Structural and topological aspects of the enumeration and hyperenumeration degrees

by

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Abstract

This thesis focuses on compatibility theory, mainly looking at enumeration reducibility and related topics. In Chapter 2 we η-representable sets. We delevope the notion of a connected approximation and use it to give a characterization of the strongly η-s-representable sets and new characterizations other classes of η representable sets. The main result involves characterizing the many-one degrees of strongly η-representable sets.

In Chapter 3 we study the enumeration degrees. In particular we present work towards understanding the $\forall\exists$ theory of the enumeration degree. We study what types of minimal pairs are possible in the enumeration degrees. These are important questions if we want to find a decision procedure for the $\forall\exists$ theory of the enumeration degrees, and if it turns out that no such procedure exists, then these questions become more significant. We prove that there are strong minimal pairs in the enumeration degrees and that there a no strong super minimal pairs in the enumeration degrees.

In Chapter 4 we explore topological aspects of enumeration reducibility. This chapter builds on work from Kihara, Ng and Pauly [26] and answers several open questions that they asked.

A point in a represented second-countable T_0 space can be identified with the set of basic open sets containing that point. By representing a point by an enumeration of the indices of the basic open sets containing that point we can consider the enumeration degrees of the points in a second-countable T_0 space. For example, the ω-product of the Sierpiński space is universal for second-countable T_0 spaces and gives us all enumeration degrees and the Hilbert cube gives us all continuous degrees.

Kihara, Ng, and Pauly [26] have studied various classes that arise from different spaces. They show that any enumeration degree is

contained in a class arising from some decidable, submetrizable space, and that no T_1-space contains all enumeration degrees. We call a class of degrees a $\mathcal{T}$ class if it comes from a $\mathcal{T}$ space. So Kihara, Ng, and Pauly show that $\mathcal{D}_e$ is not T_1. Similarly they separate T_2 classes from T_1 classes and $T_{2.5}$ classes from T_2 classes by showing that no T_2 class contains all the cylinder-cototal degrees and no $T_{2.5}$ class contains all degrees arising from $(\mathbb{N}_{\mathrm{rp}})^\omega$.

We extended these results to show that the cylinder-cototal degrees are T_2-quasi-minimal and the $(\mathbb{N}_{\mathrm{rp}})^\omega$ degrees are $T_{2.5}$ quasi-minimal. We then give separations of the $T_{2.5}$ classes from the submetrizable classes using the Arens co-d-CEA degrees and the Roy halfgraph degrees.

In Chapter 5 we present joint work with Goh, Miller and Soskova on e-pointed trees and their enumeration degrees. E-pointed trees arise in compatible model theory and have had several applications to the study of enumeration and hyperenumeration reducibility. In Chapter 6 we show how e-pointed trees can be use to show that the analogue of Selman's theorem is false for hyperenumeration reducibility. We also add to the structural knowledge of the hyperenumeration degrees by proving that they are a downwards dense degree structure.

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Contents

Chapter 1

Introduction

Coding one type of mathematical object into another is a common theme throughout mathematics. For instance, every Boolean algebra can be coded as a ring, and Fourier series are a way of coding periodic functions as vectors in ℓ_2. In computability, the basic objects studied are subsets of ω. Subsets of ω can be coded as graphs using daisy graphs: we create an isolated vertex with n many loops for each $n \in A$. Points in a second order topological space can be coded at subsets of ω via a countable basis.

Computability theory gives us a way of analyzing the effectiveness of such encodings and how easy it is to go from an encoding of a structure or set back to the original. For example, in the case of daisy graphs we can characterize the sets with computable daisy graph as the c.e. sets. In chapter 2 we consider a particular way of encoding sets into a linear order known as an η-representation. We characterize the sets that have a computable strong η-representation up to many-one degree.

In the case of topological spaces, we have a collection of points, so it is natural to consider the classes of sets that can be encoded in a particular space with fixed basis. In chapter 4 we look at the interaction between computability theoretic properties of the class of sets that arise in this way and the topological properties of the underlining space.

Turing reducibility is concerned with total functions in either ω^ω or 2^ω, but the Turing operators that give these reductions can sometimes produce partial functions. Thus it is natural to ask if Turing reducibility can be extended to partial functions. One way of doing this, independently introduced several times [12, 41, 36], is enumeration reducibility.

Enumeration reducibility has been used to capture the complexity of problems that are not easily represented by a Turing degree. For instance, the degree of difficulty of computing a copy of a structure can sometimes be an enumeration degree, but not a Turing degree [37]. Another example is the degree of difficulty of computing a presentation of a continuous function. Miller [34] characterized these degrees as the continuous degrees and showed they are a proper subclass of the enumeration degrees.

This thesis is concerned with the study of enumeration reducibility, its hyperarithmetic analogue, and how enumeration reducibility interacts with effective topology.

1.1 η-representations

In chapter 2 we look at computable η representations of sets. The idea of an η-representation was introduced by Fellner [11].

Definition 2.1.1. For a set A a linear order L is said to be an *η-representation* of A if there is a surjective function $F : \omega \to A$ such that L has order type

$$\sum_{n \in \omega} \eta + F(n)$$

where η is the order type of $\mathbb{Q}$. We say L is a *strong* η-representation if the function F is strictly increasing and an *increasing* η-representation if F is non-decreasing. If a set A has a computable (strong, increasing) η-representation then we say A is *(strongly, increasingly) η-representable*. A degree is (strongly, increasingly) η-representable if it contains a set that is (strongly, increasingly) η-representable.

Characterizations of general η-representable sets [10] and of increasingly η-representable sets [21] are known, but there is no known characterization of the strongly η-representable sets. This has been an open question since they were first introduced. Notably, strong η-representations are the only type of η-representation that has a unique order type for any representation of a set A and their study predates the study of general η-representations.

We look at a simpler version of this question, and study η-s-representable sets. A set is η-s-representable if it has a computable η-representation with computable successor relation. The hope is that we can find a characterization for the strongly η-s-representable sets and relativise this to a characterization of the η-s-representable sets. There is reason to believe that this is a good approach, the previous characterizations of η-representable sets and increasingly η-representable sets can be viewed as relativizations of characterizations of η-s-representable sets. As we show in Theorem 2.2.2, this is because the constructions involved in these characterization create the blocks of the η-representation in isolation, and thus the block relation is $\mathbf{0}'$-computable.

