 1, \ldots, m$,

(i) $\langle \mu_1, \ldots, \mu_{i-1} \rangle \notin T$;

(ii) $\{\nu_i : \langle \mu_1, \ldots, \mu_{i-1} \rangle \subseteq \nu \in T\}$ is exactly the family of μ_i.

Note that $T' := \{\nu : \nu \subseteq \mu \text{ for some } \mu \in T\}$ is a tree under the inclusion ordering, and that T' always branches according to transformation families. T is the set of maximal members of T' and can be identified with the set of branches of T'. Since T and T' uniquely determine one another, we abuse terminology and call T a **formal transformation tree in** x. Letting id denote the identity homomorphism, by convention $T := \{id\}$ is the unique formal transformation tree of height 0.

Given a set S of formal transformation sequences in (x, y), an admissible transformation ν is an **interior member** of S if $\nu = \mu_i$ for some $\langle \mu_1, \ldots, \mu_m \rangle \in S$ and $1 \leq i < m$; S **respects** y if no blowup substitution $b_\infty^{i,n+1}$, $1 \leq i \leq n$, is an interior member of S.

Theorem 3.4. *For every $n \in \mathbb{N}$ and nonzero $f \in \mathcal{F}_{n+1}$, there is a formal transformation tree T in (x, y) such that μf is normal for all $\mu \in T$. (We say T* **normalizes** *f.) Moreover, T respects y and for each $\mu = \langle \mu_1, \ldots, \mu_m \rangle \in T$, if $\mu_m = b_\infty^{i,n+1}$ and $\mu f(x, y) = x^\alpha y^d u(x, y)$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $d \in \mathbb{N}$ and $u(x, y)$ a unit, then $d \geq \alpha_i$.*

Proof. The proof is by induction on $n \in \mathbb{N}$. If $n = 0$, then $f(y)$ is normal. So fix $n > 0$ and assume the normalization theorem holds for all power series in $\mathcal{F}_n$. The following lemma is needed for both the current proof and for later use.

Lemma 3.5. *Let $x := (x_1, \ldots, x_n)$ and $z := (x_{n+1}, \ldots, x_{n+m})$, and let $f \in \mathcal{F}_{n+m}$. There is a formal transformation tree T in x such that for each $\mu \in T$ there is an $\alpha \in \mathbb{N}^n$ and a $g \in \mathcal{F}_{n+m}$ such that $\mu f(x, z) = x^\alpha g(x, z)$ and $g(0, z) \neq 0$.*

Proof (Speissegger). Write $f(x, z) = \sum_{\beta \in \mathbb{N}^m} f_\beta(x) z^\beta$ and note that $f_\beta(x) \in \mathcal{F}_n$ for each β. From the Noetherianity of $\mathbb{R}[\![x, z]\!]$, there is a finite $B \subseteq \mathbb{N}^m$ such that

$$f(x, z) = \sum_{\beta \in B} f_\beta(x) z^\beta u_\beta(x, z),$$

where for each $\beta \in B$, $f_\beta(x) \neq 0$ and $u_\beta(x, z) \in \mathbb{R}[\![x, z]\!]$ is a unit . Let

$$F(x) := \prod_{\beta \in B} f_\beta(x) \cdot \prod \{f_\beta(x) - f_\gamma(x) : \beta, \gamma \in B,\ \beta <_{lex} \gamma,\ f_\beta \neq f_\gamma\},$$

where $<_{lex}$ is the lexicographical ordering on $\mathbb{N}^m$. By the induction hypothesis in the proof of Theorem 3.4, there is a formal transformation tree T in x normalizing $F(x)$. Fix $\mu \in T$. By Lemma 3.2, the set $\{\mu f_\beta : \beta \in B\}$ is normal. So for each $\beta \in B$, $\mu f_\beta(x) = x^{\alpha_\beta} v_\beta(x)$ for some $\alpha_\beta \in \mathbb{N}^n$ and unit $v_\beta(x)$, and $\alpha := \min\{\alpha_\beta : \beta \in B\}$ is well defined. Let

$$g(x,z) := \sum_{\beta \in B} x^{\alpha_\beta - \alpha} z^\beta v_\beta(x) u_\beta(x,z),$$

and observe that $\mu f(x,z) = x^\alpha g(x,z)$, $g(x,y) \in \mathbb{R}[\![x,z]\!]$ and $g(0,z) \neq 0$. Finally, $g(x,z) \in \mathcal{F}$ since $\mu f(x,z) \in \mathcal{F}$ and $\mathcal{F}$ is closed under monomial factorization. □

To prove Theorem 3.4 apply Lemma 3.5 to $f(x,y)$ to get a formal transformation tree S in x such that for each $\mu \in S$, $\mu f(x,y) = x^\alpha g(x,y)$ for some $\alpha \in \mathbb{N}^n$ and $g(x,y) \in \mathcal{F}$ such that $g(0,y) \neq 0$. Now just proceed as in the proof of [17, Theorem 2.5] to construct a tree S_μ normalizing μf. Then $T := \{\langle \mu, \nu \rangle : \mu \in S, \nu \in S_\mu\}$ normalizes f, where $\langle \mu, \nu \rangle$ denotes the concatenation of the sequences μ and ν. Since substitutions of the form $b_\infty^{i,n+1}$ are only used in the proof of [17, Lemma 2.11], T respects y, and using the notation of the "moreover" clause of the theorem we are proving, $d \geq \alpha_i$ in this case. □

3.2 Preparing functions from q.a. IF-systems over $\mathbb{R}$

For a C^∞ function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^{n+1}$ is an open neighborhood of the origin, and for $(r,s) \in \mathbb{R}_+^n \times \mathbb{R}_+$ such that $B_{(r,s)} \subseteq U$, let $\mathrm{IFT}(f,r,s)$ be shorthand for the following statement:

> $f(0) = 0$ and for all $(a,b) \in B_{(r,s)}$, $\frac{\partial f}{\partial y}(a,b) \neq 0$ and $f(a,b) \neq 0$ if $|b| = s$.

Implicit Function Theorem. *Let $f : U \to \mathbb{R}$ be C^∞, where $U \subseteq \mathbb{R}^{n+1}$ is an open neighborhood of the origin. First, if $f(0) = 0$ and $\frac{\partial f}{\partial y}(0) \neq 0$, there is an $(r,s) \in \mathbb{R}_+^n \times \mathbb{R}_+$ such that* $\mathrm{IFT}(f,r,s)$*. Second, if* $\mathrm{IFT}(f,r,s)$ *then $\{(x,y) \in \mathrm{int}(B_{(r,s)}) : f(x,y) = 0\}$ is the graph of a C^∞ function $g : \mathrm{int}(B_r) \to \mathrm{int}(B_s)$.*

Proof. By the local version of the implicit function theorem, there is a unique C^∞ function g defined on a neighborhood $V \subseteq \mathbb{R}^n$ of the origin such that $f(x, g(x)) = 0$ and $g(0) = 0$. Simply choose (r, s) so that $B_{(r,s)} \subseteq V$, $\frac{\partial f}{\partial y} \neq 0$ on $B_{(r,s)}$, and $g(B_r) \subseteq \mathrm{int}(B_s)$. For any $a \in \mathrm{int}(B_r)$, since $\frac{\partial f}{\partial y}(a, y) \neq 0$ on $\mathrm{int}(B_s)$, then $\{y \in \mathrm{int}(B_s) : f(a, y) = 0\} = g(a)$ by Rolle's theorem. Therefore $f(a, b) \neq 0$ for all $(a, b) \neq B_{(r,s)}$ such that $|b| = s$, so $\mathrm{IFT}(f, r, s)$ holds.

Now suppose that (r, s) is any tuple for which $\mathrm{IFT}(f, r, s)$ holds. Let g be the C^∞ function implicitly defined by f in a neighorhood of the origin. Since the local implicit function theorem can be applied to any point $p \in \mathrm{int}(B_{(r,s)})$ for which $f(p) = 0$, the graph of the function g extends all the way to the boundary of $\mathrm{int}(B_{(r,s)})$. But since $f(a, b) \neq 0$ for all $(a, b) \in B_{(r,s)}$ such that $|b| = s$, we must have that that domain of g extends to all of $\mathrm{int}(B_r)$ and $g(\mathrm{int}(B_r)) \subseteq \mathrm{int}(B_s)$. To conclude, note that by Rolle's theorem, $\{(x, y) \in \mathrm{int}(B_{(r,s)}) : f(x, y) = 0\} = \mathrm{graph}(g\big|_{\mathrm{int}(B_r)})$. □

If $\mathrm{IFT}(f, r, s)$, there is an $(r', s') > (r, s)$ such that $\mathrm{IFT}(f, r', s')$, so g extends uniquely to B_r; note that $g(B_r) \subseteq \mathrm{int}(B_s)$.

Definition 3.6. A quasianalytic system of functions $\mathcal{R} = \bigcup_{n,r} \mathcal{R}_{n,r}$ over K is called a **quasianalytic implicit function system** (or a **q.a. IF-system** for short) if $\mathcal{R}$ is closed under differentiation and composition (both as defined in Definition 2.6 but replacing $\mathcal{B}_r$ with B_r), and also

(i) *monomial factorization*: for all $n \in \mathbb{N}$ and $r \in K_+^{n+1}$, if $f \in \mathcal{R}_{n+1,r}$ is such that $\widehat{f}(x, y) = y\, G(x, y)$ for some $G(x, y) \in \mathbb{R}[\![x, y]\!]$, then $f(x, y) = y\, g(x, y)$ for some $g \in \mathcal{R}_{n+1,r}$;

(ii) *implicit functions*: for all $n \in \mathbb{N}$, $(r, s) \in K_+^n \times K_+$ and $f \in \mathcal{R}_{n+1,(r,s)}$ such that $\mathrm{IFT}(f, r, s)$, if $g : B_r \to \mathbb{R}$ is the C^∞ function implicitly defined by $f(x, g(x)) = 0$ and $g(0) = 0$, then $g \in \mathcal{R}_{n,r}$.

Examples 3.7. 1. Given any Weierstrass system $\mathcal{S} = \bigcup_{n,r} \mathcal{S}_{n,r}$, define $\mathcal{R} := \bigcup_{n,r} \mathcal{R}_{n,r}$ by $\mathcal{R}_{n,r} := \{f\big|_{B_r} : f \in \mathcal{S}_{n,r}\}$. Then $\mathcal{R}$ is a q.a. IF-system.

2. Given a sequence of real numbers $M = (M_0, M_1, M_2, \ldots)$ such that $1 \leq M_0 \leq M_1 \leq M_2 \leq \cdots$, $\sum_{i=0}^\infty M_i/M_{i+1} = \infty$, and the sequence $M_i/i!$ is log-convex, define $\mathcal{R} := \bigcup_{n,r} \mathcal{R}_{n,r}$ by letting $\mathcal{R}_{n,r}$ be the collection of

all functions $f : B_r \to \mathbb{R}$ which extend to a C^∞ function $f : U \to \mathbb{R}$ for some neighborhood U of B_r in which there is an $A > 0$ such that

$$\left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \leq A^{|\alpha|+1} M_{|\alpha|} \text{ for all } x \in U \text{ and } \alpha \in \mathbb{N}^n.$$

Then $\mathcal{R}$ is q.a. IF-system over $\mathbb{R}$ and is called the **Denjoy-Carlemann class** defined by M. (See [17] for more details.)

3. Given a polynomially bounded o-minimal expansion $\mathcal{M}$ of the real field, let K be the field of definable constants of $\mathcal{M}$, and for $n \in \mathbb{N}$ and $r \in K_+^n$ let $\mathcal{R}_{n,r}$ be the collection of all functions $f : B_r \to \mathbb{R}$ which extend to a C^∞ function $f : U \to \mathbb{R}$ for some neighborhood U of B_r in which the graph of $f : U \to \mathbb{R}$ is definable in $\mathcal{M}$ without parameters. Then $\mathcal{R} := \bigcup_{n,r} \mathcal{R}_{n,r}$ is a q.a. IF-system over K. (This follows from C. Miller [14].)

For the rest of this section, fix a q.a. IF-system $\mathcal{R}$ over $\mathbb{R}$.

The objective of this section is to show that for each $n \in \mathbb{N}$ and $r \in K_+^{n+1}$, if $f \in \mathcal{R}_{n+1,r}$ then f is prepared on a B_r.

Note that $\widehat{\mathcal{R}}$, the image of $\mathcal{R}$ under the the Taylor map at the origin $\widehat{\ } : \mathcal{R} \to \mathbb{R}[\![\overline{x}]\!]$, is a ring of power series such as considered in Section 3.1. The notions of "formal admissible transformation", "formal transformation tree", etc. are defined relative to $\widehat{\mathcal{R}}$.

Definition 3.8. A function $\mu \in \mathcal{R}_n^n$ is an **admissible transformation in** x if $\widehat{\mu}$ is a formal admissible transformation in x. A finite sequence $\mu = \langle \mu_1, \ldots, \mu_m \rangle$ of admissible transformations in x is a **transformation sequence in** x, and we also write μ for the function $\mu_1 \circ \cdots \circ \mu_m$.

A set T of transformation sequences in x is a **full transformation tree in** x if $\widehat{T} := \{\widehat{\mu} = \langle \widehat{\mu}_1, \ldots, \widehat{\mu}_m \rangle : \mu = \langle \mu_1, \ldots, \mu_m \rangle \in T\}$ is a formal transformation tree in x. For a set S of transformation sequences in (x, y), S **respects** y if $\widehat{S}$ does.

If there is any ambiguity about which q.a. IF-system is being considered, we shall clarify the above terminology by saying "$\mathcal{R}$-admissible transformation," "$\mathcal{R}$-transformation sequence," and "full $\mathcal{R}$-transformation tree."

For each $i \geq 1$ and $m \geq i$ let $\Pi_i : \mathbb{R}^m \to \mathbb{R}$ denote the ith coordinate projection function $(x_1, \ldots, x_m) \mapsto x_i$. When working with a function $f(x_1, \ldots, x_m)$ for some $m \geq n$, we shall identify an admissible transformation $\mu \in \mathcal{R}_n^n$ in x with the function in $\mathcal{R}_m^m$ given by $(x_1, \ldots, x_m) \mapsto (\Pi_1 \circ \mu(x), \ldots, \Pi_n \circ \mu(x), x_{n+1}, \ldots, x_m)$. Thus we may speak of transformation sequences and trees in x even if the dimension of the ambient space m is greater than n.

Definition 3.9. Consider a sequence of functions $\mu = \langle \mu_1, \ldots, \mu_m \rangle$ where for $i = 1, \ldots, m$, $U_i \subseteq X_i$ and $\mu_i : U_i \to X_{i-1}$ for some sets $X_0, X_1, \ldots, X_m$. A set $A \subseteq X_m$ is **μ-admissible** if $\mu_{i+1} \circ \cdots \circ \mu_m(A) \subseteq U_i$ for all $i = 1, \ldots, m$. We not only use angle brackets to denote a sequence of functions, but we also use them to denote concatenation of such sequences; that is, if $\nu = \langle \nu_1, \ldots, \nu_l \rangle$ is a sequence of functions, where for $i = 1, \ldots, l$, $V_i \subseteq Y_i$ and $\nu_i : V_i \to Y_{i-1}$ for some sets $X_m = Y_0, Y_1, \ldots, Y_l$, let $\langle \mu, \nu \rangle := \langle \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_l \rangle$.

Definition 3.10. Let $s \in \mathbb{R}_+^n$ and $f \in \mathcal{R}_{n,s}$. We say that f is **normal on** B_r, where $r \leq s$, if $f(x) = x^\alpha u(x)$ on B_r for some $u \in \mathcal{R}_{n,r}$ which is a unit on B_r.

We now state the geometric form of Theorem 3.4.

Lemma 3.11. *For any $n \in \mathbb{N}$ and nonzero $f \in \mathcal{R}_{n+1}$, there is a full transformation tree T in (x, y) and a map $\epsilon : T \to \mathbb{R}_+^{n+1}$ such that for each $\mu \in T$, $B_{\epsilon(\mu)}$ is $\langle f, \mu \rangle$-admissible and $f \circ \mu$ is normal on $B_{\epsilon(\mu)}$. Moreover, T respects y and for each $\mu = \langle \mu_1, \ldots, \mu_m \rangle \in T$, if $\mu_m = b_\infty^{i,n+1}$ and $f \circ \mu(x, y) = x^\alpha y^d u(x, y)$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $d \in \mathbb{N}$ and $u(x, y)$ a unit, then $d \geq \alpha_i$.*

Proof. By Theorem 3.4 there is a formal transformation tree S normalizing $\widehat{f}(x, y)$ which respects y. Fix a full transformation tree T such that $\widehat{T} = S$. Let $\mu = \langle \mu_1, \ldots, \mu_m \rangle \in T$. We write $\widehat{\mu}$ for the corresponding homomorphism of $\widehat{\mathcal{R}}$. Since f and each μ_i are all defined on neighborhoods of the origin and each μ_i is a continuous map such that $\mu_i(0) = 0$, there is an $\epsilon(\mu) \in \mathbb{R}_+^{n+1}$ such that $B_{\epsilon(\mu)}$ is $\langle f, \mu \rangle$-admissible. Since $\widehat{\mu}\widehat{f}(x, y) = x^\alpha y^d \widehat{u}(x, y)$ for some $\alpha \in \mathbb{N}^n$, $d \in \mathbb{N}$ and $\widehat{u} \in \widehat{\mathcal{R}}_{n+1}$ such that $\widehat{u}(0) \neq 0$, the Taylor series of some $u \in \mathcal{R}_{n+1}$, we have that $f \circ \mu(x, y) = x^\alpha y^d u(x, y)$ and $u(0) \neq 0$. By possibly shrinking $\epsilon(\mu)$ we may make u a unit on $B_{\epsilon(\mu)}$. □

Lemma 3.12. *Let T be a full transformation tree in x and $\epsilon : T \to \mathbb{R}^n_+$ be such that $B_{\epsilon(\mu)}$ is μ-admissible for each $\mu \in T$. There is a finite $S \subseteq T$ such that $\bigcup_{\mu\in S} \mu(B_{\epsilon(\mu)})$ is a neighborhood of the origin.*

Proof. By induction on $\mathrm{ht}(T)$. If $\mathrm{ht}(T) = 0$ there is nothing to do, so suppose $\mathrm{ht}(T) = 1$. Then T is a transformation family, so there are three cases. If $T = \{t_\theta\}$, then $t_\theta(B_{\epsilon(t_\theta)})$ is a neighborhood of the origin since t_θ is a local homeomorphism. If $T = \{p^m_{i,\sigma} : \sigma \in \{1, -1\}\}$, then $p^m_{i,\sigma}(B_{\epsilon(p^m_{i,\sigma})}) \cap \{x \in \mathbb{R}^n : \sigma x_i \geq 0\}$ is a neighborhood of origin in the closed half-space $\{x \in \mathbb{R}^n : \sigma x_i \geq 0\}$, so $\bigcup_{\sigma\in\{1,-1\}} p^m_{i,\sigma}(B_{\epsilon(p^m_{i,\sigma})})$ is a neighborhood of the origin. If $T = \{b^{i,j}_\lambda : \lambda \in \mathbb{R}\cup\{\infty\}\}$ for some $1 \leq i < j \leq n$, then the compactness of the real projective line supplies a finite $\Lambda \subseteq \mathbb{R}\cup\{\infty\}$ such that $\bigcup_{\lambda\in\Lambda} b^{i,j}_\lambda(B_{\epsilon(b^{i,j}_\lambda)})$ is a neighborhood of the origin.

So assume $\mathrm{ht}(T) > 1$. Let $T_1 := \{\mu_1 : (\mu_1, \dots, \mu_m) \in T\}$, a transformation tree of height 1. For each $\nu \in T_1$ let $T[\nu] := \{(\mu_2, \dots, \mu_m) : (\nu, \mu_2, \dots, \mu_m) \in T\}$, a transformation tree of height less than $\mathrm{ht}(T)$. By the induction hypothesis, for each $\nu \in T_1$ there is a finite $S[\nu] \subseteq T[\nu]$ such that $U_\nu := \bigcup_{\mu\in S[\nu]} \mu(B_{\epsilon(\nu\circ\mu)})$ is a neighborhood of the origin. Let $\delta(\nu) \in \mathbb{R}^n_+$ be ν-admissible and such that $B_{\delta(\nu)} \subseteq U_\nu$. Again by the induction hypothesis, there is a finite $S_1 \subseteq T_1$ such that $\bigcup_{\nu\in S_1} \nu(B_{\delta(\nu)})$ is a neighborhood of the origin, so $S := \{\nu\circ\mu : \nu \in S_1,\ \mu \in S[\nu]\}$ is as desired. □

Any subset of a full transformation tree is called a **transformation tree**.

Corollary 3.13. *For any $n \in \mathbb{N}$ and nonzero $f \in \mathcal{R}_{n+1}$ there is a finite transformation tree T in (x, y) and a map $\epsilon : T \to \mathbb{R}^{n+1}_+$ such that $\bigcup_{\mu\in T} \mu(B_{\epsilon(\mu)})$ is a neighborhood of the origin, and for each $\mu \in T$, $B_{\epsilon(\mu)}$ is $\langle f, \mu\rangle$-admissible and $f \circ \mu$ is normal on $B_{\epsilon(\mu)}$. Moreover, T respects y, and if $\mu = \langle\mu_1, \dots, \mu_m\rangle \in T$ is such that $\mu_m = b^{i,n+1}_\infty$ for some $1 \leq i \leq n$ and $f \circ \mu(x, y) = x^\alpha y^d u(x, y)$ for some $\alpha \in \mathbb{N}^n$, $d \in \mathbb{N}$ and unit $u \in \mathcal{R}_{n+1}$, then $d \geq \alpha_i$.*

Proof. Simply apply Lemma 3.11 and then Lemma 3.12. □

Our next task is to interpret Corollary 3.13 as a preparation theorem. We will need some easy facts about the images and preimages of $\mathcal{L}'_{\mathcal{R}}$-cylinders under admissible transformations.

Remark 3.14. (i) If $\mu \in \mathcal{R}_{n+1}^{n+1}$ is an admissible transformation not of the form $\mu = b_\infty^{i,n+1}$, and $C \subseteq \mathbb{R}^{n+1}$ is an $\mathcal{L}'_\mathcal{R}$-cylinder, then both $\mu(C)$ and $\mu^{-1}(C)$ are finite unions of $\mathcal{L}'_\mathcal{R}$-cylinders.

(ii) For all $r \in \mathbb{R}_+^{n+1}$ and $1 \leq i \leq n$, $b_\infty^{i,n+1}(B_r)$ and $(b_\infty^{i,n+1})^{-1}(B_r)$ are finite unions of $\mathcal{L}'_\mathcal{R}$-cylinders.

(iii) Let $C \subseteq \mathbb{R}^{n+1}$ be an $\mathcal{L}'_\mathcal{R}$-cylinder, let $s_1(x), \ldots, s_n(x), t_1(x)$ and $t_2(x)$ be $\mathcal{L}'_\mathcal{R}$-terms, and let $L(x,y) := (s_1(x), \ldots, s_n(x), t_1(x)y + t_2(x))$. Then $L^{-1}(C)$ is a finite union of $\mathcal{L}'_\mathcal{R}$-cylinders.

