A SET-THEORETIC APPROACH TO SOME PROBLEMS IN MEASURE THEORY

by

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Mome dajdži, skulptoru Alji Kučukaliću

koji je mrzio brojeve i koji me je naučio

da je matematika umjetnost.

To my uncle, the sculptor Aljo Kučukalić,

who dispised numbers and who taught me

that mathematics is an art.

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Contents

Abstract

In this presentation we concern ourselves with a problem in measure theory, which we consider from the set-theoretical point of view. The question asks if all compact spaces supporting a non-separable Radon measure can map onto the product of uncountably many copies of the unit interval.

In the Chapter I, we give some motivation and formulate the problem. We also give basic definitions and results that are going to be used in the work to follow.

In Chapter II, we give all positive results that we are aware of.

In Chapter III, we construct two counter-examples. The first construction uses $\diamondsuit$ and produces an S-space. The second construction uses CH and produces a non-trivial HS+HL space. We also give some discussion of another counter-example, the L-space of Kunen.

In the last chapter, we show that a counter-example can exist independently of the size of the continuum. We also show how our $\diamondsuit$ construction from Chapter III can be done using forcing. At the end, we show a forcing construction of another well-known topological space, the S-space of Ostaszewski. The construction can be done in an extension by just one Cohen real.

Chapter I

Introduction

In this Chapter we formulate the problem we shall be working on for the rest of this presentation and we give some easy observations about the problem. The uncredited results in this Chapter are either a part of the general knowledge, or trivial, or both. None of the non-trivial results are due to the author.

§**1.1. The Problem**. All spaces we shall consider are Hausdorff. A mapping is a continuous function.

It is easy to see that every compact space with no isolated points maps onto $[0,1]^{\omega}$. Then the next question to ask is what kind of compact spaces maps onto $[0,1]^{\omega_1}$. The situation here is surprisingly difficult, and this question was a long term open question in general topology. Finally, a characterization was given by Shapirovskiĭ in [22]. He showed that a compact space maps onto $[0,1]^{\tau}$ for an infinite cardinal τ iff there is a closed subspace of the given space in which every point has uncountable relative π-character. However, even though the notion of the π-character is rather natural (see Juhász[10]), it is often not easy to check relative π-characters of the points of a given space. Therefore, one asks if an easier characterization can be given for spaces which have some additional structure connected with their topology.

A very natural structure to look at is a measure, since there is a strong representation theorem of Maharam ([16]) linking arbitrary finite measures to the product measure on products of the set $2 = \{0,1\}$.

Roughly speaking then, the question is to charazterize those compact spaces which support a measure and which can map onto $[0,1]^{\omega_1}$. Before giving a better formulation of this question, we need to discuss the above mentioned theorem and some other general properties of measures.

A *measure* is a non-negative countably additive function on a σ-algebra. Elements of that σ-algebra are said to be *measurable.* A measure is uniquely determined by the values it assigns to a set of generators of the σ-algebra. That is, any non-negative countably additive function on a set of generators of a σ-algebra, can be uniquely extended to a measure on the generated σ-algebra. We shall make no difference between such a function and the measure that is obtained by extending that function to the generated σ-algebra.

Note that we have excluded from our definition all infinite measures. Not the least important reason for this is that all the measures we shall be talking about will be finite anyway! Restricting ourselves in this manner allows us to take some things for granted, like that all measure algebras are ccc. See §2.3, for example.

A *measure algebra* is a Boolean algebra with a measure on it.

An example of a measure is a measure defined on a Boolean algebra of subsets of a given topological space. If X is a topological space and μ a measure on a family of subsets of X, we say that X, μ is a *measure space.*

A measure on a Boolean algebra is said to be *complete* if for all a in the algebra, if the measure of a is 0 and $b < a$, then the measure of b is defined and equal to 0. For such measures we define the *measure algebra* to be the Boolean algebra of measurable elements modulo the elements of measure 0. Since we are restricting ourselves to complete measures, every measure algebra is also a Boolean algebra.

If X, μ is a measure space and μ is a complete measure on X, then in the above described manner, μ induces a measure algebra on X. This is the only kind of a measure algebra that we shall be working with. Actually, it is true that every measure algebra can be represented as a measure algebra of the measurable sets in a topological space (see [7]).

Suppose X, μ is a measure space. Then for every subset A of X, one can define the outer measure $\mu^*(A)$ as the infimum of all $\mu(E)$, where E is a μ-measurable set containing A. We can use this to define a new measure on X, as follows.

If A is a subset of X for which there is a μ-measurable E such that $\mu(E \Delta A) = 0$, we define $\hat{\mu}(A) = \mu(E)$. Then we extend $\hat{\mu}$ to the σ-algebra generated by sets A of the above form. It is easily seen that the so defined $\hat{\mu}$ is a complete measure on X and that all μ-measurable sets are $\hat{\mu}$-measurable with the same measure. The so defined $\hat{\mu}$ is a minimal complete measure extending μ and it is easy to see that $\hat{\mu}$ is, up to an isomorphism, the unique measure with this property. The measure $\hat{\mu}$ is called the *completion* of μ and μ is *complete* if it is its own completion. We say that two measures are *equivalent* if they have the same completition. We make no difference between equivalent measures.

Suppose that for some index set I, X_i for i in I, are topological spaces. The *product topology* on the Cartesian product $X = \Pi_{i \in I} X_i$ is the topology whose basis consists of the sets of the form $O = \Pi_{i \in I} O_i$, where all O_i are open and for only finitely many i in I, $O_i \neq X_i$. Any time we speak of products of spaces, we shall have in mind such a product topology on the Cartesian product of the underlying sets. Suppose in addition, that for each i in I, μ_i is a probability measure on X_i. The *product measure* on X is the completition of the measure generated by the sets of the form $E = \Pi_{i \in I} E_i$, where all E_i are measurable and for only finitely many i, E_i is not equal to X_i. To such a set E, the product measure assigns the value $\Pi_{i \in I} \mu_i(E_i)$.

The product $\Pi_{i \in I} S_i$ in which all factors S_i are actually the same set S, is denoted by S^I. The only products of this form that we shall deal with, will have either $S = [0, 1]$ or $S = 2$.

Set $\{0, 1\}$ can be assigned a probability measure by assigning the measure $1/2$ to both $\{0\}$ and $\{1\}$. This induces a product measure on the product 2^I. If we speak of the space 2^I as of a measure space, that is this product measure that we mean.

Now we state the representation theorem of Maharam:

Maharam's Theorem. If μ is a finite measure on the space X, then X is a countable union of measurable subspaces on which the measure algebras induced by μ are either finite or measure algebra isomorphic to the measure algebra of 2^κ for some infinite cardinal κ.★

The theorem actually applies to all σ-finite measures, which is an easy consequence of the theorem as we stated it. It also has been generalized to other measures, see [5]. We shall only need the Theorem in the form we stated it. The proof of this theorem is not relevant to our further discussion, so we do not present it here. The original proof is in [16], and a longer version of it can be found in [5].

If X is a measure space, we refer to a decomposition of $X = \cup_{i\in\omega} E_i$ as in the Maharam's theorem, as a Maharam's decomposition of X. Every such a decomposition induces a countable sequence of cardinals $\langle \kappa_i : i \in \omega \rangle$ such that for every $i \in \omega$, the measure algebra of $E_i, \mu \restriction E_i$ is isomorphic to the measure algebra of 2^{κ_i}, if κ_i is infinite, and otherwise $\kappa_i = 0$ and the measure algebra of E_i is finite. Note that 2^I and 2^J are isomorphic as measure algebras, if and only if the cardinalities of I and J are the same. Therefore, the set $\{\kappa_i : i \in \omega\}$ does not depend on the choice of $E_i (i \in \omega)$. We refer to this set as to the *cardinal representation* of the measure algebra of X, μ. The supremum of the cardinal representation of a measure algebra is called its *Maharam's dimension.* If it is uncountable, we say that the measure is *non-separable*, or that the measure algebra of X, μ is non-separable.

We say that a measure μ on the space X is *outer regular for sets from $\mathcal{S}$* if $\mathcal{S}$ is a subfamily of the family of μ-measurable subsets of X, and if the measure of every measurable set E in X is the infimum of the measures of those elements of $\mathcal{S}$ which are supersets of E.

We can in an analogous way define what it means for a measure to be inner regular for a family $\mathcal{S}$. For finite measures, inner regularity for a family $\mathcal{S}$ is exactly the same as the outer regularity for the family of the complements of the sets in $\mathcal{S}$, and we shall be working with finite measures only.

If a measure μ on a compact space X is outer regular for open sets and the points have finite measures, then the measure must be finite and we can apply Maharam's Theorem. If we then know that the measure algebra of X, μ is non-separable, can we claim that X maps onto $[0,1]^{\omega_1}$? To be more precise, we define:

A complete finite measure μ on a space X is said to be *Radon* if it is defined on the Borel sets and has the property that the measure of each Borel set is the supremum of the measures of its compact subsets.

and then ask:

> Suppose X is compact and supports a Radon probability measure μ such that the measure algebra of X, μ is not separable; does this imply that X can be mapped onto $[0,1]^{\omega_1}$?

From now on we shall refer to this question as the Question H. In particular, Haydon in 1980 asked whether such an implication might follow from something like $MA + \neg CH$, see Fremlin [4].

§**1.2. Basic Stuff.** We first note that we can restrict ourselves to considering a more speciffic class of measure spaces, the ones in which the measure is homogeneous.

The measure μ on a space X is said to be *homogeneous* if for every measurable subset Y of X for which $\mu(Y) > 0$, the measure algebra of $Y, \mu \restriction Y$ is isomorphic to the measure algebra of X, μ.

Note that for infinite κ, the product measure on 2^{κ} is homogeneous. Therefore, any Maharam's decomposition of a finite measure space X, μ, consists of sets on which the restricted measure μ is homogeneous.

If μ is a Radon probability measure on a compact space X and the measure algebra of X, μ is non-separable, let us consider a subset E of X which has a positive measure and on which the restricted measure algebra is isomorphic to the one of 2^{κ} for some $\kappa > \omega$. The existence of such a set, recall, follows from Maharam's Theorem. Since the measure μ is Radon, there will be a closed in X subset C of E which has a positive measure. By the homogenouity of the measure algebra of $E, \mu \restriction E$, the measure algebra of μ restricted to C will be measure algebra isomorphic to the measure algebra of 2^{κ}.

Now, if C can map onto $[0,1]^{\omega_1}$, the map can be extended to X. To see that, simply apply Tietze Extension Theorem, to each coordinate α in ω_1:

Tietze's Extension Theorem. A continuous map from a closed subspace of a normal space onto $[0,1]$, can be extended to the entire space.★

The proof of this theorem can be found in any topology textbook, like [2].

To conclude, we see that if there is a counter-example to Question H, then there is one which supports a non-separable Radon probability measure with the measure algebra isomorphic to the one of 2^κ, for some $\kappa > \omega$. Actually, more is true:

Let ϕ, ψ be predicates applicable to topological spaces. We say that ϕ is *hereditary to the subspaces which satisfy* ψ if, for any topological space X which satisfies ϕ, every subspace of X which satisfies ψ, also satisfies ϕ.

Lemma 1.2.1. Let ϕ be a property of topological spaces hereditary to closed subspaces. In the class of the spaces which satisfy ϕ, Question H is equivalent to its restriction to the compact Radon probability measure spaces whose measure algebras are measure algebra isomorphic to the measure algebra of 2^κ, for some $\kappa > \omega$.

Proof. It is enough to show that in the class of spaces which satisfy ϕ, if there is a counter-example to the Question H, then there is one whose measure algebra is isomorphic to the one of 2^κ, for an uncountable κ.

Given a counter-example X, μ which satisfies ϕ, there is a closed subspace F of X such that the measure algebra of $F, \mu \restriction F$ is isomorphic to the one of 2^κ, for some $\kappa > \omega$. As in the above discussion, F cannot map onto $[0,1]^{\omega}1$, by Tietze Extension Theorem. Also, F satisfies ϕ, since that property is hereditary to closed subspaces. Therefore, F is a counter-example itself.★

Many properties we shall have a chance to consider, like being 1^{st} countable or hereditary Lindelöf, are hereditary to the closed subspaces. We shall frequently make use of Lemma 1.2.1 in Chapter II. It is an advantage to work with homogeneous measures, since it allows us to take some things for granted –like that points have measure 0, or that we can go back and forth from the given measure space to 2^{κ}, where κ is Maharam's dimension of the space.

Next we give a characterization of mappings onto $[0,1]^{\kappa}$ in terms of the existence of certain sequences of pairs of closed sets:

A family $\{(K_{\alpha}^{0}, K_{\alpha}^{1}) : \alpha \in \kappa\}$ is said to be a *dyadical system* if each K_{α}^{0} and K_{α}^{1} are disjoint closed sets and for every $\{\alpha_0, ..., \alpha_{n-1}\}$ in κ and a function ϕ in 2^{n}, the intersection $\cap_{i<n} K_{\alpha_i}^{\phi(i)}$ is non-empty.

The usual notion of a dyadical system is more general than the one we just gave, by not requiring that all pairs generating the system are formed of closed sets. For our purposes the above definition is enough.

We also recall the well known

Urysohn's Lemma. For any two disjoint closed subsets C and K of a normal space X, there is a mapping f from X into $[0,1]$, such that $f(C) = 0$ and $f(K) = 1$.★

Lemma 1.2.2 A compact space X maps onto $[0,1]^{\kappa}$ iff there is in X, a dyadical system of cardinality $\kappa + \omega$.

Proof. For κ finite, note that $[0,1]$ maps onto $[0,1]^{\omega}$.

Now assume κ is infinite, so $\kappa+\omega=\kappa$.

If f maps X onto $[0,1]^{\kappa}$, for every α in κ and $i\in 2$, let K_{α}^{i} be the set $f^{-1}(\{y\in[0,1]^{\kappa}: y(\alpha)=i\})$.

In the other direction, suppose $\{(K_{\alpha}^{0},K_{\alpha}^{1}):\alpha\in\kappa\}$ is a dyadical system in X. Urysohn's Lemma implies for each α, the existence of a mapping f_{α} from X onto $[0,1]$ such that $f_{\alpha}(K_{\alpha}^{i})=i$ for $i\in 2$. For $x\in X$, let $f(x)=\langle f_{\alpha}(x):\alpha\in\kappa\rangle$.★

In compact spaces with no isolated points, one can construct a countable dyadical system by a simple induction. The argument only breaks down at the first limit stage.

A closer look at the representation we get from Maharam's Theorem, makes one think that one could easily pull a dyadical system out of the generating sequence of measurable sets:

Let X,μ be a measure space such that the measure algebra is isomorphic to the measure algebra of 2^{κ} for some infinite κ. For any isomorphism f from the measure algebra of X to the measure algebra of 2^{κ}, consider a sequence $\langle H_{\alpha}:\alpha<\kappa\rangle$ such that the equivalence class of H_{α} maps onto $\{f\in 2^{\kappa}: f(\alpha)=0\}$. We refer to every such sequence as to a *generating sequence* of the measure space. Note that any generating sequence generates the measure algebra.

Suppose that X,μ and $\langle H_{\alpha}:\alpha<\kappa\rangle$ are as above. If A is any subset of X, let us denote A by A^{0} and the complement $X\setminus A$ of A by A^{1}. Then $\langle H_{\alpha}:\alpha<\kappa\rangle$ are *probabilistically independent*, which means that for any $\{\alpha_0, ..., \alpha_{n-1}\}$ in κ, and for any ϕ in 2^n, the μ measure of $\cap_{i<n} H_{\alpha_i}^{\phi(i)}$ is $1/2^n$. In particular, every such intersection is non-empty.

So, if it, for example, would happen that a space X has a generating sequence $\langle H_\alpha : \alpha < \kappa \rangle$ consisting of clopen sets, then the sequence $\langle (H_\alpha^0, H_\alpha^1) : \alpha < \kappa \rangle$ would be a dyadical system, so the space would map onto $[0,1]^\kappa$. An example of such a space is 2^κ. Excluding simple-minded generalizations, we do not know if there are other examples of this phenomenon. It certainly is more restrictive than we would like it to be. The strong requirement that H_α are always clopen, can be weakened by requiring that for each α, there is a pair of disjoint closed sets K_α^0 and K_α^1 such that the symmetric difference between H_α^i and K_α^i has measure 0, for each $i \in 2$. However, this cannnot happen, for example, in any connected space in which non-empty open sets have positive measure, like $[0,1]^\kappa$.

What we can hope for then, is that for ω_1 of the $\alpha < \kappa$, there are pairs of disjoint closed sets K_α^0, K_α^1 which are close enough to H_α and its complement, that the independence of the H_α will assure that the $\{(K_\alpha^0, K_\alpha^1) : \alpha < \omega_1\}$ form a dyadical system. For Radon measures, this approach indeed works if κ is greater that the cardinality of the continuum, as we shall explain in the next Chapter. But, in ZFC, this cannot resolve the question for $\kappa \leq 2^\omega$, as there are counter-examples known.

Note the following fact:

Lemma 1.2.3. For a compact X, the measure μ in X is Radon iff every measurable subset of X has an F_σ subset of the same measure.

Proof. If the measure is Radon:

Given a measurable set E, let C_n be a closed subset of E such that the measure of $E \setminus C_n$ is less than 1/n. Then $\cup_{n\in\omega} C_n$ is an F_σ subset of E with the same measure as E.

For the other direction, just reverse the argument. ★

Of course, F_σ sets are not necessarily closed, and, in fact, the only compact spaces in which every F_σ set is closed are finite ones!

§**1.3. More Basic Stuff.** The first look at the Question H leaves one wondering if the class of compact spaces which fulfill the hypotheses of the question is empty. After all, the only natural examples that we have of compact spaces which support a non-separable probability measure, are uncountably many copies of $[0, 1]$, or uncountably many copies of the two point set $2 = \{0, 1\}$ with their usual product measures. While it is obvious that the corresponding measure algebras are non-separable, one needs to check if all open sets get measurable.