We characterize the sets with computable strong η-representation and computable block relations as precisely those in $\mathbf{SSILM}(\mathbb{Q})^{\mathbf{0}'}$, a class introduced by Kach and Turetsky when characterizing the increasingly η-representable sets [21]. It is known that the stongly η-representable sets are a stricly larger class than the sets in $\mathbf{SSILM}^{\mathbf{0}'}$ [9]. This means than a characterization of the strongly η-s-representable sets may not relativise, and that the any characterization must allow for blocks merging.

With this idea of blocks merging in mind, we come up with the notion of a *connected approximation*. We use connected approximations to come up with new characterizations of the increasingly η-representable and η-representable sets, and to give a characterization of the strongly-η-s-representable sets. We do not know if this characterization relativises or not.

Towards characterizing the degrees of strongly-η-representable sets we prove that any dense enough set, if it has a computable increasing η-representation then it is in $\mathbf{SSILM}(\mathbb{Q})^{\mathbf{0}'}$. This allows us to characterize the many-one degrees with computable strong η-representations as precisely the degrees with sets in $\mathbf{SSILM}(\mathbb{Q})^{\mathbf{0}'}$.

1.2 Enumeration reducibility

Most of this thesis is focused on enumeration reducibility and classes of enumeration degrees. Enumeration reducibility ($\leq_e$) is a reducibility that captures the notion of how difficult it is to enumerate a given set of numbers. There are several definitions, but the one we find most useful is the one given by Friedberg and Rogers [12].

Definition 1.2.1. For sets $A, B \subseteq \omega$ we say that $A \leq_e B$ if there is a c.e. set of axioms W such that:

$$n \in A \iff \exists u[\langle n, u\rangle \in W \wedge D_u \subseteq B]$$

Here $(D_u)_u$ is the collection of all finite sets given by strong indexes.

One useful property of this definition is that it gives us a collection of enumeration operators $(\Psi_e)_e$. We define $A = \Psi_e(B)$ if $A \leq_e B$ via the eth c.e. set W_e. Enumeration reducibility is a reducibility on the positive information about a set. This can be seen by the fact that if $A \subseteq B$ then $\Psi_e(A) \subseteq \Psi_e(B)$.

Enumeration reducibility is a pre-order and the equivalence classes form an upper semi-lattice $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets and joins given by the usual operation. There is also an enumeration jump given by $A \mapsto K_A \oplus \overline{K_A} = \bigoplus_e \Psi_e(A) \oplus \bigoplus_e \overline{\Psi_e(A)}$. Like with Turing jump, we have that $A <_e A'$.

One aspect of enumeration reducibility that has been well studied is its relationship with Turing reducibility. The Turing degrees embed into the enumeration degrees via the map induced by $A \mapsto A \oplus \overline{A}$. This follows from the fact that $A \oplus \overline{A} \leq_e B \oplus \overline{B} \iff A \leq_T B$. This embedding is known to be a proper embedding [33], and the Turing and enumeration jump coincide on these degrees. The image of the Turing degrees is known as the total degrees:

Definition 1.2.2. We say that a set A is *total* if $\overline{A} \leq_e A$. We say that A is cototal if $A \leq_e \overline{A}$. A degree is *total* (*cototal*) if it contains a total (cototal) set.

It is known that the total degrees are a proper subclass of the cototal degrees and that the cototal degrees are a proper subclass of all enumeration degrees [1].

While there are many similarities between these classes of degrees they are structurally different. One notable difference is the fact that, while there are minimal Turing degrees, Gutteridge [16] proved that the enumeration degrees are downwards dense. Gutteridge's proof does not relativize though, and later Cooper [8] showed that there are empty intervals in the enumeration degrees.

We have seen that Turing reducibility can be defined in terms of enumeration reducibility. An important early result of Selman [41] shows how to define enumeration reducibility in terms of Turing reducibility.

Theorem 1.2.3 (Selman's Theorem)**.** *$A \leq_e B$ if and only if, for all X if $B \leq_e X \oplus \overline{X}$ then $A \leq_e X \oplus \overline{X}$.*

This theorem states that an enumeration degree is uniquely determined by the class total degrees above it. This means that the total degrees form an automorphism base for the enumeration degrees.

1.3 The theory of the enumeration degrees

Slaman and Woodin [42] have proven that the full theory of the enumeration degrees is one-equivalent to the theory of second order arithmetic, so we know that is as complicated as possible.

When looking at the theory of the enumeration degrees with bounded quantifier complexity, there are two main results. First, Lagemann [28] showed that every finite lattice embeds into the enumeration degrees, thus the $\exists$-theory of the e-degrees is decidable. Later Kent [24] proved that the $\exists\forall\exists$-theory is undecidable. It is an open question if the $\exists\forall$-theory of the enumeration degrees is decidable. For a partial order, it is known that the $\exists\forall$-theory is equivalent to the following question:

Question 1.3.1 (Generalized extension of embeddings)**.** Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_0, \ldots, \mathcal{Q}_{k-1}$ is it true that every embedding of P into $\mathcal{D}$ can be extended to $\mathcal{Q}_i$ for some $i < k$?

The case when $k = 1$ is known as the extension of embedding problem. Lempp, Slaman and Soskova [29] proved that the extension of embeddings problem is decidable for the e-degrees. If there one wants to show that there is an algorithm to solve the generalized extension of embeddings problem then a first step might be to show what extensions can be forbidden by an embedding of the diamond that preserve $\mathbf{0}$.

We make some steps in this direction in Chapter 3. In Chapter 3 we look at different types of minimal pairs.

Definition 3.1.1. In an upper semilattice with least element $\mathbf{0}$ a pair $\mathbf{a}, \mathbf{b} > \mathbf{0}$ is a:

- *minimal pair* if $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.
- *strong minimal pair* if it is a minimal pair, and for all $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ we have $\mathbf{x} \vee \mathbf{b} = \mathbf{a} \vee \mathbf{b}$.
- *super minimal pair* if both $\mathbf{a}, \mathbf{b}$ and $\mathbf{b}, \mathbf{a}$ are strong minimal pairs.
- *strong super minimal pair* if it is a minimal pair, and for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ and $\mathbf{0} < \mathbf{y} \leq \mathbf{b}$ we have $\mathbf{x} \vee \mathbf{y} = \mathbf{a} \vee \mathbf{b}$.