Definition 3.15. Define the **exceptional set** $\mathcal{E}(\mu)$ of an admissible transformation $\mu \in \mathcal{R}_{n+1}^{n+1}$ by induction on $n \in \mathbb{N}$: if $\mu(x,y) = (\nu(x), y)$ for an admissible transformation $\nu \in \mathcal{R}_n^n$, then $\mathcal{E}(\mu) := \mathcal{E}(\nu) \times \mathbb{R}$. Otherwise,

(i) $\mathcal{E}(t_\theta) := \emptyset$;

(ii) $\mathcal{E}(p_{i,\sigma}^m) := \{(x,y) \in \mathbb{R}^{n+1} : x_i = 0\}$;

(iii) $\mathcal{E}(b_\lambda^{i,n+1}) := \{(x,y) \in \mathbb{R}^{n+1} : x_i = 0 \text{ or } y = 0\}$.

If T is a transformation tree of height 1, then for all $\mu, \nu \in T$, $\mathcal{E}(\mu) = \mathcal{E}(\nu)$; define $\mathcal{E}(T)$ to be this common set.

Definition 3.16. Consider a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and a set $A \subseteq \mathbb{R}^{n+1}$. In the definition of "$f$ is prepared on A" found in Definition 2.3, if instead of (2.6) we have

$$f(x,y) = a(x)(y - \theta(x))^d u(x, y - \theta(x))$$

on C, where $a(x)$, $\theta(x)$ and $u(x,y)$ are $\mathcal{L}'_\mathcal{R}$-terms, $d \in \mathbb{N}$, and $u(x,y)$ is a positive $\mathcal{R}$-unit on $\{(x, y - \theta(x)) : (x,y) \in C\}$, then f is **$\mathbb{N}$-prepared on** A. If we allow $d \in \mathbb{Z}$, f is **$\mathbb{Z}$-prepared** on A.

Lemma 3.17. *Let $n \in \mathbb{N}$ and consider a function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^{n+1}$, and also an $A \subseteq U$ which is a finite union of $\mathcal{L}'_\mathcal{R}$-cylinders. Let T be a finite transformation tree of height 1, and for each $\mu \in T$ let $C_\mu \subseteq \mathbb{R}^{n+1}$ be a finite union of $\mathcal{L}'_\mathcal{R}$-cylinders which is $\langle f, \mu \rangle$-admissible. Suppose that $A \subseteq \bigcup_{\mu \in T} \mu(C_\mu)$.*

Suppose that for each $\mu \in T$, $f \circ \mu$ is $\mathbb{Z}$-prepared on $\mu^{-1}(A) \cap C_\mu$, and if $\mu = b_\infty^{i,n+1}$ for some $1 \leq i \leq n$ then $C_\mu = B_r$ for some $r \in \mathbb{R}_+^{n+1}$ and

$f \circ \mu\big|_{\mu^{-1}(A)\cap B_r}$ extends to an $f_\mu \in \mathcal{R}_{n+1,r}$ which is normal on B_r, say of the form $f_\mu = x^\alpha y^d u(x,y)$. Then f is $\mathbb{Z}$-prepared on $A \backslash \mathcal{E}(T)$.

If we further suppose that for each $\mu \in T$, $f \circ \mu$ is $\mathbb{N}$-prepared on $\mu^{-1}(A) \cap C_\mu$, and in the case that $\mu = b_\infty^{i,n+1}$ we have $d \geq \alpha_i$, then f is $\mathbb{N}$-prepared on $A \backslash \mathcal{E}(T)$.

Proof. Using Remark 3.14, the fact that $\mathcal{E}(T)$ is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders, and the fact that the collection of finite unions of $\mathcal{L}'_{\mathcal{R}}$-cylinders is a boolean algebra, we may reduce to the case that $T = \{\mu\}$ for a single admissible transformation μ, that $A = \mu(C)$ for a single $\mathcal{L}'_{\mathcal{R}}$-cylinder C on which $f \circ \mu$ is of a single $\mathbb{Z}$-prepared form, and that $A \cap \mathcal{E}(\mu) = \emptyset$. Furthermore, if μ is an admissible transformation in x, it is not hard to see that then f is $\mathbb{Z}$-prepared on A, so we may assume that μ properly involves y. The proof now breaks down into cases.

Consider the case that $\mu = b_\infty^{i,n+1}$ and $f \circ \mu\big|_{\mu^{-1}(A)\cap B_r}$ extends to an $f_\mu \in \mathcal{R}_{n+1,r}$ which is normal on B_r, say of the form $f_\mu = x^\alpha y^d u(x,y)$. Then $f(x,y) = x^\alpha y^{d-\alpha_i} u(x_1, \ldots, x_i/y, \ldots, x_n, y)$ on A, which is $\mathbb{Z}$-prepared. Furthermore, if $d \geq \alpha_i$ then $f(x,y)$ is $\mathbb{N}$-prepared on this set.

So now assume that μ is not of the form $b_\infty^{i,n+1}$. There are two cases, either C is thin or fat. First suppose that C is thin, say $C = \{(x, s(x)) : x \in B\}$ for some quantifier free definable set $B \subseteq \mathbb{R}^n$ and term $s(x)$, and let $t(x)$ be the term such that for all $x \in B$, $f \circ \mu(x, s(x)) = t(x)$. If $\mu(x,y) = (x, \nu(x,y))$ for some $\nu \in \mathcal{R}_{n+1}$ (such as when $\mu = b_\lambda^{i,n+1}$ for some $\lambda \in \mathbb{R}$ or $\mu = t_\theta$ for some $\theta \in \mathcal{R}_n$), then $A = \mu(C) = \{(x, \nu(x, s(x)) : x \in B\}$. So B is the base of A and for all $x \in B$, $f(x, \nu(x, s(x))) = f \circ \mu(x, s(x)) = t(x)$. On the other hand, if $\mu(x,y) = (\nu(x), y)$ for some $\nu \in \mathcal{R}_n^n$ such that $\nu^{-1}(x)$ is a tuple of $\mathcal{L}'_{\mathcal{R}}$-terms (such as when $\mu = p_{i,\sigma}^m$, since $(p_{i,\sigma}^m)^{-1}(x) = (x_1, \ldots, \sigma \sqrt[m]{\sigma x_i}, \ldots, x_n)$), then $A = \mu(C) = \{(\nu(x), s(x)) : x \in B\} = \{(x, s \circ \nu^{-1}(x)) : x \in \nu(B)\}$. So the quantifier free definable set $\nu(B) = \{x : \nu^{-1}(x) \in B\}$ is the base of A and for all $x \in \nu(B)$, $f(x, s \circ \nu^{-1}(x)) = t \circ \nu^{-1}(x)$. Hence if C is thin, $\{(x, f(x,y)) : (x,y) \in A\}$ agrees with the graph of a term, so f is prepared on A.

So now suppose that C is fat, say $C = \{(x,y) \in B \times \mathbb{R} : s(x) < y < t(x)\}$ for some terms $s(x)$ and $t(x)$ (the other forms are handled similarly), and that

$$f \circ \mu(x,y) = a(x)(y - \theta(x))^d u(x, y - \theta(x)),$$

on C, where $d \in \mathbb{Z}$. If $\mu = t_\psi$ then on A,

$$f(x, y) = a(x)(y - (\psi(x) + \theta(x)))^d u(x, t - (\psi(x) + \theta(x))).$$

If $\mu = b_\lambda^{i,n+1}$ then on A,

$$\begin{aligned} f(x, y) &:= a(x)\left(\frac{y}{x_i} - \lambda - \theta(x)\right)^d u\left(x, \frac{y}{x_i} - \lambda - \theta(x)\right), \\ &:= \frac{a(x)}{x_i^d}(y - x_i(\lambda + \theta(x)))^d u\left(x, \frac{y - x_i(\lambda + \theta(x))}{x_i}\right). \end{aligned}$$

If $\mu = p_{i,\sigma}^m$ then on A,

$$f(x, y) = a \circ (p_{i,\sigma}^m)^{-1}(x)(y - \theta \circ (p_{i,\sigma}^m)^{-1}(x))^d u((p_{i,\sigma}^m)^{-1}(x), y - \theta \circ (p_{i,\sigma}^m)^{-1}(x)).$$

In each case f is $\mathbb{Z}$-prepared. If $d \in \mathbb{N}$, f is $\mathbb{N}$-prepared. □

Proposition 3.18. *Let $f \in \mathcal{R}_{n+1,r}$ for some $n \in \mathbb{N}$ and $r \in \mathbb{R}_+^{n+1}$. Then f is $\mathbb{N}$-prepared on B_r.*

Proof. The proof is by induction on $n \in \mathbb{N}$. Let $f \in \mathcal{R}_{n+1,r}$ and let $F \in \mathcal{R}_{n+1,s}$ be such that $s > r$ and $f = F\big|_{B_r}$. It suffices to show that F is $\mathbb{N}$-prepared on a neighborhood of B_r. To prove the result for a particular value of n it suffices to show that each $f \in \mathcal{R}_{n+1}$ is $\mathbb{N}$-prepared on some neighborhood of the origin, since we may simply apply this to $F(x+a, y+b)$ for each (a, b) in B_r and then invoke the compactness of B_r to obtain the finitely many cylinders required for a preparation.

The result for $n = 0$ is trivial, since then $f \in \mathcal{R}_1$, so f is normal about the origin, so in particular f is $\mathbb{N}$-prepared. So let $n > 0$ and assume the lemma holds for all functions in $\mathcal{R}_n$. By Corollary 3.13 there is a finite transformation tree T respecting y and an $\epsilon : T \to \mathbb{R}_+^{n+1}$ such that for each $\mu \in T$, $B_{\epsilon(\mu)}$ is $\langle f, \mu\rangle$-admissible, $f \circ \mu$ is normal on $B_{\epsilon(\mu)}$, say $f \circ \mu(x, y) = x^\alpha y^d u(x, y)$, and $U := \bigcup_{\mu \in T} \mu(B_{\epsilon(\mu)})$ is a neighborhood of the origin. Moreover, if $\mu = \langle \mu_1, \ldots, \mu_m\rangle$ with $\mu_m = b_\infty^{i,n+1}$ then $d \geq \alpha_i$. We show by induction on $\mathrm{ht}(T)$ that f is $\mathbb{N}$-prepared on U.

If $\mathrm{ht}(T) = 0$, f is normal on U, so suppose $\mathrm{ht}(T) > 0$. Let $T_1 := \{\mu_1 : \langle \mu_1, \ldots, \mu_m\rangle \in T\}$ and for each $\mu \in T_1$ let $T[\mu] := \{\nu : \mu \circ \nu \in T\}$. Since $\mathrm{ht}(T[\mu]) < \mathrm{ht}(T)$ for each $\mu \in T_1$, by the induction hypothesis $f \circ \mu$ is $\mathbb{N}$-prepared on $U[\mu] := \bigcup_{\nu \in T[\mu]} \nu(B_{\epsilon(\mu\circ\nu)})$. Since T is with respect to y,

f and T_1 satisfies the hypothesis of Lemma 3.17 (for $\mathbb{N}$-preparation), so f in $\mathbb{N}$-prepared on $U\backslash\mathcal{E}(T_1)$. So it suffices to show that f is $\mathbb{N}$-prepared on $U\cap\mathcal{E}(T_1)$.

Note that each $\mathcal{E}(T_1)$ is the union of sets of the form $E_i := \{(x,y) \in \mathbb{R}^{n+1} : x_i = 0\}$ for some $1 \le i \le n+1$, that f is prepared on $U \cap E_{n+1}$ since then f is a function in x only, and that by the induction hypothesis, f is $\mathbb{N}$-prepared on the compact set $\overline{U}\cap E_i$ for any $i = 1,\dots,n$. Hence f is $\mathbb{N}$-prepared on $U\cap\mathcal{E}(T_1)$. □

3.3 Preparing certain fractional analytic functions

For this section, fix a Weierstrass system $\mathcal{R}$ over $\mathbb{R}$.

We shall show how to $\mathbb{Z}$-prepare functions of the form $f(x, g(x)/y, y)$, where $f \in \mathcal{R}_{n+2}$ and $g \in \mathcal{R}_n$.

Definition 3.19. Consider an open neighorhood $U \subseteq \mathbb{C}^n$ of the origin in $\mathbb{C}^n$ and a holomorphic function $f : U \to \mathbb{C}$. Let $r \in \mathbb{R}^n_+$, and let B be either $\mathcal{B}_r$ or $\operatorname{int}(\mathcal{B}_r)$. Assume that $B \subseteq U$.

We say that f is a **unit on** B if $f(x) \neq 0$ for all $x \in B$. We say that f is **normal on** B if $f(x) = x^\alpha u(x)$ on some open neighborhood $V \subseteq U$ of B, where $\alpha \in \mathbb{N}^n$ and $u : V \to \mathbb{C}$ is a holomorphic unit on B.

Consider another holomorphic function $g : U \to \mathbb{C}$. If there is a neighborhood $V \subseteq U$ of B and a holomorphic $h : V \to \mathbb{C}$ such that $f(x) = g(x)h(x)$ on V, then g **divides** f **on** B.

For a set $A \subseteq U$, let $Z_A(f)$ denote the germ of the sets $\{x \in V : f(x) = 0\}$ for all neighborhoods $V \subseteq U$ of A. If A is open, we simply consider $Z_A(f)$ to be the set $\{x \in A : f(x) = 0\}$.

Lemma 3.20. *Let $B := \operatorname{int}(\mathcal{B}_r)$ for some $r \in \mathbb{R}^n_+$, let $f, g, h : B \to \mathbb{C}$ be holomorphic, and let $E := \{x \in B : h(x) = 0\}$. If $Z_{B\backslash E}(f) \subseteq Z_{B\backslash E}(g)$, and g and h are both normal on B, then f is normal on B.*

Proof. If $a \in B$ is such that $f(a) = 0$, then either $g(a) = 0$ or $h(a) = 0$, so $Z_B(f) \subseteq Z_B(g) \cup Z_B(h) = Z_B(gh)$. Now, gh is normal on B, say $g(x)h(x) = x^\alpha u(x)$ for some $\alpha \in \mathbb{N}^n$ and unit u on B. We may choose $\beta \in \{0,1\}^n$ so that $\beta \le \alpha$, $Z_B(f) \subseteq Z_B(x^\beta)$, and $Z_B(f) \nsubseteq Z_B(x^\gamma)$ for all $\gamma \le \beta$ not equal to β. Let $I := \{i \in \{1,\dots,n\} : \beta_i \neq 0\}$. If $I = \emptyset$, then f is a unit on B and we are done, so we may assume that $I \neq \emptyset$.

We claim that $Z_B(f) = Z_B(x^\beta)$. Consider $i \in I$. By the minimality of β there is an $a \in B$ such that $a_i \neq 0$ and $f(a) = 0$. Since the regular points of $Z_B(f)$ are dense in $Z_B(f)$, there is a regular $b \in Z_B(f)$ such that $b_i \neq 0$. So for some open neighborhood V of b, $Z_V(f)$ is an $(n-1)$-dimensional complex manifold which is a subset of $Z_V(x_i)$. Since $Z_V(x_i)$ is also an $(n-1)$-dimensional complex manifold, it follows that $Z_V(f) = Z_V(x_i)$, so $f(x)\big|_{x_i=0} = 0$ on V. But then $f(x)\big|_{x_i=0} = 0$ on B. Since $i \in I$ was arbitrary, $Z_B(f) = Z_B(x^\beta)$, proving the claim.

Note that $f(x)\big|_{x_i=0} = 0$ on B iff x_i divides f on B. By applying this observation and the above claim repeatedly to $f(x)$, then $f(x)/x_i$, then $f(x)/x_i^2$, etc., for all $i \in I$, we obtain a $\gamma \in \mathbb{N}^n$ such that $\gamma_i > 0$ iff $i \in I$ and $f(x) = x^\gamma u(x)$ for some holomorphic unit u on B, as desired. □

In Lemmas 3.22 and 3.23 and Proposition 3.24 we shall be interested in the following situation: we have a full transformation tree T in (x,y), a $g \in \mathcal{R}_n$, and a map $\epsilon : T \to \mathbb{R}_+^{n+2}$. For $i = 1,\dots,n+2$ let $\epsilon_i := \Pi_i \circ \epsilon$ and $\epsilon' := (\epsilon_1,\dots,\epsilon_{n+1})$. For each $\mu \in T$ and $i = 1,\dots,n+1$ let $\mu_i := \Pi_i \circ \mu$ and $\mu' := (\mu_1,\dots,\mu_n)$. If for each $\mu \in T$, $\mathcal{B}_{\epsilon'(\mu)}$ is μ-admissible and $\mu'(\mathcal{B}_{\epsilon'(\mu)}) \subseteq \mathcal{B}_{r(g)}$, then we define the complex wedge

$$\mathcal{W}(\epsilon,\mu) := \left\{(x,y) \in \mathcal{B}_{\epsilon'(\mu)} : \left|\frac{g \circ \mu'(x,y)}{\mu_{n+1}(x,y)}\right| < \epsilon_{n+2}(\mu),\ \ \mu_{n+1}(x,y) \neq 0\right\}, \tag{3.1}$$

and also the real wedge $W(\epsilon,\mu) := \mathcal{W}(\epsilon,\mu) \cap \mathbb{R}^{n+1}$.

Remark 3.21. Lemma 3.11 is the geometric interpretation of Theorem 3.4 for q.a. IF-systems. Theorem 3.4 can also be interpreted for Weierstrass systems: namely, in Lemma 3.11 simply replace B_r with $\mathcal{B}_r$.

Similarly, Lemma 3.5 also has an obvious geometric interpretation for the Weierstrass system $\mathcal{R}$, which we do not state but use in the proofs of Lemmas 3.22 and 3.23 below.

Lemma 3.22. *Let $\lambda \in \mathbb{R}$, $f \in \mathcal{R}_{n+1}$ and $g \in \mathcal{R}_n$. Define*

$$\varphi(x,y) := (x,\ g(x)/y - \lambda y)$$

on its natural domain $\{(x,y) \in \mathcal{B}_{r(g)} \times \mathbb{C} : y \neq 0\}$, and let $F := f \circ \varphi$. There is a full transformation tree T in x and an $\epsilon : T \to \mathbb{R}_+^{n+2}$ such that for each

$\mu \in T$, $\mathcal{W}(\epsilon,\mu)$ is $\langle f,\varphi,\mu\rangle$-admissible and there is a $\Phi_\mu \in \mathcal{R}_{n+1}$ such that $\mathcal{W}(\epsilon,\mu) \subseteq \mathcal{B}_{r(\Phi_\mu)}$ and

$$Z_{\mathcal{W}(\epsilon,\mu)}(F\circ\mu) \subseteq Z_{\mathcal{W}(\epsilon,\mu)}(\Phi_\mu).$$

Proof. As described in Remark 3.21, use Lemma 3.5 to obtain a full transformation tree T in x and an $\epsilon' = (\epsilon_1,\dots,\epsilon_{n+1}) : T \to \mathbb{R}_+^{n+1}$ such that for each $\mu \in T$, $\mathcal{B}_{\epsilon'(\mu)}$ is $\langle f,\mu\rangle$-admissible and

$$f\circ\mu(x,y) = x^\alpha h(x,y) \tag{3.2}$$

on $\mathcal{B}_{\epsilon'(\mu)}$ for some $\alpha \in \mathbb{N}^n$ and $h \in \mathcal{R}_{n+1}$ regular in y, say of order d. Fix $\mu \in T$, and so also the corresponding α, h and d. By possibly shrinking $\epsilon'(\mu)$, Weierstrass preparation gives

$$h(x,y) = w(x,y)u(x,y) \tag{3.3}$$

on $\mathcal{B}_{\epsilon'(\mu)}$, where w is a Weierstrass polynomial in y of order d and u is a unit. Note that $w,u \in \mathcal{R}_{n+1}$.

By possibly shrinking $\epsilon_{n+1}(\mu)$ further and choosing $\epsilon_{n+2}(\mu)$ sufficiently small, we may in fact assume that h, w and u are defined on $\mathcal{B}_{\epsilon_\lambda(\mu)}$, where

$$\epsilon_\lambda(\mu) := (\epsilon_1(\mu),\dots,\ \epsilon_n(\mu),\ \epsilon_{n+2}(\mu) + |\lambda|\epsilon_{n+1}(\mu)).$$

Let $\epsilon(\mu) := (\epsilon'(\mu),\epsilon_{n+2}(\mu))$. So if $(a,b) \in \mathcal{W}(\epsilon,\mu)$ is such that $F\circ\mu(a,b) = 0$, then by (3.2)

$$0 = f(\mu'(a),\ g\circ\mu'(a)/b - \lambda b) = a^\alpha h(a,\ g\circ\mu'(a)/b - \lambda b),$$

so either $a^\alpha = 0$ or $w(a,\ g\circ\mu'(a)/b - \lambda b) = 0$ by (3.3). Letting $\beta = (\beta_1,\dots,\beta_n)$, where

$$\beta_i := \begin{cases} 1, & \text{if } \alpha_i > 0, \\ 0, & \text{if } \alpha_i = 0, \end{cases}$$

(just to reduce redundancies) and noting that $y^d w(x,\ g\circ\mu'(x)/y - \lambda y)$ is a polynomial in y with coefficients in $\mathcal{R}_n$, we see that

$$\Phi_\mu(x,y) := x^\beta y^d w(x, g\circ\mu'(x)/y - \lambda y)$$

is a function as desired. □

Lemma 3.23. *Let $f \in \mathcal{R}_{n+2}$ and $g \in \mathcal{R}_n$. Define*

$$\varphi(x,y) := (x,\ g(x)/y,\ y)$$

on its natural domain $\{(x,y) \in \mathcal{B}_{r(g)} \times \mathbb{C} : y \neq 0\}$, and let $F := f \circ \varphi$. There is a full transformation tree T in x and an $\epsilon : T \to \mathbb{R}_+^{n+2}$ such that for each $\mu \in T$, $\mathcal{W}(\epsilon,\mu)$ is $\langle f, \varphi, \mu\rangle$-admissible and there is a $\Phi_\mu \in \mathcal{R}_{n+1}$ such that $\mathcal{W}(\epsilon,\mu) \subseteq \mathcal{B}_{r(\Phi_\mu)}$ and

$$Z_{\mathcal{W}(\epsilon,\mu)}(F \circ \mu) \subseteq Z_{\mathcal{W}(\epsilon,\mu)}(\Phi_\mu).$$

Proof. Let $z := x_{n+2}$. As described in Remark 3.21, use Lemma 3.5 to obtain a full transformation tree S in x and a $\rho = (\rho_1, \ldots, \rho_{n+2}) : S \to \mathbb{R}_+^{n+2}$ such that for each $\nu \in S$, $\mathcal{B}_{\rho(\nu)}$ is $\langle f, \nu\rangle$-admissible and $f \circ \nu(x,y,z) = x^\alpha h(x,y,z)$ on $\mathcal{B}_{\rho(\nu)}$ for some $\alpha \in \mathbb{N}^n$ and $h \in \mathcal{R}_{n+2}$ such that $h(0,y,z) \neq 0$. Instead of considering each $\nu \in S$ to be a member of $\mathcal{R}_{n+2}^{n+2}$, we consider ν to be a member of $\mathcal{R}_{n+1}^{n+1}$, as we may since S is a transformation tree in x; so $\nu(x,y) = (\nu'(x), y)$ where $\nu' := \Pi \circ \nu$. Fix $\nu \in S$, and so also the corresponding α and h, and let $H(x,y) := h(x, g \circ \nu'(x)/y, y)$ on its natural domain. Note that $F \circ \nu(x,y) = x^\alpha H(x,y)$.