If f is a finite partial function from κ to 2, then $[f]$ denotes the set of all functions $g : \kappa \to 2$ which contain f as a subset. The product topology on 2^κ is generated by all sets of the form $[f]$ for a finite partial function f from κ to 2, as its clopen basis. If the domain of f has size n, we say $[f]$ is *of dimension* n. The product measure on 2^κ is the completition of the measure which assigns $1/2^n$ to each $[f]$ of dimension n.

Lemma 1.3.1. The product measure on 2^κ or $[0,1]^\kappa$ is Borel.

Proof. Let us just deal with 2^κ, a similar argument can be applied to $[0,1]^\kappa$.

An open set O in 2^κ is the union of countably many open sets $O_n (n \in \omega)$, each of which is the union of basic clopen sets of a fixed dimension, n. It is enough to show that each O_n is measurable. We shall do so by induction on n. Case $n = 0$ is trivial.

Suppose $O_n = \cup_{\alpha<\omega_1}[s_\alpha]$, where for each α, s_α has a domain of size $n \geq 1$. Let μ_α stand for the minimum of the domain of s_α.

In the case that μ_α are unbounded, the inner measure of O_n is 1, so it is measurable (the measure is complete!).

If μ_α are bounded, then they take on only countably many values. So we can split O_n into a countable union of sets, each of which is the union of a family of sets $[s_\alpha]$ for which μ_α is constant. Therefore it is enough for us to work with the case that all μ_α are the same ordinal, β.

For $\alpha < \omega_1$, let t_α denote the restriction of s_α on its domain without β. We may as well assume that all s_α assign the same value to β, say value 0. Then $\cup_{\alpha<\omega_1}[s_\alpha] = [(\beta, 0)] \cap \cup_{\alpha<\omega_1}[t_\alpha]$. But the last union is measurable, by the induction hypothesis.★

Note that this proof works not just for products of $[0,1]$ or 2, but for products of any 2^{nd} countable space.

How complicated must a space be in order to support a non-separable Radon measure? For one thing, it cannot be 2^{nd} countable:

If X, μ is a measure space, then the measure algebra of X, μ supports a metric, given by $d(E,F) = \mu(E\Delta F)$, for E and F in the measure algebra. A family $\mathcal{F}$ of measurable sets in X, μ is said to be *dense* in the measure algebra, if it is dense in the metric d of the measure algebra. That is, $\mathcal{F}$ has the property that for every $\epsilon > 0$ and every measurable E subset of X, there is an element D of $\mathcal{F}$ such that $\mu(E\Delta F) < \epsilon$.

So, for example, closed sets form a dense family in a Radon probability space.

In the light of this new structure on a measure space, the metric, Maharam's Theorem can be stated in a stronger version:

Maharam's Theorem Revisited. If μ is a finite measure on the space X, then the measure algebra of X, μ is either finite, or isometric to the measure algebra of 2^{κ}, for some infinite cardinal κ.★

The Theorem so stated does not need any new proof, as the measure algebra isomorphism established in the proof of the original version, happens also to be an isometry.

Note that the measure algebra of 2^κ does not have any dense subfamilies of size less than κ: if $\langle H_\alpha : \alpha < \kappa\rangle$ is a generating sequence, then for $\alpha \neq \beta$, no measurable set E can be less than 1/8 far from both H_α and H_β.

Then, the above version of Maharam's theorem implies that Maharam's dimension of a homogeneous measure algebra is the same as the density degree of that algebra as a metric space. So, for example, no measure algebra which is non-separable in the sense of Maharam's dimension, can be separable as a metric space.

We use that fact to prove the following:

Lemma 1.3.2. No 2^{nd} countable compact space X supports a non-separable Radon measure.

Proof. Fix a countable basis $\mathcal{B} = \{B_n : n \in \omega\}$ of X such that $\mathcal{B}$ is closed under finite unions. We show that $\mathcal{B}$ is dense in the measure:

Given an n in ω and a measurable E subset of X, fix an open O containing E and a closed C contained in E such that $\mu(O\Delta C) < 1/n$. For every x in C, fix a set B_x from $\mathcal{B}$ such that $x \in B_x \subset O$. Then there are finitely many $\{x_0, .., x_{n-1}\}$ in C such that C is contained in $U = \cup_{i<n} B_{x_i}$. Then $U \in \mathcal{B}$ and $\mu(E\Delta B) < 1/n$.★

Examples are known, under CH, of 1^{st} countable spaces which support non-separable compact spaces (see [13] and Chapter III). For all we know, it might be that there is a 1^{st} countable example like that which is constructible just from

ZFC. In the next Chapter we present a result which says that such an example must have a large weight.

The 1^{st} countable examples from [13] and Chapter III have a stronger property: they are hereditarily Lindelöf (HL). In other words, every open family in either of these spaces has a countable subfamily with the same union. A result of Kunen that we present in Chapter II, shows that under MA restricted to measure algebras and not CH, no HL space can support a non-separable Radon measure.

The counter-examples to Question H that we know, are all subsets of 2^{ω_1}. How large are these subsets? The following Lemma shows that any counter-example must be in some sense "small" compared to 2^{ω_1}:

If A is a subset of a product space $\Pi_{i \in I} X_i$, then for every $i \in I$ and for each $x \in X_i$, one defines the x-th *section* of $A_x = \{y \in \Pi_{i \neq j \in I} : (x, y) \in A\}$. If I only has two elements, so the product space is of the form $X \times Y$, it is usual to denote sections with respect to $x \in X$ by E_x, and the sections with respect to $y \in Y$, by E^y.

By the *i-th projection* $\pi_i(A)$ of A we mean the set of all x in X_i for which there is an $a \in A$ whose i-th coordinate is x. Equivalently, the i-th projection of A is the set of all x in X_i for which A_x is non-empty.

A major result on the relation of the measures on a product space with the measures on the factors of the product, is given by the Fubini's Theorem. The proof of this theorem can be found in any measure theory textbook, like [8].

Fubini's Theorem. If μ is a finite measure on a space X and ν is a finite measure on a space Y, then for every $\mu \times \nu$ mesurable subset E of $X \times Y$,

$$(\mu \times \nu)(E) = \int_Y \mu(E^y) d\nu = \int_X \nu(E_x) d\mu. \bigstar$$

The only instance of this theorem that we shall need is the one concerning products of the form 2^κ. We give below a brief discussion of this.

For a subset A of κ, let μ_A denote the product measure on 2^A. Note that there is a close connection between the projection on A and the measure μ_A: for a measurable subset E of 2^A, the inverse projection $(\pi_A)^{-1}(E)$ of E is a measurable subset of 2^κ, and $\mu_\kappa((\pi_A)^{-1}(E)) = \mu_A(E)$.

Now, if E is a measurable subset of 2^κ and A, B are disjoint subsets of κ such that $A \cup B = \kappa$, then Fubini's Theorem implies that $\mu_\kappa(E) = \mu_A(\pi_A(E)) \times \mu_B(\pi_B(E))$.

Lemma 1.3.3. If a closed subset X of 2^{ω_1} does not admit a continuous map onto 2^{ω_1}, then the product measure of X is 0 and X is nowhere dense in 2^{ω_1}.

Proof. Any basic clopen set in 2^{ω_1} is easily seen to contain a copy of 2^{ω_1}.

If X is a closed set of positive measure in 2^{ω_1}, then Fubini's Theorem implies that co-countably many projections of X must have measure 1. ★

This then says that one cannot construct a counter-example by simply taking a convenient positive measure set in 2^{ω_1}, getting the non-separability of the measure simply by the homogeneouity of the product measure on 2^{ω_1}.

§**1.4. What is known?** We would like here to summarize what is known about the Question H.

The Question H is open in ZFC. Kunen showed that, under CH, there is a counter-example which is, in addition, a compact L-space (hereditarily Lindelöf (HL) but not hereditarily separable (HS)); see [13,9]. Even at the risk of losing completness, we choose to refer the reader to [13] for the proof of this result, rather than attempting a presentation here which would necessarily be weaker in style than the original one is. We do give a brief discussion of the result from [13] at the end of Chapter III.

In Chapter III, we show that, assuming $\diamondsuit$, there is another counter-example which is an S-space (HS, but not HL). Then, assuming just CH, we construct a third counter-example which is both HS and HL. In Chapter IV, we show that the HS+HL counter-example constructed in Chapter III, is indestructible by Cohen reals. Therefore, we know that a counter-example is consistent with any size of the continuum. In the same Chapter we show that the S-space from Chapter III can be constructed just by ω_1 Cohen reals and CH, so the $\diamondsuit$ is not really needed.

Neither of the above mentioned examples could be constructed in ZFC, since under $MA + \neg CH$, there are neither compact L-spaces (Juhász) nor compact S-spaces (Szentmiklóssy) (see [19,25]). Furthermore, Fremlin in [4] shows that under $MA + \neg CH$, the measure algebra of a compact HL (equivalently, HS) Radon measure space is separable . A stronger assertion is true: under $MA_{(measure\ algebras)} + \neg CH$ the measure algebra of any Radon measure on a compact HL space is separable. This is an unpublished result of Kunen, and we include the proof here, in Chapter II.

We also show that, just in ZFC, there cannot be any counter-examples with Maharam's dimension greater than the continuum. This result is presented in Chapter II. This result has been independently proved by Haydon ([9]), but it also follows from a result of Shapirovskiĭ ([21,22]). In our process of rediscovering the wheel, we came up with the proof we show in the next Chapter.

It would be interesting to know if ther can be an HS counter-example in a model of $MA_{(measure\ algebras)} + \neg CH$, as this weaker version of MA does not imply that compact HS and HL are equivalent.

Another interesting question is if PFA has anything to do with Question H.

Chapter II

Positive Results

In this Chapter, we present some results which give sufficient conditions for the answer to the Question H to be positive. Or, the other way around, these results tell us how NOT to look for the counter-examples to the Question H. The first result is a result of Haydon, which also follows from results of Shapirovskiĭ, and which says that every compact space which supports a Radon measure of Maharam dimension greater than the size of the continuum, maps onto $[0,1]^{\omega_1}$. Next, we give a result of Kunen, which shows that under $MA_{(measure\ algebras)}(\omega_1)+\neg CH$, no hereditarily Lindelöf compact space supports a non-separable Radon measure. A consequence of this is an earlier result of Fremlin, that under $MA+\neg CH$, no hereditarily separable or hereditarily Lindelöf compact space, supports a non-separable Radon measure.

Finally, we give a result that shows that, under $MA_{(measure\ algebras)}(\lambda)+\neg CH$, for a 1^{st} countable compact space to support a non-separable Radon measure, it has to have a weight larger than either ω_ω or λ.

§**2.1. No Counter-examples of large Maharam Dimension.** The result that we present here tells us that if a compact space happens to support a Radon measure algebra of large Maharam dimension, then the space must be able to map onto $[0,1]^{\omega_1}$. This is a result of Haydon in [9], but we give here a proof which slightly differs from the original argument. A similar result was also shown by Shapirovskiĭ in [21,22].

A major ingredient of the proof is an application of Maharam's Theorem. We shall need some conventions regarding measure algebra isomorphisms:

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are the measure algebras of measure spaces X_1, μ_1 and X_2, μ_2 respectively. If $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a measure algebra isomorphism, and H_1 is a positive measure subset of X_1, while H_2 is a positive measure subset of X_2, then by $f(H_1) = H_2$, we mean that f maps the equivalence class of H_1 to the equivalence class of H_2. Similarly, we define what we mean by $f^{-1}(H_2) = H_1$.

Suppose that $\mathcal{K}$ is a family of pairs $(K^0_\alpha, K^1_\alpha)(\alpha < \lambda)$ of disjoint sets in X_1. A *finite Boolean combination* of the sets from $\mathcal{K}$, is any set F of the form $F = \cap_{i<n} K^{\phi(i)}_\alpha$, where $\phi \in 2^n$ and $K_0, \ldots, K_{n-1} \in \mathcal{K}$. If $\mathcal{K}$ is actualy a family of pairs of sets of positive μ_1 measure, and F is a finite Boolean combination as above, we may refer to the *corresponding Boolean combination in* $\mathcal{M}_2$, which is the set $f(F) = \cap_{i<n} f(K^{\phi(i)}_i)$. Note that every Boolean combination of the sets from $\mathcal{M}_1$ has the same μ_1-measure as the μ_1-measure of the corresponding Boolean combination of the sets from $\mathcal{M}_2$.

We also need a couple of facts about products of the form 2^I, for an index set I:

Recall from Chapter I, that, if J is a subset of I, then the projection $\pi^I_J : 2^I \to 2^J$ is the function which to every f in 2^I assigns its restriction to J. Note that this is a continuous function, and, by compactness, it is also closed. For a given A, a subset of 2^I, and an x in 2^J, A_x denotes the set of all elements of A, whose restriction on J is x.

Also recall that, for a finite function g from a cardinal λ to 2, $[g]$ denotes the set of all functions in 2^λ which extend g. If f is a measure algebra isomorphism from a measure algebra $\mathcal{M}$ onto the measure algebra of 2^λ and sets $H_\alpha(\alpha < \lambda)$ are such that $f(H_\alpha) = [(\alpha, 0)]$, then the sequence $\langle H_\alpha : \alpha < \lambda \rangle$ is said to be a generating sequence of the measure in $\mathcal{M}$.

Theorem 2.1.1. (Haydon) If a compact space X supports a Radon measure μ of Maharam's dimension greater than $\mathfrak{c}$, then X maps onto $[0,1]^{\omega_1}$.

Proof. By Lemma 1.2.1, we can assume that the measure algebra of X, μ is isomorphic to that of 2^λ, for some $\lambda > \mathfrak{c}$. In X, we shall construct a dyadical system of length $\mathfrak{c}^+$. Then, by Lemma 1.2.2, not only that there is a mapping of X onto $[0,1]^{\omega_1}$, but there is one onto $[0,1]^{\mathfrak{c}^+}$ as well.

First, fix a generating sequence of measurable sets $\langle H_\alpha : \alpha < \lambda \rangle$ in X and denote by f the measure algebra isomorphism that gives rise to this generating sequence. For each $\alpha < \lambda$, fix a closed subset K^0_α of H_α and a closed subset K^1_α of $X \setminus H_\alpha$, such that both $\mu(H_\alpha \setminus K^0_\alpha)$ and $\mu((X \setminus H_\alpha) \setminus K^1_\alpha)$, are less than 1/8. Note that for $\alpha \neq \beta < \lambda$, and for any $i, j \in 2$, the intersection $K_\alpha^i \cap K_\beta^j$ has positive measure.

The dyadical system that we need is going to be a subset of $\mathcal{K} = \{(K_\alpha^0, K_\alpha^1) : \alpha < \lambda\}$. We shall actually get more than we need: not only that all the relevant finite Boolean combinations are going to be non-empty, but they are going to have positive measure.

For $\alpha < \lambda$, denote by C_α^0 and C_α^1 the sets $f^{-1}(K_\alpha^0)$ and $f^{-1}(K_\alpha^1)$, respectively.

Note that, for any $\alpha < \lambda$, there is a countable subset S_α of λ, such that both C_α^0 and C_α^1 are in the σ-algebra generated by $\{[(\beta, 0)] : \beta \in S_\alpha\}$, modulo a set of measure 0. In other words, C_α^0 and C_α^1 are simply the inverse projections of their projections on 2^{S_α}. By Lemma 1.2.3, we can assume that all sets C_α^0, C_α^1 are F_σ. We shall only need to know that they are Borel.

As λ is greater than the cardinality of the continuum, we can conclude that there is a subset I of λ, such that I has cardinality $\mathfrak{c}^+$, and that the sets S_α for $\alpha \in I$, form a Delta-system. Denote by R the root of the Delta-system.

Let us use the letter ν to denote the product measure on 2^λ. For a subset A of λ, let ν_A denote the product measure on 2^A.

If the root R is empty, then for any finitely many distinct $\alpha_0, \ldots, \alpha_{n-1}$ in I and a ϕ in 2^n, the ν-measure of the intersection $\cap_{i<n} C_\alpha^{\phi(i)}$ is simply the product of all $\nu_{S_{\alpha_i}} C_{\alpha_i}^{\phi(i)}$, which is positive.

If R is non-empty, note that for each $\alpha \in I$, the projection of C^0_α on 2^R is a Borel subset of 2^R. Since R is countable, there are only $\mathfrak{c}$ many Borel subsets of 2^R. So we may as well assume that I is thin enough, so that all C^0_α, for $\alpha \in I$, have the same projection on 2^R. Let B_0 denote that common projection. In the same manner, we assume that for all $\alpha \in I$, the projection of C^1_α to 2^R is a fixed set B_1.

Now note that B_0 and B_1 must intersect in a set of a positive measure:

Take $\alpha \neq \beta$ in I, then:

$$0 < \nu(C^0_\alpha \cap C^1_\beta) \leq \nu((\pi_R)^{-1}(\pi_R(C^0_\alpha \cap C^1_\beta))) = \nu_R(\pi_R(C^0_\alpha \cap C^1_\beta)) \leq$$

$$\leq \nu_R(\pi_R(C^0_\alpha) \cap \pi_R(C^1_\beta)) = \nu_R(B_0 \cap B_1).$$

Then, an application of Fubini's Theorem gives that for any $\alpha_0 \neq \ldots \neq \alpha_{n-1}$, the measure of the intersection $\cap_{i<n} C^{\phi(i)}_{\alpha_i}$ is not less than $\nu_R(F_0 \cap F_1) \times \Pi_{i<n}\nu_{(S_{\alpha_i} \setminus R)}(\pi^\lambda_{(S_{\alpha_i} \setminus R)} C^{\phi(i)}_{\alpha_i})$, which is positive.★

This argument can be generalized to a result about mappings onto cubes of higher dimensions, see §2.4 for a remark on this.