It is clear that if $\mathbf{a}$ and $\mathbf{b}$ are distinct minimal degrees then $\mathbf{a}$ and $\mathbf{b}$ form a strong super minimal pair, so the question of the existence of these types of minimal pairs is only of interest in upper semilattices with downward density, like the enumeration degrees, the enumeration degrees below $\mathbf{0}'$ ($\mathcal{D}_e(\leq \mathbf{0}')$) and the c.e. Turing degrees. In the case of the c.e. degrees, recent work by Cai, Liu, Liu, Peng and Yang [6] proves that there are no strong minimal pairs.

In the case of the enumeration degrees, the motivation for studying strong super minimal pairs came from an idea towards an algorithm to decide the $\exists\forall$ theory of the e-degrees that Lempp, Slaman and Soskova had that required these to exist. We put an end to this idea by proving that the e-degrees do not have any strong super minimal pairs in Chapter 3.

Following on from this we asked if it was possible to find a strong or super minimal pairs in the enumeration-degrees. It still remains an open question if there are any super minimal pairs in the enumeration degrees, but we give a construction of a strong minimal pair in Chapter 3.

It was known earlier, though unpublished, that $\mathcal{K}$-pairs could be used to show that there are strong minimal pairs in the enumeration degrees. $\mathcal{K}$-pairs are a useful tool in the study of the enumeration degrees and were first introduce by Kalimullin [23] when he proved that the jump is definable. They have since been used to prove that the total degrees are definable [13], and we make use of $\mathcal{K}$-pairs in our proof that there are no strong super minimal pairs. We include this use of $\mathcal{K}$-pairs to construct a strong minimal pair in Chapter 3.

The new construction of a strong minimal pair that we give is more direct than the one using $\mathcal{K}$-pairs. We modify our construction to build a strong minimal pair A, B where A is Σ^0_2 and B is Π^0_2. The $\mathcal{K}$-pair construction of a strong minimal pair gives a pair A, B where A is Π^0_2 and $B = \emptyset'$. This means that both sides of a strong minimal pair can be Σ^0_2, however, it is open whether or not there is a strong minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$.

1.4 Topological classes of enumeration degrees

A subclass of the cototal degrees that has been studied is the continuous degrees. These were introduced by Miller [34] as a way of characterizing the degree of difficulty of producing a representation of a point in a computably represented metric.

The question of whether there are points whose degree spectra have no least element was asked by Pour-El and Lempp (Specifically for the space of continuous functions on $\mathbb{R}$). It was known that for spaces like $\omega^\omega, 2^\omega$ and $\mathbb{R}$ every point has a least Turing degree, but not for spaces like Hilbert's cube $[0,1]^\omega$ or $C[0,1]$. Miller [34] answered this question in the affirmative, showing that the Turing degrees are not sufficient to capture points in $[0,1]^\omega$, but that the enumeration degrees are. Later, Andrews, Ganchev, Kuyper, Lempp, Miller, Soskova and Soskova [1] proved that every continuous degree is cototal.

The continuous degrees turn out to arise in other natural ways, for instance Andrews, Igusa, Miller and Soskova [2] characterize the continuous degrees in purely degree theoretic terms as the almost total degrees.

Kihara and Pauly [27] extended this idea and study the degrees of points in arbitrary represented topological spaces. Of particular interest to us are the degrees that arise when using the notion of a countably based space:

Definition 4.1.1. A cb_0 space $\mathcal{X}$ is a second countable T_0 space given with a listing of a basis $(\beta_e)_e$. Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ and a point $x \in X$ the coded neighborhood filter of x is $\mathrm{NBase}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$. We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{NBase}(x) \in \mathbf{a}]\}$.

Kihara and Pauly [27] showed that $\mathcal{D}_e$ is the class of degrees of the ω-product of Sierpiński space $\mathbb{S}^\omega$, where $\mathbb{S} = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$. This a universal second countable T_0 space, so we can relate the study of classes of enumeration degrees to the study of second countable T_0 spaces.

Kihara, Ng and Pauly [26] looked at many cb_0 spaces from classical topology to expand the zoo of enumeration degrees. They discovered some new classes, as well as spaces that give rise to some previously studied classes. Some new classes that are of particular interest in this thesis are the cocylinder degrees, the doubled co-CEA degrees, the Arens co-d-CEA degrees and the Roy halfgraph above degrees. We also look at the degrees of the relatively prime integer topology, $\mathbb{N}_{\mathrm{rp}}$.

Kihara, Ng and Pauly [26] also looked at topological separation axioms and how they interact with classes of enumeration degrees. The separation axioms that we explore are as follows.

Definition 4.1.2. A topological space is considered

- T_0 (Kolmogorov) if for any $x \neq y$ there is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$. In other words, points can be distinguished by the topology.
- T_1 (Fréchet) if for any $x \neq y$ there are open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Equivalently if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \in V$.

- $T_{2.5}$ (Urysohn) if for any $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.
- *Submetrizable* if there is a coarser topology on the space that is metrizable. In other words, if $\mathcal{X} = (X, (\beta_e)_e)$ is submetrizable then there is a collection of $\mathcal{X}$-open sets $(\alpha_e)_e$ such that $(X, (\alpha_e)_e)$ is metrizable.

Kihara, Ng and Pauly [26] showed that for any enumeration degree $\mathbf{a}$ there is a decidable, effectively submetrizable cb$_0$ space $\mathcal{X}$ such that $\mathbf{a} \in \mathcal{D}_\mathcal{X}$. Similarly there is a (non-decidable) metric space $\mathcal{Y}$ such that $\mathbf{a} \in \mathcal{D}_\mathcal{Y}$. So even if we require the topology to be decidable, the non-metrizable separation axioms do not give rise to any new classes of degrees like the continuous degrees.

These non-metrizable separation axioms may not give us new classes of degrees, but we can use the separation axioms to classify classes of degrees.