We now use a method of Parusinski's to study the complex roots of H, and so also of $F \circ \nu$. Fix a $\lambda \in \mathbb{R}$ such that $h(x, y + \lambda z, z)$ is regular in z, which may be done since $h(0,y,z) \neq 0$. By possibly shrinking $\rho(\nu)$, Weierstrass preparation gives

$$h(x, y + \lambda z, z) = w(x,y,z)\, u(x,y,z) \tag{3.4}$$

on $\mathcal{B}_{\rho(\nu)}$, where w is a Weierstrass polynomial in z and u is a unit. Let $\delta(\nu) := (\delta_1(\nu), \ldots, \delta_{n+2}(\nu))$, where $\delta_i(\nu) := \rho_i(\nu)$ for $1 \leq i \leq n$ and $\delta_{n+1}(\nu), \delta_{n+2}(\nu) > 0$ are chosen so that $\delta_{n+2}(\nu) + |\lambda|\delta_{n+1}(\nu) \leq \rho_{n+1}(\nu)$ and $\delta_{n+1}(\nu) \leq \rho_{n+2}(\nu)$. Consider $(a,b) \in \mathcal{W}(\delta,\nu)$ such that $H(a,b) = 0$, and let $c := g \circ \nu'(a)/b - \lambda b$. So $\lambda b^2 + cb - g \circ \nu'(a) = 0$ and $h(a, c + \lambda b, b) = 0$, so by (3.4) $w(a,c,b) = 0$. Since the polynomials in y, $\lambda y^2 + cy - g \circ \nu'(a)$ and $w(a,c,y)$, have a common root b, they have a common factor, so

$$\psi(a,c) := \mathrm{Res}_y(w(a,c,y), \lambda y^2 + cy - g \circ \nu'(a)) = 0,$$

where Res_y denotes the resultant with respect to y. Note that $\psi \in \mathcal{R}_{n+1}$. The above argument shows that

$$Z_{\mathcal{W}(\delta,\nu)}(H) \subseteq Z_{\mathcal{W}(\delta,\nu)}(\Psi),$$

where $\Psi(x, y) := \psi(x,\, g \circ \nu'(x)/y - \lambda y)$, and so

$$Z_{\mathcal{W}(\delta,\nu)}(F \circ \nu) \subseteq Z_{\mathcal{W}(\delta,\nu)}(\Phi_\nu), \tag{3.5}$$

where $\Phi_\nu(x, y) := x^\alpha \Psi(x, y)$.

By applying Lemma 3.22 to Φ_ν, there is a full transformation tree S_ν in x and a $\delta_\nu : S_\nu \to \mathbb{R}_+^{n+2}$ such that for each $\eta \in S_\nu$, $\mathcal{W}(\delta_\nu, \eta)$ is $\langle \Phi_\nu, \nu \rangle$-admissible and there is a $\Phi_{\nu\circ\eta} \in \mathcal{R}_{n+1}$ such that $\mathcal{W}(\delta_\nu, \eta) \subseteq \mathcal{B}_{r(\Phi_{\nu\circ\eta})}$ and

$$Z_{\mathcal{W}(\delta_\nu,\eta)}(\Phi_\nu \circ \eta) \subseteq Z_{\mathcal{W}(\delta_\nu,\eta)}(\Phi_{\nu\circ\eta}). \tag{3.6}$$

For each $\nu \in S$, by possibly refining δ_ν we may assume that δ_ν is "compatible" with δ in the sense that for each $\eta \in S_\nu$, $\nu(\mathcal{B}(\delta'_\nu, \eta)) \subseteq \mathcal{B}(\delta', \nu)$ and $\nu(\mathcal{W}(\delta_\nu, \eta)) \subseteq \mathcal{W}(\delta, \nu)$. Then (3.5) and (3.6) show that

$$Z_{\mathcal{W}(\delta_\nu,\eta)}(F \circ \nu \circ \eta) \subseteq Z_{\mathcal{W}(\delta_\nu,\eta)}(\Phi_{\nu\circ\eta}). \tag{3.7}$$

Let $T := \{\nu \circ \eta : \nu \in S, \eta \in S_\nu\}$, and let $\epsilon : T \to \mathbb{R}_+^{n+2}$ be given by $\epsilon(\nu \circ \eta) := \delta_\nu(\eta)$ for $\nu \in S$ and $\eta \in S_\nu$. With this new notation (3.7) becomes

$$Z_{\mathcal{W}(\epsilon,\mu)}(F \circ \mu) \subseteq Z_{\mathcal{W}(\epsilon,\mu)}(\Phi_\mu)$$

for each $\mu \in T$, as desired. $\square$

We now state a main result.

Proposition 3.24. *Let $n \in \mathbb{N}$, $f \in \mathcal{R}_{n+2}$ and $g \in \mathcal{R}_n$. Define*

$$\varphi(x, y) := (x,\, g(x)/y,\, y)$$

on its natural domain $\{(x, y) \in \mathcal{B}_{r(g)} \times \mathbb{C} : y \neq 0\}$, and let $F := f \circ \varphi$. For $r \in \mathbb{R}_+^{n+1}$ and $s > 0$ let $W_{(r,s)} := \{(x, y) \in B_r : y \neq 0,\, |g(x)/y| < s\}$.

There are $r \in \mathbb{R}_+^{n+1}$, $s > 0$, a finite transformation tree T' in (x, y) and an $\epsilon : T' \to \mathbb{R}_+^{n+2}$ such that

(i) *T' respects y;*

(ii) $W_{(r,s)} \subseteq \bigcup_{\mu \in T'} \mu(W(\epsilon, \mu))$;

(iii) *for each $\mu \in T'$, $W(\epsilon, \mu)$ is $\langle f, \varphi, \mu \rangle$-admissible and $F \circ \mu\big|_{W(\epsilon,\mu)}$ extends to a function $F_\mu \in \mathcal{R}_{n+1,\epsilon'(\mu)}$ which is normal on $B_{\epsilon'(\mu)}$.*

Proof. By Lemma 3.23 there is a full transformation tree S in x and a $\delta : S \to \mathbb{R}_+^{n+2}$ such that for each $\nu \in S$, $\mathcal{W}(\delta,\nu)$ is $\langle f,\varphi,\nu\rangle$-admissible and there is a $\Phi_\nu \in \mathcal{R}_{n+1}$ such that $\mathcal{W}(\delta,\nu) \subseteq \mathcal{B}_{r(\Phi_\nu)}$ and

$$Z_{\mathcal{W}(\delta,\nu)}(F \circ \nu) \subseteq Z_{\mathcal{W}(\delta,\nu)}(\Phi_\nu). \tag{3.8}$$

Let $\nu \in S$, considered to be a function in $\mathcal{R}_{n+1}^{n+1}$, and define

$$\Psi_\nu(x,y) := \Phi_\nu(x,y) \cdot g \circ \nu'(x) \cdot y \cdot (y - g \circ \nu'(x)), \tag{3.9}$$

where $\nu' := \Pi \circ \nu$. By Remark 3.21 there is a full transformation tree S_ν in (x,y) respecting y and a map $\delta'_\nu : S_\nu \to \mathbb{R}_+^{n+1}$ such that for each $\eta \in S_\nu$, $\mathcal{B}_{\delta'_\nu(\eta)}$ is $\langle \Psi_\nu, \eta\rangle$-admissible and $\Psi_\nu \circ \eta$ is normal on $\mathcal{B}_{\delta'_\nu(\eta)}$. We may assume that $\mathcal{B}_{\delta'_\nu(\eta)}$ is $\langle f,\varphi,\nu,\eta\rangle$-admissible. Let $\delta_\nu(\eta) := (\delta'_\nu(\eta), \delta_{n+2}(\nu))$. Let $T := \{\nu \circ \eta : \nu \in S, \eta \in S_\nu\}$, and let $\epsilon : T \to \mathbb{R}_+^{n+3}$ be given by $\epsilon(\nu \circ \eta) := \delta_\nu(\eta)$ for $\nu \in S$ and $\eta \in S_\nu$. We claim that by possibly refining ϵ, some finite $T' \subseteq T$ and some sufficiently small $r \in \mathbb{R}_+^{n+1}$ and $s > 0$ satisfy the conclusion of the proposition.

To see this let $\mu \in T$, say $\mu = \nu \circ \eta$ with $\nu \in S$ and $\eta \in S_\nu$, and write $\mu' := \Pi \circ \mu$ and $\mu_{n+1} := \Pi_{n+1} \circ \mu$. By Lemma 3.2 and (3.9), $g \circ \mu'(x,y) = (xy)^\alpha u(x,y)$ and $\mu_{n+1}(x,y) = (xy)^\beta v(x,y)$ for some units u and v on $\mathcal{B}_{\epsilon'(\mu)}$ and $\alpha = \alpha(\mu), \beta = \beta(\mu) \in \mathbb{N}^{n+1}$ such that either $\alpha \geq \beta$ or $\alpha \leq \beta$. Note that

$$\begin{aligned} F \circ \mu(x,y) &= f\left(\mu'(x,y),\, \frac{g \circ \mu'(x,y)}{\mu_{n+1}(x,y)},\, \mu_{n+1}(x,y)\right), \\ &= f\left(\mu'(x,y),\, (xy)^{\alpha-\beta}\frac{u(x,y)}{v(x,y)},\, (xy)^\beta v(x,y)\right). \end{aligned} \tag{3.10}$$

If $\alpha \geq \beta$ and $\alpha \neq \beta$, then by possibly shrinking $\epsilon'(\mu)$ we may assume that $|(xy)^{\alpha-\beta}u/v| < \epsilon_{n+2}(\mu)$ on $\mathcal{B}_{\epsilon'(\mu)}$, and so $\mathcal{W}(\epsilon,\mu) = \{(x,y) \in \mathcal{B}_{\epsilon'(\mu)} : \mu_{n+1}(x,y) \neq 0\}$. By (3.10), $F \circ \mu(x,y)$ extends to an analytic function $F_\mu \in \mathcal{R}_{n+1}$ on $\mathcal{B}_{\epsilon'(\mu)}$. By (3.8),

$$Z_{\mathcal{W}(\epsilon,\mu)}(F \circ \mu) \subseteq Z_{\mathcal{W}(\epsilon,\mu)}(\Phi_\nu \circ \eta).$$

Since $\Phi_\nu \circ \eta$ and μ_{n+1} are both normal on $\mathcal{B}_{\epsilon'(\mu)}$, then F_μ is normal on $\mathcal{B}_{\epsilon'(\mu)}$ by Lemma 3.20.

If $\alpha = \beta$, simply shrink $\epsilon_{n+2}(\mu)$ so that $|u/v| > \epsilon_{n+2}(\mu)$ on $\mathcal{B}_{\epsilon'(\mu)}$, so $\mathcal{W}(\epsilon,\mu) = \emptyset$.

If $\alpha \leq \beta$ and $\alpha \neq \beta$, then simply shrink $\epsilon'(\mu)$ so that for all $(x, y) \in \mathcal{B}_{\epsilon'(\mu)}$ such that $\mu_{n+1}(x, y) \neq 0$,

$$\left| \frac{g \circ \mu'(x, y)}{\mu_{n+1}(x, y)} \right| = \left| \frac{u(x, y)}{(xy)^{\beta - \alpha} v(x, y)} \right| > \epsilon_{n+2}(\mu),$$

so $\mathcal{W}(\epsilon, \mu) = \emptyset$.

Now, by Lemma 3.12 there is a finite $T'' \subseteq T$ such that $B_r \subseteq \bigcup_{\mu \in T''} \mu(B_{\epsilon'(\mu)})$ for some $r \in \mathbb{R}_+^{n+2}$, so in particular $W_{(r,s)} \subseteq \bigcup_{\mu \in T'} \mu(B_{\epsilon'(\mu)})$ for any $s > 0$. Letting $s := \min\{\epsilon_{n+2}(\mu) : \mu \in T''\}$, it follows that $W_{(r,s)} \subseteq \bigcup_{\mu \in T''} \mu(W(\epsilon, \mu))$, so

$$W_{(r,s)} \subseteq \bigcup_{\mu \in T'} \mu(W(\epsilon, \mu)),$$

where $T' := \{\mu \in T'' : \alpha(\mu) \geq \beta(\mu), \alpha(\mu) \neq \beta(\mu)\}$, since $W(\epsilon, \mu) = \emptyset$ for any $\mu \in T'' \backslash T'$. Therefore T' is the desired transformation tree. □

We now set out on the task of using Proposition 3.24 to $\mathbb{Z}$-prepare functions of the form $F(x, y) = f(x, g(x)/y, y)$. To make the induction go through, we shall consider a slightly more general form for F.

For Lemma 3.26 and Proposition 3.27 let $n \in \mathbb{N}$ and consider the following situation:

(i) $r' = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ and $g, L_1, L_2 \in \mathcal{R}_{n,r'}$;

(ii) $L(x, y) := L_1(x)y + L_2(x)$ on $B_{r'} \times \mathbb{R}$;

(iii) $\varphi(x, y) := (x,\ g(x)/L(x, y),\ L(x, y))$ on $\text{dom}(\varphi) := \{(x, y) \in B_{r'} \times \mathbb{R} : L(x, y) \neq 0\}$;

(iv) C is an $\mathcal{L}'_{\mathcal{R}}$-cylinder such that $C \subseteq \text{dom}(\varphi)$, and both C and $\varphi(C)$ are bounded;

(v) $r = (r_1, \ldots, r_{n+2}) \in \mathbb{R}_+^{n+2}$ is such that $\Pi(r) = r'$, and $f \in \mathcal{R}_{n+2,r}$ is such that $\overline{\varphi(C)} \subseteq \text{int}(B_r)$;

(vi) $F := f \circ \varphi$, so

$$F(x, y) = f\left(x,\ \frac{g(x)}{L_1(x)y + L_2(x)},\ L_1(x)y + L_2(x)\right). \tag{3.11}$$

Note that $\text{dom}(F) = \{(x, y) \in B_{r'} \times \mathbb{R} : L(x, y) \neq 0, |g(x)/L(x, y)| \leq r_{n+1}, |L(x, y)| \leq r_{n+2}\}$.

Definition 3.25. For a set $A \subseteq \mathbb{R}^n$ and functions $f, g : A \to \mathbb{R}$, f is **equivalent** to g on A, written $f \sim g$ on A, if there are $0 < a < b$ such that for all $x \in A$,

$$\begin{aligned} af(x) \le g(x) \le bf(x) \quad &\text{if} \quad f(x) \ge 0, \\ bf(x) \le g(x) \le af(x) \quad &\text{if} \quad f(x) < 0. \end{aligned}$$

If $\epsilon > 0$ is such that $1 - \epsilon \le a$ and $b \le 1 + \epsilon$, we write $f \sim_\epsilon g$ on A.

Lemma 3.26. *If $L(x, y) \sim \psi(x)$ on C for some $\mathcal{L}'_{\mathcal{R}}$-term $\psi(x)$, then F is $\mathbb{N}$-prepared on C.*

Proof. Fix $0 < a < b$ such that on C,

$$\begin{aligned} a\psi(x) \le L(x, y) \le b\psi(x) \quad &\text{if} \quad \psi(x) \ge 0, \\ b\psi(x) \le L(x, y) \le a\psi(x) \quad &\text{if} \quad \psi(x) < 0. \end{aligned}$$

Since $L(x, y) \neq 0$ for all $(x, y) \in C$, $\{(x, y) \in C : \psi(x) > 0\}$ and $\{(x, y) \in C : \psi(x) < 0\}$ cover C. By considering each of these sets separately we may assume that ψ has constant sign on C, and since both cases are handled similarly, we may assume that $\psi > 0$ on C.

For each $\lambda \in [a, b]$ let $C_\lambda := \left\{(x, y) \in C : \frac{1}{2}\lambda\psi(x) < L(x, y) < \frac{3}{2}\lambda\psi(x)\right\}$. Letting $\Lambda = \{a = \lambda_1 < \ldots < \lambda_k = b\}$ where the λ_i are chosen so that $\frac{\lambda_{i+1}}{\lambda_i} < 3$ for $i = 1, \ldots, k - 1$, we have $C \subseteq \bigcup_{\lambda \in \Lambda} C_\lambda$. So by considering each C_λ separately for each $\lambda \in \Lambda$, without loss of generality we may assume that $a = 1/2$ and $b = 3/2$.

Since $\varphi(C)$ is bounded we may fix an $M > 0$ such that $|x_1|, \ldots, |x_n|, \left|\frac{g(x)}{L(x,y)}\right|$, $|L(x, y)| \le M$ for all $(x, y) \in C$. Therefore on C,

$$\left|\frac{g(x)}{\psi(x)}\right| = \left|\frac{L(x, y)}{\psi(x)} \cdot \frac{g(x)}{L(x, y)}\right| < \frac{3}{2}M,$$

$$|\psi(x)| \le 2|L(x, y)| \le 2M,$$

$$\left|\frac{L(x, y) - \psi(x)}{\psi(x)}\right| \le \frac{1}{2}.$$

Let $s := (\frac{3}{2}M, M, \ldots, M, 2M, \frac{1}{2})$, a tuple in $\mathbb{R}_+^{n+3}$. Consider the maps $\varphi_1 : C \to \mathbb{R}^{n+3}$ and $\varphi_2 : \mathbb{R}^{n+2} \times (-1, 1) \to \mathbb{R}^{n+2}$ defined by

$$\begin{aligned} \varphi_1(x, y) &:= \left(\frac{g(x)}{\psi(x)},\, x,\, \psi(x), \frac{L(x, y) - \psi(x)}{\psi(x)} \right), \\ \varphi_2(w, x, y, z) &:= \left(x,\, \frac{w}{1+z},\, y(1+z) \right). \end{aligned}$$

We see that $\overline{\varphi_1(C)} \subseteq \mathcal{B}_s$, $\varphi_2\big|_{\mathcal{B}_s} \in \mathcal{R}_{n+3,s}$ and $\varphi\big|_C = \varphi_2 \circ \varphi_1$. Since $f \circ \varphi_2$ is $\mathcal{R}$-analytic on the compact set $\overline{\varphi_1(C)}$, by Proposition 3.18 $f \circ \varphi_2$ is $\mathbb{N}$-prepared on a neighborhood of $\overline{\varphi_1(C)}$.

Let $C' \subseteq \mathbb{R}^{n+3}$ be a typical cylinder given by this preparation and suppose

$$f \circ \varphi_2(w, x, y, z) = a(w, x, y)(z - \theta(w, x, y))^d u(w, x, y, z - \theta(w, x, y))$$

on C', where $d \in \mathbb{N}$. Note that $F(x, y)$ is prepared on $\{(x, y) \in \varphi_1^{-1}(C') : L_1(x) = 0\}$ since it is a term in x. Letting $\varphi_1'(x) := \left(\frac{g(x)}{\psi(x)}, x, \psi(x)\right)$, on $\{(x, y) \in \varphi_1^{-1}(C') : L_1(x) \neq 0\}$ we have

$$\begin{aligned} F(x, y) &= a \circ \varphi_1'(x) \left(\frac{L(x, y) - \psi(x)}{\psi(x)} - \theta \circ \varphi_1'(x) \right)^d \\ &\quad u \left(\varphi_1'(x),\ \frac{L(x, y) - \psi(x)}{\psi(x)} - \theta \circ \varphi_1'(x) \right), \\ &= a \circ \varphi_1'(x) \left(\frac{L_1(x)}{\psi(x)} \right)^d \left(y - \frac{L_2(x) + \psi(x)(1 + \theta \circ \varphi_1'(x))}{L_1(x)} \right)^d \\ &\quad u \left(\varphi_1'(x),\ y - \frac{L_2(x) + \psi(x)(1 + \theta \circ \varphi_1'(x))}{L_1(x)} \right), \end{aligned}$$

which is $\mathbb{N}$-prepared. To finish note that $\varphi_1^{-1}(C')$ is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders by Remark 3.14. □

Proposition 3.27. *The function F is $\mathbb{Z}$-prepared on C.*

Proof. By induction on $n \in \mathbb{N}$. Consider $n = 0$; so $F(y) = f(g/(L_1 y + L_2), L_1 y + L_2)$ for some $g, L_1, L_2 \in \mathbb{R}$. If $L_1 = 0$, F is constant. If $g = 0$, then F is $\mathcal{R}$-analytic on the compact set $\overline{C}$, so we are done by Proposition

3.18. So suppose $g \neq 0$ and $L_1 \neq 0$. Since $\varphi(C)$ is bounded, there is an $M > 0$ such that $|g/(L_1 y + L_2)|, |L_1 y + L_2| \leq M$ for all $y \in C$, so $0 < |g|/M \leq |L_1 y + L_2| \leq M$ on C. Therefore F is $\mathcal{R}$-analytic on the compact set $\overline{C}$, so we are again done by Proposition 3.18.

So let $n > 0$. On the set $\{(x, y) \in C : L_1(x) = 0\}$, F is prepared since it is a term in x alone, so we may assume that $L_1(x) \neq 0$ for all $x \in \Pi(C)$. For each $\epsilon > 0$ let

$$\begin{aligned} C_\epsilon &:= \{(x, y) \in C : |L(x, y)|, |g(x)/L(x, y)| < \epsilon\}, \\ C'_\epsilon &:= \{(x, y) \in C : |L(x, y)| \geq \epsilon\}, \\ C''_\epsilon &:= \{(x, y) \in C : |g(x)/L(x, y)| \geq \epsilon\}, \end{aligned}$$

and note that $C = C_\epsilon \cup C'_\epsilon \cup C''_\epsilon$.

Let $\epsilon > 0$. First, note that φ is $\mathcal{R}$-analytic on $\overline{C'_\epsilon}$, so $F = f \circ \varphi$ is $\mathcal{R}$-analytic on the compact set $\overline{C'_\epsilon}$, so by Proposition 3.18 F is $\mathbb{N}$-prepared on C'_ϵ. Next, since $\varphi(C)$ is bounded we may fix an $M > 0$ such that $\epsilon \leq |g(x)/L(x, y)| \leq M$ for all $(x, y) \in C''_\epsilon$, so up to subcylindering C''_ϵ to account for signs, $L(x, y) \sim \sigma g(x)$ on C''_ϵ for some $\sigma \in \{-1, 1\}$. Hence by Lemma 3.26 F in $\mathbb{N}$-prepared on C''_ϵ.