§**2.2. No HL Counter-examples under** $MA+\neg CH$. In this section we give an unpublished result of Kunen, which shows that, if $MA_{(measure\ algebras)}(\omega_1)+\neg CH$ is true, no hereditarily Lindelöf compact space can support a non-separable Radon measure. As a corollary, we get an earlier result of Fremlin: under

$MA + \neg CH$, no compact space which is either hereditarily Lindelöf, or hereditarily separable, can support a non-separable Radon measure.

An easy way to know that a given compact space does not map onto $[0,1]^{\omega_1}$, is to see that the space is either hereditarily separable, or hereditarily Lindelöf:

A space is said to be *separable* if it has a countable dense subset. A space is said to be *hereditarily separable (HS)* if every subspace of the space is separable.

A space is *Lindelöf* if every of its open covers has a countable subcover. A space is *hereditarily Lindelöf (HL)* if all of its subspaces are Lindelöf. Equivalently, all open families of subsets of X, have a countable subfamily with the same union.

Both HS and HL are preserved under continuous mappings:

Lemma 2.2.1. Suppose X maps onto Y. If X is hereditarily separable (hereditarily Lindelöf), then so is Y.

Proof. Suppose that f maps X onto Y.

Let C denote an arbitrary subset of Y. If D is a dense subset of $f^{-1}(C)$, then $f(D)$ is dense in C.

Let $\{O_i : i \in I\}$ an open family in Y. For any subset J of I, if $\cup_{i \in J} f^{-1}(O_i) = \cup_{i \in I}(O_i)$, then $\cup_{i \in J} O_i = \cup_{i \in I} O_i$.★

The cube $[0,1]^{\omega_1}$ is neither HS nor HL:

Lemma 2.2.2. $[0,1]^{\omega_1}$ is neither HS nor HL.

Proof. The set of all functions f in $[0,1]^{\omega_1}$ for which there is exactly one coordinate $\alpha < \omega_1$ such that $f(\alpha) > 1/2$, has no countable dense subset.

Similarly, the open family consisting of all sets $O_\alpha = \{f \in [0,1]^{\omega_1} : f(\alpha) < 1/2\}$, for $\alpha < \omega_1$, has no countable subfamily with the same union.★

A note to clarify these notions is that, being compact, $[0,1]^{\omega_1}$ is certainly Lindelöf. It is actually also separable. It is the "hereditary" part that fails.

Anyway, we conclude:

Lemma 2.2.3. No space which is either hereditarily Lindelöf or hereditarily separable, can map onto $[0,1]^{\omega_1}$.★

Actually, all counter-examples that we know to the Question H, that is, compact spaces which support a non-separable Radon measure, but still don't map onto $[0,1]^{\omega_1}$, are either HS or HL (or both). The result that we are about to present implies that none of these examples can survive $MA + \neg CH$.

Let us, then, pass on to the presentation of the result:

Theorem 2.2.4. **(Kunen)** Assuming the $MA(\omega_1)$ for measure algebras and $\neg CH$, the measure algebra of any Radon probability measure on a compact hereditarily Lindelöf space is separable.

The theorem is proved by refuting the HL in an arbitrary compact space which supports a non-separable Radon probability measure. It is enough to construct an uncountable strictly decreasing sequence of closed sets. This is an easy observation of Sierpinski in [24]:

Lemma 2.2.5. (Sierpinski) A compact space X is HL iff there is no uncountable strictly increasing sequence of open sets in X. Equivalently, there is no uncountable strictly decreasing sequence of closed sets in X.

Proof. In the direction from left to right, if U_α for $\alpha < \omega_1$ were strictly increasing and open, then $U = \cup_{\alpha<\omega_1} U_\alpha$ would have an open cover $\{U_\alpha : \alpha < \omega_1\}$ with no countable subcover.

In the other direction, if Z is a subset which is not Lindelöf, fix an open cover $\{V_\alpha : \alpha < \kappa\}$ of Z which does not have a countable subcover. Then κ is uncountable. For $\alpha < \omega_1$, fix an open set O_α in X with $O_\alpha \cap Z = V_\alpha$. Set $U_\alpha = \cup\{V_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$. Then $\langle U_\alpha : \alpha < \omega_1 \rangle$ is a strictly increasing sequence of open sets.

The last assertion of the Lemma is clear, by taking complements. ★

For the proof of Theorem 2.2.4, one actually only needs the left to right implication of this Lemma.

Let X be any topological space. If $\mathcal{H}$ is a family of subsets of X, then $\Sigma(H)$ denotes the σ-algebra generated by $\mathcal{H}$. A sequence $\langle K_\alpha : \alpha < \omega_1 \rangle$ is said to be *pseudo-independent (PI)* iff for all $\alpha < \omega_1$ and all non-empty sets B in the $\Sigma(\{K_\beta : \beta < \alpha\})$, B is not a proper subset of K_α.

The main ingredient of the proof of the Theorem is the fact that in a compact space which supports a non-separable Radon probability measure, there is a PI sequence in which all elements are closed G_δ positive measure sets. To prove it one only uses the axioms of ZFC. The $MA_{(measure\ algebras)}(\omega_1) + \neg CH$ is used to construct the desired decreasing sequence of closed sets, using the PI sequence just mentioned.

We proceed by presenting several lemmas that are going to be used to find the PI sequence as above. From now on, let X stand for a compact space with a Radon probability measure μ defined on X.

The first of the lemmas to follow is a well known fact about Z-sets.

A subset K of X is said to be a *Z-set*, if there is a map f from X into $[0,1]$ with $K = f^{-1}(\{0\})$.

Lemma 2.2.6. A subset of a compact space is a Z-set iff it is a closed G_δ set.

Proof. If f is a map from a compact space X into $[0,1]$ and $K = f^{-1}(\{0\})$, then K is closed by continuity. But also, $K = \cap_{n\in\omega\setminus\{0\}} f^{-1}([0,1/n])$, so K is a G_δ set.

In the other direction, suppose $K = \cap_{n\in\omega} U_n$, where each U_n is open and K closed. By Urysohn Lemma, choose for each n a map f_n from X into $[0, 2^{-n-1}]$ such that $f_n(K) = \{0\}$ and for all x in $U_n \setminus K$, $f_n(x) = 2^{-n-1}$. Let f be the sum of all f_n, that is the function that assigns to every x in X the sum of the convergent series $\Sigma_{n\in\omega} f_n(x)$. Then $f(x) = 0$ iff $x \in K$.★

Note that Urysohn's Lemma implies that for any closed subset C of a compact space X, there is mapping f of X into $[0, 1]$ such that $f(A) = 0$. But this is not quite saying that every such A is a Z-set.

The PI sequence desired will be constructed by induction on countable ordinals. Lemma 2.2.6 has a corollary which will be used to do the first ω steps of the induction:

Corollary. If $\langle K_n : n \in \omega\rangle$ are closed G_δ sets in X, then there is a map f from X into $[0, 1]^\omega$ such that for every set B in the σ-algebra $\mathcal{H} = \Sigma(\{K_n : n \in \omega\})$ generated by the sets $K_n (n \in \omega)$, there is a subset A of $[0, 1]^\omega$ for which $B = f^{-1}(A)$.

Proof. For each n choose a map $f_n : X \to [0, 1]$ such that $K_n = f_n^{-1}(\{0\})$. Let f be the "product" of all f_n, that is the function from X to $[0, 1]^\omega$ which assigns $\langle f_n(x) : n \in \omega\rangle$ to every $x \in X$. This f is easily seen to be continuous.

The collection $\{f^{-1}(A) : A \subseteq [0, 1]^\omega\}$ is a σ-algebra and contains all $K_n (n \in \omega)$. So it contains $\mathcal{H}$.★

The infinite stages of the induction require a more subtle argument:

Lemma 2.2.7. If X, μ is a Radon measure space, then the closed G_δ sets form a dense family in the measure algebra.

Proof. Given a measurable E subset of X and an $\epsilon > 0$. Find a closed $F \subset E$ such that $\mu(E \setminus F) < \epsilon$. Now, by induction on $n \in \omega$, find a sequence $\langle O_n : n \in \omega \rangle$ of open supersets of F, so that for each n, O_n contains the closure of O_{n+1} and the measure $\mu(O_n \setminus F) < 1/(n+1)$. The set $K = \cap_{n \in \omega}$ is a closed G_δ which is less than ϵ far from E.★

Lemma 2.2.8. If f is a map from X to $[0,1]^\omega$, then there are disjoint closed G_δ,of positive measure, subsets H and K of X, such that $f(H) = f(K)$.

Proof. For Borel subsets A of $[0,1]^\omega$, define $\lambda(A) = \mu(f^{-1}(A))$. It is easily seen that λ extends to a Radon probability measure on $[0,1]^\omega$. Note that $[0,1]^\omega$ is 2^{nd} countable, so that Lemma 1.3.2 implies that the measure algebra of $[0,1]^\omega, \lambda$ is separable.

Fix a sequence $\langle E_n : n \in \omega \rangle$ which is dense in the measure space of λ. Since μ is non-separable, $\langle f^{-1}(E_n) : n \in \omega \rangle$ is not dense in the measure space of μ. Fix a Borel set R in X and a positive c such that for all $n \in \omega$, the μ-distance between R and $f^{-1}(E_n)$ is at least $2c$. Therefore, for all Borel subsets A of $[0,1]^\omega$, the distance between $f^{-1}(A)$ and R is not less than $2c$.

By Lemma 2.2.7, there is a closed G_δ set M in X, whose distance from R is less than c. For that M, M is a proper subset of $f^{-1}(f(M))$. Actually, their difference must have measure at least c. Define H to be the difference, $f^{-1}(f(M)) \setminus M$.

By compactness of M, $f(M)$ is closed in $[0,1]^\omega$. Since $[0,1]^\omega$ is 2^{nd} countable, $f(M)$ is a G_δ in $[0,1]^\omega$. Therefore H is a closed G_δ set in X.

To find K, consider $B = f(H)$. Set $K = f^{-1}(B) \cap M$, therefore K is disjoint from H. Note that $f(H) = f(K) = B$. If $A = f(M) \setminus B$, then $K = M \setminus f^{-1}(A)$, so the measure of K is at least c.

The only problem now is that K might not be G_δ. To fix that, we change both H and K a little bit, using a so called cushion argument. By Lemma 2.2.6, we can find a function ϕ from X to the unit interval, such that M is $\phi^{-1}(\{0\})$. Note then that H is $f^{-1}(f(M)) \cap \phi^{-1}((0,1])$. If we define H_η as $f^{-1}(f(M)) \cap \phi^{-1}([\eta, 1])$, for positive η, then H is an increasing union of closed G_δ sets H_η. Therefore, there is an ϵ for which H_ϵ has positive measure.

Now, $K_\eta = f^{-1}(f(H_\eta)) \cap M$ is a closed G_δ set and $f(K_\eta) = f(H_\eta)$. Since K is an increasing union of K_η for $\eta > 0$, we can assume that the ϵ we chose is such that both K_ϵ and H_ϵ are of positive measure.

Now reset K to K_ϵ and H to H_ϵ. ★

We are now ready for the main Lemma before the proof of the Theorem, the choice of the PI sequence of closed G_δ positive measure sets:

Lemma 2.2.9. Assume that the measure algebra of X, μ is non-separable. Then there is a pseudo-independent sequence of closed G_δ sets of positive measure in X.

Proof. We construct a PI sequence $\langle K_\alpha : \alpha < \omega_1 \rangle$ of closed G_δ positive measure sets, by induction on $\alpha < \omega_1$.

For finite n, just choose any continuous function f_n from X to [0,1] and set K_n to be $f_n^{-1}(\{0\})$.

Given $\langle K_\delta : \delta < \alpha \rangle$, set $\mathcal{I} = \Sigma(\{K_\delta : \delta < \alpha\})$. By the Corrolary to Lemma 2.2.6, fix a map $f : X \to [0,1]^\omega$ such that for all B in $\mathcal{I}$, there is a subset A of $[0,1]^\omega$ for which $B = f^{-1}(A)$. Apply Lemma 2.2.8 to this f to find two disjoint closed G_δ sets K and H, of positive measure and such that $f(K) = f(H)$. Then any non-empty $B \in \mathcal{I}$ meets H iff it meets K, so we can set $K_\alpha = K$. ★

The only other ingredient that we need is a well known consequence of $MA + \neg CH$: that all *ccc* partial orders have precaliber ω_1.

A partial order $\mathbb{P}$ is said to be *ccc* if all antichains in $\mathbb{P}$ are countable. A subset of $\mathbb{P}$ is said to have *the finite intersection property* if its every finite subset has a common extension in $\mathbb{P}$. $\mathbb{P}$ is said to be of *precaliber* ω_1 if all ω_1 sequences in $\mathbb{P}$ have an ω_1 subsequence which has the finite intersection property.

Note that there is a partial order on every Boolean algebra: an element a of the Boolean algebra is less than another element b, if $ab = a$ in the Boolean algebra. With a Boolean algebra $\mathcal{B}$, we associate a partially ordered set, namely $\mathcal{B} \setminus 0_\mathcal{B}$. Two elements are incompatible in this order, if their product is $0_\mathcal{B}$. Note that this partial order is ccc, if and only if $\mathcal{B}$ is ccc as a Boolean algebra. Whenever we talk of a Boolean algebra as a partial order, it is the above partial order that we have in mind.

Measure algebras are special kinds of Boolean algebras, so we can think of them as of partial orders, too. Recall from Chapter I that we are only concerned with finite measures. In that way, all measure algebras are ccc.

For a cardinal κ, $MA(\kappa)$ means that for any ccc partial order $\mathbb{P}$ and any family $\mathcal{D}$ of $\leq \kappa$ dense sets in $\mathbb{P}$, there is a filter in $\mathbb{P}$, which intersects each element of $\mathcal{D}$ in a non-empty set.

If $\mathcal{K}$ is a class of ccc partial orders, then by $MA_{\mathcal{K}}(\kappa)$, we mean that $MA(\kappa)$ is true restricted to the partial orders from $\mathcal{K}$. If $\mathcal{K}$ is a singleton, $\mathbb{P}$, then we write $MA_{\mathbb{P}}$ rather than $MA_{\{\mathbb{P}\}}$.

MA means that $MA(\kappa)$ is true for all $\kappa < \mathfrak{c}$. We similarly define $MA_{\mathcal{K}}$.

The above mentioned consequence of $MA + \neg CH$ only needs $MA(\omega_1) + \neg CH$ and remains true relativized to measure algebras: assuming just the restriction of $MA(\omega_1)$ to measure algebras and $\neg CH$, we can conclude that all measure algebras have precaliber ω_1. We proceed by giving the proof of this fact.

Lemma 2.2.10. Assume $MA_{(measure\ algebras)}(\omega_1) + \neg CH$. Then every measure algebra has precaliber ω_1.

Proof. Let us consider a sequence $\langle p_\alpha : \alpha < \omega_1 \rangle$ in an arbitrary measure algebra $\mathcal{M}$. For $\alpha < \omega_1$, let C_α denote the set of all elements p of $\mathcal{M}$, which are compatible with p_γ for some $\gamma \geq \alpha$. Then, obviously, if α is less than β, then F_α contains F_β. Actually, we claim that there must be an α such that for all β greater than α, $F_\beta = F_\alpha$. (*)

If it was not so, then for every $\alpha < \omega_1$, we could find a q_α in $F_\alpha \setminus F_{\alpha+1}$. Such a q_α would have to be compatible with p_α, but not with any p_β for $\beta > \alpha$. In particular, for all $\alpha < \beta < \omega_1$, q_α and q_β would be incompatible. But this cannot happen, as $\mathcal{M}$ is ccc.

So, fix an α which satisfies $(*)$. Now consider another partial order, $\mathbb{P}$, which consists of all elements of $\mathcal{M}$ which extend p_α and in which the order is inherited from $\mathcal{M}$. In other words, $\mathbb{P}$ is $\mathcal{M}$ restricted to p_α. In particular, $\mathbb{P}$ is also a measure algebra.

For countable $\beta \geq \alpha$, consider the set

$$D_\beta = \{p \in \mathbb{P} : \exists \gamma \geq \beta (p \leq p_\gamma)\}.$$

Each of these sets is dense in $\mathbb{P}$: for a p in $\mathbb{P}$, p is $\leq$ than p_α, so p is in F_α. By $(*)$, p is in F_β too. Any q which is a common extension of p and p_γ for a $\gamma \geq \beta$, is also an element of D_β.

Now use $MA_{(measure\ algebras)}(\omega_1)$ to find a filter G in $\mathbb{P}$ which intersects all D_β, for $\beta \geq \alpha$.

Then the set

$$A = \{\gamma < \omega_1 : \exists p \in G (p \leq p_\gamma)\}$$

is unbounded, and the set $\{p_\gamma : \gamma \in A\}$ has the finite intersection property, since G does.★

Proof of Theorem 2.2.4. $(MA_{(measure\ algebras)}(\omega_1)+\neg CH)$. Let X be a compact space and μ a Radon probability measure on X with the measure algebra of X, μ non-separable.

By Lemma 2.2.9, fix an ω_1-sequence $\langle K_\delta : \delta < \omega_1 \rangle$ of pseudo-independent closed G_δ sets of positive measure. By Lemma 2.2.10, there is an uncountable subsequence $\langle K_{\alpha_\xi} : \xi < \omega_1 \rangle$ of $\langle K_\alpha : \alpha < \omega_1 \rangle$ which has the finite intersection property.

Set $F_\eta = \cap_{\eta<\xi} K_{\alpha_\eta}$, for $\eta < \omega_1$. Then each F_η is a nonempty closed set. Since $F_{\eta+1} = F_\eta \cap K_{\alpha_\eta}$, the pseudo-independence of $\langle K_\alpha : \alpha < \omega_1 \rangle$ implies that the sequence $\langle F_\eta : \eta < \omega_1 \rangle$ is strictly decreasing. By Lemma 2.2.5, X is not hereditarily Lindelöf. ★

A consequence of this result, is an earlier result of Fremlin:

Corollary. (Fremlin) If $MA + \neg CH$ is true, and if X is a compact space which supports a non-separable Radon measure, then X is neither hereditarily Lindelöf, nor hereditarily separable.