Definition 4.1.3. Given a collection of cb$_0$ spaces $\mathcal{T}$ we say that a class $\mathcal{C}$ of enumeration degrees is $\mathcal{T}$ if there is some $\mathcal{X} \in \mathcal{T}$ such that $\mathcal{D}_\mathcal{X} = \mathcal{C}$.

We have the same implications of the separation axioms, but because multiple different cb$_0$ spaces may give rise to the same class of degrees, it is not clear that these implications are strict. In fact, Kihara, Ng and Pauly [26] considered another separation axiom, the notion of a T_D space. For second countable spaces T_D lies strictly between T_0 and T_1. However they show that for any T_D cb$_0$ space $\mathcal{X}$ there is a T_1 space $\mathcal{Y}$ such that $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$, so this is a case of two topological separation axioms that are distinct for second countable spaces, but not for classes of enumeration degrees.

Kihara, Ng and Pauly [26] gave some separations for this classification of classes of degrees. They showed that $\mathcal{D}_e$ is T_0 but not T_1, that cylinder-cototal degrees are T_1 but not T_2, and that $\mathcal{D}_{\mathbb{N}^\omega_{\mathrm{rp}}}$ is T_2 but not $T_{2.5}$. They did not show $T_{2.5}$ and submetrizable are different notions for classes of degrees and asked as a question if there is a $T_{2.5}$ class that is not submetrizable. They also showed that the degrees of the Gandy-Harrington topology do not arise from any metrizable space, giving a separation between submetrizable and metrizable for classes of degrees.

Kihara, Ng and Pauly [26] suggested some candidates for classes that could be $T_{2.5}$ but not submetrizable. They introduced the Arens co-d-CEA degrees and the Roy halfgraph above degrees. Both classes arise from spaces that are $T_{2.5}$ but not submetrizable.

In chapter 4 we answer several questions from Kihara, Ng and Pauly's paper [26]:

- We prove that the Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable, separating classifications for classes of degrees. In the proof of these results we introduce a general method that could be used to get similar results. As part of the general method we introduce the notion of a space being effectively submetrizable.

- We show that the doubled co-CEA degrees are not $T_{2.5}$, giving a quasi-polish separation of T_2 and $T_{2.5}$ for classes of degrees. We prove the Arens co-d-CEA degrees and the Roy halfgraph degrees are distinct classes of degrees, neither contained in the other.

- We improve on two of the separations given by Kihara, Ng and Pauly, showing that the cylinder-cototal degrees are T_2-quasi-minimal and that $\mathcal{D}_{\mathbb{N}_{\text{rp}}}$ is $T_{2.5}$-quasi-minimal.

In Chapter 4 we also consider the degrees that can arise from decidable, metrizable cb_0-spaces. Kihara and Pauly [27] observed that if one takes as a basis the balls of rational radius centered at points in the chosen countable dense set, then the degree of a point with the cb_0 representation will coincide the degree of that point as Miller [34] defined it. It is notable that a basis taken in this way will always be decidable, but we prove that a decidable basis need not arise in this way. We give an example of a decidable, metrizable cb_0-space that has a point with a degree that in not continuous. We do not know if all enumeration degrees are degrees of some decidable, metrizable cb_0-space.

1.5 E-pointed trees

E-pointed trees were studied by McCarthy [32] and were used to characterize the cototal enumeration degrees.

Definition 5.1.1. A tree T is *e-pointed* if for every path $P \in [T]$ we have that $T \leq_e P$. We say T is *uniformly* e-pointed if there is a single enumeration operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy [32] was interested in e-pointed trees on Cantor space. He proved that every e-pointed tree on $2^{<\omega}$, possibly with dead ends, is a cototal set, and characterized the cototal degrees as the degrees of uniformly e-pointed trees on $2^{<\omega}$ without dead ends. E-pointed trees have been used in computable model theory, notably Montalbán [35] used them to prove that the degree spectrum of a structure is never the upward closure of an F_σ set unless it is an enumeration cone. In Chapter 6 we give another application of e-pointed trees, this time on Baire space, in the study of the hyperenumeration degrees.

In Chapter 5 we look at e-pointed trees that are subsets of ω^ω. To distinguish them from the e-pointed trees on Cantor space we will refer to them as Baire e-pointed trees. The results in this chapter were obtained in collaboration with Jun Le Goh, Joseph Miller and Mariya Soskova [14].

The degrees containing Baire e-pointed trees turn out to characterize a strictly larger set of degrees that comes from the notion of hyperenumeration reducibility. We prove that every Baire e-pointed tree is a hypercototal set, and that every hypercototal enumeration degree contains a uniformly e-pointed tree. We define hyperenumeration enumeration reducibility and hypercototal sets in Section 1.6 as well as the new results about hyperenumeration reducibility that we prove in Chapter 6.

Unlike in the case of e-pointed trees on 2^ω, we prove in Chapter 5 that there is a hypercototal enumeration degree that is not the degree of any Baire e-pointed tree *without* dead ends. Requiring the e-pointed tree to have no dead ends may reduce the class of degrees, but we prove that there is a uniformly Baire e-pointed tree without dead ends that is not of cototal degree, so it is a strictly larger class than the degrees of e-pointed trees on 2^ω. It is an open question whether every Baire e-pointed tree without dead ends must be enumeration equivalent to a uniformly Baire e-pointed tree without dead ends.

Also of interest to us in Chapter 5 are introenumerable sets. These are a variation on introreducible sets.

Definition 5.1.2. A set A is *introenumerable* if for all infinite $S \subseteq A$, $A \leq_e S$. A set A is *uniformly* introenumerable if there is an enumeration operator, Ψ_e, such that $A = \Psi_e(S)$ for all infinite $S \subseteq A$.

These were first introduced by Jockusch [20], although with a slightly different definition. Introenumerable sets have been studied by Greenberg, Harrison-Trainor, Patey and Turetsky [15] who showed, among other results, that both Jockusch's and our definitions are equivalent in the uniform case. From recent conversations with Turetsky we have established that the two definitions are also equivalent in the non-uniform case.