Therefore it suffices to show that F is $\mathbb{Z}$-prepared on C_ϵ for some $\epsilon > 0$. For $a \in \mathbb{R}^n$ let $s_a(x) := x + a$, and if $t(x, y) = t_1(x) y + t_2(x)$ for some $t_1, t_2 \in \mathcal{R}_{n,s}$ and $s \in \mathbb{R}^n_+$, let

$$A_t(x, y) := \left(x, \frac{y - t_2(x)}{t_1(x)}\right)$$

on its domain $\{(x, y) \in B_s \times \mathbb{R} : t_1(x) \neq 0\}$. Note that $F \circ A_L(x, y) = f(x, g(x)/y, y)$ on the $\mathcal{L}'_{\mathcal{R}}$-cylinder $A_L^{-1}(C) = \{(x, L(x, y)) : (x, y) \in C\}$.

Claim 1. There is a finite $B' \subseteq B_{r'}$ such that for each $b \in B'$ there is a finite transformation tree $T(b)$ in (x, y) respecting y, a $\delta_b : T(b) \to \mathbb{R}^{n+2}_+$, and an $(r(b), s(b)) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ such that

(i) $\{(x, y) \in B_{r(b)} \times \mathbb{R} : y \neq 0,\ |g(x+b)/y|, |y| < s(b)\} \subseteq \bigcup_{\mu \in T(b)} \mu(W(\delta_b, \mu))$,

(ii) for each $\mu \in T(b)$, $W(\delta_b, \mu)$ is $\langle f, \varphi, A_L, s_{(b,0)}, \mu\rangle$-admissible and $F \circ A_L \circ s_{(b,0)} \circ \mu\big|_{W(\delta_b,\mu)}$ extends to a function in $\mathcal{R}_{n+1,\delta'_b(\mu)}$ which is normal on $B_{\delta'_b(\mu)}$;

(iii) $B_{r'} \subseteq \bigcup_{b \in B'} B_{r(b)}(b)$.

Proof. By Proposition 3.24, for each $b \in B_{r'}$ there is a $T(b)$, δ_b, and $(r(b), s(b))$ satifying (i) and (ii). The existence of a finite $B' \subseteq B$ satisfying (iii) follows from the compactness of $B_{r'}$. □

We now extend the scope of what is considered to be an "admissible transformation": in addition to functional translations, power substitutions and blowup substitutions, for the rest of the proof of this proposition let us also consider the affine transformations A_t to be "admissible". Let $\{A_t\}$ be the "family" of A_t, and extend the definition of "transformation tree" accordingly.

Claim 2. Suppose there is a finite transformation T in (x, y) (in the new sense of the word) respecting y and a $\delta : T \to \mathbb{R}_+^{n+2}$ such that for each $\mu \in T$, $W(\delta, \mu)$ is $\langle f, \varphi, \mu\rangle$-admissible and $F \circ \mu\big|_{W(\delta,\mu)}$ extends to a function in $\mathcal{R}_{n+1,\delta'(\mu)}$ which is normal on $B_{\delta'(\mu)}$. Then F is $\mathbb{Z}$-prepared on $U := \bigcup_{\mu \in T} \mu(W(\delta, \mu))$.

Proof. By induction on $\text{ht}(T)$; let us call the induction on n the "outer" induction and the induction on $\text{ht}(T)$ the "inner" induction. We are done if $\text{ht}(T) = 0$, so assume that $\text{ht}(T) > 0$. Let $T_1 := \{\mu_1 : \langle \mu_1, \dots, \mu_m\rangle \in T\}$, and for each $\mu \in T_1$ let $T[\mu] := \{\nu : \mu \circ \nu \in T\}$.

Fix $\mu \in T_1$. If $\mu = b_\infty^{i,n+1}$ for some $i = 1, \dots, n$, then since T respects y, we have $\mu \in T$ and $F \circ \mu$ is normal on $B_{\delta'(\mu)}$. In any other case, $F \circ \mu$ is of the same form as F, as given in (3.11) (this is the reason why we consider the slightly more general form $f(x, g(x)/L(x, y), L(x, y))$ and not just $f(x, g(x)/y, y))$. Since $\text{ht}(T[\mu]) < \text{ht}(T)$, by the inner induction hypothesis $F \circ \mu$ is $\mathbb{Z}$-prepared on $U[\mu] := \bigcup_{\nu \in T[\mu]} \nu(W(\delta, \mu \circ \nu))$, say

$$F \circ \mu(x, y) = a(x)(y - \theta(x))^d u(x, y - \theta(x))$$

with $d \in \mathbb{Z}$. If T_1 is a transformation family of a functional translation, power substitution, or blowup substitution, then by Lemma 3.17 F is $\mathbb{Z}$-prepared on $U \setminus \mathcal{E}(T_1)$. Since $U \cap \mathcal{E}(T_1)$ is a union of sets of the form $\{(x, y) \in U : x_i = 0\}$ for some $i = 1, \dots, n+1$, and doing such a substitution $x_i = 0$ either makes F a function in x alone or a function in (x, y) of the same form as given in (3.11) but in one less variable, we see that F is $\mathbb{Z}$-prepared on $\{(x, y) \in U : x_i = 0\}$ by the outer induction hypothesis. On the other hand, if $T_1 = \{A_t\}$, where $t(x, y) = t_1(x)y + t_2(x)$ for some $t_1, t_2 \in \mathcal{R}_n$, then for all $(x, y) \in U$, $t_1(x) \neq 0$

and

$$\begin{aligned} F(x,y) &= a(x)(t_1(x)y + t_2(x) - \theta(x))^d u(x, t_1(x)y + t_2(x)), \\ &= \frac{a(x)}{t_1(x)^d}\left(y - \frac{\theta(x) - t_2(x)}{t_1(x)}\right)^d u\left(x, y - \frac{\theta(x) - t_2(x)}{t_1(x)}\right), \end{aligned}$$

which is $\mathbb{Z}$-prepared. □

To complete the proof of Proposition 3.27, simply apply Claim 1 and let $\epsilon := \min\{s(b) : b \in B'\}$. Since $B_{r'} \subseteq \bigcup_{b\in B'} B_{r(b)}(b)$, $C_\epsilon \subseteq \bigcup_{b\in B'} \bigcup_{\mu\in T(b)} A_L \circ s_{(b,0)} \circ \mu(W(\delta_b, \mu))$. So letting $T'(b) := \{A_L \circ s_{(b,0)} \circ \mu : \mu \in T(b)\}$, applying Claim 2 to each $F \circ s_{(b,0)}$ and $T'(b)$ shows that F is $\mathbb{Z}$-prepared on C_ϵ. □

3.4 Proof of the Main Theorem over $\mathbb{R}$

For this section, fix a Weierstrass system $\mathcal{R}$ over $\mathbb{R}$.

Lemma 3.28. *Let $\theta(x)$ be an $\mathcal{L}'_{\mathcal{R}}$-term, $A \subseteq \mathbb{R}^{n+1}$ be a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders and $\epsilon > 0$. There are $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \ldots, C_k$ covering A such that for each $C \in \{C_1, \ldots, C_k\}$ one of the following holds:*

1. *$y \sim_\epsilon \theta(x)$ on C;*
2. *$y - \theta(x) = a(x)\, u(x,y)$ on C, where $u(x,y)$ is an $\mathcal{R}$-unit on C and $a(x)$ is an $\mathcal{L}'_{\mathcal{R}}$-term;*
3. *$y - \theta(x) = y\, u(x,y)$ on C, where $u(x,y)$ is an $\mathcal{R}$-unit on C.*

Proof. By subcylindering we may assume that θ has constant sign on A. If $\theta = 0$ on C we are in case 3, so assume that $\theta > 0$ on C (the case $\theta < 0$ is similar). Let $0 < a' < a < 1 < b < b'$ be such that $1 - \epsilon \le a'$ and $b' \le 1 + \epsilon$, and let

$$\begin{aligned} C_1 &:= \{(x,y) \in C : a'\theta(x) < y < b'\theta(x)\}, \\ C_2 &:= \{(x,y) \in C : -b'\theta(x) < y < a\,\theta(x)\}, \\ C_3 &:= \{(x,y) \in C : |y| > b\,\theta(x)\}. \end{aligned}$$

Note that each of these sets is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders and they cover C. For $i = 1, 2, 3$ we are in case i on C_i, since $y \sim_\epsilon \theta(x)$ on C_1, on C_2

$$y - \theta(x) = \theta(x)\left(\frac{y}{\theta(x)} - 1\right)$$

and

$$0 < 1 - a < \left| \frac{y}{\theta(x)} - 1 \right| < 1 + b',$$

and on C_3

$$y - \theta(x) = y \left(1 - \frac{\theta(x)}{y} \right)$$

and

$$0 < 1 - \frac{1}{b} < \left| 1 - \frac{\theta(x)}{y} \right| < 1 + \frac{1}{b}.$$

□

Lemma 3.29. *Let $\theta_1(x)$ and $\theta_2(x)$ be $\mathcal{L}'_{\mathcal{R}}$-terms and $A \subseteq \mathbb{R}^{n+1}$ be a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders. There are $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \ldots, C_k$ covering A such that for each $C \in \{C_1, \ldots, C_k\}$ either*

1. $y - \theta_2(x) = a(x)\, u(x, y - \theta_1(x))$ *on* C, *or*
2. $y - \theta_1(x) = a(x)\, u(x, y - \theta_2(x))$ *on* C, *or*
3. $y - \theta_2(x) = (y - \theta_1(x))\, u(x, y - \theta_1(x))$ *on* C, *or*
4. $y - \theta_1(x) = (y - \theta_2(x))\, u(x, y - \theta_2(x))$ *on* C,

where $a(x)$ is an $\mathcal{L}'_{\mathcal{R}}$-term, $u(x, y)$ is an $\mathcal{R}$-unit on $\{(x, y - \theta_1(x)) : (x, y) \in C\}$ in cases 1 and 3, and $u(x, y)$ is an $\mathcal{R}$-unit on $\{(x, y - \theta_2(x)) : (x, y) \in C\}$ in cases 2 and 4.

Proof. By subcylindering we may assume that $\theta_1 - \theta_2$ has constant sign on C. Since we are in case (ii) if $\theta_1 = \theta_2$ on C, and the other two cases are symmetric, we may assume that $\theta_1 > \theta_2$ on C.

Choose $a, b \in \mathbb{R}$ such that $\frac{1}{2} < a < 1 < b < 1 + a$ and consider the following sets, each of which is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders:

$$\begin{aligned}
C_1 &:= \{(x, y) \in C : \theta_1(x) - a(\theta_1(x) - \theta_2(x)) < y < \theta_1(x) + a(\theta_1(x) - \theta_2(x))\}, \\
C_2 &:= \{(x, y) \in C : \theta_2(x) - a(\theta_1(x) - \theta_2(x)) < y < \theta_2(x) + a(\theta_1(x) - \theta_2(x))\}, \\
C_3 &:= \{(x, y) \in C : y < \theta_1(x) - b(\theta_1(x) - \theta_2(x))\}, \\
C_4 &:= \{(x, y) \in C : y > \theta_2(x) + b(\theta_1(x) - \theta_2(x))\}.
\end{aligned}$$

Note that by the choice of a and b, $C = \bigcup_{i=1}^{4} C_i$. We show that for $i = 1, \ldots, 4$, we are case i on C_i.

On C_1, $\left|\frac{y-\theta_1(x)}{\theta_1(x)-\theta_2(x)}\right| < a < 1$, so

$$y - \theta_2(x) = (\theta_1(x) - \theta_2(x))\left(1 + \frac{y - \theta_1(x)}{\theta_1(x) - \theta_2(x)}\right)$$

is as in case 1. C_2 is similar.

On C_3, $-1 < -\frac{1}{b} < \frac{\theta_1(x)-\theta_2(x)}{\theta_1(x)-y} < 0$, so

$$y - \theta_2(x) = (y - \theta_1(x))\left(1 + \frac{\theta_1(x) - \theta_2(x)}{y - \theta_1(x)}\right)$$

is as in case 3. C_4 is similar. □

Definition 3.30. Let $\epsilon > 0$ and consider a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ which is prepared on $A \subseteq \mathbb{R}^{n+1}$: say there are $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \ldots, C_k \subseteq \mathbb{R}^{n+1}$ covering A such that for each $C \in \{C_1, \ldots, C_k\}$, if C is thin then the graph of $f\big|_C$ is given by a term in x, and if C is fat then

$$f(x, y) = a(x)|y - \theta(x)|^q u(x, |y - \theta(x)|^{1/p})$$

on C. If for each such $\theta(x)$ which is not identically zero on $\Pi(C)$ we have that $y \sim_\epsilon \theta(x)$ on C, then we say that f is **ϵ-prepared** on A.

The functions $f_1, \ldots, f_m : \mathbb{R}^{n+1} \to \mathbb{R}$ are **simultaneously prepared** on $A \subseteq \mathbb{R}^{n+1}$ if there is a common cylinder covering of A preparing each $f_1, \ldots, f_m$ and the $\theta(x)$'s given by this preparation are uniform for all i. More precisely, there are $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \ldots, C_k$ covering A such that for each $C \in \{C_1, \ldots, C_k\}$, if C is thin then the graph of $f\big|_C$ is given by a term in x, and if C is fat then for $i = 1, \ldots, m$,

$$f_i(x, y) = a_i(x)|y - \theta(x)|^{q_i} u_i(x, |y - \theta(x)|^{1/p_i}), \tag{3.12}$$

on C.

These adjectives can be combined with the various special types of preparations, such as with "simultaneously ϵ-$\mathbb{Z}$-prepared" for example.

Remark 3.31. Consider (3.12). We may choose $p, p'_1, \ldots, p'_m \in \mathbb{N}_+$ and $q'_1, \ldots, q'_m \in \mathbb{Z}$ such that $q_i = q'_i/p$ and $1/p_i = p'_i/p$ for $i = 1, \ldots, m$. Then by replacing $u_i(x, y)$ with $u_i(x, y^{p'_i})$, we may assume that

$$f_i(x, y) = a_i(x)|y - \theta(x)|^{q'_i/p} u_i(x, |y - \theta(x)|^{1/p})$$

for all $i = 1, \ldots, m$.

Corollary 3.32. *If $\epsilon > 0$, $A \subseteq \mathbb{R}^{n+1}$ is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders, and $f_1, \ldots, f_m : \mathbb{R}^{n+1} \to \mathbb{R}$ are prepared on A, then $f_1, \ldots, f_m$ are simultaneously ϵ-prepared on A. The same also holds for "$\mathbb{N}$-prepared" and "$\mathbb{Z}$-prepared" in place of "prepared.".*

Proof. We prove the corollary by induction on $m \geq 1$. By Lemma 3.28 there is a finite collection $\mathcal{C}$ of $\mathcal{L}'_{\mathcal{R}}$-cylinders covering $\mathbb{R}^{n+1}$ such that on any fat cylinder $C \in \mathcal{C}$,

$$f_1(x, y) = a_1(x)|y - \theta_1(x)|^{q_1} u_1(x, |y - \theta_1(x)|^{1/p_1})$$

where $y \sim_\epsilon \theta_1(x)$ on C whenever $\theta_1(x)$ is not identically zero. By the induction hypothesis we may further subcylinder and obtain that for any fat cylinder $C \in \mathcal{C}$,

$$f_i(x, y) = a_i(x)|y - \theta_2(x)|^{q_i} u_i(x, |y - \theta_2(x)|^{1/p_2}), \text{ for } i = 2, \ldots, m,$$

where $y \sim_\epsilon \theta_2(x)$ on C whenever $\theta_2(x)$ is not identically equal to zero.

By using the Lemma 3.29 to further subcylinder, we may assume, for example, that

$$y - \theta_1(x) = (y - \theta_2(x))v(x, y - \theta_2(x))$$

and $y > \theta_2(x)$ on C with v a positive $\mathcal{R}$-unit on $\{(x, y - \theta_2(x)) : (x, y) \in C\}$ (the other cases given by Lemma 3.29 and the other possible sign conditions for $y - \theta_2(x)$ and v are handled similarly). Then

$$f_1(x, y) = a_1(x)|y - \theta_2(x)|^{q_1} v(x, y - \theta_2(x))^{q_1} u_1(x, |y - \theta_2(x)|^{1/p_1} v(x, y - \theta_2(x))^{1/p_1})$$

on C. Since $v(x, y)$ is an $\mathcal{R}$-unit on $\{(x, y - \theta_2(x)) : (x, y) \in C\}$, then so is $v(x, y)^q$ for any $q \in \mathbb{Q}$. So by letting $p, q'_1, p'_1, p'_2 \in \mathbb{Z}$ $(p > 0)$ be such that $q_1 = q'_1/p$, $1/p_1 = p'_1/p$ and $1/p_2 = p'_2/p$, and letting $U_1(x, y) := v(x, y^p)u_1(x, y^{p'_1} v(x, y^{p'_1})^{1/p_1})$ and $U_i(x, y) := u_i(x, y^{p'_2})$ for $i = 2, \ldots, m$, all of which are $\mathcal{R}$-units on $\{(x, (y - \theta_2(x))^{1/p}) : (x, y) \in C\}$, we have

$$f_i(x, y) = a_i(x)|y - \theta_2(x)|^{q'_i/p} U_i(x, |y - \theta_2(x)|^{1/p})$$

on C for all $i = 1, \ldots, m$. □

Lemma 3.33. *Let $A \subseteq \mathbb{R}^{n+1}$ be a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders, $g = (g_1, \ldots, g_m) : A \to \mathbb{R}^m$ be a bounded function such that g_i is prepared on A for each $i = 1, \ldots, m$, and $f \in \mathcal{R}_m$ be such that $g(A) \subseteq B_{r(f)}$. Then $f \circ g$ is prepared on A.*

Proof. By Corollary 3.32, $g_1, \ldots, g_m$ are simultaneously prepared on A. Consider a cylinder C given by this preparation. If C is thin, then each $g_i\big|_C$ is given by a term in x, and hence so is $f \circ g\big|_C$. So it suffices to consider the case that C is fat and for $i = 1, \ldots, m$,

$$g_i(x, y) = a_i(x)|y - \theta(x)|^{m_i/p} u_i(x, |y - \theta(x)|^{1/p})$$

on C.

Since on C each g_i is bounded and $u_i(x, |y - \theta(x)|^{1/p})$ is bounded below by a positive constant, then $a_i(x)|y - \theta(x)|^{m_i/p}$ is bounded on C. So by viewing each $a_i(x)|y - \theta(x)|^{m_i/p}$ as $\frac{|y-\theta(x)|^{m_i/p}}{1/a_i(x)}$ when $m_i > 0$ and $\frac{a_i(x)}{|y-\theta(x)|^{-m_i/p}}$ for $m_i \leq 0$, we can write $f(x, y) = F \circ \varphi(x, y)$ on C, where φ is a bounded function on C given by

$$\varphi(x, y) = \left(a(x), \frac{|y - \theta(x)|^{1/p}}{\bar{b}(x)}, \frac{\bar{c}(x)}{|y - \theta(x)|^{1/p}} \right)$$

for tuples of $\mathcal{L}'_{\mathcal{R}}$-terms $a(x) = (a_1(x), \ldots, a_{k_1}(x))$, $\bar{b}(x) = (b_1(x), \ldots, b_{k_2}(x))$ and $\bar{c}(x) = (c_1(x), \ldots, c_{k_3}(x))$, and $F : \mathbb{R}^{k_1+k_2+k_3} \to \mathbb{R}$ is $\mathcal{R}$-analytic on $\overline{\varphi(C)}$. Here $\frac{|y-\theta(x)|^{1/p}}{\bar{b}(x)} := \left(\frac{|y-\theta(x)|^{1/p}}{b_1(x)}, \ldots, \frac{|y-\theta(x)|^{1/p}}{b_{k_2}(x)} \right)$ and $\frac{\bar{c}(x)}{|y-\theta(x)|^{1/p}}$ is defined similarly. Let $k := k_1 + k_2 + k_3$.

By further subcylindering, without loss of generality we may assume that $|b_1(x)| \leq \cdots \leq |b_{k_2}(x)|$ and $|c_1(x)| \geq \cdots \geq |c_{k_3}(x)|$ on C. Just to reduce subscripts, put $b(x) := b_1(x)$ and $c(x) := c_1(x)$. Note that for each i, $|b(x)/b_i(x)| \leq 1$ and $|c_i(x)/c(x)| \leq 1$ on C. So by writing

$$\frac{|y - \theta(x)|^{1/p}}{b_i(x)} = \frac{b(x)}{b_i(x)} \frac{|y - \theta(x)|^{1/p}}{b(x)}, \text{ and}$$

$$\frac{c_i(x)}{|y - \theta(x)|^{1/p}} = \frac{c(x)}{c_i(x)} \frac{c(x)}{|y - \theta(x)|^{1/p}},$$

then by lengthening the tuple $a(x)$ to include the $b(x)/b_i(x)$'s and the $c_i(x)/c(x)$'s and by modifying F appropriately we may assume that

$$\varphi(x, y) = \left(a(x), \frac{|y - \theta(x)|^{1/p}}{b(x)}, \frac{c(x)}{|y - \theta(x)|^{1/p}} \right).$$

By further subcylindering we may assume that $|c(x)| \leq |b(x)|$ on C (the case $|b(x)| \geq |c(x)|$ can be handled similarly).

By Proposition 3.27 and Corollary 3.32 we may ϵ-$\mathbb{Z}$-prepare the function $F(x_1, \ldots, x_{k-2}, x_k, x_{k-1}/x_k)$ on $\tau(C)$ for any $\epsilon \in (0,1)$ we wish, where $\tau(x,y) := \left(a(x), \frac{c(x)}{b(x)}, \frac{|y-\theta(x)|^{1/p}}{b(x)}\right)$. Let C' be a cylinder given by this preparation and suppose

$$F(x_1, \ldots, x_{k-2}, x_k, x_{k-1}/x_k) = A(x')(x_k - \psi(x'))^d u(x', x_k - \psi(x')),$$

on C', where $x' := (x_1, \ldots, x_{k-1})$. So on $\tau^{-1}(C') \cap C$, which is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders, we have

$$\begin{aligned} f \circ g(x,y) &= A \circ \tau'(x) \left(\frac{|y-\theta(x)|^{1/p}}{b(x)} - \psi \circ \tau'(x) \right)^d \\ &\qquad u\left(\tau'(x), \frac{|y-\theta(x)|^{1/p}}{b(x)} - \psi \circ \tau'(x)\right), \end{aligned}$$

where $\tau'(x) := (a(x), c(x)/b(x))$.