Proof. That X is not hereditarily Lindelöf, we conclude directly from Theorem 2.2.4. On the other hand, under $MA + \neg CH$, every compact hereditarily separable space is hereditarily Lindelöf ([19,25]), so X cannot be hereditarily separable either.★

We note that, independently of the size of the continuum, Theorem 2.2.4 is indeed stronger than its above Corollary. That is so because full MA is stronger than its restriction to the measure algebras. One of the consequences of full $MA + \neg CH$ that does not follow from $MA_{(measure\ algebras)} + \neg CH$, is the above mentioned equivalence of HL and HS for compact spaces. So, it is still conceivable that there is an HS counter-example to Question H, consistent with $MA_{(measure\ algebras)} + \neg CH$.

§**2.3. 1^{st} countable Counter-examples.** By Theorem 2.2.4, we know that if $MA_{(measure\ algebras)}(\omega_1) + \neg CH$ is true, there cannot be any HL counter-examples to Question H. But a weaker condition than HL implies that a compact space does not map onto $[0,1]^{\omega_1}$:

A family $\mathcal{F}$ of open sets in a space X is a *point-base* for a point x in X, if every element of $\mathcal{F}$ contains x and every open set containing x, is a superset to an element of $\mathcal{F}$.

A space is *1^{st} countable*, if every point in the space has a countable point-base.

It is easy to see that every regular HL space is also 1^{st} countable. In fact, for compact spaces X, X is HL iff all closed sets in X are G_δ. To know that a compact space does not map onto $[0,1]^{\omega_1}$, it is enough to check that it is 1^{st} countable. However, the argument is not as easy as in the case of HL or HS. While it is obviously true that $[0,1]^{\omega_1}$ is not 1^{st} countable, it is also true that

1^{st} countability is not necessarily preserved by continuous mappings. The reason that a 1^{st} countable space cannot map onto $[0,1]^{\omega_1}$ is a property implied by the 1^{st} countability, countable tightness. This property is preserved under continuous mappings and is not possessed by $[0,1]^{\omega_1}$:

A point x in a space X has *countable tightness* if any time that x is in the closure of a subset S of X, there is a countable subset T of S, such that x is also in the closure of T.

A space has *countable tightness* if all the points in the space do.

Lemma 2.3.1. Every 1^{st} countable space has countable tightness.

Proof. Let X be a 1^{st} countable space and $x \in X$. Fix a countable point-base $\mathcal{B}$ at x. If x is in the closure of a set S, choose a countable $T \subset S$ which intersects every element of $\mathcal{B}$, so x is in the closure of T.★

Lemma 2.3.2. Countable tightness is preserved by continuous mappings.

Proof. Suppose f maps a countably tight space X onto a Y, and $y \in Y$ is in the closure of a subset S of Y. Choose an x such that $f(x) = y$. Then x is in the closure of the preimage of S, $f^{-1}(S)$. Let T be a countable subset of $f^{-1}(S)$ whose closure contains x. Then $f(T)$ is a countable subset of S, whose closure contains y.★

Lemma 2.3.3. $[0,1]^{\omega_1}$ is not countably tight.

Proof. The point which has all coordinates 0, is in the closure of the set S of all points which have only finitely many coordinates equal to 0, but not in the closure of any countable subset T of S.★

Corollary. No 1^{st} countable space maps onto $[0,1]^{\omega_1}$.★

So, the next question to answer is, if there may be a 1^{st} countable counterexample to Question H, which survives $MA+\neg CH$. Subject to a condition on the weight, a negative answer was given by Fremlin ([4]). We now present that result. The present approach, due to to Kunen, differs from the original one and makes it clear that $MA_{(measure\ algebras)}+\neg CH$ is all that is needed.

To describe the condition on the weight, we use the following notation, due to Shelah.

If $\kappa, \lambda, \mu, \nu$ and o are cardinals, then

$$\text{cov}(\kappa, \lambda, \mu, \nu) \leq o$$

means that there is a family $\mathcal{F}$ of $\leq o$ subsets of κ, each of cardinality $< \lambda$, such that every subset of κ of cardinality $< \mu$ is covered by $< \nu$ elements of $\mathcal{F}$.

The particular instance of this notation that we shall use, is $\text{cov}(\kappa, \omega_1, \omega_1, 2) \leq \lambda$. This means that there is a family $\mathcal{F}$ of countable subsets of κ, whose size is $\leq \lambda$ and such that every countable subset of κ is covered by an element of $\mathcal{F}$.

Note that $\text{cov}(\kappa, \omega_1, \omega_1, 2) \leq \lambda$ implies $\text{cov}(\mu, \omega_1, \omega_1, 2) \leq \lambda$, for every $\mu < \kappa$.

We shall need the following observation:

Lemma 2.3.4. For a partial order $\mathbb{P}$ and a cardinal λ: if $MA_{\mathbb{P}}(\lambda)$ is true, and $\mathbb{Q}$ is a suborder of $\mathbb{P}$ which is dense in $\mathbb{P}$, then $MA_{\mathbb{Q}}(\lambda)$ is also true.

Proof. Every dense subset of $\mathbb{Q}$ is also dense in $\mathbb{P}$, and the intersection of a filter in $\mathbb{P}$ with $\mathbb{Q}$, is a filter in $\mathbb{Q}$.★

We can now prove the announced result:

Theorem 2.3.6. Assume $MA_{(measure\ algebras)}(\lambda) + \neg CH$ and, also, that $\mathrm{cov}(\kappa, \omega_1, \omega_1, 2) \leq \lambda$. Then no 1^{st} countable compact space whose weight is $\leq \kappa$ can support a non-separable Radon measure.

Proof. Let X, μ be a compact Radon measure space of weight $\leq \kappa$. Let us fix a basis $\mathcal{B}$ of X which has size κ. Then, for some $o \leq \lambda$, there is a family $\{B_\alpha : \alpha < o\}$ in $[\mathcal{B}]^\omega$, such that every element of $[\mathcal{B}]^\omega$ is covered by a B_α.

Suppose that the measure algebra of X, μ is non-separable. Note that being a 1^{st} countable space of weight less than κ is hereditary to any kind of subsets, so by Lemma 1.2.1, we can assume that the measure algebra of X, μ is homogeneous.

We force with $\mathbb{P} = \{p \subset X : p \text{ closed } \wedge \mu(p) > 0\}$. Since the measure is Radon, P is dense in the measure algebra of X, μ. By Lemma 2.3.4 and $MA_{(measure\ algebras)}(\lambda)$, we conclude that $MA_{\mathbb{P}}(\lambda)$ is true.

If G is a $\mathbb{P}$-generic filter, then $\cap G$ is non-empty. This is so because X is compact and every filter has the finite intersection property. On the other hand,

$\cap G$ cannot have more than one element. The reason is that, for any $x \neq y$ in X, the set of all elements of $\mathbb{P}$ which either contain exactly one of x, y, or do not contain any of them, is dense in $\mathbb{P}$. So, there is a point x in X, such that $\{x\} = \cap G$. We claim that x cannot have countable character.

To see this, we show that for each $\alpha < o$, the set

$$D_\alpha = \{p \in \mathbb{P} : (p \cap (\cap B_\alpha) = \emptyset)) \vee (\exists a \neq b(a, b \in (\cap B_\alpha) \cap p))\}$$

is dense in $\mathbb{P}$. Once we know that, since $MA_{\mathbb{P}}(o)$, we know that G could have been chosen to intersect each D_α. Then, if α is such that $x \in \cap B_\alpha$, the p from $D_\alpha \cap G$ witnesses that there is another point $y \neq x$ in the intersection $\cap B_\alpha$. This could not happen if B_α contained a point-base for x. But a B_α covers every element of $[\mathcal{B}]^\omega$, so x has no countable point-base.

Let us then show that each D_α is dense in $\mathbb{P}$. This is where we are going to use the assumption that the measure μ is non-separable.

Take a p in $\mathbb{P}$, and assume that $p \cap (\cap B_\alpha)$ is not empty, but that it contains exactly one point. Let $\{E_n : n \in \omega\}$ enlist all the sets $B \cap p$ for a B in B_α. Since the measure algebra of X, μ is non-separable, there is a subset q of p which has positive measure and which is at a distance $> \epsilon$ from each E_n, for some positive ϵ.

If there is an n such that $q \setminus E_n$ is of positive measure, then we can find an r which is a closed subset of q and which is disjoint from E_n. Such an r is an extension of p which is in D_α.

Otherwise, each $E_n \cap q$ has measure greater than ϵ, so the measure of $(\cap B_\alpha) \cap q$ is positive. By the homogeneouity of the measure algebra, this intersection must contain more than one point. But it contradicts our assumption on p.★

Note that the same proof works to show that there must be in X a point of uncountable π-character. This is not of much use. Namely, in order to to use Shapirovskiĭ's result and conclude that X maps onto $[0,1]^{\omega_1}$, we would need to know that there is in X an entire closed subspace of points each of which has uncountable RELATIVE π-character.

A question to ask, of course, is if there may be any cardinals κ for which it is true that $\text{cov}(\kappa, \omega_1, \omega_1, 2) < \mathfrak{c}$, otherwise we have been wasting time in proving the previous Theorem. Here is a Lemma which shows that all cardinals less than ω_ω have the right covering property, if they are less than $\mathfrak{c}$:

Lemma 2.3.7. For every cardinal κ less than ω_ω, $\text{cov}(\kappa, \omega_1, \omega_1, 2) \leq \kappa$.

Proof. We prove this for each ω_n, by induction on $n < \omega$. We construct a C_n in $[\omega_n]^\omega$, of size ω_n, so that each element of $[\omega_n]^\omega$ is covered by an element of C_n.

Trivially, the statement is true for $n = 0$, by taking $C_0 = \{\omega\}$. Given C_n. For each $\xi < \omega_{n+1}$, choose a 1-1 function f_ξ which maps ξ into ω_n. Set $C_{n+1} = \{f_\xi^{-1}(C) : \xi < \omega_{n+1} \wedge C \in C_n\}$.★

§**2.5. Remarks.** Even though the question of Haydon is only concerned with mappings onto $[0,1]^{\omega_1}$, the material in this section calls for a few words on mappings on cubes of larger dimension. So, for a moment, let us consider the following situation: X, μ is a compact Radon probability measure space of Maharam dimension $\lambda > \omega$, and we want to know for which $\kappa > \omega$, X maps onto $[0,1]^\kappa$.

Trivially, we are only interested in those κ which are less or equal than λ. By the same Delta-system argument of the proof of Theorem 2.0.0, we can see that if κ is regular, $\leq \lambda$ and such that for all $o < \kappa$, the cardinality of $[o]^\omega$ is less than κ, then X maps onto $[0,1]^\kappa$. So, for many λ, X will map on $[0,1]^\lambda$ itself!

We also remark that the arguments of sections 2.3 and 2.4 can be generalized, using cardinal functions which extend the notions of 1^{st} countability and hereditary Lindelöfness.

Finally, we would like to mention another class of spaces which at first look like they would have something to do with the Question H. That is the linearly ordered spaces, LOTS. There is a result of Mardešić and Papić ([17]) which implies that every compact LOTS which supports a totally finite measure, is hereditarily Lindelöf. (In fact, this was generalized in [15] to any LOTS.) Therefore, a natural way to build a non-MA counter-example to the Question H, would be to construct a non-separable Radon measure on a compact LOTS. However, the following Lemma shows that that is not possible:

Lemma 2.5.1. The measure algebra of any finite Radon measure on a compact LOTS is separable.

Proof. Given a compact LOTS, X and a finite Radon measure μ on it. We can assume that μ is a probability.

It is easy to see that X has to have a minimal and a maximal element, by compactness. So, $X = [a, b]$ for some $a, b \in X$.

For an $x \in X$, define $f(x) = \mu[a, x]$. We shall show that f induces a measure-preserving transformation into the unit interval with a Radon measure. Then, the measure algebra of X, μ must be separable.

So, define a measure in λ in [0,1] in the following way:

Given an open set (a, b) in [0,1]. If $(a, b) \cap f(X)$ is empty, let $\lambda((a, b)) = 0$. Otherwise, let $c = \inf((a, b) \cap f(X))$ and $d = \sup((a, b) \cap f(X))$. Find a decreasing sequence $\langle c_n \rangle_{n \in \omega}$ of numbers in $(a, b) \cup f(X)$ converging to c, and an increasing sequence $\langle d_n \rangle_{n \in \omega}$ in $(a, b) \cap f(X)$ converging to d. For each n, choose x_n, y_n in X such that $f(x_n) = c_n$ and $f(y_n) = d_n$. Define $\lambda((a, b)) = \mu(\cup_{n \in \omega}(x_n, y_n))$. Extend the so defined λ to a Radon measure in [0,1], in the usual way.

To show that f is measure preserving, we need to see that every measurable set E in X is the inverse image of a (measurable) set in [0,1], modulo a set of μ-measure 0. Since the measure is Radon, it is enough to show this for closed sets in X. In fact, it is just enough to show this for sets E of the form $E = [a, x]$.

Let $s = f(x)$ and $B = f^{-1}([0, s])$. Then note that B is a superset of $[a, x]$ and that B must be an interval. So, $B = [a, y]$ or $B = [a, y)$ for some $y \geq x$. But then $\mu((x, y)) = 0$, so $\mu(B \setminus [a, x]) = 0$.★

Chapter III

Counter-examples

In this chapter, we construct two examples of a compact, 0-dimensional space which supports a Radon probability measure whose measure algebra is isomorphic to the measure algebra of 2^{ω_1}. The first construction uses $\diamondsuit$ to produce an S-space with no convergent sequences in which every perfect set is a G_δ. We show that space with these properties must be both hereditarily normal and hereditarily countably paracompact. The second space is constructed under CH and is both HS and HL. Also see [1]. The last section of the chapter is a short overview of another CH counter-example, which is the L-space of Kunen.

§**3.0. Introduction.** The constructions that we would like to present are quite long, and we break them into several sections. They can be both described using the same notation and, the HS+HL space is, actually, a modification of the S-space. So, we take this section to give an introduction to both of these results.

The first known counter-example to Question H, is an L-space constructed by Kunen in [13]. Also see [9]. These constructions assume CH. See §3.5 for a discussion. In the previous chapter, we gave a result showing that such a counter-example could not exist under $MA+\neg CH$, or even under $MA_{(measure\ algebras)}+$

$\neg CH$. We also know that the full $MA + \neg CH$ would destroy an HS counter-example, too. But, we do not know if just the $MA_{(measure\ algebras)} + \neg CH$ would do so.

We first describe how to get the S-space counter-example.

We also discuss some additional properties of the space, for example, the property that every perfect set is a G_δ, whereas no point is a G_δ.

Then later, assuming just CH, we modify the construction to get another counter-example which is both HS and HL.

As we explained before, neither of the above mentioned examples could be constructed in ZFC.

The following theorem details the properties of the S-space.

Theorem 3.1.1. If $\diamondsuit$ holds, then there is a compact, 0-dimensional, hereditarily separable space X and a Radon probability measure μ on X such that

1. X has weight ω_1, and every point in X has character ω_1.
2. There are no convergent sequences in X.
3. Every perfect subset of X is a G_δ set.
4. The measure algebra of X, μ is isomorphic to the measure algebra of 2^{ω_1} with the usual product measure.

From now till the end of the construction, X stands for the S-space we are constructing.

We do not need to state in the Theorem that X cannot be mapped onto $[0,1]^{\omega_1}$, since that follows from hereditary separability. See Lemma 2.2.1. Likewise, part (1) of the Theorem implies that X is not hereditarily Lindelöf, since in a hereditarily Lindelöf space, every point is a G_δ. Also, (1) implies that the cardinality of X is 2^{ω_1}.

Our proof is patterned after the result, announced by Федорчук in [3], that under $\diamondsuit$, there is a compact S-space of size 2^{ω_1}, although we do not follow exactly the method indicated in [3]. By induction on $\alpha \leq \omega_1$, we shall construct a closed $X_\alpha \subseteq 2^\alpha$. If $\alpha < \beta$, then X_α will be the projection of X_β. The X of the Theorem will be X_{ω_1}. To make sure that the space is hereditarily separable, we shall use $\diamondsuit$ to capture all candidates for an ω_1 left separated sequence. This method will also capture all ω-sequences as well, so that Part (2) of the Theorem 3.1.1 will essentially come for free. To guarantee Part (3), we use $\diamondsuit$ a second time to control the perfect sets.

Since X will be compact and 0-dimensional, it will be possible to define the measure by its values on the clopen sets, as we now explain.

The *Baire sets* in a topological space are the least σ-algebra containing the clopen sets, and the *Borel sets* are the least σ-algebra containing the open sets.

Suppose that μ is a finitely additive probability measure defined on the clopen subsets of X. Then, in the standard way (see Chapter I), μ defines an outer measure, from which we define an extension of μ to a countably additive probability measure $\hat{\mu}$ on some σ-algebra, $\mathcal{S}$. $\mathcal{S}$ certainly contains all Baire sets. It need not contain all Borel sets, but if it does, then $\hat{\mu}$ is automatically a Radon measure.

So, why should all Borel sets be $\hat{\mu}$-measurable? In an HL space (as in §3.5 or [11]), this is trivial, since the Borel and Baire sets coincide. In fact, for any compact X, X is HL iff all closed sets are G_δ sets iff all Borel sets are Baire. Also, for any compact X, a closed set is Baire iff it is a G_δ. In our particular space, (1) says that each point is a non-G_δ and hence non-Baire. However, (4) implies that the measure algebra is non-atomic, so that points have measure 0; in particular, they are $\hat{\mu}$-measurable. Now, suppose K is any closed set, and let I be the set of isolated points of K. By HS, I is countable, and hence $\hat{\mu}$-measurable (and of measure 0). By (2), either I is finite and $K = I$ or I is infinite and $K \backslash I$ is perfect, and hence a G_δ by (3). In either case, K is the union of a measurable set and a Baire set, and hence measurable. A subtle point: of course, we must verify (4), or at least that $\hat{\mu}$ gives points measure 0, *without* needing that every Borel set is $\hat{\mu}$-measurable; but we do that in §3.2.