We show that the introenumerable degrees lie between the cototal and hypercototal degrees. We prove that a uniformly e-pointed tree on 2^ω without dead ends is an introenumerable set and construct a uniformly introenumerable set that is not of cototal enumeration degree. Every set S is enumeration equivalent to the set of finite increasing enumerations of subsets of S. These form an ω-branching tree whose paths are all enumerations of infinite subsets of S. If S is (uniformly) introenumerable then this tree will be (uniformly) e-pointed. To complete the separation we give a construction of a uniformly e-pointed tree without dead ends that is not enumerations equivalent to any introenumerable set. It is an open question whether there is a degree that contains an introenumerable set, but does not contain any uniformly introenumerable set.

We link these results back to those in Chapter 4 by proving that all these classes are all T_1 classes of degrees.

1.6 Hyperenumeration reducibility

Sanchis [39] introduced the notion of hyperenumeration reducibility $\leq_{he}$, an analogue of enumeration reducibility relating to hyperarithmetic reducibility rather than Turing reducibility.

Definition 1.6.1. [Sanchis 1978 [39]] We say that $A \leq_{he} B$ if there is a c.e. set W such that

$$n \in A \iff \forall f \in \omega^\omega \exists u \in \omega, x \prec f[\langle n, x, u \rangle \in W \wedge D_u \subseteq B]$$

Sanchis proved that $\leq_{he}$ is a pre-order, giving rise to the hyperenumeration degrees $\mathcal{D}_{he}$. These have a similar relationship with the hyperarithmetic degrees to the relationship the enumeration degrees have with the Turing degrees. We can define notions of hypertotal and hypercototal.

Definition 5.2.3. We say that a set A is *hypertotal* if $\overline{A} \leq_{he} A$. We say that A is *hypercototal* if $A \leq_{he} \overline{A}$. A degree is *hypertotal* (*hypercototal*) if it contains a hypertotal (hypercototal) set.

Sanchis [39] proved that the map $A \mapsto A \oplus \overline{A}$ induces an embedding of the hyperarithmetic degrees into the hyperenumeration degrees as the hypertotal degrees. The main result of Sanchis' paper was proving that there is a hyperenumeration degree that is not a hypertotal degree.

In Chapter 6 we look at a couple of aspects of the relationships between Turing reducibility and enumeration reducibility and see if they also hold for the relationship between hyperarithmetic reducibility and hyperenumeration reducibility.

First we consider Selman's theorem. If Selman's theorem held, it would allow us distinguish hyperenumeration degrees by the set of hypertotal degrees above them, and allow us to define hyperenumeration reducibility in terms of the degree of difficulty of producing a Π^1_1 presentation of a set. However, it turns out that Selman's theorem fails for hyperenumeration reducibility.

Corollary 6.2.8. *There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.*

The proof of this works by constructing a uniformly e-pointed tree without dead ends that is not of hypertotal degree.

The second aspect of enumeration reducibility that we explore for hyperenumeration reducibility is downwards density. In this case we are successful in adapting Gutteridge's original proof [16] to this context. In the process, we describe a problem that arises when trying to do priority constructions for hyperenumeration reducibility and give a method that can be used to solve this problem in cases like downwards density.

Because Selman's theorem fails for the *he*-degrees, it makes sense to question if this is the right notion of hyperenumeration reducibility even if it has some applications, like characterizing the degrees of e-pointed trees. We look at some other natural reducibilities that could be considered hyperarithmetic analogues of enumeration reducibility, and we consider their relationship to hyperarithmetic and enumeration reducibility.

Chapter 2

A Characterization of the Strongly η-Representable Many-One Degrees

2.1 Introduction and History

In this chapter we study η-representations. The definition of an η-representation was first introduced by Fellner [11].

Definition 2.1.1. For a set A a linear order L is said to be an *η-representation* of A if there is a surjective function $F : \omega \to A$ such that L has order type

$$\sum_{n \in \omega} \eta + F(n)$$

where η is the order type of $\mathbb{Q}$. We say L is a *strong* η-representation if the function F is strictly increasing and an *increasing* η-representation if F is non-decreasing. If a set A has a computable (strong, increasing) η-representation then we say A is *(strongly, increasingly) η-representable*. A degree is (strongly, increasingly) η-representable if it contains a set that is (strongly, increasingly) η-representable.

Note that an η-representation of A cannot tell us if 0 or 1 is in A so we will assume that $0, 1 \notin A$ when we are talking about representations of A.

In his thesis, Fellner [11] introduced the notion of a strong η-representation (predating the introduction of general η-representations) and proved that every set with a computable strong η-representation is Δ^0_3 and that every Σ^0_2 and every Π^0_2 set is strongly η-representable.

For the case of general η-representations we first look at the following definitions.

Definition 2.1.2. For any linear order L the successor relation S_L on L is defined by $S_L(x, y) \iff |[x, y]| = 2$. The block relation B_L is given by $B_L(x, y) \iff [x, y]$ and $[y, x]$ are finite. A block of size n in L is a collection $x_0 <_L \cdots <_L x_{n-1}$ such that $B_L(x_0, y) \rightarrow \bigvee_{i<n} y = x_i$.

For any linear order L, one can see that S_L is Π^0_1 in L and B_L is Σ^0_2 in L. Feiner [10] proved the following:

Theorem 2.1.3. *For a linear order L, the set $\{n : L$ has a block of size $n\}$ is Σ^0_3 in L.*

For an η-representation L of a set A, we have $A = \{n : L \text{ has a block of size } n\}$. This gives us the following.

Corollary 2.1.4. *If a set A has a computable η-representation then A is Σ^0_3.*

Coles, Downey and Khoussainov [7] show the reverse of theorem 2.1.3 is true for general linear orders.

Theorem 2.1.5. *For any Σ^0_3 set A there is a computable linear order L, such that $A = \{n : L$ has a block of size $n\}$.*

Fellner [11] showed that every strongly η-representable set is Δ^0_3 and went on to conjecture that every Δ^0_3 set has a strong η-representation. However, Lerman [30] later showed that this is not the case.

Theorem 2.1.6 (Lerman [30])**.** *There is a Δ^0_3 set with no computable η-representation.*

Lerman also characterized the m-degrees with computable η-representations showing that they are the Σ_3^0 degrees:

Theorem 2.1.7 (Lerman [30])**.** *If A is Σ_3^0 then $A \oplus \omega$ has a computable η-representation.*

This left open the questions of what are the (strongly) η-representable sets and what are the strongly η-representable degrees. In the case of η-representations, Harris [17] came up with a characterization involving limitwise monotonic functions. Limitwise monotonic functions were first introduced by Khoussainov, Nies and Shore [25].