For simplicity of notation, let us assume $C = \tau^{-1}(C') \cap C$. We are done if ψ is identically zero on C, so we assume otherwise. For simplicity we also assume that $y > \theta(x)$ on C (the case $y < \theta(x)$ is similar). Let $h(x,y) := |y - \theta(x)|^{1/p}/b(x)$. Since we can have $h \sim_\epsilon \psi$ on C for any $\epsilon > 0$ of our choosing, we can have $1 - \epsilon < h/\psi < 1 + \epsilon$ on C for some $\epsilon \in (0,1)$. Note that

$$h - \psi = \frac{h^p - \psi^p}{\sum_{i=1}^p h^{p-i}\psi^{i-1}} = \frac{h^p - \psi^p}{h^{p-1}} \cdot \frac{1}{\sum_{i=1}^p (\psi/h)^{i-1}},$$

and $\sum_{i=1}^p (1-\epsilon)^{i-1} < \sum_{i=1}^p (\psi/h)^{i-1} < \sum_{i=1}^p (1+\epsilon)^{i-1}$, so $\sum_{i=1}^p (\psi/h)^{i-1}$ is a unit on C. Letting

$$\theta'(x) := \theta(x) + (b(x)\psi(x))^p$$

and

$$v(x,y) := \left(\sum_{i=1}^p (b(x)\psi(x))^{i-1}/y^{i-1} \right)^{-1},$$

we have

$$\begin{aligned} f \circ g(x,y) &= \frac{A \circ \tau'(x)}{b(x)^d}(y - \theta'(x))^d (y-\theta(x))^{d(1-p)/p} v(x, (y-\theta(x))^{1/p})^d \\ &\qquad u\left(\tau'(x), \frac{(y-\theta'(x))(y-\theta(x))^{(p-1)/p}}{b(x)} v(x, (y-\theta(x))^{1/p})\right) \end{aligned}$$

on C. Now apply Lemma 3.29 to $y - \theta(x)$ and $y - \theta'(x)$ to finish preparing $f \circ g$. $\square$

Proof of the Main Theorem over $\mathbb{R}$. We show that every $\mathcal{L}'_{\mathcal{R}}$-term is prepared. This follows directly from the following claims.

Claim 1. If $f, g : \mathbb{R}^{n+1} \to \mathbb{R}$ are prepared, then $f \cdot g$, f/g and $\sqrt[m]{f}$ are prepared.

Claim 2. If $f_1, f_2 : \mathbb{R}^{n+1} \to \mathbb{R}$ are prepared, then $f_1 + f_2$ is prepared.

Claim 3. If $g = (g_1, \ldots, g_m) : \mathbb{R}^{n+1} \to \mathbb{R}^m$ and each g_i is prepared, and $f : \mathbb{R}^m \to \mathbb{R}$ is a restricted $\mathcal{R}$-function, then $f \circ g$ is prepared.

Claim 1 is obvious by simultaneous preparation, so we prove Claims 2 and 3.

Proof of Claim 2. Simultaneously prepare f_1 and f_2 to obtain $\mathcal{L}'_{\mathcal{R}}$-cylinders $C_1, \ldots, C_k$ covering $\mathbb{R}^{n+1}$ such that on each fat $C \in \{C_1, \ldots, C_k\}$,

$$f_i(x, y) = a_i(x)|y - \theta(x)|^{q_i} u_i(x, |y - \theta(x)|^{1/p}),$$

for $i = 1, 2$. Fix C and let $0 < \epsilon < M$ be such that $\epsilon < u_i(x, |y - \theta(x)|^{1/p}) < M$ on C. Let

$$\begin{aligned} C_{f_1 \sim f_2} &:= \left\{ (x, y) \in C : \frac{\epsilon}{2M} \leq \left| \frac{a_1(x)}{a_2(x)} \right| |y - \theta(x)|^{q_1 - q_2} \leq \frac{2M}{\epsilon} \right\}, \\ C_{f_1 \gg f_2} &:= \left\{ (x, y) \in C : \left| \frac{a_1(x)}{a_2(x)} \right| |y - \theta(x)|^{q_1 - q_2} \geq \frac{2M}{\epsilon} \right\}, \\ C_{f_1 \ll f_2} &:= \left\{ (x, y) \in C : \left| \frac{a_1(x)}{a_2(x)} \right| |y - \theta(x)|^{q_1 - q_2} \leq \frac{\epsilon}{2M} \right\}. \end{aligned}$$

On $C_{f_1 \sim f_2}$, $f_1 + f_2 = f_1(1 + f_2/f_1)$ and f_2/f_1 is prepared and bounded. By applying Lemma 3.33 to compose f_2/f_1 with the function $t \mapsto 1 + t$, $1 + f_2/f_1$ is prepared, and so $f_1(1 + f_2/f_1)$ is too.

On $C_{f_1 \gg f_2}$, $1 + f_2/f_1$ is a unit so $f_1 + f_2 = f_1(1 + f_2/f_1)$ is prepared, and on $C_{f_1 \ll f_2}$, $f_1/f_2 + 1$ is a unit so $f_1 + f_2 = f_2(f_1/f_2 + 1)$ is prepared.

Proof of Claim 3. Claim 2 shows that

$$\{(x, y) \in \mathbb{R}^{n+1} : -1 \leq g_i(x) \leq 1 \text{ for } i = 1, \ldots, m\}$$

is a finite union of $\mathcal{L}'_{\mathcal{R}}$-cylinders since, for instance, we may prepare $1 - g_i(x)$ and so cylinderwise we have

$$\begin{aligned} g_i(x,y) \leq 1 \quad &\text{iff} \quad 0 \leq 1 - g_i(x,y) = a(x)|y-\theta(x)|^q u(x, |y-\theta(x)|^{1/p}), \\ &\text{iff} \quad a(x) \geq 0. \end{aligned}$$

Thus we can decompose $\mathbb{R}^{n+1}$ into finitely many $\mathcal{L}'_{\mathcal{R}}$-cylinders such that on each of these cylinders either $f \circ g(x,y) = 0$ or $|g_1|, \ldots, |g_m| \leq 1$. In the latter case simply apply Lemma 3.33. □

Chapter 4

Obtaining the preparation theorem for general Weierstrass systems

The primary purpose of this chapter is to prove the Main Theorem for a Weierstrass system $\mathcal{R}$ over a field K, where K need not be all of $\mathbb{R}$. Before explaining how we will prove this, let us discuss the necessity of the proof.

Fix a Weierstrass system $\mathcal{R}$ over a subfield K of $\mathbb{R}$. Let $\mathcal{S}$ be the smallest Weierstrass system over $\mathbb{R}$ containing $\mathcal{R}$, as given by Proposition 2.13.

Let $f \in \mathcal{R}_{n,r}$, and suppose we want to normalize f on B_r. Fix $s > r$ and $F \in \mathcal{R}_{n,s}$ such that $f = F\big|_{\mathcal{B}_r}$. For each $a \in B_r$, Lemma 3.11 supplies a full $\mathcal{S}$-transformation tree $S(a)$ and a map $\epsilon_a : S(a) \to \mathbb{R}^n_+$ such that for each $\mu \in S(a)$, $f \circ \mu$ is normal on $B_{\epsilon_a(\mu)}$ and $\mathcal{B}_{\epsilon_a(\mu)}$ is $\langle F, \mu \rangle$-admissible. If needed we may shrink $\epsilon_a(\mu)$ and so assume that each $\epsilon_a(\mu) \in \mathbb{Q}^n_+$ (we want $\epsilon_a(\mu)$ to be in K^n_+ at the very least). Let $S := \{s_a \circ \mu : a \in B_r, \mu \in S(a)\}$ and define $\epsilon : S \to \mathbb{Q}^n_+$ by $\epsilon(s_a \circ \mu) := \epsilon_a(\mu)$ for $\mu \in S(a)$. By Lemma 3.12, each $V(a) := \bigcup_{\mu \in S(a)} \mu(B_{\epsilon_a(\mu)})$ is a neighborhood of the origin, so $V := \bigcup_{a \in B_r} s_a(V(a))$ is a neighborhood of B_r. By the compactness of the sets involved, it follows that for some finite $S' \subseteq S$, $B_r \subseteq \bigcup_{\mu \in S'} \mu(B_{\epsilon(\mu)})$. But it is completely unclear that we may take S' to be a set of $\mathcal{R}$-transformation sequences. In fact, if for each $a \in B_r \cap K^n$ we let $T(a)$ be the set of all $\mathcal{R}$-transformation sequences in $S(a)$, which is obtained by only including the blowup substitutions $b^{i,n}_\lambda$ with $\lambda \in K \cup \{\infty\}$, and if we put $T := \{s_a \circ \mu : a \in B_r \cap K^n, \mu \in T(a)\}$, it is not clear that $B_r \subseteq \bigcup_{\mu \in T} \mu(B_{\epsilon(\mu)})$, nor is it even clear that $\bigcup_{\mu \in T(a)} \mu(B_{\epsilon_a(\mu)})$ is a neighborhood of the origin for $a \in B_r \cap K^n$.

The reason for this lack of clarity is that the tree $T(a)$ is constructed from its root to its leaves, but the neighborhoods on which the admissible transformations comprising $T(a)$ are applied are constructed from the leaves to the root. The next section remedies this problem by simply specifying the set on which an admissible transformation is applied at the time it is added to the transformation tree being constructed; each admissible transformation is the master of its own domain, so to speak, and is not at the mercy of all future transformations. We do this by normalizing f on B_r directly, without recourse to a local normalization theorem.

The proof of our normalization theorem, Theorem 4.2, follows the procedure given in [17, Theorem 2.5], but the rank upon which they inducted is defined globally on compact sets, not just at a point. This idea works out quite easily, but with one major drawback: because of the nonlocal nature of the proof, I found it necessary to use linear transformations of the form $(x, y) \mapsto (x+\lambda y, y)$, $\lambda \in K^n$, to make f regular in y, so the coordinate transformations given by Theorem 4.2 can not be unwound to give a preparation theorem in the original coordinates. Instead, we obtain some useful consequences of Theorem 4.2 in Sections 4.2 and 4.3 which, when coupled with the special case of this preparation theorem proved in Chapter 3, enables us to deduce the Main Theorem in 4.4 by a simple model theoretic argument.

4.1 A normalization theorem for q.a. IF-systems over K

Fix a q.a. IF-system $\mathcal{R}$ over a subfield K of $\mathbb{R}$. We shall use the following definition of an admissible transformation, which is more inclusive than the definition of Chapter 3.

Definition 4.1. For $(r, s) \in K_+^n \times K_+$, a function $t \in \mathcal{R}_{n+1,(r,s)}^{n+1}$ is an **admissible transformation in** (x, y) if either there is an admissible transformation $s \in \mathcal{R}_{n,r}^n$ in x such that $t(x, y) = (s(x), y)$ on $B_{(r,s)}$ or if t is one of the following four types of transformations on $B_{(r,s)}$:

(i) **linear transformation**: for $\lambda \in K^n$,

$$l_\lambda(x, y) := (x + \lambda y, y);$$

(ii) **general translation**: for any $a \in K^n$ and $\theta \in \mathcal{R}_{n,r}$,

$$t_{(a,\theta)}(x,y) := (x+a, y+\theta(x));$$

(iii) **power substitution**: for $m \in \mathbb{N}_+$, $1 \le i \le n$ and $\sigma \in \{-1,1\}$,

$$p^m_{i,\sigma}(x,y) := (x_1,\dots,\sigma(\sigma x_i)^m,\dots,x_n,y);$$

(iv) **blowup substitution**: for $1 \le i \le n$,

$$\begin{array}{rcl} b_0^{i,n+1}(x,y) & := & (x, x_i y), \\ b_\infty^{i,n+1}(x,y) & := & (x_1,\dots,x_i y,\dots,x_n,y). \end{array}$$

It is convenient to distinguish two types of general translations:

(i) **point translation**: for $(a,b) \in K^n \times K$,

$$s_{(a,b)}(x,y) := (x+a, y+b);$$

(ii) **functional translation**: for $\theta \in \mathcal{R}_{n,r}$,

$$t_\theta(x,y) := (x, y+\theta(x)).$$

Also, we consider $b_\lambda^{i,n+1}$ to be the composition of admissible transformations $b_0^{i,n+1} \circ s_{\lambda e_{n+1}}$, where e_{n+1} is the $(n+1)$-rst standard basis vector.

A sequence $\langle \mu_1,\dots,\mu_m\rangle$ of admissible transformations μ_i is a **transformation sequence** and is identified with the map $\mu_1 \circ \cdots \circ \mu_m$.

For a field $L \subseteq \mathbb{R}$, a **compact L-box** is a set A of the form $[a_1,b_1]\times\cdots\times[a_n,b_n] \subseteq \mathbb{R}^n$, where $a_i < b_i$ and $a_i, b_i \in L$ for all $i = 1,\dots,n$. If $a \in \operatorname{int}(A)$ then A is **about** a. If $a \in L^n$ and $A = B_r(a)$ for some $r \in L^n_+$, then A is **centered about** a.

The main task of this section is to prove the following.

Theorem 4.2. *For any $n \in \mathbb{N}$, compact K-box $A \subseteq \mathbb{R}^n$, open neighborhood U of A, and $f : U \to \mathbb{R}$ which is $\mathcal{R}$-analytic on U and not identically zero on A, we can associate a rank $\mathrm{h}_A(f) \in (\mathbb{N}\cup\{\infty\})^{m_n}$ depending on A and $f\big|_A$, where m_n only depends on n and h_A satisfies the following:*

(i) *if* $h_A(f) = 0$, *where* 0 *denotes the tuple of all zeros, then* A *is centered about the origin and* f *is normal on* A*;*

(ii) *if* A *is not centered about the origin or* f *is not normal on* A*, then there is a finite set* T *of admissible transformations in* x *and for each* $\mu \in T$ *there is a finite collection* $\mathcal{C}(\mu)$ *of* $\langle f, \mu\rangle$*-admissible compact* $\mathbb{Q}$*-boxes* $B \subseteq \mathbb{R}^n$ *such that* $A \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ *and for each* $\mu \in T$ *and* $B \in \mathcal{C}(\mu)$, $h_B(f \circ \mu) < h_A(f)$, *where* $<$ *denotes the lexicographical ordering on* $(\mathbb{N} \cup \{\infty\})^{m_n}$*.*

Corollary 4.3. *Let* A *and* $f : U \to \mathbb{R}$ *be as in the hypothesis of Theorem 4.2. Then there is a finite set* T *of transformation sequences in* x *and for each* $\mu \in T$ *there is a finite collection* $\mathcal{C}(\mu)$ *of* $\langle f, \mu\rangle$*-admissible compact* $\mathbb{Q}$*-boxes* $B \subseteq \mathbb{R}^n$ *centered about the origin such that* $A \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ *and for all* $\mu \in T$ *and* $B \in \mathcal{C}(\mu)$, $f \circ \mu$ *is normal on* B*.*

Let $\mathcal{C} := \{\mathcal{C}(\mu) : \mu \in T\}$*. We say that* $(T, \mathcal{C})$ *normalizes* f *on* A*.*

Proof. This follows immediately from Theorem 4.2 by inducting on $\mathrm{h}_A(f)$. □

To prove the theorem we need some lemmas.

Lemma 4.4. *For a nonzero* $f \in \mathcal{R}_1$, $\{a \in B_{r(f)} : f(a) = 0\} \subseteq K$*. In particular,* K *is real closed, since* $K[x_1] \subseteq \mathcal{R}_1$ *and* $K \subseteq \mathbb{R}$*.*

Proof. Let $a \in B_{r(f)}$ be a zero of f. We may assume that $a \in \mathrm{int}(B_{r(f)})$. Since $f \neq 0$, by quasianalyticity there is an $i \in \mathbb{N}$ such that $f^{(i)}(a) = 0$ but $f^{(i+1)}(a) \neq 0$, so we may assume that $f(a) = 0$ and $f'(a) \neq 0$. Fix $b \in K$ and $r \in K_+^n$ such that $a \in B_r(b) \subseteq B_{r(f)}$ and $f'(x + b) \neq 0$ for all $x \in B_r$. By closure under composition, $f \circ s_b\big|_{B_r} \in \mathcal{R}_1$, and by Rolle's Theorem, $\{x \in B_r : f \circ s_b(x) = 0\} = \{a - b\}$. By closure under implicit functions, and hence also inverse functions of one variable, $(f \circ s_b)^{-1}\big|_{B_s} \in \mathcal{R}_1$ for some $s > 0$. Therefore by closure under composition, $a - b = (f \circ s_b)^{-1}(0) \in K$, so $a = (a - b) + b \in K$. □

Lemma 4.5. *Let* $f : \mathbb{R}^{n+1} \to \mathbb{R}$ *be* $\mathcal{R}$*-analytic at* $(a, b) \in \mathbb{R}^n \times \mathbb{R}$*, and suppose that* $f(a, b) = 0$ *and* $\frac{\partial f}{\partial y}(a, b) \neq 0$*. Let* g *be the* C^∞ *function defined implictly in a neighborhood of* a *by* $f(x, g(x)) = 0$ *and* $g(a) = b$*. Then* g *is* $\mathcal{R}$*-analytic at* a*.*

Proof. Choose $(r,s) \in K_+^n \times K_+$ and $(c,d) \in K^n \times K$ such that $(a,b) \in \operatorname{int}(B_{(r,s)}(c,d))$ and $f \circ s_{(c,d)}\big|_{B_{(r,s)}} \in \mathcal{R}_{n+1}$. Since by closure under composition $f \circ s_{(c,d)} \circ s_{(c',d')}\big|_{B_{(r',s')}} \in \mathcal{R}_{n+1}$ for all $(c',d') \in B_{(r,s)} \cap K^{n+1}$ and (r',s') such that $B_{(r',s')}(c',d') \subseteq B_{(r,s)}$, we may assume that (c,d) is as close to (a,b) as we wish. So from this and the continuity of g we may assume that

(i) $\frac{\partial f}{\partial y}(x,y) \neq 0$ for all $(x,y) \in B_{(r,s)}(c,d)$;

(ii) there is an $(r',s') \in K_+^n \times K_+$ such that $(a,b) \in \operatorname{int}(B_{(r',s')}(c,g(c))) \subseteq B_{(r,s)}(c,d)$ and $g(B_{r'}(c)) \subseteq \operatorname{int}(B_{s'}(g(c)))$.

By (i), $g(c)-d = \{y \in B_s : f(c,y+d) = 0\}$. Since the function $y \mapsto f(c,y+d)$ is in $\mathcal{R}_{1,s}$, $g(c)-d \in K$ by Lemma 4.4, so $g(c) \in K$.

Therefore $f \circ s_{(c,g(c))}\big|_{B_{(r',s')}} \in \mathcal{R}_{n+1}$. Since $f \circ s_{(c,g(c))}(0) = 0$, $\frac{\partial f \circ s_{(c,g(c))}}{\partial y}(x,y) \neq 0$ for all $(x,y) \in B_{(r',s')}$, and $g(B_{r'}(c)) \subseteq \operatorname{int}(B_{s'}(g(c)))$, by closure under implicit functions $g(x+c)-g(c)\big|_{B_{r'}}$ is in $\mathcal{R}_n$, and hence so is $g(x+c)\big|_{B_{r'}}$. Since $a \in B_{r'}(c)$, this shows that g is $\mathcal{R}$-analytic at a. □

Proof of Theorem 4.2. Let $n = 1$, so $A = [a,b]$ for some $a,b \in K$ with $a < b$. We define $h_A(f) := 0$ if A is centered about the origin and f is normal on A; define $\mathrm{h}_A(f) := 1$ otherwise.

Since $f \neq 0$, f has finitely many zeros $c_1 < \ldots < c_k$ in $[a,b]$, all of which are in K by Lemma 4.4. So each s_{c_i} is an admissible transformation and $f \circ s_{c_i}$ is normal in a neighborhood of the origin. The result for $n = 1$ easily follows.

So let $n \geq 1$ and inductively assume that Theorem 4.2 holds for n. We prove the theorem for f and A with $A \subseteq \mathbb{R}^{n+1}$.

Define $\mathcal{F}_A(f)$ to be the set of all (h,g) such that h is a function of x which is $\mathcal{R}$-analytic on a neighborhood of $\Pi(A)$, g is a function of (x,y) which is $\mathcal{R}$-analytic on a neighborhood of A, and $f = hg$ on A. We shall follow the following convention: whenever we choose an $(h,g) \in \mathcal{F}_A(f)$ we shall assume that h is $\mathcal{R}$-analytic on $\Pi(U)$ and g is $\mathcal{R}$-analytic on U, which is permissible since we may shrink U about A. For $(a,b) \in A$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, define

$$\begin{aligned} \operatorname{ord}_{(a,b)}(f) &:= \inf\left\{ i \in \mathbb{N} : \frac{\partial^i f}{\partial y^i}(a,b) \neq 0 \right\} \in \mathbb{N} \cup \{\infty\}; \\ \operatorname{ord}_A(f) &:= \sup\{\operatorname{ord}_{(a,b)}(f) : (a,b) \in A\}; \\ \operatorname{Ord}_A(f) &:= \min\{\operatorname{ord}_A(g) : (h,g) \in \mathcal{F}_A(f)\}. \end{aligned}$$

Let $\widehat{f}_{(a,b)}(x,y) := \widehat{f \circ s_{(a,b)}}(x,y)$.

For $j = 1, \ldots, 4$ we shall define a tuple $\mathrm{i}_A^j(f) \in (\mathbb{N} \cup \{\infty\})^{l_j}$ and put

$$\mathrm{h}_A(f) := (\mathrm{Ord}_A(f), \mathrm{i}_A^1(f), \ldots, \mathrm{i}_A^4(f)).$$

So the following claim proves the theorem in the case that $\mathrm{Ord}_A(f) = \infty$.

Claim 0. There is a finite set T of linear transformations of the form l_λ, where $\lambda \in \mathbb{Q}^n$, such that for each $\mu \in T$ there is an $\langle f, \mu \rangle$-admissible compact $\mathbb{Q}$-box A_μ such that $\mathrm{Ord}_{A_\mu}(f \circ \mu) < \infty$ and $A \subseteq \bigcup_{\mu \in T} \mu(A_\mu)$.

Proof. Let $(a,b) \in A$. Since U is a neighborhood of A and l_λ is a homeomorphism for any $\lambda \in \mathbb{R}^n$, if we can find a $\lambda \in \mathbb{Q}^n$ such that $\frac{\partial^d f \circ l_\lambda}{\partial y^d}(l_\lambda^{-1}(a,b)) \neq 0$ for some $d \in \mathbb{N}$, then for any sufficiently small $\mathbb{Q}$-box B about $l_\lambda^{-1}(a,b)$ we will have that $\mathrm{Ord}_B(f \circ l_\lambda) \leq d$, B is $\langle f, l_\lambda \rangle$-admissible and $l_\lambda(B)$ is a neighborhood of (a,b). Since A is compact, this will suffice to prove the claim.

Now, $\frac{\partial^d f \circ l_\lambda}{\partial y^d}(l_\lambda^{-1}(a,b)) \neq 0$ for some $d \in \mathbb{N}$ iff $(\widehat{f \circ l_\lambda})_{l_\lambda^{-1}(a,b)}(0,y) \neq 0$. So since $l_\lambda \circ s_{l_\lambda^{-1}(a,b)} = s_{(a,b)} \circ l_\lambda$, it suffices to show that $\widehat{f}_{(a,b)} \circ l_\lambda(0,y) \neq 0$. Now,

$$\begin{aligned} \widehat{f}_{(a,b)}(x + \lambda y, y)\big|_{x=0} &= \sum_{(\alpha,i) \in \mathbb{N}^{n+1}} \frac{1}{\alpha! i!} \cdot \frac{\partial^{|\alpha|+i} f}{\partial x^\alpha \partial y^i}(a,b) \lambda^\alpha y^{|\alpha|+i}, \\ &= \sum_{i=0}^{\infty} p_i(a,b,\lambda) y^i, \end{aligned}$$

where

$$p_i(a,b,\lambda) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq i} \frac{1}{\alpha!(i - |\alpha|!)} \cdot \frac{\partial^i f}{\partial x^\alpha \partial y^{i-|\alpha|}}(a,b) \lambda^\alpha.$$

By quasianalyticity, $\widehat{f}_{(a,b)}(x,y) \neq 0$, so there is a least $d \in \mathbb{N}$ such that $p_d(a,b,z)$ is a nonzero polynomial in z. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and the zero set of $p_d(a,b,z)$ is closed and nowhere dense in $\mathbb{R}^n$, we may fix a $\lambda \in \mathbb{Q}^n$ such that $p_d(a,b,\lambda) \neq 0$. But then $\widehat{f}_{(a,b)} \circ l_\lambda(0,y) \neq 0$, as required. □

Because of Claim 0, it suffices to show the theorem for f and A such that $\mathrm{Ord}_A(f) < \infty$.