Since the measure is determined by its values on the clopen sets, it is easily constructed by induction. When we construct X_α, we also decide what the measure is of all its clopen subsets. Then when we get to $X = X_{\omega_1}$, we will also have μ and hence $\hat{\mu}$, and as we pointed out above, $\hat{\mu}$ will almost automatically be a Radon measure. We still have to ensure that the resulting measure algebra is isomorphic to the measure algebra of 2^{ω_1}. To do that, we shall simply make it everywhere non-separable; the isomorphism will then follow by Maharam's Theorem. However, doing this requires an additional complication on the construction of the X_α, which makes verifying that we can get an S-space somewhat harder. We comment on this further at the end of §3.4.

We proceed by listing a number of requirements on the X_α and μ_α which, if met, guarantee that the resulting (X, μ) satisfies Theorem 3.1.1. In §3.2, we describe the basic requirements, which guarantee that X will have every point of character ω_1 and a Baire measure μ such that the resulting measure algebra is isomorphic to the measure algebra of 2^{ω_1}. In §3.3, we describe some additional requirements on the construction which will guarantee the rest of the properties of X. In §3.4, we verify that all the requirements of §§3.2,3 can be simultaneously fulfilled, thus proving Theorem 3.1.1; we also explain there why a space with the properties of Theorem 3.1.1 must be hereditarily normal and hereditarily countably paracompact. In §3.5, we explain how to use CH and modify the construction to get a space which is both HS and HL.

There is a somewhat simpler class of compact S-spaces [6,12,19] which require only CH to construct; but these have points of countable character, so we could not use them to prove Theorem 3.1.1.

§**3.2. The Basic Stuff.** For $\alpha \leq \beta$, define $\pi^\beta_\alpha : 2^\beta \to 2^\alpha$ by $\pi^\beta_\alpha(f) = f \restriction \alpha$. We shall choose X_α for $\alpha \leq \omega_1$ and A_α, B_α for $\alpha < \omega_1$ so that:

R1.1. X_α is a closed subset of 2^α, and $\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $\alpha < \beta \leq \omega_1$.

R1.2. For $\alpha < \omega_1$, A_α and B_α are closed in X_α, $A_\alpha \cup B_\alpha = X_\alpha$, and $X_{\alpha+1} = A_\alpha \times \{0\} \cup B_\alpha \times \{1\}$. Here, we identify $2^{\alpha+1}$ with $2^\alpha \times \{0,1\}$.

R1.3. For $n < \omega$, $X_n = A_n = B_n = 2^n$. For $\alpha \geq \omega$, A_α and B_α have no isolated points.

We remark that this may be viewed as an inductive construction. The choice of A_α and B_α determines $X_{\alpha+1}$ from X_α, and, by **R1.1**, X_γ is determined from the earlier X_α at limit γ. **R1.3** avoids some trivialities; it implies that for $\alpha \geq \omega$, X_α has no isolated points; also, $X_\omega = 2^\omega$.

We also choose μ_α for $\alpha < \omega_1$ so that:

R1.4. μ_α is a finitely additive probability measure on the clopen subsets of X_α, and $\mu_\alpha = \mu_\beta(\pi^\beta_\alpha)^{-1}$ whenever $\alpha < \beta \leq \omega_1$.

Again, this may be viewed as an inductive construction. For limit γ, μ_γ is determined from the earlier μ_α. As described in the Introduction, each μ_α extends to a $\hat{\mu}_\alpha$ on some σ-algebra, $\mathcal{S}_\alpha$ which contains all the Baire sets. For $\alpha < \omega_1$, the Baire and Borel sets coincide, but this need not hold for $\alpha = \omega_1$.

At a successor stage, in the case that $\hat{\mu}_\alpha(A_\alpha \cap B_\alpha) > 0$, there is some freedom in defining $\mu_{\alpha+1}$, in that the measure on $A_\alpha \cap B_\alpha$ can be distributed arbitrarily over $(A_\alpha \cap B_\alpha) \times \{0\}$ and $(A_\alpha \cap B_\alpha) \times \{1\}$. For the purpose of this paper, the equitable distribution will work. That is,

R1.5. For each $\alpha < \omega_1$, and each Borel $D \subseteq A_\alpha \cap B_\alpha$, $\hat{\mu}_{\alpha+1}(D \times \{0\}) = \hat{\mu}_{\alpha+1}(D \times \{1\}) = \frac{1}{2}\hat{\mu}_\alpha(D)$.

In terms of the clopen sets, this defines $\mu_{\alpha+1}$ from μ_α. Specifically, every clopen $K \subseteq X_{\alpha+1}$ can be written uniquely as $K = K_0 \times \{0\} \cup K_1 \times \{1\}$, where K_0 is a relatively clopen subset of A_α and K_1 is a relatively clopen subset of B_α. Then $\mu_{\alpha+1}(K) = \hat{\mu}_\alpha(K_0) + \hat{\mu}_\alpha(K_1) - \frac{1}{2}\hat{\mu}_\alpha(K_0 \cap B_\alpha) - \frac{1}{2}\hat{\mu}_\alpha(K_1 \cap A_\alpha)$. In the special case that K is the inverse projection of a clopen set $L \subset X_\alpha$, then $L = K_0 \cup K_1$ and $K_0 \cap B_\alpha = K_1 \cap A_\alpha = K_0 \cap K_1$, so $\mu_{\alpha+1}(K) = \hat{\mu}_\alpha(K_0) + \hat{\mu}_\alpha(K_1) - \hat{\mu}_\alpha(K_0 \cap K_1) = \hat{\mu}_\alpha(K_0 \cup K_1) = \mu_\alpha(L)$. Thus, $\mu_{\alpha+1} = \mu_\alpha(\pi_\alpha^{\alpha+1})^{-1}$. This shows that condition **R1.4** gets preserved at successor stages. It is also easy to verify:

Lemma 3.2.1. Requirements **R1.1 - 1.5** imply that

1. For $\alpha \leq \omega$, $\hat{\mu}_\alpha$ is the usual product measure on X_α.
2. For all α, $\hat{\mu}_\alpha$ gives each non-empty clopen set positive measure
3. For all $\alpha \geq \omega$, $\hat{\mu}_\alpha$ gives each point measure 0. ★

R1.1 - R1.5 do not prevent us from choosing $A_\alpha \cap B_\alpha$ to be of measure 0 (or even a singleton) for all $\alpha \geq \omega$. In that case, the measure algebra on $X = X_{\omega_1}$ would be isomorphic to the measure algebra of 2^ω. To prevent this, we shall demand

R1.6. For each $\alpha < \omega_1$ and each closed $J \subseteq X_\alpha$, if $\hat{\mu}_\alpha(J) > 0$, then there is a $\beta \geq \alpha$ such that $\hat{\mu}_\beta((\pi_\alpha^\beta)^{-1}(J) \cap A_\beta \cap B_\beta) > 0$.

Lemma 3.2.2. Requirements **R1.1 - 1.6** imply that the measure algebra of $X, \hat{\mu}$ is isomorphic to the measure algebra of 2^{ω_1} (with the usual product measure).

Proof. Let $\mathcal{B}$ be the measure algebra; so, elements of $\mathcal{B}$ are equivalence classes of $\hat{\mu}$-measurable subsets of X. For each $b \in \mathcal{B}$, let $\mathcal{B} \restriction b$ be the algebra $\{a \in \mathcal{B} : a \leq b\}$ (so, b is the 1 of $\mathcal{B} \restriction b$). By Maharam's Theorem, the lemma will follow if we can show that for all $b \neq 0$, $\mathcal{B} \restriction b$ is not separable.

For $\alpha < \omega_1$, let $\mathcal{B}_\alpha$ be the sub-algebra of $\mathcal{B}$ generated by all $(\pi_\alpha^{\omega_1})^{-1}(H)$, for H a Borel (= Baire) subset of X_α. Then $\mathcal{B}$ is an ascending union, $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$. If $\mathcal{B} \restriction b$ were separable, we may fix $\alpha < \omega_1$ such that $b \in \mathcal{B}_\alpha$ and $\mathcal{B} \restriction b = \mathcal{B}_\alpha \restriction b$. Then, there is a closed $J \subseteq X_\alpha$ such that $\hat{\mu}_\alpha(J) > 0$ and $[J] \leq b$; applying **R1.6** and **R1.5** to this J yields a contradiction. ★

The following requirement implies (actually is equivalent to) that every point in X_{ω_1} has character ω_1.

R1.7. For each $\alpha < \omega_1$ and each $y \in X_\alpha$, there is a β with $\alpha \leq \beta < \omega_1$ and $A_\beta \cap B_\beta \cap (\pi_\alpha^\beta)^{-1}\{y\} \neq \emptyset$.

Lemma 3.2.4. Requirements **R1.1 - 1.7** imply that every point in X_{ω_1} has character ω_1.

Proof. If $x \in X_{\omega_1}$ had countable character, then there would be a $\alpha < \omega_1$ such that $(\pi_\alpha^{\omega_1})^{-1}\{x \restriction \alpha\} = \{x\}$. Applying **R1.7** to $y = x \restriction \alpha$ yields a contradiction. ★

Requirements **R1.1 - R1.7** are consistent with having $A_\alpha = B_\alpha = X_\alpha$ for all α, whence X will simply be 2^{ω_1} with the usual product measure. We cannot get an S-space unless $A_\alpha \cap B_\alpha$ is nowhere dense in X_α for all but countably many α.

§**3.3. How to use** $\diamondsuit$**.** We begin with some notation on sequences. Suppose that $\vec{y} = \langle y_\xi : \xi < \beta \rangle$ is a β-sequence of distinct points in some space. Then a point x is a *limit point* of $\vec{y}$ iff for all neighborhoods U of x, $\exists \xi (y_\xi \in U \wedge x \neq y_\xi)$. Now, suppose that $\vec{Y} = \langle Y_\xi : \xi < \beta \rangle$ is a β-sequence of disjoint *sets*. We shall call x a *strong limit point* of $\vec{Y}$ iff for all neighborhoods U of x, $\exists \xi (Y_\xi \subseteq U \wedge x \notin Y_\xi)$. This implies that however we choose points $y_\xi \in Y_\xi$, x will be a limit point of $\langle y_\xi : \xi < \beta \rangle$.

For ordinals $\alpha < \beta$, if $\vec{f} = \langle f_\xi : \xi < \beta \rangle$ is a β-sequence in 2^β, we shall use $\vec{f} \restriction \alpha$ for the α-sequence in 2^α, $\langle f_\xi \restriction \alpha : \xi < \alpha \rangle$. Assuming $\diamondsuit$, we can fix $\vec{f^\alpha}$ for each $\alpha \in [\omega, \omega_1)$, such that $\vec{f^\alpha}$ is an α-sequence in 2^α and such that whenever $\vec{g}$ is an ω_1-sequence in 2^{ω_1}, $\{\alpha : \vec{g} \restriction \alpha = \vec{f^\alpha}\}$ is stationary.

For each $\beta \leq \omega_1$, we may postulate

R2.1(β). For all $\alpha \leq \beta$: If the f^α_ξ, for $\xi < \alpha$ are all distinct points in X_α, $h \in X_\beta$, and $h \restriction \alpha$ is a limit point of $\vec{f^\alpha}$, then h is a strong limit point of $\langle (\pi^\beta_\alpha)^{-1}(f^\alpha_\xi) : \xi < \alpha \rangle$.

The reason that $\diamondsuit$ is relevant to capturing sequences in 2^{ω_1} is given by the following lemma, which states that some basic properties of such sequences reflect on a club (closed-unbounded set).

Lemma 3.3.1. Suppose that $\vec{f}$ is an ω_1-sequence of distinct points in 2^{ω_1}. Then there is a club $C \subseteq \omega_1$ such that for all $\gamma \in C$,

1. The $f_\xi \upharpoonright \gamma$ for $\xi < \gamma$ are all distinct.
2. If $\eta \geq \gamma$, then $f_\eta \upharpoonright \gamma$ is a limit point of $\vec{f} \upharpoonright \gamma$.

Proof. Let $Fn(\gamma, 2)$ be the set of finite partial functions from γ to 2, and, for $s \in Fn(\gamma, 2)$, let Z_s be the basic clopen subset of 2^γ, $\{f \in 2^\gamma : f \upharpoonright dom(s) = s\}$. To achieve (1), make sure that whenever $\gamma \in C$,

$$\forall \xi < \eta < \gamma\, \exists s \in Fn(\gamma, 2)(f_\xi \in Z_s \wedge f_\eta \notin Z_s) \quad .$$

To achieve (2), make sure that whenever $\gamma \in C$ and $s \in Fn(\gamma, 2)$,

$$\sup\{\xi < \gamma : f_\xi \in Z_s\} < \gamma \quad \Longleftrightarrow \quad \sup\{\xi < \omega_1 : f_\xi \in Z_s\} < \gamma \quad .$$

In particular, if $\eta \geq \gamma$, $s \in Fn(\gamma, 2)$, and $f_\eta \in Z_s$, then $\sup\{\xi < \omega_1 : f_\xi \in Z_s\} \geq \eta$, so there are unboundedly many $\xi < \gamma$ such that $f_\xi \in Z_s$. ★

Lemma 3.3.2. Assuming the requirements **R1.1–1.6** and **R2.1**(ω_1), X is hereditarily separable.

Proof. If not, then let $\vec{f}$ be a left-separated ω_1–sequence in X. Let C be as in Lemma 3.3.1 and fix $\gamma \in C$ such that $\vec{f} \upharpoonright \gamma = \vec{f}^\gamma$. Let $h = f_\gamma$. By Lemma 3.3.1,

$h \restriction \gamma$ is a limit point of $\vec{f} \restriction \gamma$, so h is a strong limit point of $\langle (\pi^{\omega_1}_\alpha)^{-1}(f^\gamma_\xi) : \xi < \gamma \rangle$, so h is a limit point of $\langle f_\xi : \xi < \gamma \rangle$, which is impossible if $\vec{f}$ is left-separated. ★

By a *sequence* we mean an ω-sequence. One could ensure that the space has no convergent sequences by a similar use of $\diamondsuit$ to capture ω-sequences, but it turns out that this comes for free because of the way we captured ω_1-sequences.

Lemma 3.3.3. Assuming **R1.1–R1.7** and **R2.1**(ω_1), there are no convergent sequences in X.

Proof. Suppose $\vec{x}$ is a discrete sequence in X and y is a limit point of $\vec{x}$. Let $\vec{f}$ be any ω_1-sequence of points in X such that $f_n = x_n$ for $n < \omega$ and $f_\xi(0) \neq y(0)$ for all $\xi \geq \omega$. Fix an $\alpha < \omega_1$ such that $\pi^{\omega_1}_\alpha(\vec{f}) = \vec{f^\alpha}$ and note that $y \restriction \alpha$ is a limit point of $\vec{f^\alpha}$. Then, every point in $(\pi^{\omega_1}_\alpha)^{-1}(y \restriction \alpha)$ is a strong limit point of $\langle (\pi^{\omega_1}_\alpha)^{-1}(f^\alpha_\xi) : \xi < \alpha \rangle$, and hence a limit point of $\langle f_\xi : \xi < \alpha \rangle$. Since we have separated the f_ξ for $\xi \geq \omega$ away from y at co-ordinate 0, every point in $(\pi^{\omega_1}_\alpha)^{-1}(y \restriction \alpha)$ is a limit point of $\vec{x}$. By **R1.7**, $(\pi^{\omega_1}_\alpha)^{-1}(y \restriction \alpha)$ will not be a singleton, so $\vec{x}$ does not converge to y. ★

We shall now replace requirement **R2.1** by a somewhat more complicated requirement on the A_α and B_α, so that it will be clear that this is something which may be ensured during the inductive construction. First, note that there is no problem at limits.

Lemma 3.3.4. If γ is a limit and **R2.1**(β) holds for all $\beta < \gamma$, then **R2.1**(γ) holds as well. ★

However, there is a problem at successors. Suppose **R2.1**(β) holds and $h \in X_{\beta+1}$. If $h \restriction \beta \notin A_\beta \cap B_\beta$, then **R2.1**$(\beta+1)$ will hold at the point h, since at h, the projection $\pi_\beta^{\beta+1}$ is locally a homeomorphism. But, if $h \restriction \beta \in A_\beta \cap B_\beta$, then **R2.1**$(\beta+1)$ will fail, for example, if for some $\alpha < \beta$, the f_ξ^α, for $\xi < \alpha$ are all distinct points in X_α and each $(\pi_\alpha^\beta)^{-1}(f_\xi^\alpha)$ meets both A_β and B_β. To avoid this, we assume

R2.2. For all $\alpha \le \beta < \omega_1$: If the f_ξ^α, for $\xi < \alpha$ are all distinct points in X_α, $h \in A_\beta \cap B_\beta$, and h is a strong limit point of $\langle (\pi_\alpha^\beta)^{-1}(f_\xi^\alpha) : \xi < \alpha \rangle$, and U is any neighborhood of h, then $\exists \xi ((\pi_\alpha^\beta)^{-1}(f_\xi^\alpha) \subseteq U \wedge (\pi_\alpha^\beta)^{-1}(f_\xi^\alpha) \cap A_\beta = \emptyset)$ and $\exists \xi ((\pi_\alpha^\beta)^{-1}(f_\xi^\alpha) \subseteq U \wedge (\pi_\alpha^\beta)^{-1}(f_\xi^\alpha) \cap B_\beta = \emptyset)$.

Lemma 3.3.5. Assuming the requirements **R1.1–1.6** and **R2.2**, X is hereditarily separable and has no convergent sequences.

Proof. By induction on β, verify **R2.1**(β), and then apply Lemma 3.3.2 and Lemma 3.3.3. ★

We may now regard **R2.1** as obsolete, having been replaced by **R2.2**.