Definition 2.1.8. A function $F : \omega \to \omega$ is *limitwise monotonic* if there is a computable function $f : \omega^2 \to \omega$ such that $F(n) = \lim_s f(n, s)$ and for all n, s, $f(n, s) \leq f(n, s+1)$.

By the limit lemma, if F is limitwise monotonic then F is Δ_2^0, and hence if $A =$ range(F) then A is Σ_2^0.

Limitwise monotonic functions have been used to solve questions computable model theory ([18], [23], [25]). In particular Coles, Downey and Khoussainov [7] proved that for any computable η-like linear order (a class that includes computable η-representations) that the set $\{n : L \text{ has a block of size } n\}$ is the range of a $\mathbf{0}'$-limitwise monotonic function. Harris [17] showed the reverse direction holds for computable η-representations.

Theorem 2.1.9. *A set A is η-representable if and only if A is the range of a $\mathbf{0}'$-limitwise monotonic function.*

The construction of the η-representation L is performed uniformly, constructing linear orders $L_n \cong \eta + F(n)$ and taking $L = \sum_n L_n$. From this, it can be seen that if A is the range of a strictly increasing $\mathbf{0}'$-limitwise monotonic function, then A is strongly η-representable. However, Harris [17] showed that this is not a characterization of the strongly η-representable sets.

Harris also showed that the degrees with computable strong η-representations are not trivial.

Theorem 2.1.10 (Harris [17])**.** *There is a Δ_3^0 degree that does not contain a set with a computable strong η-representation.*

Kach and Turetsky [21] modified the notion of limitwise monotonic to give the following:

Definition 2.1.11. A function $F : \mathbb{Q} \to \omega$ is *support (strictly) increasing limitwise monotonic function on* $\mathbb{Q}$ if there is computable $f : \mathbb{Q} \times \omega \to \omega$ such that

- $F(q) = \lim_s f(q, s)$.
- For all q, s $f(q, s) \leq f(q, s + 1)$.
- The set $S := \{q \in \mathbb{Q} : F(q) \neq 0\}$ has order type ω.
- $F \upharpoonright S$ is (strictly) increasing.

One can relativize this to a degree $\mathbf{d}$ by allowing f to be $\mathbf{d}$-computable. They define $\mathbf{SILM^d}(\mathbb{Q})$ to be the set of A such that A is the range of a $\mathbf{d}$-support increasing limitwise monotonic function on $\mathbb{Q}$ and $\mathbf{SSILM^d}(\mathbb{Q})$ to be the set of A such that A is the range of a $\mathbf{d}$-support strictly increasing limitwise monotonic function on $\mathbb{Q}$.

Kach and Turetsky were able to get the following result about increasing η-representations.

Theorem 2.1.12. *A set A has a computable increasing η-representation if and only if $A \in \mathbf{SILM^{0'}}(\mathbb{Q})$.*

Similarly to the case of η-representable degrees, Kach and Turetsky proved that every Δ^0_3 degree has a computable increasing η-representation.

Like in the case of theorem 2.1.9, the proof of Theorem 2.1.12 gives us that if $A \in \mathbf{SSILM^{0'}}(\mathbb{Q})$ then A is strongly η-representable. The converse, however, is not true in general.

Theorem 2.1.13 (Turetsky [9])**.** *There is a set $A \notin \mathbf{SSILM^{0'}}(\mathbb{Q})$ with a computable strong η-representation.*

This is close to a characterization of strongly η-representable sets. In section 2.4 we are able to prove that for dense enough sets this is a characterization.

Corollary 2.4.2. *Suppose $g : \omega \to \omega$ is a $\mathbf{0}'$-computable increasing function. If a set A has a strong η-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in \mathbf{SSILM}^{0'}(\mathbb{Q})$.*

Using this we are then able to characterize the sets with computable strong η-representations up to many-one degree.

Corollary 2.4.5. *The following coincide.*

- *The m-degrees of sets with computable strong η-representations.*
- *The m-degrees of sets in $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$.*
- *The m-degrees of sets with Δ^0_2 strong η-s-representations.*

We make some progress towards characterizing the η-representable sets as well in this chapter. To simplify the problem, in Section 2.2 we look at η-s-representations, which are computable η-representations with computable successor relation. We observe that all existing characterizations are relativizations of characterizations of η-s-representable sets, and explore why this is the case. We also give a characterization of the sets in $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$ in terms of η-representation and explain why this means the existing tools will not be able to give a characterization of the strongly η-representable sets.

This leads onto Section 2.3 where we introduce the notion of a connected approximation and use this to characterize the strongly η-s-representable sets, as well as giving new characterizations for most of the other classes of sets discussed here. It is an open question if our characterization of the strongly η-s-representable sets relativizes to a characterization of the strongly η-representable sets.

2.2 η-s-Representations

The existing characterizations of sets with computable η-representations and with computable increasing η-representations both involve relativizing some construction to $\mathbf{0}'$ and make use of the fact that $\mathbf{0}'$ can compute the successor relation on any computable linear order. For this reason we propose the following definition.

Definition 2.2.1. A (strong) η-s-representation of a set A is a computable (strong) η-representation L where the successor relation S_L is also computable.

The hope is that we can find a characterization of strongly η-s-representable sets and turn it into a characterization of the strongly η-representable sets. Towards this idea we have the following theorem.

Theorem 2.2.2. *If L is a $\mathbf{0}'$-computable η-representation of some set A and the block relation $B_L \leq_T \mathbf{0}'$ then there is a computable linear order D such that $D \cong L$ and $B_D \leq_T \mathbf{0}'$.*

Proof. Using that L is Δ^0_2 we can approximate L in stages. We keep track of the blocks that $\emptyset'_s$ thinks are in L_s and build corresponding blocks in D_s. When we see two blocks in L_s change order or merge, we keep the representative of the block with the smallest member (in the sense of $<_{\mathbb{N}}$) and remove the other one by densifying (i.e. adding points so that the block becomes part of a copy of $\mathbb{Q}$). Then we add a new block in the correct place if needed.

More formally, let $(L_s, <_s, B_s)_s$ be a sequence of linear orders with block relation that has limit $(L, <_L, B)$ where each $L_s \subseteq L_{s+1}$ is a subset of ω.