If $\mathrm{Ord}_A(f) = 0$ we define $\mathrm{i}_A^1(f) = \mathrm{i}_A^3(f) = \mathrm{i}_A^4(f) = 0$ and $\mathrm{i}_A^2(f) = \mathrm{h}_{\Pi(A)}(h)$, where $(h,g) \in \mathcal{F}_A(f)$ is chosen so that $\mathrm{ord}_A(g) = 0$ and $\mathrm{h}_{\Pi(A)}(h)$ is minimal.

Since $\mathrm{ord}_A(g) = 0$, g is a unit on A, so applying the inductive hypothesis to h on $\Pi(A)$ proves the theorem in this case.

So fix a positive integer d, and inductively assume that $\mathrm{h}_A(f)$ has been defined for all f and $A \subseteq \mathbb{R}^{n+1}$ for which $\mathrm{Ord}_A(f) < d$. Let $\mathrm{P}^1_A(f)$ be shorthand for the following statement:

> A is centered about the origin, $\mathrm{Ord}_A(f) = d$, and for some $(h, g) \in \mathcal{F}_A(f)$ with $\mathrm{ord}_A(g) = d$,
>
> $$g(x, y) = \sum_{i=0}^{d-2} g_i(x)y^i + y^d u(x, y)$$
>
> on A, where u is a unit on A, and $\mathrm{ord}_{(a,b)}(g) < d$ for all $(a, b) \in A$ such that $b \neq 0$.

Note that unlike the analagous property in [17], we do not require that $g_0(0) = \cdots = g_{d-2}(0) = 0$. Define $\mathrm{i}^1_A(f) = 0$ if $\mathrm{P}^1_A(f)$ holds, and $\mathrm{i}^1_A(f) = 1$ otherwise. The following claim proves the theorem in the case that $\mathrm{Ord}_A(f) = d$ and $\mathrm{P}^1_A(f) = 1$.

Claim 1. Suppose $\mathrm{Ord}_A(f) = d$. There is a finite set T of general translations $t_{(a,\theta)}$, where $a \in \mathbb{Q}^n$, such that for each $\mu \in T$ there is a finite collection $\mathcal{C}(\mu)$ of $\langle f, \mu\rangle$-admissible compact $\mathbb{Q}$-boxes about the origin such that $A \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ and for each $\mu \in T$ and $B \in \mathcal{C}(\mu)$, either $\mathrm{Ord}_B(f \circ \mu) < d$ or $\mathrm{P}^1_B(f \circ \mu)$ holds.

Proof. Fix $(h, g) \in \mathcal{F}_A(f)$ such that $\mathrm{ord}_A(g) = d$. Let $A' := \{(a, b) \in A : \mathrm{ord}_{(a,b)}(g) = d\}$. Note that A' is compact since $(a, b) \in A'$ iff $(a, b) \in A$ and $\frac{\partial^i g}{\partial y^i}(a, b) = 0$ for all $i = 0, \ldots, d-1$.

Given any open neighborhood V of A', $U \backslash A'$ is a neighborhood of the compact set $A \backslash V$, so there is a finite collection $\mathcal{C}(id)$ of $\mathbb{Q}$-boxes such that $\bigcup\{B : B \in \mathcal{C}(id)\} \subseteq U$ is a neighborhood of $A \backslash V$ and $\mathrm{Ord}_B(f) < d - 1$ for each $B \in \mathcal{C}(id)$. So it suffices to construct $(T, \mathcal{C})$ such that $\bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ is a neighborhood of A'.

Fix $(a, b) \in A'$. Since A' is compact, it suffices to construct $(T, \mathcal{C})$ such that $\bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ is a neighborhood of (a, b).

By Lemma 4.5, the C^∞ function $\varphi(x)$ defined implicitly in a neighborhood of a by $\frac{\partial^{d-1} g}{\partial y^{d-1}}(x, \varphi(x)) = 0$ and $\varphi(a) = b$ is $\mathcal{R}$-analytic at a. So there is an

$a' \in K^n$ such that $\theta(x) := \varphi(x + a') \in \mathcal{R}_n$ and $a \in \operatorname{int}(B_{r(\theta)}(a'))$. Let $b' := \varphi(a')$. By closure under composition, we may assume that $a' \in \mathbb{Q}^n$ and that a' is as close to a as we wish, so b' is also as close to b as we wish. Hence there is an $(r, s) \in \mathbb{Q}_+^n \times \mathbb{Q}_+$ such that $(a, b) \in \operatorname{int}(t_\varphi(B_{(r,s)}(a', b')))$ and $t_\varphi(B_{(r,s)}(a', b')) \subseteq U$. Since (a, b) and (a', b') are both on the graph of φ, for any $\epsilon > 0$, $(a, b) \in \operatorname{int}(t_\varphi(B_{(r,\epsilon)}(a', b'))) = \operatorname{int}(t_{(a',\theta)}(B_{(r,\epsilon)}))$. So we may shrink $s > 0$ as needed. Note that

$$
\begin{aligned}
g \circ t_{(a',\theta)}(x, y) &= g(x + a', y + \theta(x)), \\
&= \sum_{i=0}^{d-2} \frac{1}{i!} \frac{\partial^i g}{\partial y^i}(x + a', \theta(x)) y^i + y^d u(x, y)
\end{aligned}
$$

for some u which is $\mathcal{R}$-analytic on $B_{(r,s)}$. Since $u(x, 0) = \frac{1}{d!} \frac{\partial^d g}{\partial y^d}(x + a', \theta(x)) \neq 0$ for all x in the compact set B_r, by possibly shrinking s, u is a unit on $B_{(r,s)}$. To finish, note that for all $(a'', b'') \in B_{(r,s)}$ such that $b'' \neq 0$, $t_{(a',\theta)}(a'', b'') \notin A'$, so $\operatorname{ord}_{(a'',b'')}(g \circ t_{(a',\theta)}) < d$. □

Let $\mathrm{P}_A^2(f)$ be shorthand for the following statement:

> $\mathrm{P}_A^1(f)$ holds, and there is an $(h, g) \in \mathcal{F}_A(f)$ such that h is normal on $\Pi(B)$ and
>
> $$g(x, y) = \sum_{i \in I} x^{\alpha_i} y^i v_i(x) + y^d u(x, y)$$
>
> for some $I \subseteq \{0, \ldots, d - 2\}$, where each v_i is a unit on $\Pi(B)$ and u is a unit on B, and $\{d!\alpha_i/(d - i) : i \in I\}$ is a linearly ordered subset of $\mathbb{N}^n \backslash \{0\}$.

If $\mathrm{P}_A^1(f)$ does not hold, then define $\mathrm{i}_A^2(f) := \infty$. So suppose $\mathrm{P}_A^1(f)$ holds, witnessed by $(h, g) \in \mathcal{F}_A(f)$. Let $I := \{i \leq d - 2 : g_i(x) \neq 0\}$, $J := \{(i, j) \in I^2 : i < j,\ g_i(x)^{d!/(d-i)} \neq g_j(x)^{d!/(d-j)}\}$ and

$$\Phi(x) := h(x) \cdot \prod_{i \in I} g_i(x) \cdot \prod_{(i,j) \in J} \left(g_j(x)^{d!/(d-j)} - g_i(x)^{d!/(d-i)} \right).$$

Note that the definition of Φ does not depend on the choice of $(h, g) \in \mathcal{F}_A(f)$ witnessing $\mathrm{P}_A^1(f)$, so we may define $\mathrm{i}_A^2(f) := \mathrm{h}_{\Pi(A)}(\Phi)$.

Claim 2. Suppose that $\mathrm{P}^1_A(f)$ holds and that Φ is not normal on $\Pi(A)$. Then there is a finite set T of admissible transformations in x and for each $\mu \in T$ there is a finite collection $\mathcal{C}(\mu)$ of $\langle f, \mu\rangle$-admissible compact $\mathbb{Q}$-boxes $B \subseteq \mathbb{R}^n$ such that $A \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ and for each $\mu \in T$ and $B \in \mathcal{C}(\mu)$, either $\mathrm{Ord}_B(f \circ \mu) < d$ or $\mathrm{i}^2_B(f \circ \mu) < \mathrm{i}^2_A(f)$.

Proof. By the induction hypothesis in n there is a finite set T of admissible transformations in x and for each $\mu \in T$ there is a finite collection $\mathcal{C}'(\mu)$ of $\langle \Phi, \mu\rangle$-admissible compact $\mathbb{Q}$-boxes $B' \subseteq \mathbb{R}^n$ such that $\Pi(A) \subseteq \bigcup\{\mu(B') : \mu \in T, B' \in \mathcal{C}'(\mu)\}$ and $\mathrm{h}_{B'}(\Phi \circ \mu) < \mathrm{h}_{\Pi(A)}(\Phi)$ for each $\mu \in T$ and $B' \in \mathcal{C}'(\mu)$. Let $\mathcal{C}(\mu) := \{B' \times \Pi_{n+1}(A) : B' \in \mathcal{C}'(\mu)\}$.

We now consider each $\mu \in T$ to be a function on $B' \times \mathbb{R}$ which acts trivially in the last coordinate. Let $\mu \in T$ and $B \in \mathcal{C}(\mu)$. Either $\mathrm{Ord}_B(f \circ \mu) < d$ or $\mathrm{Ord}_B(f \circ \mu) = d$. If the latter case, $\mathrm{P}^1_B(f \circ \mu)$ holds and $\mathrm{i}^2_B(f \circ \mu) < \mathrm{i}^2_A(f)$. □

Claim. If $\mathrm{P}^1_A(f)$ holds and Φ is normal on A, then $\mathrm{P}^2_A(f)$ holds.

To prove the claim, suppose $\mathrm{P}^1_A(f)$ holds and Φ is normal on A. By Lemma 3.2, there is an $(h, g) \in \mathcal{F}_A(f)$ such that h is normal on $\Pi(A)$ and

$$g(x, y) = \sum_{i \in I} x^{\alpha_i} y^i v_i(x) + y^d u(x, y)$$

for some $I \subseteq \{0, \ldots, d-2\}$, where each v_i is a unit on $\Pi(B)$ and u is a unit on B, and $\{d!\alpha_i/(d-i) : i \in I\}$ is a linearly ordered subset of $\mathbb{N}^n$. If $\alpha_i \neq 0$ for each $i \in I$, then $\mathrm{P}^2_A(f)$ holds.

So suppose for a contradiction that $\alpha_i = 0$ for some $i \in I$; let $k \in I$ be least such that $\alpha_k = 0$. Then for some sufficiently small $s \in \mathbb{Q}_+$, $\sum_{i \in I, i \geq k} x^{\alpha_i} y^{i-k} v_i(x) + y^{d-k} u(x, y)$ is a unit on $\Pi(A) \times B_s$, so $\mathrm{ord}_{\Pi(A) \times B_s}(g) = k < d$. But letting $A' := A \backslash \Pi(A) \times \mathrm{int}(B_s)$, since $\mathrm{P}^1_A(f)$ holds, we have $\mathrm{ord}_{A'}(g) < d$. Hence $\mathrm{Ord}_A(f) \leq \mathrm{ord}_A(g) < d$. But this contradicts our assumption that $\mathrm{P}^1_A(f)$ holds, which in particular states that $\mathrm{Ord}_A(f) = d$. This proves the claim.

So by Claims 0, 1 and 2 we have reduced the proof of the theorem to the case that $\mathrm{P}^2_A(f)$ holds. Define $\mathrm{P}^3_A(f)$ to be the following statement:

$\mathrm{P}^2_A(f)$ holds and $\alpha_i/(d-i) \in \mathbb{N}^n$ for each $i \in I$.

If $\mathrm{Ord}_A(f) = d$ but $\mathrm{P}^2_A(f)$ does not hold, define $\mathrm{i}^3_A(f) := \infty$. If $\mathrm{P}^2_A(f)$ holds, let $\mathrm{i}^3_A(f)$ be the cardinality of the set

$$\{j \in \{1, \dots, n\} : d - i \text{ does not divide } \alpha_{ij} \text{ for some } i \in I\}.$$

So if $\mathrm{i}^3_A(f) = 0$, then $\mathrm{P}^3_A(f)$ holds.

Claim 3. Suppose $\mathrm{P}^2_A(f)$ holds but $\mathrm{P}^3_A(f)$ does not. Then there is a $j \in \{1, \dots, n\}$ and an $m \in \mathbb{N}_+$ such that for each $\sigma \in \{-1, 1\}$ there is an $\langle f, p^m_{j,\sigma}\rangle$-admissible compact $\mathbb{Q}$-box A_σ such that $A \subseteq \bigcup\{p^m_{j,\sigma}(A_\sigma) : \sigma \in \{-1, 1\}\}$ and $\mathrm{i}^3_{A_\sigma}(f \circ p^m_{j,\sigma}) < \mathrm{i}^3_A(f)$.

Proof. Let $j \in \{1, \dots, n\}$ be such that that $d - i$ does not divide α_{ij} for some $i \in I$. Then for $\sigma \in \{-1, 1\}$, $p^{d!}_{j,\sigma}$ and $A_\sigma := \{(x, y) : p^{d!}_{j,\sigma}(x, y) \in A\}$ do the job. □

If $\mathrm{Ord}_A(f) = d$ but $\mathrm{P}^3_A(f)$ does not hold, define $\mathrm{i}^4_A(f) := \infty$. If $\mathrm{P}^3_A(f)$ holds, witnessed by $(h, g) \in \mathcal{F}_A(f)$, then using the notation introduced in the definition of $\mathrm{P}^2_A(f)$, we define $\mathrm{i}^4_A(f) := \sum_{i \in I} |\alpha_i|$. This is well-defined since for each $(h, g) \in \mathcal{F}_A(f)$ witnessing $\mathrm{P}^3_A(f)$, g is uniquely determined up to multiplication by a unit in x.

Claim 4. Suppose that $\mathrm{P}^3_A(f)$ holds and that f is not normal on A. Then there is a $j \in \{1, \dots, n\}$ such that by letting $T := \{b^{j,n+1}_0, b^{j,n+1}_\infty\}$, for each $\mu \in T$ there is a finite collection $\mathcal{C}(\mu)$ of $\langle f, \mu\rangle$-admissible compact $\mathbb{Q}$-boxes $B \subseteq \mathbb{R}^{n+1}$ centered about the origin such that $A \subseteq \{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ and for each $\mu \in T$ and $B \in \mathcal{C}(\mu)$, either

(i) $f \circ \mu$ is normal on B, or

(ii) $\mathrm{Ord}_B(f \circ \mu) < d$, or

(iii) $\mathrm{i}^4_B(f \circ \mu) < \mathrm{i}^4_A(f)$.

Proof. We shall prove a slightly weaker form of the claim. For each $\mu \in T$ we will construct a finite set $\mathcal{C}'(\mu)$ of compact sets $A' \subseteq \mathbb{R}^{n+1}$ with the following property:

($*$) if $a = (a_1, \dots, a_{n+1}) \in A'$ and $\delta = (|a_1|, \dots, |a_{n+1}|)$, then $B_\delta \subseteq A'$.

It will be be clear from the construction of A' that $\mathrm{Ord}_{A'}(f\circ\mu)$ and $\mathrm{i}^4_{A'}(f\circ\mu)$ can be defined even though A' is not necessarily a box. The collection $\mathcal{C} = \bigcup\{\mathcal{C}'(\mu) : \mu \in T\}$ will satisfy all the conclusions of the claim except that each $A' \in \mathcal{C}$ is not necessarily a $\mathbb{Q}$-box.

This will suffice to prove the claim, since for each $\mu \in T$, $A' \in \mathcal{C}'(\mu)$ and $a \in A'$, we have $B_\delta \subseteq A'$, where $\delta := (|a_1|,\ldots,|a_{n+1}|)$. Since $\mu(B_\delta) \subseteq \mu(A') \subseteq U$, there is an $\epsilon \in \mathbb{Q}^{n+1}_+$ such that $\epsilon > \delta$ and $\mu(B_\epsilon) \subseteq U$. By choosing ϵ sufficiently close to δ, we can ensure that for each of the three properties (i), (ii) and (iii) listed in the conclusion of the claim, if $f\circ\mu$ and A' have that property then so do $f\circ\mu$ and B_ϵ. Since $a \in A'$ was arbitrary, $a \in \mathrm{int}(B_\epsilon)$ and A' is compact, this will show that A' can be covered by finitely many of such boxes B_ϵ, proving the claim.

Write $h(x) = x^\beta v(x)$ and $g(x,y) = \sum_{i\in I} x^{\alpha_i} y^i v_i(x) + y^d u(x,y)$. Let $k \in I$ be least such that $\alpha_k/(d-k) \le \alpha_i/(d-i)$ for all $i \in I$, and let $I' := \{i \in I : \alpha_i/(d-i) = \alpha_k/(d-k)\}$. Fix $j \in \{1,\ldots,n\}$ such that $\alpha_{kj} > 0$. We consider the blowup substitutions $b^{j,n+1}_\lambda$ for $\lambda \in \{0,\infty\}$.

First consider $b^{j,n+1}_\infty$. On the set $(b^{j,n+1}_\infty)^{-1}(A)$ we have $h\circ b^{j,n+1}_\infty(x) = x^\beta y^{\beta_j} v\circ b^{j,n+1}_\infty(x,y)$ and $g\circ b^{j,n+1}_\infty(x,y) = y^d g_\infty(x,y)$, where

$$g_\infty(x,y) := \sum_{i\in I} x^{\alpha_i} y^{\alpha_{ij}+i-d} v_i \circ b^{j,n+1}_\infty(x,y) + u\circ b^{j,n+1}_\infty(x,y).$$

For all $i \in I$, $\alpha_{ij}/(d-i) \ge \alpha_{kj}/(d-k) \ge 1$, so $\alpha_{ij}+i-d \ge 0$. From this we see that g_∞ is $\mathcal{R}$-analytic on $(b^{j,n+1}_\infty)^{-1}(A)$ and that $\alpha_{ij} > 0$, so there is an $\epsilon_\infty > 0$ such that $g_\infty(x,y) \ne 0$ for all $(x,y) \in (b^{j,n+1}_\infty)^{-1}(A)$ with $|x_i| \le \epsilon_\infty$. Therefore $f\circ b^{j,n+1}_\infty$ is normal on the compact set $A_\infty := \{(x,y) \in (b^{j,n+1}_\infty)^{-1}(A) : |x_i| \le \epsilon_\infty\}$. Note for later that $b^{j,n+1}_\infty(A_\infty) = \{(x,y) \in A : |y| \ge |x_j|/\epsilon_\infty\}$.

Now consider $b^{j,n+1}_0$. For each $i \in I$ let $\beta_i := \alpha_i + (i-d)e_j$, where e_j is the jth standard unit basis vector, and note that the linear ordering of the $\alpha_i/(d-i)$'s are preserved by the $\beta_i/(d-i)$'s. Note that $h\circ b^{j,n+1}_0 = h$ and that $g\circ b^{j,n+1}_0(x,y) = x_j^d g_0(x,y)$, where

$$g_0(x,y) := \sum_{i\in I} x^{\beta_i} y^i v_i(x) + y^d u(x, x_j y).$$

Case 1. $\beta_i \ne 0$ for all $i \in I$.

Then $\mathrm{i}^4_{(b^{j,n+1}_0)^{-1}(A)}(f\circ b^{j,n+1}_0) = \sum_{i\in I}|\beta_i| < \sum_{i\in I}|\alpha_i| = \mathrm{i}^4_A(f)$. But the set $(b^{j,n+1}_0)^{-1}(A)$ is not compact, since it is unbounded in y. This is not a problem, though, since we may simply truncate the set: define $A_0 := \{(x,y) \in$

$(b_0^{j,n+1})^{-1}(A) : |y| \leq 2/\epsilon_\infty\}$. Then $b_0^{j,n+1}(A_0) = \{(x,y) \in A : |y| \leq 2|x_j|/\epsilon_\infty\}$, so $A \subseteq b_\infty^{j,n+1}(A_\infty)) \cup b_0^{j,n+1}(A_0)$, as desired. Using the fact that A is centered about the origin, it is easy to see that A_0 and A_∞ have property $(*)$ (draw a picture).

Case 2. $\beta_i = 0$ for some $i \in I$.

Then $\beta_i = 0$ for all $i \in I'$ and $\beta_i \neq 0$ for all $i \in I \backslash I'$. Therefore $\beta_i \neq 0$ for all $i \in I$ such that $i < k$; so

$$g_0(x,y) = \sum_{i \in I, i<k} x^{\beta_i} y^i v_i(x) + y^k u_0(x,y),$$

where

$$u_0(x,y) := \sum_{i \in I, i \geq k} x^{\beta_i} y^{i-k} v_i(x) + y^{d-k} u(x, x_j y).$$

Since $\beta_k = 0$ there is an $\epsilon_0 > 0$ such that $u_0(x,y) \neq 0$ for all $(x,y) \in (b_0^{j,n+1})^{-1}(A)$ with $|y| \leq \epsilon_0$. Define $A_0 := \{(x,y) \in (b_0^{j,n+1})^{-1}(A) : |y| \leq \epsilon_0\}$, and note that $\mathrm{Ord}_{A_0}(f \circ b_0^{j,n+1}) = k < d$ and that $b_0^{j,n+1}(A_0) = \{(x,y) \in A : |y| \leq \epsilon_0 |x_j|\}$. To finish, we must fill in the gap between $b_0^{j,n+1}(A_0)$ and $b_\infty^{j,n+1}(A_\infty)$.

For any nonzero $\lambda \in \mathbb{R}$,

$$g_0(x, y+\lambda) = (y^d + d\lambda y^{d-1}) u(x, x_j(y+\lambda)) + \sum_{i=0}^{d-2} h_i(x) y^i,$$

for some h_i which are $\mathcal{R}$-analytic on $\Pi(\{(x,y) : (x, x_j(y+\lambda)) \in A\})$. Therefore $\frac{\partial^{d-1} g_0}{\partial y^{d-1}}(x,\lambda) = \frac{\partial^{d-1} g_0}{\partial y^{d-1}}(x, y+\lambda)\Big|_{y=0} = d! \lambda u(x, x_j \lambda) \neq 0$. So letting $A_{(0,\infty)} := \{(x,y) \in (b_0^{j,n+1})^{-1}(A) : \epsilon_0/2 \leq |y| \leq 2/\epsilon_\infty\}$, we have $\mathrm{Ord}_{A_{(0,\infty)}}(f \circ b_0^{j,n+1}) < d$ and $b_0^{j,n+1}(A_{(0,\infty)}) = \{(x,y) \in A : \epsilon_0 |x_j|/2 \leq |y| \leq 2|x_j|/\epsilon_\infty\}$. Clearly $A \subseteq b_0^{j,n+1}(A_0) \cup b_0^{j,n+1}(A_{(0,\infty)}) \cup b_\infty^{j,n+1}(A_\infty)$. It is also easy to see that A_0, $A_{(0,\infty)}$ and A_∞ have property $(*)$.