We must still make sure that each perfect set F becomes a G_δ. To do this, we arrange for F to become $(\pi_\alpha^{\omega_1})^{-1}(H)$ for some $\alpha < \omega_1$ and some perfect H in X_α.

A *perfect set* is a closed set with no isolated points.

We shall need another definition:

If K is a closed subset of a space Z and $g : Z \to Y$, we shall call g *irreducible on* K iff $g(L)$ is a proper subset of $g(K)$ for all proper closed subsets L of K.

Note that if F is a closed subset of Z, then $F \subseteq g^{-1}(g(F))$; and if g is irreducible on $g^{-1}(g(F))$, then $F = g^{-1}(g(F))$.

So, we shall arrange that for every perfect F, there will be an $\alpha < \omega_1$ such that $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(F))$. Then $F = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(F))$ will be a G_δ. As in the usual inductive construction of the projective cover, irreducibility of $\pi_\alpha^{\omega_1}$ can be guaranteed during the inductive construction by a condition on the A_α and B_α, which we now explain.

If $A \subseteq H \subseteq Y$, where Y is any space, A is called *regular closed* in H iff $A = cl_H(int_H(A))$; here cl_H and int_H denote the closure and interior operators relative to the subspace H. We call $A, B \subseteq H$ *complementary* regular closed sets in H iff they are each regular closed in H, $A \cup B = H$, and $A \cap B$ is nowhere dense in H; this implies that $A \cap B$ is the common boundary of A and B in H, and that the natural projection from $A \times \{0\} \cup B \times \{1\}$ onto H is irreducible.

Lemma 3.3.6. Suppose H is closed in X_α and, for all $\beta \geq \alpha$, $A_\beta \cap (\pi_\alpha^\beta)^{-1}(H)$ and $B_\beta \cap (\pi_\alpha^\beta)^{-1}(H)$ are complementary regular closed subsets of $(\pi_\alpha^\beta)^{-1}(H)$. Then $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(H)$.

Proof. By induction on $\beta \le \omega_1$, show that π^β_α is irreducible on $(\pi^\beta_\alpha)^{-1}(H)$.★

Using $\diamondsuit$, we can fix closed sets F_α in 2^α, for $\alpha < \omega_1$, such that for every closed F in 2^{ω_1}, $\{\alpha : \pi^{\omega_1}_\alpha(F) = F_\alpha\}$ is stationary. Once we have constructed X_α, where $\omega \le \alpha < \omega_1$, we shall define Q_α to be F_α if F_α is perfect and a subset of X_α, and set $Q_\alpha = X_\alpha$ otherwise. So, Q_α is always a perfect subset of X_α.

We now note that the property of not having isolated points is reflected on a club:

Lemma 3.3.7. If F is a perfect subset of 2^{ω_1}, then there is a club C such that for all $\alpha \in C$, $\pi^{\omega_1}_\alpha(F)$ is a perfect subset of 2^α.★

R2.3. If $\alpha \ge \omega$ then for all $\beta \ge \alpha$

- **a.** $A_\beta \cap B_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ is nowhere dense in $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$.
- **b.** $A_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ and $B_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ are complementary regular closed subsets of $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$.

Actually, **R2.3a** is redundant, given **R2.3b**, but when we verify in §3.4 that **R2.3** can be accomplished, it will be easier to handle **R2.3a** before considering **R2.3b**.

It is clear from the preceding lemmas that

Lemma 3.3.8. Assuming **R1.1–R1.3** and **R2.3**, all perfect subsets of X are G_δ. ★

§**3.4. Putting it Together.** We are now done with the proof of Theorem 3.1.1, assuming that the construction can be made to meet all of our requirements. Examining them, it appears that only **R1.6**, **R1.7**, **R2.2**, and **R2.3** are are non-trivial; the rest just detail how the final X and μ are completely determined by the choice of A_α and B_α for $\omega \leq \alpha < \omega_1$. Let $S_\alpha = A_\alpha \cap B_\alpha$. Then **R1.6** and **R1.7** only involve what happens eventually, and only involve the S_α; they ensure that every closed set of positive measure gets split (so the measure algebra will be non-separable) and that every point eventually gets split (so points get uncountable character). **R2.3a** must be met at every stage α, but only involves S_α. Finally, **R2.2** and **R2.3b** must also be met at each stage, and actually involve A_α and B_α, so we consider them last, after fixing S_α.

Partition $\omega_1 \setminus \omega$ into two disjoint uncountable sets, $EVEN$ and ODD. We shall handle **R1.6** in $EVEN$ and **R1.7** in ODD. Applying CH, for $\beta \in EVEN$, choose $\delta_\beta \leq \beta$ and a closed $J_\beta \subseteq 2^{\delta_\beta}$ so that for each $\alpha < \omega_1$ and each closed $J \subseteq 2^\alpha$, there is a $\beta \geq \alpha$ such that $\beta \in EVEN$ and $\delta_\beta = \alpha$ and $J_\beta = J$. Again applying CH, for $\beta \in ODD$, choose $\delta_\beta \leq \beta$ and $p_\beta \in 2^{\delta_\beta}$ so that for each $\alpha < \omega_1$ and each $p \in 2^\alpha$, there is a $\beta \geq \alpha$ such that $\beta \in ODD$ and $\delta_\beta = \alpha$ and $p_\beta = p$.

These choices are made ahead of time, before constructing the X_α. Once X_β is constructed, let $K_\beta = (\pi^\beta_{\delta_\beta})^{-1}(J_\beta)$ if $J_\beta \subseteq X_{\delta_\beta}$, and set $K_\beta = X_\beta$ otherwise. Use N_α to denote the set $\{f^\alpha_\xi : \xi < \alpha\}$.

Now, determine S_β as follows.

For $\beta \in EVEN$, choose S_β so that

Σ1. If $\hat{\mu}_\beta(K_\beta) > 0$, then $\hat{\mu}_\beta(K_\beta \cap S_\beta) > 0$.

Σ2. For all $\alpha \in (\omega, \beta]$, $S_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ is nowhere dense in $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$.

Σ3. For all $\alpha \in (\omega, \beta]$, $S_\beta \cap (\pi^\beta_\alpha)^{-1}(N_\alpha) = \emptyset$.

It is clear that Σ1 will guarantee requirement **R1.6** and Σ2 will guarantee **R2.3a** for $\beta \in EVEN$. Σ3 will make it possible to choose A_β and B_β later. First, we must see that such a choice of S_β is possible. For each $\alpha \in (\omega, \beta]$, choose a countable $D_\alpha \subseteq (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ which is dense in $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$. For each $\alpha \in (\omega, \beta]$, let $E_\alpha = N_\alpha \cap X_\alpha$. Let $L = K_\beta$ if $\hat{\mu}_\beta(K_\beta) > 0$; set $L = X_\beta$ otherwise. Let $M = L \setminus \bigcup_{\alpha \in [\omega, \beta)}(D_\alpha \cup (\pi^\beta_\alpha)^{-1}(E_\alpha))$. M is a positive measure set minus a countable union of measure 0 sets (since by Lemma 3.2.1, points have measure 0). Thus, M has positive measure, and we may choose S_β to be any closed subset of M of positive measure.

For $\beta \in ODD$, choose S_β as follows: If $p_\beta \in X_{\delta_\beta}$, choose S_β to be any singleton from $(\pi^\beta_\alpha)^{-1}\{p_\beta\}$. If not, set S_β to be any singleton. It is clear that this choice will guarantee requirement **R1.7**. It also guarantees **R2.3a** for $\beta \in ODD$, since $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$ has no isolated points.

Proof of Theorem 3.1.1. As noted above, we are done provided we can show that at each stage β we can can satisfy **R2.2** and **R2.3b**. We already have S_β, and we must find A_β and B_β. Fix a strictly decreasing sequence of clopen sets in X_β, $\langle V_n : n \in \omega \rangle$, such that $V_0 = X_\beta$ and $\cap_{n \in \omega} V_n = S_\beta$. If $\phi : \omega \to \omega$ is any strictly increasing function with $\phi(0) = 0$, we may set $A_\beta = S_\beta \cup \bigcup_n (V_{\phi(2n)} \backslash V_{\phi(2n+1)})$ and $B_\beta = S_\beta \cup \bigcup_n (V_{\phi(2n+1)} \backslash V_{\phi(2n+2)})$. With any choice of ϕ, $A_\beta \cap B_\beta = S_\beta$, and the "trivial" conditions **R1.2** and **R1.3** are met. We shall show that if ϕ grows "fast enough", both **R2.2** and **R2.3b** will be met – essentially by the same argument.

To handle **R2.3b**, fix, for each $\alpha \leq \beta$, a countable dense subset T_α of $S_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$. Since $S_\beta \cap (\pi^\beta_\alpha)^{-1}(Q_\alpha)$ is nowhere dense in $(\pi^\beta_\alpha)^{-1}(Q_\alpha)$, there is for each element t of T_α a sequence $\vec{s_t}$ of distinct points in $(\pi^\beta_\alpha)^{-1}(Q_\alpha) \setminus S_\beta$ converging to t. Note that **R2.3.b** will hold if for all $\alpha \leq \beta$ and $t \in T_\alpha$, both $A_\beta \setminus B_\beta$ and $B_\beta \setminus A_\beta$ contain infinitely many elements of $\vec{s_t}$.

To simplify the notation, we re-index all the $\langle (\pi^\beta_\alpha)^{-1}(f^\alpha_\xi) : \xi < \alpha \rangle$ for $\alpha < \beta$ which are relevant to **R2.2**, as well as all the sequences $\vec{s_t}$ just chosen. We then have countably many sequences, $\vec{Y^i}$ $(i \in \omega)$. Each $\vec{Y^i}$ is a sequence of disjoint closed sets (identifying each of the sequences of points $\vec{s_t}$ with the corresponding sequence of singletons). Since the notion of strong limit point does not depend on the order type of the sequence, we may as well assume that each $\vec{Y^i}$ is an ω-sequence, so $\vec{Y^i} = \langle Y^i_j : j < \omega \rangle$. If $i \in \omega$ and U is a clopen subset of X_β, let $R(i, U)$ be the assertion that for each point $h \in S_\beta$, *if* h is a strong limit point of $\vec{Y^i}$, *and* U is a neighborhood of h, *then* there are infinitely many n such that

$$\exists j (Y^i_j \subseteq U \cap (V_{\phi(2n)} \backslash V_{\phi(2n+1)})) \quad \wedge \quad \exists j (Y^i_j \subseteq U \cap (V_{\phi(2n+1)} \backslash V_{\phi(2n+2)})) \quad .$$

Then both **R2.2** and **R2.3b** will hold if $R(i,U)$ holds for each i,U. To accomplish this we shall, for each i,U, find a $\psi_{i,U}:\omega\to\omega$ such that $R(i,U)$ holds whenever $\phi(\ell+1)\geq\psi_{i,U}(\phi(\ell))$ for all but finitely many ℓ; this will be sufficient to be able to define ϕ, since there are only countably many i and U.

So, fix i and U. We must find $\psi=\psi_{i,U}$ such that for each point $h\in S_\beta$, *if* h *is a strong limit point of* $\vec{Y^i}$, *and* U *is a neighborhood of* h, *then* for each m, $\psi(m)>m$ and

$$\exists j(Y^i_j\subseteq U\cap(V_m\backslash V_{\psi(m)}))\quad .$$

Now, fix m, and assume that there is some $h\in U\cap S_\beta$ such that h is a strong limit point of $\vec{Y^i}$, since otherwise our condition is vacuous. Now, fix j with $Y^i_j\subseteq(U\cap V_m)$ and $h\notin Y^i_j$. Note that $Y^i_j\cap S_\beta=\emptyset$; this is obvious when $\vec{Y^i}$ comes from one of the $\vec{s_t}$, (since none of the points in $\vec{s_t}$ is in S_β) or when $\beta\in ODD$ (since then $S_\beta=\{h\}$). When $\beta\in EVEN$, it follows from item $\Sigma 3$ in our choice of S_β above. By compactness, there must be an $r>m$ such that $Y^i_j\cap V_r=\emptyset$, and we choose such an r for $\psi(m)$. ★

If we only wanted to construct an S-space, then we could have made all the S_β singletons. That would simplify the proof – especially in the above discussion of **R2.2**, where the U could always be one of the V_n, and ϕ could be chosen by a simple diagonal argument. Making all the S_β singletons would also force the measure algebra to be separable.

It is not clear whether one could do the above construction under CH, without $\diamondsuit$. Even without the measure, it is already a well known open question whether CH alone implies the existence of a compact S-space of size greater than ω_1.

In answer to a question of Peter Nyikos, we now show that our space is hereditarily normal. It is also hereditarily countably paracompact. Furthermore, both of these properties follow from the properties of the space stated in Theorem 3.1.1.

Theorem 3.4.1. Suppose that X is compact, X has no uncountable discrete subsets and no convergent sequences, and every perfect subset of X is a G_δ. Then X is hereditarily normal and hereditarily countably paracompact.

The proof of Theorem 3.4.1 seems somewhat simpler if we follow M. E. Rudin and express our properties in terms of "shrinkings" of countable (or finite) covers.

Lemma 3.4.2. A space Z is normal and countably paracompact iff for all $\beta \leq \omega$ and all open covers of Z, $\{U_n : n < \beta\}$, there are closed $H_n \subseteq U_n$ such that $\{H_n : n < \beta\}$ covers Z.★

For $U_n, H_n (n < \beta)$ as above, one says that $\{H_n : n < \beta\}$ is a *shrinking* of $\{U_n : n < \beta\}$.

From Lemma 3.4.2, it is easy to prove the following well-known result.

Lemma 3.4.3. Suppose that the space Z has a locally finite cover by closed sets, each of which is normal and countably paracompact in its relative topology. Then Z is normal and countably paracompact.★

Now, note, as we did in §1, that the hypotheses on X in Theorem 3.4.1 imply that every infinite closed subset of X is the union of a perfect set, which we denote now by $ker(X)$, and a countable set of isolated points. If X is finite, let $ker(X) = \emptyset$. Then $ker(X)$ is always a G_δ.

Lemma 3.4.4. If X is as in Theorem 3.4.1 and $p \in X$, then $X \setminus \{p\}$ is normal and countably paracompact.

Proof. Applying Lemma 3.4.2, let $\beta \leq \omega$ and let $\{U_n : n \leq \beta\}$ be an open cover of $X \setminus \{p\}$. If $p \in ker(X \setminus U_n)$ for each n, then $\{p\}$ would be the intersection of countably many G_δ sets and hence a G_δ, which is impossible, since p would then be the limit of a convergent sequence. So, fix i such that p is isolated in $X \setminus U_i$. Let $U_i' = U_i \cup \{p\}$, and let $U_n' = U_n$ for $n \neq i$. Then the U_n' form an open cover of X, which is compact, so we may shrink the U_n' to closed H_n' in X and then let $H_n = H_n' \cap X \setminus \{p\}$. ★

Since normality and countable paracompactness are hereditary to closed subsets, we now know that whenever K is closed in X, $K \setminus \{p\}$ is normal and countably paracompact. Actually, the proof of 3.4.4 shows that $K \setminus \{p\}$ it is countably *compact*, although we do not need that fact here.

Proof of Theorem 3.4.1. It is enough to prove that $X\backslash H$ is normal and countably paracompact whenever H is closed. Since $ker(H)$ is a G_δ, we may find a countable locally finite (in $X\backslash H$) cover of $X\backslash H$ by closed (in $X\backslash H$) sets, where each set in the cover is of the form K or $K\backslash\{p\}$, with K closed in X. Thus, by Lemma 3.4.3, $X\backslash H$ is normal and countably paracompact. ★

Actually, $X\backslash H$ is countably compact iff $ker(H)$ is clopen.

§**3.5. Getting the HS+HL Space.** To make the space HL also, we simply make *every* closed set a G_δ – not just the perfect sets. This time, we only need CH.

Theorem 3.5.1. If CH holds, then there is a compact 0-dimensional, hereditarily separable and hereditarily Lindelöf space X and a Radon probability measure μ on X such that the measure algebra of X, μ is isomorphic to the measure algebra of 2^{ω_1} with the usual product measure. X, μ also have the property that all measure 0 sets are second countable in their relative topology.

To prove this theorem, we modify the construction from the previous proof. HS+HL is guaranteed by a similar use of irreducible maps, as explained by the next lemma.

Lemma 3.5.2. Assume just **R1.1** (X is closed in 2^{ω_1} and X_α is its projection on 2^α). Then

a. X is HL iff for all closed $H \subseteq X$, there is an $\alpha < \omega_1$ for which $H = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$.

b. X is HL+HS iff for all closed $H \subseteq X$, there is an $\alpha < \omega_1$ for which $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$.

Proof. Part (a) follows from the fact that HL is equivalent to all closed sets being G_δ sets (for compact X).

For the $\Leftarrow$ of Part (b), irreducibility implies that $H = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$, so X is HL by Part (a). Since the irreducible preimage of a separable space is separable, all closed subsets of X are separable, which implies that X is HS, since X is first countable.

For the $\Rightarrow$ of Part (b), assume X is HS+HL. Let H be closed in X, and let E be a countable dense subset of H. Applying Part (a), we may fix $\alpha < \omega_1$ such that $H = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$ and $\{e\} = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(e))$ for all $e \in E$. If F is any closed subset of H such that $\pi_\alpha^{\omega_1}(F) = \pi_\alpha^{\omega_1}(H)$, then for each $e \in E$, $\pi_\alpha^{\omega_1}(e) \in \pi_\alpha^{\omega_1}(F)$, so $e \in F$; hence $E \subseteq F$, so $F = H$. Thus, $\pi_\alpha^{\omega_1}$ is irreducible on $H = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$. ★

Now, to prove Theorem 3.5.1, we delete **R1.7** (which gave points uncountable character), and replace it by **R4.1** below, which will have just the opposite effect. We also delete **R2.1-R2.2**, which relied on $\diamondsuit$. Requirement **R2.3**, which guaranteed irreducibility of maps, remains the same as it was, but the Q_α will have a different meaning.