Define $D_0 = \emptyset$. We will keep a follower function f_s from the blocks of D_s to a corresponding element in L_s that represents the block. At stage s, for any $b_i, b_j \in \mathrm{dom}(f_s)$ if we have $f_s(b_i) <_{\mathbb{N}} f_s(b_j)$ and $f_s(b_i) <_s f_s(b_j)$ but $f_s(b_i) >_{s+1} f_s(b_j)$ then in D_{s+1} we will remove b_j from $\mathrm{dom}(f_{s+1})$. Similarly if $f_s(b_i) >_s f_s(b_j)$ but $f_s(b_i) <_{s+1} f(b_j)$ or $\neg B_s(f_s(b_i), f_s(b_j))$ but $B_{s+1}(f_s(b_i), f_s(b_j))$.

Next, for each block $b \in \mathrm{dom}(f_s)$ that has not been removed we make sure it has the correct size. Let $y = \min_{\mathbb{N}}\{x : B_s(x, f_s(b)) \wedge \neg B_{s+1}(x, f_s(b))\}$. If y exists, remove points from the end of b so that it has size $|\{x <_{\mathbb{N}} y : B_s(x, f_s(b))\}|$. Now we add points to the end of the block so that the block of $f_s(b)$ in L_{s+1} will have the same size as b does in D_{s+1}. Then, in case small numbers have been added we set $f_{s+1}(b) = \min_{\mathbb{N}}\{x : B_{s+1}(x, f_s(b))\}$.

Then for each block c in L_{s+1} that does not have an element in $\mathrm{range}(f_s)$ we create a corresponding block b in D_{s+1} of the same size as c and set $f_{s+1}(b) = \min_{\mathbb{N}}(c)$. Finally we densify; for all adjacent x, y which are not part of the same block in $\mathrm{dom}(f_{s+1})$, we add a new point between x and y. We now have D_{s+1}.

Now the verification. It is clear that D is a computable linear order. We need to make sure it has the right order type. At each stage we densify around the points that are not part of a block, so between adjacent blocks we must have order type η.

Claim 2.2.2.1. *For every block $c \in L$ there is a unique block $b \in D$ that has the same length as c.*

Proof. Let $n = \max_{\mathbb{N}}(c) + 1$. There is a stage t such that for all $s \geq t$, $B_s \restriction n = B$ and $<_s\restriction n = <_L\restriction n$. At this stage t there will be a b such that $f_t(b) = \min_{\mathbb{N}}(c)$. By our choice of t we have $f_s(b) = \min_{\mathbb{N}}(c)$ and $|b| \geq |c|$ for all $s \geq t$ as there can be no reason to destroy b and we will never see any number smaller than n leave c.

Given any $s > t$ and $m = \min_{\mathbb{N}}\{x \geq_{\mathbb{N}} n : \exists r > s[B_r(f_r(b), x)]\}$ there is a stage $r > s$ such that $B_r(f_r(b), x) \wedge \neg B_{r+1}(f_{r+1}(b), x)$. So at stage $r+1$ we will have $|b| = |c|$ and as s is arbitrary, we have $|b| \leq |c|$ in D. □

Claim 2.2.2.2. *For every block $b \in D$ there is a block $c \in L$ and t such that $|b| = |c|$ and for all $s \geq t$, $f_s(b) = \min(c)$. Furthermore, if $b_i <_D b_j$ then for the corresponding blocks $c_i, c_j \in L$ we have $c_i <_L c_j$.*

Proof. Consider a block b. Suppose $f_s(b) = n$ and $f_t(b) = m$ for $s < t$. Then it must be that $f_s(b) \geq f_t(b)$. So $\lim_s f_s(b)$ exists. If $x = \lim_s f_s(b)$ and c is the block of x in L then by the same argument as above we have that $|b| = |c|$.

If $x = \lim_s f_s(b_i)$ and $y = \lim_s f_s(b_j)$ and $b_i <_D b_j$ then $x <_L y$ as otherwise we would have removed one of the blocks. □

So we can see that there is an order preserving bijection F from the blocks of D to the blocks of L with $|b| = |F(b)|$. Hence the order type of D is the same as that of L.

From the construction, if a point is removed from a block then it is never put back in a block at a later stage. So $\mathbf{0}'$ can compute the set of points in D that are not in blocks.

As D is computable, $\mathbf{0}'$ can also compute the successor relation on D. From both of these, $\mathbf{0}'$ can compute the block relation. □

Theorem 2.2.2 is not quite what we would like as it requires the block relation to be $\mathbf{0}'$-computable. However, this is a property that occurs if the blocks are created in isolation and never merged. This is precisely what happens in the constructions given in the proofs of the characterizations of η-representable and increasingly η-representable sets.

Theorem 2.2.3. *A set A is in* $\mathbf{SSILM}(\mathbb{Q})$ *if and only if there is a strong η-s-representation of A with computable block relation.*

Proof. For the left to right direction we observe that the usual construction (unrelativizing the one given in [21]) has computable block relation as the blocks that are created are never merged.

For the other direction, since we can compute if two blocks are actually the same block we can make sure we only assign one follower to each block. □

By combining Theorems 2.2.3 and 2.2.2 we get a characterization of $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$ in terms of computable η-representations:

Corollary 2.2.4. *A set A is in* $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$ *if and only if there is a strong η-representation of A with $\mathbf{0}'$-computable block relation.*

Theorem 2.1.13 states that there are strongly η-representable sets which are not in $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$, so as a result any characterization of the sets with strong η-representations must involve merging blocks as part of the construction.

2.3 Connected Approximations

The limit lemma says that we can approximate any Δ^0_2 set A with a computable sequence $(A_n)_n$ such that $A(x) = \lim_n A_n(x)$. Limitwise monotonic functions are one way of building on this idea. From what we have seen, the problem with trying to use these to characterize strongly η-representable sets is that each limit of a sequence $F(q) = \lim_s f(q, s)$

is taken in isolation, and there is no natural way of merging sequences. So we propose a different way of approximating sets that captures the idea of merging sequences.