□

This completes the proof of Theorem 4.2. □

4.2 Some consequences of the normalization theorem

Definition 4.6. Let $A \subseteq \mathbb{R}^n$ and $\mathcal{F}$ be a collection of functions $f : U_f \to \mathbb{R}$ such that $A \subseteq U_f \subseteq \mathbb{R}^n$ for all $f \in \mathcal{F}$. Then $V_A(\mathcal{F}) := \{a \in A : f(a) = 0 \text{ for all } f \in \mathcal{F}\}$, the **variety of $\mathcal{F}$ on** A. We write $V_A(f_1, \dots, f_m)$ for $V_A(\{f_1, \dots, f_m\})$. If $U_f = A$ for all $f \in \mathcal{F}$ we simply write $V(\mathcal{F})$ for $V_A(\mathcal{F})$.

Lemma 4.7. *Let $\mathcal{R}$ be a q.a. IF-system over K, and let $f \in \mathcal{R}_{n,r}$. Then $V(f) \cap K^n$ is dense in $V(f)$.*

Proof. Let $a \in V(f)$. By property (iv) of Definition 2.1, we may assume that $a \in \operatorname{int}(B_r)$. Let $A \subseteq B_r$ be a compact K-box about a. We must show that $A \cap V(f) \cap K^n \neq \emptyset$. Let $(T, \mathcal{C})$ normalize f on A, as given by Corollary 4.3. Since $A \subseteq \bigcup\{B : \mu \in T, B \in \mathcal{C}(\mu)\}$, we may fix $\mu \in T$ and $B \in \mathcal{C}(\mu)$ such that $a = \mu(b)$ for some $b \in B$. Since $f \circ \mu(b) = 0$ and since $f \circ \mu(x) = x^\alpha u(x)$ for some $\alpha \in \mathbb{N}^n$ and unit u on B, $\alpha_i \neq 0$ and $b_i = 0$ for some $i = 1, \dots, n$. Since K is dense in $\mathbb{R}$ and μ is continuous, there is a $b' \in B \cap K^n$ such that $b'_i = 0$ and $\mu(b') \in A$. Hence $f \circ \mu(b') = 0$, and by closure under composition $\mu(b') \in K^n$, so $\mu(b') \in A \cap V(f) \cap K^n$. □

Lemma 4.8. *Let $\mathcal{R}$ be a q.a. IF-system over K, and let $\mathcal{F} \subseteq \mathcal{R}_{n,r}$. Then $V(\mathcal{F}) = V(\mathcal{F}')$ for some finite $\mathcal{F}' \subseteq \mathcal{F}$.*

Proof. We may assume that $\mathcal{F}$ contains a function which is not identically zero, else the result is trivial. The proof now proceeds by induction on $n \geq 1$.

For $n = 1$ choose a nonzero $f \in \mathcal{F}$. Since $V(\mathcal{F}) \subseteq V(f)$ and the latter is a finite set, the result is trivial. So consider $n > 1$, and again pick a nonzero $f \in \mathcal{F}$. Let $(T, \mathcal{C})$ normalize f on B_r, as given by Corollary 4.3. Since $V(\mathcal{F}) = \bigcup\{\mu(V_{\mu^{-1}(B_r) \cap B}(\mathcal{F} \circ \mu)) : \mu \in T, B \in \mathcal{C}(\mu)\}$, where $\mathcal{F} \circ \mu := \{g \circ \mu : g \in \mathcal{F}\}$, it suffices to prove the result for each $V_B(\mathcal{F} \circ \mu)$. So we may assume that f is normal on B_r. But then since f is normal, $V(\mathcal{F}) \subseteq V(f) \subseteq \{x \in B_r : x_i = 0\}$ for some $i = 1, \dots, n$, and we are done by the induction hypothesis. □

Remark 4.9. If we let $\mathcal{R}$ and $\mathcal{F}$ be as in Lemma 4.8, and let $\mathcal{F}' = \{f_1, \dots, f_m\}$, then $V(\mathcal{F}) = V(f_1, \dots, f_m) = V(f_1^2 + \cdots + f_m^2)$, so by Lemma 4.7, $V(\mathcal{F}) \cap K^n$ is dense in $V(\mathcal{F})$.

Lemma 4.10. *Let $\mathcal{R}$ be a q.a. IF-system over K, and let E be a field such that $K \subseteq E \subseteq \mathbb{R}$. Define*

$$\mathcal{S}_0 := L := \bigcup_{m\in\mathbb{N}} \bigcup_{s\in K_+^m} \{f(a) : f \in \mathcal{R}_{m,s}, a \in E^m \cap B_s\},$$

and for $n \in \mathbb{N}_+$ and $r \in L_+^n$ define

$$\mathcal{S}_{n,r} := \bigcup_{m\in\mathbb{N}} \bigcup_{\substack{(r',s)\in K_+^{n+m} \\ r'\geq r}} \{f(x,a)\big|_{B_r} : f \in \mathcal{R}_{n+m,(r',s)}, a \in E^m \cap B_s\}.$$

Then $\mathcal{S} := \bigcup_{n\in\mathbb{N},r\in L_+^n} \mathcal{S}_{n,r}$ is the smallest q.a. IF-system containing $E \cup \mathcal{R}$.

Proof. The facts that $\mathcal{S}$ contains $\mathcal{R}$ and that $\mathcal{S}$ is contained in any q.a. IF-system containing $E \cup \mathcal{R}$ are clear, and verifying that $\mathcal{S}$ is a q.a. IF-system is done just as in Proposition 2.13, with the verfication of closure under implicit functions being similar the verification of closure under Weierstrass preparation, except we must now also show the following two things: $\mathcal{S}$ is quasianalytic and $\mathcal{S}$ is closed under monomial factorization.

Claim. If $r \in K_+^n$ and $f \in \mathcal{S}_{n,r}$ is such that $\widehat{f} \in \widehat{\mathcal{R}}_n$, then $f \in \mathcal{R}_{n,r}$.

To show the claim, fix $F \in \mathcal{R}_{n+m,(r,s)}$ such that $f(x) = F(x,a)$ for some $a \in B_s \cap E^n$. It suffices to show that $f(x) = F(x,b)$ for some $b \in B_s \cap K^m$. Write $z := (x_{n+1}, \ldots, x_{n+m})$. Since $\widehat{f}(x) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(0,a)x^\alpha \in \widehat{\mathcal{R}}_n \subseteq K[\![x]\!]$, $\mathcal{F} := \{\frac{\partial^\alpha F}{\partial x^\alpha}(0,z) - \frac{\partial^{|\alpha|}f}{\partial x^\alpha}(0,a) : \alpha \in \mathbb{N}^n\} \subseteq \mathcal{R}_{m,s}$. By applying Remark 4.9 to $\mathcal{F}$, there is a sequence $a_i \in B_s \cap K^m$, $i \in \mathbb{N}$, converging to a such that $\widehat{F}(x,a_i) = \widehat{F}(x,a)$ for all $i \in \mathbb{N}$. Note that $F(x,a_i) \in \mathcal{R}_{n,r}$ for each i. Since $\mathcal{R}$ is quasianalytic and $\widehat{F}(x,a_i) = \widehat{F}(x,a_j)$ for all $i,j \in \mathbb{N}$, $F(x,a_i) = F(x,a_j)$ for all $i,j \in \mathbb{N}$. But for any $b \in B_r$, $F(b,a) = \lim_{i\to\infty} F(b,a_i)$. So in fact, $F(x,a) = F(x,a_i)$ for each $i \in \mathbb{N}$, proving the claim.

To show that $\mathcal{S}$ is quasianalytic, let $f \in \mathcal{S}_{n,r}$ be such that $\widehat{f} = 0$. By enlarging r we may assume that $r \in K_+^n$. Since $\widehat{f} \in \widehat{\mathcal{R}}_n$, by the claim $f \in \mathcal{R}_{n,r}$. Since $\mathcal{R}$ is quasianalytic, $f = 0$ as desired.

To show that $\mathcal{S}$ is closed under monomial factorization, let $f \in \mathcal{S}_{n+1,r}$ be such that y divides $\widehat{f}$ in $\mathbb{R}[\![x,y]\!]$. We may assume that $r \in K_+^{n+1}$. Fix $F \in \mathcal{R}_{n+1+m,(r,s)}$ and an $a \in B_s \cap E^m$ such that $f(x,y) = F(x,y,a)$. Since

y divides $\widehat{f}$, $F(x,0,a) = f(x,0) = 0$. So by replacing F with $F(x,y,z) - F(x,0,z)$, we may assume that both $f(x,y) = F(x,y,a)$ and $F(x,0,z) = 0$. Therefore y divides $\widehat{F}(x,y,z)$, so by closure under monomial factorization in $\mathcal{R}$, there is a $G \in \mathcal{R}_{n+1+m,(r,s)}$ such that $F(x,y,z) = yG(x,y,z)$. So $g(x,y) := G(x,y,a) \in \mathcal{S}_{n+1,r}$, and $f(x,y) = y\,g(x,y)$, as desired. □

Corollary 4.11. *If $\mathcal{R}$ is a q.a. IF-system, then $\mathbb{R}_{\mathcal{R}}$ is a polynomially bounded o-minimal structure having $\mathbb{Q}$ as its field of definable exponents.*

We recall for the reader the meaning of the terminology in this corollary. Let $\mathcal{M}$ be an expansion of the real field. $\mathcal{M}$ is **polynomially bounded** if for every function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$ *with parameters* there is an $n \in \mathbb{N}$ and an $a > 0$ such that $|f(t)| \leq t^n$ for all $t > a$. $\mathcal{M}$ is **o-minimal** if every set $A \subseteq \mathbb{R}$ definable in $\mathcal{M}$ *with parameters* is a finite union of points $\{a\}$ and intervals (a,b), where $a, b \in M$. The **field of definable exponents** of $\mathcal{M}$ is the set of all $\lambda \in \mathbb{R}$ such that $t \mapsto t^\lambda$ is definable in $\mathcal{M}$ *with parameters.* If $\mathcal{R}$ is a a q.a. IF-system, and $\mathcal{S}$ is the smallest q.a IF-system over $\mathbb{R}$ containing $\mathcal{R}$, Lemma 4.10 shows that the set of $\mathcal{L}_{\mathcal{R}}(\mathbb{R})$-formulas is exactly the set of $\mathcal{L}_{\mathcal{S}}$-formulas. So the corollary is really a statement about $\mathcal{S}$.

Proof of Corollary 4.11. Let $\mathcal{S}$ be the smallest q.a. IF-system over $\mathbb{R}$ containing $\mathcal{R}$. By [17], $\mathbb{R}_{\mathcal{S}}$ is a polynomially bounded o-minimal structure having $\mathbb{Q}$ as its field of definable exponents. Since $\mathbb{R}_{\mathcal{R}}$ is a reduct of $\mathbb{R}_{\mathcal{S}}$, so is $\mathbb{R}_{\mathcal{R}}$. □

Let $\mathcal{R}$ and $\mathcal{S}$ be as in the proof of the corollary. The proof in [17] also shows that $\mathbb{R}_{\mathcal{S}}$ is model complete. In the next section we trace through the construction in [17] to show that with the help of Theorem 4.2, it shows that $\mathcal{R}_{\mathcal{R}}$ is model complete.

4.3 Model completeness of $\mathbb{R}_{\mathcal{R}}$

Fix a q.a. IF-system $\mathcal{R}$ over a subfield K of $\mathbb{R}$, and let $\mathcal{S}$ be the smallest q.a. IF-system over $\mathbb{R}$ containing $\mathcal{R}$.

In [17] they show that if $\Lambda(\mathcal{S}) := (\Lambda_n(\mathcal{S}) : n \in \mathbb{N})$, where $\Lambda(\mathcal{S}) := \{A \subseteq [-1,1]^n : A \text{ is } \mathcal{S}\text{-semianalytic}\}$, then every $\Lambda(\mathcal{S})$-set has the Gabrielov property (see Van den Dries and Speissegger [22]). [22, Corollary 2.9] shows that $\mathbb{R}_{\mathcal{S}}$ is therefore o-minimal and model complete; here they need $\mathcal{S}$ to be over $\mathbb{R}$ to get o-minimality, but the proof of [22, Corollary 2.9] does not need

$\mathcal{S}$ to be over $\mathbb{R}$ to get model completeness. Thus showing that $\Lambda(\mathcal{R})$ has the Gabrielov property will show that $\mathbb{R}_{\mathcal{R}}$ is model complete.

A careful reading of the argument in [17] shows that the only place they need the fact that $\mathcal{S}$ is over $\mathbb{R}$ is to prove [17, Corollary 4.4], which we now restate for reference:

> Let $A \subseteq \mathbb{R}^n$ be bounded and $\mathcal{S}$-semianalytic. Then there are $n_i \geq n$ and trivial $\mathcal{S}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{n_i}$ for $i = 1, \dots, k$, each Δ-definable from A, such that
>
> $$A = \Pi(N_1) \cup \cdots \cup \Pi(N_k),$$
>
> and for each i, the set $\Pi(N_i)$ is a manifold and $\Pi : N_i \to \Pi(N_i)$ is a diffeomorphism. In particular, A has dimension.

In [17] this is proved by showing a local version of the same fact, [17, Proposition 3.8], which is proved by their local normalization theorem over $\mathbb{R}$, [17, Theorem 2.5]. When working over a general K, one may simply prove [17, Corollary 4.4] for $\mathcal{R}$ directly, without recourse to a local result, by using the proof of [17, Proposition 3.8] but using Theorem 4.2 of this thesis in place of [17, Theorem 2.5]. It is tempting to leave this modification of the proof of [17, Corollary 4.4] as an exercise for the reader since it is extremely straightforward. But the model completeness result it proves is essential to our proof of the preparation theorem for $\mathbb{R}'_{\mathcal{R}}$ when $\mathcal{R}$ is a general Weierstrass system, so we shall walk through some of the details of the modification. We shall not concern ourselves with the notion of Δ-definability.

Lemma 4.12. *Let $A \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}$ be as in the hypothesis of Theorem 4.2. Suppose that A is not centered about the origin or f is not normal on A, and let $(T, \mathcal{C})$ be the finite collection of $\mathcal{R}$-admissible transformations and collection of $\mathbb{Q}$-boxes given by Theorem 4.2. Fix $\mu \in T$, $B \in \mathcal{C}(\mu)$ and $q \in \mathbb{N}$.*

(i) *If $\mu = p^m_{i,\sigma}$, then $\mathrm{h}_B((x_i^q f) \circ \mu) < \mathrm{h}_A(f)$.*

(ii) *If $\mu = b_0^{i,j}$ or $\mu = b_\infty^{i,j}$, then $\mathrm{h}_B((x_i^q f) \circ \mu) < \mathrm{h}_A(f)$ and $\mathrm{h}_B((x_j^q f) \circ \mu) < \mathrm{h}_A(f)$.*

Proof. This is a restatement of [17, Lemma 2.13] to our context. It is proved in the same way. □

Definition 4.13. A set $A \subseteq \mathbb{R}^n$ is a **basic $\mathcal{R}$-set** if there is an $r \in K_+^n$ and $f, g_1, \ldots, g_k \in \mathcal{R}_{n,r}$ such that

$$A = \{x \in B_r : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}.$$

A finite union of basic $\mathcal{R}$-sets is an **$\mathcal{R}$-set**.

A set $A \subseteq \mathbb{R}^n$ is **$\mathcal{R}$-semianalytic at** $a \in \mathbb{R}^n$ if there is a $b \in K^n$ and an $r \in K_+^n$ such that $a \in \operatorname{int}(B_r(b))$ and $(A - b) \cap B_r$ is an $\mathcal{R}$-set. The set A is **$\mathcal{R}$-semianalytic** if it is $\mathcal{R}$-semianalytic at a for all $a \in \mathbb{R}^n$. If in addition A is a manifold, then A is an **$\mathcal{R}$-semianalytic manifold**.

For $f = (f_1, \ldots, f_k) \in \mathcal{R}_{n,r}^k$, $A \subseteq B_r$ and a sign condition $\sigma \in \{-1, 0, 1\}^k$, define $B_A(f, \sigma) := \{x \in A : \operatorname{sign} f_i(x) = \sigma_i \text{ for } i = 1, \ldots, k\}$.

For a map $\lambda : \{1, \ldots, m\} \to \{1, \ldots, n\}$, let $\Pi_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ be the projection $\Pi_\lambda(x) := (x_{\lambda(1)}, \ldots, x_{\lambda(m)})$.

Let $r \in K_+^n$. A set $M \subseteq B_r$ is **$\mathcal{R}$-trival**, if one of the following holds:

(i) $M = B_{B_r}((x_1, \ldots, x_n), \sigma)$ for some sign condition $\sigma \in \{-1, 0, 1\}^n$, or

(ii) there is a permutation λ of $\{1, \ldots, n\}$, an $\mathcal{R}$-trivial $N \subseteq B_s$ and a $g \in \mathcal{R}_{n-1,s}$, where $s = (r_{\lambda(1)}, \ldots, r_{\lambda(n-1)})$, such that $g(B_s) \subseteq (-r_{\lambda(n)}, r_{\lambda(n)})$ and $\Pi_\lambda(M) = \operatorname{graph}(g\big|_N)$.

An $\mathcal{R}$-semianalytic manifold $M \subseteq \mathbb{R}^n$ is called **trivial** if $M = N + a$ for some $\mathcal{R}$-trivial $N \subseteq \mathbb{R}^n$ and $a \in K^n$.

Proposition 4.14. *Let $A \subseteq \mathbb{R}^n$ be bounded and $\mathcal{R}$-semianalytic. Then there are $n_i \geq n$ and trivial $\mathcal{R}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{n_i}$ for $i = 1, \ldots, k$ such that*

$$A = \Pi(N_1) \cup \cdots \cup \Pi(N_k),$$

and for each i, the set $\Pi(N_i)$ is a manifold and $\Pi\big|_{N_i} : N_i \to \Pi(N_i)$ is a diffeomorphism.

Proof. By the definition of $\mathcal{R}$-semianalytic and the fact that A is bounded, A is the union of finitely many K-translates of $\mathcal{R}$-sets. Since $\mathcal{R}$-sets are unions of basic $\mathcal{R}$-sets, we may assume that $A = B_{B_r}(f, \sigma)$ for some $r \in K_+^n$, $f = (f_1, \ldots, f_k) \in \mathcal{R}_{n,r}^k$ and sign condition $\sigma \in \{-1, 0, 1\}^k$. Note that by property (iv) of Definition 2.1 we may assume that f is defined on some neighborhood U of B_r. But then by adding in all the functions $r_i - x_i$ and $r_i + x_i$ to the tuple f, and considering each of the the different cases $|x_i| < r_i$, $x_i = r_i$

and $x_i = -r_i$ separately, we may assume that $A = B_U(f, \sigma)$. Let $F(x) := \prod_{i=1}^k f_i(x)$. We induct on the pair $(n, \mathrm{h}_{B_r}(F))$, ordered lexicographically.

If $n = 1$, then A is a finite union of points and intervals and the result is trivial. So assume $n > 1$. If F is normal on B_r, then each f_i is normal on B_r and the result is also trivial. So we may assume that F is not normal on B_r; so $\mathrm{h}_{B_r}(F) > 0$. Fix the $(T, \mathcal{C})$ given by Theorem 4.2. So $B_r \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$ and $\mathrm{h}_B(F \circ \mu) < \mathrm{h}_{B_r}(F)$ for each $\mu \in T$ and $\mu \in \mathcal{C}(\mu)$. By the induction hypothesis, $B_B(f \circ \mu, \sigma) = \Pi(M_1) \cup \cdots \cup \Pi(M_k)$ for some trivial $\mathcal{R}$-semianalytic manifolds $M_i \subseteq \mathbb{R}^{n_i}$, for $n_i \geq n$, where each $\Pi\big|_{M_i} : M_i \to \Pi(M_i)$ is a diffeomorphism of manifolds.

The proof now breaks down into four cases, depending on what type of admissible transformations comprise T: general translations, linear transformations, power substitutions or blowup substitutions. The only difference between the situation in [17, Proposition 3.8] and the current situation is that the first case is no longer local and the induction hypotheses we can invoke in the proofs of Cases 3 and 4 are no longer local. As an example we shall verify the first case, but we refer the reader to [17] for the other cases.

Case 1: T is a collection of general translations.

Fix $\mu \in T$, say $\mu = t_{(a,\theta)}$ for some $a \in K^{m-1}$ and $\theta \in \mathcal{R}_{m-1}$, where $1 \leq m \leq n$, and let $B \in \mathcal{C}(\mu)$ (note: $B \subseteq \mathbb{R}^n$). We may suppose that μ is defined on some neighborhood V of B. For $i = 1, \ldots, k$ define

$$N_i = N_i(\mu, B) := \{(a + z_{<m}, z_m + \theta(z_{<m}), z_{>m}, z_m) : z \in M_i\},$$

where $z = (z_1, \ldots, z_{n_i})$, $z_{<m} := (z_1, \ldots, z_{m-1})$ and $z_{>m} := (z_{m+1}, \ldots, z_{n_i})$. Clearly each N_i is a trivial $\mathcal{R}$-semianalytic manifold, and since $\mu : V \times \mathbb{R}^{n_i - n} \to \mathbb{R}^{n_i}$ is a diffeomorphism, then $\Pi(N_i) = \mu(M_i)$ is a manifold and $\Pi\big|_{N_i} : N_i \to \Pi(N_i)$ is a diffeomorphism. Since

$$\begin{aligned} B_{\mu(B)}(f, \sigma) &= \mu(B_B(f \circ \mu, \sigma)), \\ &= \mu(\Pi(M_1) \cup \cdots \cup \Pi(M_k)), \\ &= \mu(\Pi(M_i)) \cup \cdots \cup \mu(\Pi(M_k)), \\ &= \Pi(N_1) \cup \cdots \cup \Pi(N_k), \end{aligned}$$

and $A \subseteq B_r \subseteq \bigcup\{\mu(B) : \mu \in T, B \in \mathcal{C}(\mu)\}$, then $A = \bigcup\{\Pi(N_i(\mu, B)) : \mu \in T, B \in \mathcal{C}(\mu)\}$.

Case 2: T is a collection of linear translations.

This is done similarly to Case 1.

Cases 3 and 4: T is a pair of power substitutions, or a pair of blowup substitutions.

See the proof of [17, Proposition 3.8]. $\square$

Proposition 4.14 and the proof in Sections 4 and 5 of [17] give the following.