Using CH, choose, for all countable $\alpha \geq \omega$, an ordinal $\delta_\alpha \leq \alpha$ and a Borel set, $J_\alpha \subseteq 2^{\delta_\alpha}$, so that for all $\gamma < \omega_1$ and each Borel set $J \subseteq 2^\gamma$, there is an $\alpha \geq \gamma$ such that $\delta_\alpha = \gamma$ and $J_\alpha = J$.

Once we have defined X_α, where $\omega \leq \alpha < \omega_1$, we define the following subsets of X_α:

$C_\alpha = (\pi^\alpha_{\delta_\alpha})^{-1}(J_\alpha)$, if $J_\alpha \subseteq X_{\delta_\alpha}$; $C_\alpha = \emptyset$ otherwise.

$K_\alpha = C_\alpha$ if C_α is closed; $K_\alpha = X_\alpha$ otherwise.

$Q_\alpha = K_\alpha \setminus \bigcup\{O : O \text{ is open } \wedge \mu_\alpha(K_\alpha \cap O) = 0\}$

$N_\alpha = (K_\alpha \setminus Q_\alpha) \cup C_\alpha$, if $\mu_\alpha(C_\alpha) = 0$; $N_\alpha = (K_\alpha \setminus Q_\alpha)$ otherwise.

Then we require

R4.1. For any $\beta \geq \alpha \geq \omega$, $A_\beta \cap B_\beta \cap (\pi^\beta_\alpha)^{-1}(N_\alpha) = \emptyset$.

Lemma 3.5.3. Assume the requirements **R1.1-R1.6**, **R2.3**, and **R4.1**. Let H be a closed subset of X. Then there is an $\alpha < \omega_1$ such that $\pi^{\omega_1}_\alpha$ is irreducible on $(\pi^{\omega_1}_\alpha)^{-1}(\pi^{\omega_1}_\alpha(H))$.

Proof. For each $\gamma < \omega_1$, let $H_\gamma = \pi^{\omega_1}_\gamma(H)$. Then the $\mu_\gamma(H_\gamma)$ form a non-increasing sequence of real numbers, so we may fix a $\gamma < \omega_1$ such that for all $\alpha \geq \gamma$, $\mu_\alpha(H_\alpha) = \mu_\gamma(H_\gamma)$. Now fix an $\alpha \geq \gamma$ such that $\delta_\alpha = \gamma$ and $J_\alpha = H_\gamma$. Then $K_\alpha = C_\alpha = (\pi^\alpha_\gamma)^{-1}(H_\gamma)$. Then H_α is a closed subset of K_α with the same measure, so $Q_\alpha \subseteq H_\alpha \subseteq K_\alpha$. Now, $\pi^{\omega_1}_\alpha$ is irreducible on $(\pi^{\omega_1}_\alpha)^{-1}(Q_\alpha)$ (by **R2.3b** and Lemma 3.3.6), and $\pi^{\omega_1}_\alpha$ is 1-1 on $(\pi^{\omega_1}_\alpha)^{-1}(H_\alpha \setminus Q_\alpha)$ (by **R4.1**). Thus, $\pi^{\omega_1}_\alpha$ must be irreducible on $(\pi^{\omega_1}_\alpha)^{-1}(H_\alpha)$ as well. ★

Including the C_α in N_α ensures that all measure 0 sets will be second countable.

Proof of Theorem 3.5.1. By Lemmas 3.5.2 and 3.5.3, we are done, provided all the requirements can be met. But the proof of this is exactly as for Theorem 3.1.1, except that now $ODD = \emptyset$ and $EVEN = \omega_1 \setminus \omega$. ★

§**3.6. Remarks on [13]** As we mentioned before, there is also a counterexample to Question H which is an L-space, see [13]. Note that the space in [13] also needed only CH and satisfied everything in Theorem 3.5.1 except being HS.

The original construction of Kunen in [13] can also be described using the notation we had for the constructions in this Chapter. So, if X is the L-space of Kunen, we can think of $X = X_{\omega_1}$ in the inductive construction which for each $\omega \leq \alpha \leq \omega_1$ gives a closed subset X_α of 2^α. As in §3.2, we can define A_α and B_α.

The construction in [13] did not use irreducibility, but rather established HL by making the measure 0 ideal and the first category ideal coincide; this was accomplished by taking $A_\beta = X_\beta$ for all β. The requirement of separability forces there to be a first category set of measure 1:

Lemma 3.6.1. In a separable Radon probability space in which points have measure 0, there is a first category set of measure 1.

Proof. Let X, μ be such a space and D a countable dense subset of X. Then $\mu(D) = 0$. So, for every $n \in \omega$, we can find an open O_n containing D such that $\mu(O_n) < 1/2^n$. Then $\cup_{n\in\omega}(X \setminus O_n)$ is a nowhere dense set of measure 1.★

Chapter IV

More Counter-examples

In this chapter we show that a counter-example to Question H can exist independently of the size of the continuum. Also, we show that the construction we had in the previous chapter of a compact S-space with a non-separable Radon probability measure on it, can actually be carried without $\diamondsuit$. The assumptions we use this time are CH and ω_1 Cohen reals. These assumptions do not imply $\diamondsuit$. Using just one Cohen real, we construct another example of an S-space, the Ostaszewski space. This space, however, cannot help building a counter-example to the Question H - we explain that at the end of §4.3.

§**4.1. The Size of the Continuum does not matter.** The counter-examples to the question H that we presented so far all depended on CH or $\diamondsuit$. So, we always had $\mathfrak{c} = \omega_1$. Now we show that a counter-example can exist no matter what the size of $\mathfrak{c}$ is.

The way we do this is to start with a model of set theory which satisfies CH, do the construction of X, μ from §3.5 in it, and then add Cohen reals. The fact that X is a compact space and μ is a non-separable Baire probability measure on X, supported by closed sets, will not change. The other properties: HS, HL and the fact that all open sets are μ-measurable, might be destroyed, as we are adding more open sets. We shall see that the properties of being HS and HL are preserved. As we explained in the previous Chapter, every Baire measure on a compact HL space is Borel, so we do not need to check the Radon property separately.

For any set J, $Fn(J,2)$ denotes the set of all finite partial functions from J to 2, ordered by reverse inclusion. If $J = \lambda$ is an infinite cardinal, by *adding λ Cohen reals* we mean forcing with $Fn(\lambda, 2)$, or, equivalently, $Fn(\lambda \times \omega, 2)$. (If $\lambda = \omega$, we more often say that *a* Cohen real is added).

Sometimes it is more convenient to think of $Fn(\lambda, \omega)$ then of $Fn(\lambda, 2)$ - the two forcing notions are easily seen to be equivalent.

Note that the partial order used to add λ Cohen reals has precaliber ω_1, for any λ.

If $M \subseteq N$ are models of set theory, and X is a 0-dimensional compact space in M, then by *X in N* we mean the space in N whose clopen algebra is isomorphic to the clopen algebra of X in M. A property of X in M is *preserved by N*, if X in N has that same property.

Lemma 4.1.1. The property of being a compact 0-dim HS space is preserved by extensions by forcings of precaliber ω_1.

Proof. Suppose M is model of set theory, $\mathbb{P}$ a forcing of precaliber ω_1 in M and X is a compact 0-dim HS space in M. We can then assume that X is a closed subspace of 2^λ, for some λ. It is then easily seen that X remains a compact 0-dimensional space in any extension by $\mathbb{P}$.

Suppose that $p \in \mathbb{P}$ forces that B_α for $\alpha < \omega_1$ is a sequence of basic clopen sets in X such that no B_α is covered by the union of all B_β for $\beta > \alpha$. For each α we can fix a p_α extending p, such that p_α decides that B_α is $[f_\alpha]$ for some f_α in $Fn(\lambda, 2)$. Since $\mathbb{P}$ has precaliber ω_1, there is in M an uncountable subset A of ω_1 such that $\{p_\alpha : \alpha \in A\}$ has the finite intersection property.

Fix an α in A and suppose that F is a finite subset of $A \setminus \alpha$. Any $q \in \mathbb{P}$ which extends p_α and all p_β for β in F, will force that $[f_\alpha]$ is not covered by the union of $[f_\beta]$ for β in F. So, in M, $[f_\alpha]$ is not covered by the union of $[f_\beta]$ for β in F.

By compactness, we conclude that $\langle [f_\alpha] \rangle_{\alpha \in A}$ refutes HS of X in M.★

An analogous argument shows also

Lemma 4.1.2. The property of being a compact 0-dimensional HL space is preserved by extensions by forcings of precaliber ω_1. ★

Theorem 4.1.3. If λ is a cardinal such that it is consistent that $\mathfrak{c} = \lambda$, then it is consistent that $\mathfrak{c} = \lambda$ and that there is a compact HS and HL space which supports a non-separable Radon probability measure.

Proof. Start with a model M which satisfies CH and do the construction from §3.5 in M. This yields a compact 0-dimensional Radon probability measure space X, μ which is both HS and HL and whose measure algebra is non-separable.

Now add λ Cohen reals, thus the forcing extension will have $\mathfrak{c} = \lambda$. By Lemmas 4.1.1 and 4.1.2, $X\mu$ will be both HS and HL in the extension, and the rest of its properties are easily seen to be preserved, as explained before. ★

Starting with Kunen's L-space and using a a similar argument, we see the following

Theorem 4.1.3. If λ is like in Theorem 4.1.2, then it is consistent that $\mathfrak{c} = \lambda$ and that there is a compact HL but not HS space which supports a non-separable Radon probability measure.★

§**4.2. The S-space Counter-example revisited.** The S-space construction that we had in the previous chapter needed $\diamondsuit$. Actually, the same result can be obtained in a model obtained by adding ω_1 Cohen reals to a model which only satisfies CH and not $\diamondsuit$. The following lemma shows that a $\diamondsuit$-sequence cannot exist in the generic extension, either.

$\diamondsuit^-$ is the statement: There are sets $\mathcal{A}_\alpha$ in $\mathcal{P}(\alpha)$ for $\alpha < \omega_1$, such that each $\mathcal{A}_\alpha$ is countable and for each $A \subseteq \omega_1$,

$$\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$$

is stationary.

Although seemingly weaker than $\diamondsuit$, $\diamondsuit^-$ is actually equivalent to $\diamondsuit$. For the proof see [14, II7.14].

Lemma 4.2.1. Forcing with a ccc partial order does not add a $\diamondsuit$-sequence.

Proof. Let $M[G]$ be a ccc forcing extension of a model M of set theory. If $A_\alpha(\alpha < \omega_1)$ is a $\diamondsuit$ sequence in $M[G]$, there is in M a function $f : \omega_1 \to [\omega_1]^\omega$, such that for every $\alpha \in \omega_1$, $A_\alpha \in f(\alpha)$. So $\diamondsuit^-$ holds in M.★

Corollary. There is a model of ZFC in which there are ω_1 Cohen reals and in which GCH holds but $\diamondsuit$ does not.

Proof. Start with a model in which GCH holds but $\diamondsuit$ does not, and add ω_1 Cohen reals.★

Now we show how to modify the construction of the S-space that we had in Chapter III, working only with CH and ω_1 Cohen reals.

Theorem 4.2.2. It is relatively consistent with GCH without $\diamondsuit$, that there is a compact, 0-dimensional, hereditarily separable space X and a Radon probability measure μ on X, such that 1-4 of Theorem 3.1.1 are satisfied.

To prove this Theorem, we repeat the construction which proved Theorem 3.1.1, but this time using just CH and ω_1 Cohen reals.

For $\gamma \in \omega_1$, we use r_γ to denote the γ-th Cohen real, that is, for n in ω, $r_\gamma(n) = \cup G(\gamma, n)$. Also, $M_\gamma = M[\{r_\beta : \beta < \gamma\}]$. Note that r_γ is Cohen generic over M_γ.

We start with a model M of GCH, and add ω_1 Cohen reals. Let G be a generic filter. The space X, μ from Theorem 4.2.1 will be constructed in $M[G]$. Note, of course, that $M[G]$ satisfies GCH.

We construct X, μ in $M[G]$. Like in Chapter III, the construction will satisfy requirements **R1.1-R1.7**, as well as modified versions of **R2.2** and **R2.3**, **R2.2$'$** and **R2.3$'$**.

Rather than using $\diamondsuit$ to choose $\overrightarrow{f^\alpha}$, as we did in §3.3, this time we use CH. For every α in $[\omega, \omega_1)$, we enumerate all α-sequences in 2^α as $\overrightarrow{f^\alpha_\zeta}, \zeta < \omega_1$. We enumerate each $\overrightarrow{f^\alpha_\zeta}$ as $\langle f^\alpha_{\zeta,\xi} : \xi < \alpha\rangle$.

We then postulate

R2.2$'$. For all $\alpha \le \beta < \omega_1$ and for all $\zeta \le \beta$: If $f^\alpha_{\zeta,\xi}$, for all $\xi < \alpha$ are all distinct points in X_α, and if $\overrightarrow{f^\alpha_\zeta}$ is in M_β, then the conclusion of **R2.2** holds with f^α_ξ replaced by $f^\alpha_{\zeta,\xi}$.

We need to require that $\overrightarrow{f^\alpha_\zeta}$ is an element of M_β in order to be able to assure this requirement later.

To see that this requirement does what we would like it to do:

Lemma 4.2.3. Assuming the requirements **R1.1-R1.6** and **R2.2$'$**, X is hereditarily separable and has no convergent sequences.

Proof. We simply follow the proof of Lemma 3.3.5, using a modified version of **R2.1**. Reformulate **R2.1**(β), for $\beta < \omega_1$, into the following

R2.1$'$ (β). For all $\alpha < \beta$ and for all $\zeta < \beta$: If $f^{\alpha}_{\zeta,\xi}$ are all distinct points in X_{α}, then **R2.1**(β) holds with $\overrightarrow{f^{\alpha}}$ replaced by $\overrightarrow{f^{\alpha}_{\zeta}}$.

Given an ω_1-sequence $\overrightarrow{f}$ of distinct points in 2^{ω_1}, note that the club C from Lemma 3.3.1 can be chosen to satisfy an additional requirement:

3. $\overrightarrow{f} \restriction \gamma$ is in M_{γ}.

Also, the counterpart of Lemma 3.3.2 is true: assuming **R1.1-R1.6** and **R2.1$'$**(ω_1), X is hereditarily separable. The reason is that, for an ω_1-sequence $\overrightarrow{f}$ of distinct points in X, we can find a $\delta \leq \gamma$ in C, where C is a club like above, such that $\overrightarrow{f} \restriction \delta = \overrightarrow{f^{\delta}_{\zeta}}$ for some $\zeta < \gamma$.Then the first γ elements of $\overrightarrow{f}$ form a dense subset of $\overrightarrow{f}$.

Now we prove that **R2.2$'$** implies **R2.1$'$**(ω_1), just like in the proof of Lemma 3.3.5.

The statement about convergent sequences is verified in a similar fashion.★

To handle perfect sets, we again use CH, and we fix for all $\alpha < \omega_1$ an enumeration $F^{\alpha}_{\xi}(\xi < \omega_1)$ of closed subsets of 2^{α}.

R2.3$'$. Suppose that $\alpha \geq \omega$, $\beta \geq \alpha$, $\xi < \beta$ and F^{α}_{ξ} is in M_{β} and is a perfect subset of X_{α}. Then

a. $A_{\beta} \cap B_{\beta} \cap (\pi^{\beta}_{\alpha})^{-1}(F^{\alpha}_{\xi})$ is nowhere dense in $(\pi^{\beta}_{\alpha})^{-1}(F^{\alpha}_{\xi})$.

b. $A_{\beta} \cap (\pi^{\beta}_{\alpha})^{-1}(F^{\alpha}_{\xi})$ and $B_{\beta} \cap (\pi^{\beta}_{\alpha})^{-1}(F^{\alpha}_{\xi})$ are complementary regular closed subsets of $(\pi^{\beta}_{\alpha})^{-1}(F^{\alpha}_{\xi})$.

As before, **R2.3′a** is implied by **R2.3′b**, but stating the requirement in this way makes it easier to handle. The reason for mentioning M_β is the same as in **R2.2′**.

To see that requirement **R2.3′** will assure that all perfect sets are G_δ, we have the following

Lemma 4.2.4. Assuming **R1.1-R1.3** and **R2.3′**, all perfect subsets of X are G_δ.

Proof. Given a perfect set F in X. Let C be a club which has properties as in Lemma 3.3.7, but in addition, for every α in C, $(\pi^{\omega_1}_\alpha)(F)$ is an element of M_α. This is possible because $(\pi^{\omega_1}_\alpha)(F)$ is determined by countably many basic clopen sets.

Suppose that $\alpha < \beta$ are elements of C and $(\pi^{\omega_1}_\alpha)(F) = F^\alpha_\xi$ for some $\xi < \alpha$. Then the requirement **R2.3′**, by Lemma 3.3.6, assures that $\pi^{\omega_1}_\beta$ is irreducible on the inverse projection $(\pi^{\omega_1}_\beta)^{-1}$ of $(\pi^\beta_\alpha)^{-1}(\pi^{\omega_1}_\alpha(F))$.

As C is a club, we can find an increasing sequence $\alpha_n (n \in \omega)$ of elements of C, such that $\pi^{\omega_1}_{\alpha_n}(F)$ is enumerated as $F^{\alpha_n}_{\xi_n}$ for some $\xi_n < \alpha_n$. Then the supremum β of α_n is an element of C.

Note then that $\pi^{\omega_1}_\beta(F) = \cap_{n\in\omega}(\pi^\beta_{\alpha_n})^{-1}(\pi^{\omega_1}_{\alpha_n}(F))$. By our above observation and the choice of α_n, $\pi^{\omega_1}_\beta$ is irreducible on $(\pi^{\omega_1}_\beta)^{-1}(F)$.★

To put everything together, we do exactly as we did in §3.4. The meaning of ODD, $EVEN$, p_β, J_β and K_β is as before. For $\alpha, \xi < \omega_1$, let $N^\alpha_\xi = \overrightarrow{f^\alpha_\xi}$ if $\overrightarrow{f^\alpha_\xi}$

is a sequence of distinct points in X_α. Otherwise $N_\xi^\alpha = \emptyset$. Similarly, for $\alpha, \xi < \omega_1$, let $Q_\xi^\alpha = F_\xi^\alpha$ if F_ξ^α is a perfect subset of X_α, and $Q_\xi^\alpha = X_\alpha$ otherwise.

For $\beta \in ODD$, we choose S_β as before. For $\beta \in EVEN$, rather than choosing S_β to satisfy $\Sigma 1 - \Sigma 3$ as for Theorem 3.1.1, we still require $\Sigma 1$ but we modify $\Sigma 2$ and $\Sigma 3$. We replace $\Sigma 2$ by $\Sigma 2'$ and $\Sigma 3$ by $\Sigma 3'$.

$\Sigma 2'$ asks that for all $\alpha, \xi \in (\omega, \beta]$, $\Sigma 2$ holds with Q_α replaced by Q_ξ^α. $\Sigma 3'$ asks that for all $\alpha, \xi \in (\omega, \beta]$, $\Sigma 3$ holds with N_α replaced by N_ξ^α.

To see that such a choice of S_β is possible, we apply exactly the same argument as for Theorem 3.1.1.

Proof of Theorem 4.2.4 Now we come to choosing A_β and B_β, given S_β. We do it generically. Basically, the function ϕ from the Proof of Theorem 3.1.1 is going to be r_β. However, it is easier to express this if we think of r_β as of a function from ω to 2. With the choice of $V_n (n \in \omega)$ as in the Proof to Theorem 3.1.1, we define

$$A_\beta = S_\beta \bigcup \cup_{n \in \omega} \{V_{n+1} \setminus V_n : r_\beta(n) = 0\}$$

and B_β is the same with "$r_\beta(n) = 0$" replaced by "$r_\beta(n) = 1$".

As before, we can reindex the relevant sequences as $\overrightarrow{Y_i}$ $(i \in \omega)$. Note that now all sequnces $\overrightarrow{Y^i}$ are elements of M_β. We only have to show that r_β grows fast enough to satisfy the equivalent of the requirement $R(i, U)$ from the proof of

Theorem 3.1.1., for all $i \in \omega$ and U a clopen subset of X_β. The new requirement is $R'(i, U)$, which requires that:

For each $h \in S_\beta$, *if* h is a strong limit point of $\vec{Y^i}$, *and* U is a neighborhood of h, *then* there are infinitely many n such that

$$\exists j \exists k (Y^i_j \subseteq U \cap \cup_{s<k}(V_{n+s+1} \setminus V_{n+s})) \wedge \forall s < k (r_\beta(n+s) = 0)$$

and the same with "$r_\beta(n+s) = 0$" replaced by $r_\beta(n+s) = 1$".

Let $\mathbb{P}$ denote Cohen forcing over M_β. Then r_β is $\mathbb{P}$ generic. We show that all relevant $R'(i, U)$ are met, by exhibiting convenient dense subsets of $\mathbb{P}$.★

§**4.3. Ostaszewski Space.** In this, last, section, we show how another well known consequence of $\diamondsuit$ can be constructed using ω_1 Cohen reals. The space we have in mind is the S-space of Ostaszewski, as constructed in [18]. It is also known (see e.g. [20]), that this space can be constructed using ω_1 Cohen reals. We show that, in fact, one Cohen real is enough.

The original construction of Ostaszewski uses a weaker version od $\diamondsuit$, so called $\clubsuit$. By a result of Shelah in [23], this version of $\diamondsuit$ does not imply CH, and actually, $\diamondsuit$ is equivalent to $\clubsuit + CH$.

A topological space is said to be *locally compact* if every point in the space has a neighboorhood whose closure is compact.

A space is *perfectly normal* if every closed subset of the space is a G_δ.

So, for example, the space we had in §3.5 is perfectly normal. There was no reason to state that as a separate property of the space in §3.5, since for compact spaces, perfect normality is equivalent with hereditary Lindelöfness.

Using ♣, Ostaszewski proved in [18] the following

Theorem 4.3.1. (Ostaszewski) There is a model of set theory in which there is a 1^{st} countable, 0-dimensional, locally compact, perfectly normal topology τ on ω_1, such that, in τ, ω_1 is a HS but not HL space.

To do the construction, we use a weaker version of the ♣ of Ostaszewski.

($♣'$) For each limit ordinal γ in ω_1, there is a cofinal ω-sequence r_γ in γ, and a function $f_\gamma : r_\gamma \to \omega$, such that for every unbounded subset X of ω_1, there is a limit ordinal γ with the property that for all n, $X \cap f_\gamma^{-1}(\{n\})$ is unbounded in γ.

We shall now show that this version of ♣ holds in any extension by a Cohen real. Actually, by a different proof, $♣'$ holds in the extension by any number of Cohen reals.

We first need this fact:

Lemma 4.3.2. Suppose that M is a model of set theory, $M[G]$ is an extension of M by a Cohen real and X is an unbounded subset of ω_1 in $M[G]$. Then there is an unbounded subset Y of X, which is in M.

Proof. The partial order used to add a Cohen real is countable. Therefore, G is countable as well. Note that

$$X = \{\alpha \in \omega_1 : (\exists p \in G) p \Vdash "\alpha \in \overset{\circ}{X}"\}.$$

Therefore there will be a $p \in G$ such that $Y = \{\alpha : p \Vdash "\alpha \in \overset{\circ}{X}"\}$ is uncountable.★

Then we have:

Lemma 4.3.3. If M is a model of set theory and $M[G]$ is a generic extension of M by a Cohen real, then $\clubsuit'$ is true in $M[G]$.

Proof. Given a limit ordinal γ in ω_1, note that $\mathbb{P}\gamma = Fn(\omega, \gamma) \times Fn(\omega, \omega)$ is countable, so forcing with it is equivalent with the Cohen forcing. Therefore, there is in $M[G]$ a pair $\langle r\gamma, t\gamma \rangle$ of functions, where $r\gamma : \omega \to \gamma$ and $t\gamma : \omega \to \omega$, which is $\mathbb{P}\gamma$-generic over M. The $r\gamma$ we are looking for, is the first coordinate of that pair.

We define $f\gamma$ by saying that for every n, $f\gamma(r\gamma(n)) = t\gamma(n)$.

Then, given an uncountable subset X of ω_1 in $M[G]$, there is a Y in M which is an uncountable subset of X. For that Y, there is a club C in M such that for each γ in C:

1. γ is a limit ordinal and
2. $Y \cap \gamma$ is unbounded in γ.

Take a γ in C. Set $A = Y \cap \gamma$, and let $n \in \omega$ and $\beta \in \gamma$. We want to show that there is an $\alpha > \beta$ which is in $A \cap f_{\gamma}^{-1}(\{n\})$. Such an α will certainly be in $X \cap f_{\gamma}^{-1}(\{n\})$ as well.

It is enough to note that the set

$$D = \{\langle p, q\rangle : (\exists k \in \mathrm{dom}(p) \cap \mathrm{dom}(q))(\exists \alpha > \beta)\alpha \in A \wedge p(k) = n \wedge q(n) = \alpha\}$$

is a dense subset of $\mathbb{P}$ and, at the same time, an element of M.★

Before proving Theorem 4.3.1, let us state two relevant definitions:

A subset A of a space is *discrete* if for every point of A, there is an open neighboorhood of that point which does not contain any other points of A.

A disjoint family of open sets *separates* points from A, if for every point in A, there is an element of the family which intersects A exactly in that point.

Proof of Theorem 4.3.1. Let M be a model of set theory and $M[G]$ an extension of M by a Cohen real. Then $M[G]$ satisfies $\clubsuit'$.

In $M[G]$, we construct a 0-dimensional, 1^{st} countable topology τ on ω_1, in which ω_1 is a locally compact, perfectly normal S-space. First, we fix an increasing enumeration $\langle \gamma_\alpha : \alpha < \omega_1\rangle$ of limit ordinals in ω_1. For each $\alpha < \omega_1$, let r_{γ_α} and f_{γ_α} be a sequence and a function as guaranteed by $\clubsuit'$.

Now, by induction on $\alpha \in \omega_1$, we construct a sequence $\langle \tau_\alpha : \alpha < \omega_1\rangle$ where for each $\alpha < \beta < \omega_1$, τ_α is a subset of τ_β. Each τ_α is a 0-dimensional, locally compact and metrizable topology. At the end, to get τ, we simply take the union of all τ_α, for $\alpha < \omega_1$. It is obvious that τ is a locally countable, 1st countable topology on ω_1. Now we explain how to choose τ_α so that τ becomes 0-dimensional and ω_1, τ becomes a perfectly normal an S-space.

For $\alpha = 0$, τ_α is simply the discrete topology on $\omega = \gamma_0$.

If α is a limit ordinal, then $\gamma_\alpha = \sup_{\beta<\alpha} \gamma_\beta$, and we let τ_α be the union of all τ_β for $\beta < \alpha$. It is then obvious that τ_α is a 0-dimensional, metrizable and locally compact topology on γ_α.

Otherwise, $\alpha = \beta + 1$ for some β. We enumerate $\gamma_\alpha \setminus \gamma_\beta$ as $\{x_n : n \in \omega\}$, and we consider the function $f = f_{\gamma_\beta}$ and the sequence $r = r_{\gamma_\beta}$. Let $\{r_k : k \in \omega\}$ be an enumeration of r. Note that r is a closed discrete subspace of $(\gamma_\beta, \tau_\beta)$. Then, in the 0-dimensional locally compact space γ_β, r can be separated by a disjoint family $\{u_k : k \in \omega\}$ of clopen compact sets, such that for each $k \in \omega$, $r_k \in u_k$.

We define the point base at each x_n to be the family of all sets O^n_m of the form

$$O^n_m = \{x_n\} \bigcup \cup \{u_k : f(r_k) = n \wedge k > m\},$$

where $n, m \in \omega$.

The above, and the requirement that $\tau_\beta \subseteq \tau_\alpha$, uniquely determine τ_α. It is easy to see that so defined τ_α is locally compact and metrizable. Let us check that it is 0-dimensional as well.

First note that we can fix a clopen basis $\mathcal{B}$ for γ_β, such that each element of $\mathcal{B}$ intersects only finitely many of the sets $\{u_k : k \in \omega\}$. Now define $\mathcal{C}$ to be $\mathcal{B}$ together with all the sets of the form $B \cap O^n_m$, where B is either in $\mathcal{B}$ or is equal to γ_α, and $n, m \in \omega$.

Now, we easily verify that every element of $\mathcal{C}$ is clopen and that $\mathcal{C}$ covers γ_α. To see that $\mathcal{C}$ is a clopen basis for γ_α, it is enough to check that for all $x \in \gamma_\beta$, and $C \in \mathcal{C}$ which does not contain x, there is a neighborhood of x which avoids C.

There are several cases to consider, and the only non-trivial one is when $x \in \gamma_\beta$ and C is of the form $B \cap O^n_m$, for some $B \in \mathcal{B}$ and $n, m \in \omega$. That case is taken care of by requiring that B can only intersect finitely many of the u_k, so we can pick a neighborhood of x and subtract all the relevant u_k.

In the so constructed topology, ω_1 is not HL or even Lindelöf: the countable limit ordinals are an open cover without a countable subcover. To prove that ω_1 is HS is harder, and that is where the $\clubsuit'$ comes in.

Given an uncountable X in ω_1, we shall not only prove that it is separable, but, moreover, that for some $\gamma < \omega_1$, the closure of $X \cap \gamma$ is the entire $\omega_1 \setminus \gamma$.

So, fix such an X, and, by $\clubsuit'$, find a limit γ such that for al $n \in \omega$, $f^{-1}_\gamma\{n\}$ is unbounded in $X \cap \gamma$. We claim that the closure of $X \cap \gamma$ contains all ordinals δ which are $\geq \gamma$.

To prove this claim, let α be such that $\gamma = \gamma_\alpha$. Do an induction on $\delta \geq \gamma$. Assume that for all $\beta \in [\gamma, \delta)$, β is in the closure of X.

Let η such that $\gamma_\eta \leq \delta < \gamma_{\eta+1}$. If $\eta = \alpha$, then $\delta = x_n$ for some $n \in \omega$. If U is a basic open neighborhood of x_n, then $U = \{x_n\} \bigcup \cup \{u_k : k > m \wedge f(r_k) = n\}$, for some $m \in \omega$. The x_n and u_n are as defined at the stage α of the construction.

By the choice of f and because $r_k (k \in \omega) \cap X$ is cofinal in $X \cap \gamma$, there is a $k > m$ such that $f(r_k) = n$ and $r_k \in X$. So, δ is in the closure of X.

If $\eta > \alpha$, we can by a similar argument show that δ is in the closure of the set $S_\eta = \gamma_\eta \setminus \gamma_\alpha$. But, by the induction hypothesis, every element of S_η is in the closure of X, so δ is as well.

This finishes the proof of the claim. The perfect normality of the space also follows from this claim, as the claim implies that every closed set in ω_1, τ is either countable or co-countable.★

We note that, exactly as in the original proof of Ostaszewski, with the additional assumption of CH, the construction can be modified so that ω_1 becomes countably compact.

The space of Ostaszewski cannot help us build a counter-example of Question H. That is, suppose that K is a compactification of the Ostaszewski space X and μ a non-separable Radon probability on it. Being locally compact, X embeds in every of it compactifications as an open, so Radon measurable, subspace. Then, since X is scattered, the measure algebra of ν restricted to X must be separable. So, just knowing what μ looks like on X, would not be enough to rebuild μ on K.

Under CH, more is true: no compactification of the Ostaszewski space can suport a non-separable Radon probability measure. If CH is true, the Ostaszewski space is countably compact and all its compactifications are equivalent to its one-point compactification.

To construct an Ostaszewski space just from CH, is a well known open question.

As pointed out to us by J. Roitman, $\clubsuit'$ alone is not enough to construct a countably compact Ostaszewski space. The reason is that adding a Cohen real does not change $\mathfrak{p} > \omega_1$, while no thin-tall locally compact scattered regular space is countably compact if $\mathfrak{p} > \omega_1$ is true.

References

[1] M. Džamonja and K. Kunen, Measures on Compact HS Spaces, *Fundamenta Mathematicae*, in print.

[2] R. Engelking, General Topology, *Wroclawska Drukarnia Naukowa*, 1977.

[3] В. В. Федорчук, О мощности наследственно цепарабельных бикомпактов, *Доклады Академии Наук СССР* 222 (1975) 302–305.

[4] D. Fremlin, Consequences of Martin's Axiom, *Cambridge University Press*, 1984.

[5] D. Fremlin, Measure Algebras, in *Handbook of Boolean Algebras* (J. D. Monk and R. Bonnet, eds.), Elsevier Science Publishers B.V., 1989, vol. 3, 877-980.

[6] A. Hajnal and I. Juhász, On first countable non–Lindelöf S–spaces, *Coll. Math. Soc. János Bolyai* 10 (1973) 837–852.

[7] P. Halmos, Boolean Algebras, *D. Van Nostrand Company, Inc.*, 1963.

[8] P. Halmos, Measure Theory, *D. Van Nostrand Company, Inc.*, 1950.

[9] R. Haydon, On dual L^1–spaces and injective bidual Banach spaces, *Israel J. Math* 31 (1978) 142–152.

[10] I. Juhász, Cardinal Functions in Topology, *Mathematical Centre Tracts 34*, Mathematisch Centrum, Amsterdam, 1975.

[11] I. Juhász, Cardinal Functions in Topology -Ten Years Later, *Mathematical Centre Tracts 123*, Mathematisch Centrum, Amsterdam, 1980.

[12] I. Juhász, K. Kunen and M.E. Rudin, Two more hereditarily separable non–Lindelöf spaces, *Can. J. Math* 28 (1976) 998–1005.

[13] K. Kunen, A compact L–space under CH, *Topology Appl.* 12 (1981) 283–287.

[14] K. Kunen, Set Theory, An Introduction to Independence Proofs, *North-Holland*, Amsterdam 1980.

[15] D. J. Lutzer and H. R. Bennett, Separability, the countable chain condition and the Lindelöf property in linearly orderable spaces, *Proc. Amer. Math. Soc.* 23 (1969), 664-667.

[16] D. Maharam, On homogeneous measure algebras, *Proc. Nat. Acad. Sci. USA* 28 (1942) 108–111.

[17] S. Mardešić and P. Papić, Continuous images of ordered compacta, the Suslin property and diadic compacta, *Glasnik Mat.* 17 (1962) 3–25.

[18] A. J. Ostaszewski, On countably compact perfectly normal spaces, *J. London Math. Soc. (2)* 14(1976), 505-516.

[19] J. Roitman, Basic S and L, in *Handbook of Set-Theoretic Topology* (K. Kunen and J. Vaughan, eds.), North-Holland, 1984, 295–326.

[20] J. Roitman, Autohomeomorphism groups of Ostaszewski spaces, *Topol. Appl.*, in print.

[21] Б. Э. Шапировский, Канонические множества и характер. Плотност и вес в бикомпактах, *Доклады Академии Наук СССР*, 1974, 218:1, 58-61.

[22] Б. Э. Шапировский, Об Отображениях на Тихоновские Кубы, *Успехи Математических Наук.* 35 (1980) НО. 3, 122-130.

[23] S. Shelah, Whitehead groups may not be free, even assuming CH, II. *Israel J. Math.* 35(1980), no.4, 257-285.

[24] W. Sierpinski, Sur l'équivalence de trois propriétés des ensembles abstraits, *Fundamenta Mathematicae* 2(1921), 179-188.

[25] Z. Szentmiklòssy, S-spaces and L-spaces under Martin's axiom, *Coll. Math. Soc. János Bolyai* 23(1978) Budapest II, 1139-1145. (North-Holland, Amsterdam 1980).