Proposition 4.15. *If $\mathcal{R}$ is a q.a. IF-system, $\mathbb{R}_{\mathcal{R}}$ is model complete.*

4.4 Completing the proof of the main theorem for general Weierstrass systems

Given a q.a. IF-system $\mathcal{R}$ over K, let $K_{\mathcal{R}}$ denote the substructure of $\mathbb{R}_{\mathcal{R}}$ with universe K; this is indeed a structure since $\mathcal{R}$ is closed under composition. We want to show that $K_{\mathcal{R}}$ is the prime model of $\mathrm{Th}(\mathbb{R}_{\mathcal{R}})$. To do so we need the following simple fact.

Lemma 4.16. *Let $\mathcal{L}$ be a first order language and $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures. If $\mathcal{N}$ is model complete, and $\mathcal{M}$ and $\mathcal{N}$ have the same existential $\mathcal{L}(M)$-theory, then $\mathcal{M} \preccurlyeq \mathcal{N}$.*

Proof. Since the negation of a universal $\mathcal{L}(M)$-sentence is equivalent to an existential $\mathcal{L}(M)$-sentence, and since $\mathcal{M}$ and $\mathcal{N}$ have the same existential theory, they must also have the same universal $\mathcal{L}(M)$-theory. Now, since $\mathcal{N}$ is model complete, for any existential $\mathcal{L}(M)$-formula $\varphi(x)$ there is a universal $\mathcal{L}(M)$-formula $\psi(x)$ such that $\mathcal{N} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$. In particular, $\mathcal{N} \models \varphi(a) \leftrightarrow \psi(a)$ for all $a \in M^n$. Since $\varphi(a)$ is an existential $\mathcal{L}(M)$-sentence and $\psi(a)$ is a universal $\mathcal{L}(M)$-sentence, $\mathcal{M} \models \varphi(a) \leftrightarrow \psi(a)$. Hence, $\mathcal{M} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$.

To prove the lemma, let φ be an arbitrary $\mathcal{L}(M)$-sentence. By writing φ in a prenex normal form, we can then iterate the above observation to find an existential $\mathcal{L}(M)$-sentence ψ such that $\mathcal{M}$ and $\mathcal{N}$ both model $\varphi \leftrightarrow \psi$. Since $\mathcal{M} \models \psi$ iff $\mathcal{N} \models \psi$, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$. $\square$

Proposition 4.17. *If $\mathcal{R}$ is a q.a. IF-system over K, then $K_{\mathcal{R}}$ is the prime model of the theory of $\mathbb{R}_{\mathcal{R}}$. It follows that $K'_{\mathcal{R}}$ is the prime model of the theory of $\mathbb{R}'_{\mathcal{R}}$.*

Proof. Let $\mathcal{M} \models \mathrm{Th}(\mathbb{R}_{\mathcal{R}})$. Since K is included in the language $\mathcal{L}_{\mathcal{R}}$, $K_{\mathcal{R}}$ embeds into $\mathcal{M}$. So we may assume that $K_{\mathcal{R}} \subseteq \mathcal{M}$. We must show that $K_{\mathcal{R}} \preccurlyeq \mathcal{M}$. Since any $\mathcal{L}_{\mathcal{R}}(K)$-formula is an $\mathcal{L}_{\mathcal{R}}$-formula, this means we must show that $\mathrm{Th}(K_{\mathcal{R}}) = \mathrm{Th}(\mathcal{M})$. But $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathbb{R}_{\mathcal{R}})$, so by Proposition 4.15 and Lemma 4.16 it suffices to show that $K_{\mathcal{R}}$ and $\mathbb{R}_{\mathcal{R}}$ have the same existential theory.

Any $\mathcal{L}_{\mathcal{R}}$-term $t(x)$ can be uniquely written as follows:

(i) $t(x) = x_i$ for some i, or $t(x) = a$ for some $a \in K$, or

(ii) $t(x) = f(t_1(x), \dots, t_m(x))$, where $t_1(x), \dots, t_m(x)$ are $\mathcal{L}_{\mathcal{R}}$-terms and f is either a restricted $\mathcal{R}$-function or is one of the arithmetic operations $+$, $\cdot$ or $-$.

Inductively define $\mathrm{lgt}(t)$, the composition length of t, by $\mathrm{lgt}(t) := 0$ if t is as in (i), and $\mathrm{lgt}(t) := 1 + \max\{\mathrm{lgt}(t_i) : 1 \le i \le m\}$ if t is as in (ii).

Let $\varphi(x)$ be a quantifier free $\mathcal{L}_{\mathcal{R}}$-formula. We want to show that $\mathbb{R}_{\mathcal{R}} \models \exists x \varphi(x)$ iff $K_{\mathcal{R}} \models \exists x \varphi(x)$. To do this we go through a series of syntactic reductions that are true of both $\mathrm{Th}(\mathbb{R}_{\mathcal{R}})$ and $\mathrm{Th}(K_{\mathcal{R}})$.

By writing $\varphi(x)$ in its disjunctive normal form and distributing the existential quantifiers across the disjunction, we may assume that $\varphi(x)$ is of the form $\varphi(x) := \bigwedge_{i=1}^{k} t_i(x) = 0 \wedge \bigwedge_{j=1}^{l} s_j(x) > 0$ for some $\mathcal{L}_{\mathcal{R}}$-terms t_i and s_j. By replacing each $s_j(x) > 0$ with $\exists y(y^2 s_j(x) - 1 = 0)$, and by lengthening the tuple x and increasing k, we may assume that $\varphi(x) := \bigwedge_{i=1}^{k} t_i(x) = 0$ for $\mathcal{L}_{\mathcal{R}}$-terms $t_1, \dots, t_k$. In particular, $\varphi(x)$ is of the following slightly more general form:

$$\varphi(x, y) := \bigwedge_{i=1}^{k} y_i = t_i(x), \tag{4.1}$$

where the y_i's are either variables or 0. For any φ as in (4.1) define $\mathrm{lgt}(\varphi) := \max\{\mathrm{lgt}(t_i) : i = 1, \dots, k\}$. For each $i = 1, \dots, k$, if $\mathrm{lgt}(t_i) > 1$ write $t_i(x) = f(t_{i1}(x), \dots, t_{im}(x))$ as in (ii). By replacing $y_i = t_i(x)$ with $\exists z_1, \dots, z_m(y_i = f(z_1, \dots, z_l) \wedge \bigwedge_{j=1}^{l} z_j = t_{i,j}(x))$, and by lengthening the tuple of variables (x, y), $\varphi(x, y)$ may be replaced by another formula of the form given in (4.1) but of lower composition length. Continuing as such we may therefore assume that each $t_i(x)$ is either a restricted $\mathcal{R}$-function, an aritmetical operation $+$, $\cdot$ or $-$, a variable, or a member of K.

Let $s_i(x, y_i) := y_i - t_i(x)$ and let $I := \{i : t_i \text{ is a restricted } \mathcal{R}\text{-function}\}$. Fix $M \in K_+$ such that for each $i \in I$, the range of $t_i(x)$ is bounded by $M > 0$. For each $i \in I$ replace each instance of y_i in $\bigwedge_{i=1}^{k} s_i(x, y_i) = 0$ with My_i. Therefore if $t_i(x)$ is a restricted $\mathcal{R}$-function, we may consider $s_i(x, y_i)$ to be a restricted $\mathcal{R}$-function, and if $t_i(x)$ is a polynomial, then so is $s_i(x, y_i)$. So by taking sums of squares we arrive at our final form: we may assume that

$$\varphi(x) := f(x) = 0 \wedge p(x) = 0, \tag{4.2}$$

where f is a restricted $\mathcal{R}$-function and $p(x) \in K[x]$.

By Lemma 4.7, $\{x \in K^n : K_{\mathcal{R}} \models \varphi(x)\}$ is dense in $\{x \in \mathbb{R}^n : \mathbb{R}_{\mathcal{R}} \models \varphi(x)\}$. It follows that $\mathbb{R}_{\mathcal{R}} \models \exists x \varphi(x)$ iff $K_{\mathcal{R}} \models \exists x \varphi(x)$, showing that $\mathbb{R}_{\mathcal{R}}$ and $K_{\mathcal{R}}$ have the same existential theory. □

We may now accomplish our goal.

Proof of the Main Theorem. Let $\mathcal{R}$ be a Weierstrass system over a field K, and let $f(x, y)$ be an $\mathcal{L}'_{\mathcal{R}}$-term. We want to prepare $f(x, y)$.

Let $\mathcal{S}$ be the smallest Weierstrass system over $\mathbb{R}$ containing $\mathcal{R}$, as given by Proposition 2.13. By Chapter 3 we may prepare f as an $\mathcal{L}'_{\mathcal{S}}$-term. Namely, there is a finite collection $\mathcal{C}$ of $\mathcal{L}'_{\mathcal{S}}$-cylinders covering $\mathbb{R}^{n+1}$ such that for each fat cylinder $C \in \mathcal{C}$,

$$f(x, y) = a(x)|y - \theta(x)|^q u(x, |y - \theta(x)|^{1/p})$$

on C, where $a(x)$ and $\theta(x)$ are $\mathcal{L}'_{\mathcal{S}}$-terms, $p \in \mathbb{N}_+$, $q \in \mathbb{Q}$, and $u(x, y)$ in a positive $\mathcal{S}$-unit on $\{(x, |y - \theta(x)|^{1/p}) : (x, y) \in C\}$. Fix positive rational numbers ϵ_1 and ϵ_2 such that $\epsilon_1 < u(x, |y - \theta(x)|^{1/p}) < \epsilon_2$ on C, and let φ_C be the $\mathcal{L}'_{\mathcal{S}}$-sentence

$$\begin{aligned} \forall x \forall y \big(\text{“}(x, y) \in C\text{”} \rightarrow (f(x, y) = a(x)|y - \theta(x)|^q u(x, |y - \theta(x)|^{1/p}) \\ \wedge \quad \epsilon_1 < u(x, |y - \theta(x)|^{1/p}) < \epsilon_2)\big) . \end{aligned}$$

Let φ be the $\mathcal{L}'_{\mathcal{S}}$-sentence

$$\forall x \forall y \left(\bigvee_{C \in \mathcal{C}} \text{“}(x, y) \in C\text{”} \right) \wedge \bigwedge_{\substack{C \in \mathcal{C} \\ C \text{ fat}}} \varphi_C .$$

Note that $\mathbb{R}'_{\mathcal{R}} \models \varphi$ and this expresses the fact that f is prepared as an $\mathcal{L}'_{\mathcal{S}}$-term. (We do not need to worry about the form of f on the thin cylinders C, since $f\big|_C = t\big|_C$ for some $\mathcal{L}'_{\mathcal{S}}$-term t simply by the definition of a thin cylinder and the fact that f is an $\mathcal{L}'_{\mathcal{S}}$-term.)

By Proposition 2.13, to each restricted $\mathcal{S}$-function $g : \mathbb{R}^n \to \mathbb{R}$ we can associate a restricted $\mathcal{R}$-function $\overline{g} : \mathbb{R}^{n+m} \to \mathbb{R}$, called a "parameterized form of g", and also an $a_g \in [-1,1]^m$ such that $g(x) = \overline{g}(x, a_g)$. Note that for any $b \in (K \cap [-1,1])^m$, $g(x,b)$ is a restricted $\mathcal{R}$-function. Each $\mathcal{L}'_{\mathcal{S}}$-sentence ψ has a parameterized form also, which is the $\mathcal{L}'_{\mathcal{R}}$-formula $\overline{\psi}(z)$ obtained by replacing every restricted $\mathcal{S}$-function $g(x)$ occuring in ψ with its parameterized form $\overline{g}(x,z)$ and adding on the the conjunction "$z \in [-1,1]^m$".

So $\varphi = \overline{\varphi}(a)$ for some $a \in [-1,1]^m$. Note that if $\mathbb{R}'_{\mathcal{R}} \models \overline{\varphi}(b)$ for some $b \in ([-1,1] \cap K)^m$, then f would be prepared as an $\mathcal{L}'_{\mathcal{R}}$-term, since each of the functions occuring in $\overline{\varphi}(b)$ would be $\mathcal{L}'_{\mathcal{R}}$-terms. But this easily follows from the elementary equivalence of $\mathbb{R}'_{\mathcal{R}}$ and $K'_{\mathcal{R}}$:

$$\begin{aligned} \mathbb{R}'_{\mathcal{R}} \models \overline{\varphi}(a) \quad & \text{so} \quad \mathbb{R}'_{\mathcal{R}} \models \exists z \overline{\varphi}(z), \\ & \text{so} \quad K'_{\mathcal{R}} \models \exists z \overline{\varphi}(z), \\ & \text{so} \quad K_{\mathcal{R}} \models \overline{\varphi}(b) \text{ for some } b \in K^m, \\ & \text{so} \quad \mathbb{R}'_{\mathcal{R}} \models \overline{\varphi}(b). \end{aligned}$$

□

4.5 Some concluding remarks

To conclude, I briefly discuss some issues for future investigation. They are only intended to be a collection of hints and have not been fully thought out (hence the phrase "future investigation"). I shall use the first person, since many of the remarks here are as much about the author's lack of knowledge as they are about his knowledge.

Let $\mathcal{R}$ be a q.a. IF-system over a field K. The elementary equivalence of $K_{\mathcal{R}}$ and $\mathbb{R}_{\mathcal{R}}$ and the close relationship between $\mathcal{R}$ and its smallest expansion $\mathcal{S}$ over $\mathbb{R}$ provide a general tool which enables us to work locally without keeping track of parameters but still obtain more robust results which do keep track of the parameters. For instance, if $\mathcal{R}$ is a Weierstrass system over K (as defined in [19] with closure under Weierstrass division, not just Weierstrass preparation), it follows from [19] that $\langle \mathbb{R}_{\mathcal{R}}, / \rangle$ admits quantifier-elimination.

But, proving the Main Theorem by using the elementary equivalence of $K_{\mathcal{R}}$ and $\mathbb{R}_{\mathcal{R}}$ does go against the spirit motivating this work, namely, an interest in effectivity questions. So a more explicit geometric proof may be more favorable. It appears that this can be done by proving what I call a "parameterized normalization theorem," which associates a function $f \in \mathcal{R}_{n,r}$ with its parameterized form $\overline{f}(x;z) := f(x+z)$, a function defined on a neighborhood of $\{0\} \times B_r$ (by Definition 2.1, property (iv)), and then goes through a detailed analysis of how the normalization procedure performed in the proof of Theorem 3.4 normalizes $\overline{f}(x;a)$ for the various $a \in B_r$ and the various $\lambda \in \mathbb{R}$ associated to the blowup substitutions $b_\lambda^{i,n}$ to show that it suffices to only consider $a \in K^n \cap B_r$ and blowup substitutions $b_\lambda^{i,n}$ with $\lambda \in K \cup \{\infty\}$. This is much more complicated than the method employed here, though, so I decided against it. The main reason for this is that the normalization procedure of Chapter 4 seems more suitable for dealing with effectivity questions than the preparation theorem itself, so this harder proof seems unwarranted.

I am primarily interested in decidability questions, so I am not particularly interested in the preparation theorem for the algebraic restricted analytic functions, since they are all definable over the real field which, as is well known, has a decidable theory. So the collection $\mathcal{R}$ should contain at least one transcendental function, which brings us to consider differentially algebraic functions. But I do not know of an effective proof of the fact that the differentially algebraic power series are closed under Weierstrass preparation, so at the moment Weierstrass preparation seems to be too strong a closure assumption.

In contrast, it is very easy to show in an effective manner that the Noetherian functions are closed under addition, multiplication, differentiation, composition and implicit functions. But they are not closed under monomial factorization; in fact, Bergeron and Reutenauer [1] showed that $(e^x - 1)/x$ is not Noetherian (but they used the name "constructible differentially algebraic"), while of course $e^x - 1$ is. Nevertheless, studying some small expansion of the Noetherian functions which is an IF-system may enable one to get an effective version of the model completeness proof of Chapter 4. But of course this is far from being clear, and because it is a model completeness result and not a quantifier elimination result, it is quite similar in spirit to what has been done in [13] and [8] and will most likely encounter similar difficulties.

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Appendix A

Differentially algebraic power series

Let K be a field of characteristic 0. Given a field extension $L \supseteq K$ we write $\mathrm{td}_K\, L$ for the transcendence degree of L over K. For $f \in K[\![x]\!]$ and $z = (z_1, \ldots, z_n)$ let $\Delta[f](z) := \{\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z) : \alpha \in \mathbb{N}^n\}$ and $\Delta[f] := \Delta[f](x)$. By definition, $f \in K[\![x]\!]$ is **differentially algebraic over** K if $\mathrm{td}_K\, K(\Delta[f]) < \infty$. Let $K[\![x]\!]^{da}$ denote the set of $f \in K[\![x]\!]$ which are differentially algebraic over K.

Lemma A.1. $K[\![\overline{x}]\!]^{da}$ *is a ring closed under differentiation and formal composition, which is the formal analogue of local composition (see Section 3.1).*

Proof. If $f, g \in K[\![x]\!]^{da}$ then $\Delta[f+g] \subseteq \mathbb{Q}(\Delta[f], \Delta[g])$, so $\mathrm{td}_K\, K(\Delta[f+g]) \leq \mathrm{td}_K\, K(\Delta[f], \Delta[g]) < \infty$. Similarly, Leibniz' rule gives $\Delta[fg] \subseteq \mathbb{Q}(\Delta[f], \Delta[g])$ so $\mathrm{td}_K\, K(\Delta[fg]) < \infty$. Also, $\Delta[\frac{\partial f}{\partial x_i}] \subseteq \Delta[f]$ for any $i \geq 1$, so $\mathrm{td}_K\, K(\Delta[\frac{\partial f}{\partial x_i}]) < \infty$. So $K[\![x]\!]^{da}$ is a ring closed under differentiation.

Let $f \in K[\![x_1, \ldots, x_m]\!]^{da}$ and $g \in (K[\![x]\!]^{da})^m$ be such that $g(0) = 0$. The chain rule gives $\Delta[f \circ g](x) \subseteq \mathbb{Q}(\Delta[f](g(x)), \Delta[g](x))$. Since $\mathrm{td}_K\, K(\Delta[f](x)) < \infty$, $\mathrm{td}_K\, K(\Delta[f](g(x))) < \infty$, so

$$\mathrm{td}_K\, K(\Delta[f \circ g](x)) \leq \mathrm{td}_K\, K(\Delta[f](g(x)), \Delta(g)(x)) < \infty.$$

□

By a proof too long to be included here, it is shown in [19] that $K[\![\overline{x}]\!]^{da}$ is closed under Weierstrass preparation. Hence $K[\![\overline{x}]\!]^{da}$ is a "formal" Weierstrass system, meaning that it is a K-algebra of formal power series over

K which is closed under differentiation, formal composition and Weierstrass preparation.

The system $\mathcal{D}$ of differentially algebraic analytic functions, as defined in Examples 2.7, is by definition the collection of all $f \in \mathcal{O}$ such that $\widehat{f} \in \mathbb{R}[\![\overline{x}]\!]^{da}$ (recall that $\mathcal{O}$ is the system of all restricted analytic functions). Consider $f \in \mathcal{O}_n$. For any $a \in \operatorname{int}(B_{n,1})$ and $p \in \mathbb{R}[X_\alpha : \alpha \in \mathbb{N}^n]$, $p(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) : \alpha \in A) = 0$ iff $p(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x+a) : \alpha \in A) = 0$. Hence $\mathcal{D}$ is also closed under translation, so $\mathcal{D}$ is a Weierstrass system.

The following characterizes $K[\![x]\!]^{da}$ as being the collection of power series satisfying certain highly determined systems of polynomial differential equations whose coefficients may always be taken to be in $\mathbb{Q}$.

Lemma A.2. *Let $f \in K[\![x]\!]$. The following are equivalent.*

(i) *f is differentially algebraic over K;*

(ii) $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\Delta[f]) < \infty$*;*

(iii) *there is an $N \in \mathbb{N}$ such that for each $\beta \in \mathbb{N}$ with $|\beta| = N$, there is a $p_\beta \in \mathbb{Q}[X_\alpha : \alpha \in \mathbb{N}^n, |\alpha| < N$ or $\alpha = \beta]$ such that $p_\beta(\Delta[f]) = 0$ and $\frac{\partial p_\beta}{\partial X_\beta}(\Delta[f]) \neq 0$;*

(iv) $\mathbb{Q}(\Delta[f])$ *is finitely generated over $\mathbb{Q}$;*

Proof. Let L be a subfield of K and suppose that $\operatorname{td}_L L(\Delta[f]) < \infty$. Fix $A \subseteq \mathbb{N}^n$ such that $\{\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : \alpha \in A\}$ is a transcendence basis of $L(\Delta[f])$ over L, and pick $N \in \mathbb{N}^n$ such that $|\alpha| < N$ for all $\alpha \in A$. For each $\beta \in \mathbb{N}^n$ such that $|\beta| = N$ there is a nonzero $p_\beta \in L[X_\alpha : \alpha \in \mathbb{N}^n, |\alpha| < N$ or $\alpha = \beta]$ such that $p_\beta(\Delta[f]) = 0$. By choosing p_β to have minimum degree in X_β we can ensure that $\frac{\partial p_\beta}{\partial X_\beta}(\Delta[f])) \neq 0$.

Since $p_\beta(\Delta[f]) = 0$, for each $i = 1, \ldots, n$, $\frac{\partial}{\partial x_i}(p_\beta(\Delta[f])) = 0$. Calculating this derivative gives

$$\frac{\partial p_\beta}{\partial X_\beta}(\Delta[f]) \cdot \frac{\partial^{N+1} f}{\partial x^{\beta+e_i}} + \sum_{|\alpha|<N} \frac{\partial p_\beta}{\partial X_\alpha}(\Delta[f]) \cdot \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+e_i}} = 0,$$

where e_i is the ith standard unit basis vector. Since $\frac{\partial p_\beta}{\partial X_\beta}(\Delta[f]) \neq 0$, this shows that $P(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : |\alpha| \leq N+1) \subseteq P(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : |\alpha| \leq N)$, where $P \subseteq L$ is the field generated by the coefficients of all the p_β's. An easy induction shows that therefore $P(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : |\alpha| \leq N + i) \subseteq P(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : |\alpha| \leq N)$ for all $i \in \mathbb{N}$. Hence $P(\Delta[f])$ is finitely generated over P. But since P is finitely generated over $\mathbb{Q}$, then $P(\Delta[f])$ is finitely generated over $\mathbb{Q}$. Since $\mathbb{Q}(\Delta[f]) \subseteq P(\Delta[f])$, then $\mathbb{Q}(\Delta[f])$ is finitely generated over $\mathbb{Q}$.

If we take L to be K, the preceding argument shows that (i) implies (iv). Clearly the converse is true, so (i) and (iv) are equivalent.

To finish, note that (iv) implies (ii), and that by taking L to be $\mathbb{Q}$ in the above argument shows that (ii) implies (iii) and that (iii) implies (iv). □

By the above lemma, if $L \supseteq K$ is a field extension, then $K[[x]]^{da} = L[[x]]^{da} \cap K[[x]]$, and there is no ambiguity in just saying "f is differentially algebraic" without the modifier "over K."

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