Definition 2.3.1. A *connected approximation* to a set A is a sequence of finite functions $(c_n)_n$ with associated sequences of finite sets $(A_{n,m})_m$ that satisfy the following:

1. $\text{range}(c_n) \subseteq \text{dom}(c_{n+1})$ for all n.
2. $A_{n,0} := \text{dom}(c_n)$, $A_{n,m+1} := c_{n+m}(A_{n,m})$.
3. The limit $A_{n,\omega} := \lim_m A_{n,m}$ always exists.
4. $A = \cup_n A_{n,\omega}$.

We can assume each c_n is coded by a canonical index for the finite set of its graph $\{\langle x, c_n(x)\rangle : x \in \text{dom}(c_n)\}$, so we can say a connected approximation $(c_n)_n$ is computable if the corresponding sequence of indices is computable.

We call a connected approximation $(c_n)_n$ *monotonic* if $c_n(x) \geq x$ for each n and $x \in \text{dom}(c_n)$, and *order preserving* if each c_n preserves $\leq$. We use the acronym MOP to denote monotonic and order preserving.

We give characterizations of all of the existing classes described so far using connected approximations.

Theorem 2.3.2. *For a set A we have the following characterizations.*

1. *A has a computable connected approximation if and only if A is Σ^0_2.*
2. *A has a computable monotonic connected approximation if and only if A is the range of a computable limitwise monotonic function.*
3. *A has a computable MOP connected approximation if and only if $A \in \mathbf{SILM}(\mathbb{Q})$.*
4. *A has a computable MOP connected approximation where each c_n is injective if and only if $A \in \mathbf{SSILM}(\mathbb{Q})$.*

Proof of (1) and (2). We will handle the first two statements together. Given a Σ^0_2 set A we can assume $A = \text{range}(F)$ for $F(n) = \lim_s f(n,s)$ where f is computable. Then we can define a connected approximation of $A = \text{range}(F)$ as follows. Let $\text{dom}(c_n) = \{f(x,n) : x < n\}$ and define $c_n(y) = f(x, n+1)$ where x is least such that $f(x,n) = y$. Clearly $(c_n)_n$ is computable and $\text{range}(c_n) \subseteq \text{dom}(c_{n+1})$. For each n, m we have that $A_{n,m} = f(B_{n,m}, n+m)$ for some $B_{n,m} \subseteq n$. We take the $B_{n,m}$ which minimizes $\sum_{x \in B_{n,m}} x$. By construction $\sum_{x \in B_{n,m}} x \geq \sum_{x \in B_{n,m+11}}$ and so the limit $B_{n,\omega} := \lim_m B_{n,m}$ exists. Hence $A_{n,\omega} = F(B_{n,\omega})$, so we have $(c_n)_n$ is a connected approximation of a subset of A. Consider an $n \in \omega$. Let $t > n$ be a stage after which $f(m,s) = F(m)$ for all $s \geq t, m \leq n$. Then $F[n] \subseteq A_{t,0}$, so $F[n] \subseteq A_{t,\omega}$. So $(c_n)_n$ is a connected approximation of A. Notice that if $f(n,s)$ is monotonic in s then $(c_n)_n$ is also monotonic.

Now consider a computable connected approximation $(c_n)_n$ of a set A. We define a computable function $f : \omega^2 \to \omega$ as follows. $f(n,0) = 0$, $t_0 = 0$. Define $f(n, s+1)$ as follows: $f(n, s+1) = c_s(f(n,s))$ if $n < t_s$. Let $m_0, \ldots, m_{k-1}$ list $\text{range}(c_s) \setminus \text{range}(c_s \circ c_{s-1})$ in order. Define $f(t_s + i, s+1) = m_i$ and $t_{s+1} = t_s + k$. For $n \geq t_{s+1}$ let $f(n, s+1) = 0$. We have that $A_{n,m} = \{f(x, n+m) : x \leq t_n\}$ and so $\text{range}(F \restriction t_n) = A_{n,\omega}$ and hence $\text{range}(F) = A$. Notice that if $(c_n)_n$ is monotonic then F is limitwise monotonic. $\square$

A similar idea works for characterizations (3) and (4), but when going from a connected approximation we need to choose rationals so that the order is preserved.

Now we give a characterization of strongly η-s-representable sets using connected approximations. In a construction of a strong η-s-representation, blocks can do two things: they can grow and they can merge. Eventually they must stop doing either of these things, but we cannot put a computable bound of how late these actions take place. However, if two blocks are, in fact, different then we will see infinitely many points go in between them. Thus, if blocks merge at a late stage then the size of the resulting block should be very large. This is the main idea behind the formula in the following characterization and the proof.

Theorem 2.3.3. *A set A has a strong η-s-representation if and only if it has a computable*

MOP connected approximation where each c_n satisfies

$$\psi(n) = \forall x \in \text{range}(c_n)[\sum_{m \in c_n^{-1}(\{x\})} (m+n) \leq x+n].$$

Proof. Suppose we have a strong η-s-representation L of A. We can assume that L has domain ω and let $L_s = L \restriction s$. Let B_s be the blocks of L_s according to S_L. For blocks $b, c \in B_s$ and $t \geq s$ we use $|b|_t$ to denote the size the block has in L_t, and we use $b <_t c$ and $b =_t c$ to denote the order of the, possibly merged, blocks in L_t.

We start with $c_0 = \emptyset$, $t_0 = 0$. At stage s we assume we are given $\text{dom}(c_s)$, t_s and a block $b_s \in B_{t_s}$. We assume that for any $c, d \in B_{t_s}$ with $c <_{t_s} d \leq_{t_s} b$ we have $|c|_{t_s} < |d|_{t_s}$ and there are at least s many points between c and d in L_{t_s}. We also assume $\text{dom}(c_s) = \{|b|_{t_s} : b \in B_{t_s} \wedge b \leq_{t_s} b_s\}$.

We let $b_{s+1} = \max_{<_{t_s}} B_{t_s}$. Search for a $t > t_s$ such that for every $c, d \in B_t$ with $c <_t d \leq_t b_{s+1}$ we have $|c|_t < |d|_t$ and there are at least $s+1$ many points between c and d in L_{t_s}. The fact that L is a strong η-s-representation guarantees that we will find such a t. We let $t_{s+1} = t$ and $\text{dom}(c_{s+1}) = \{|b|_t : b \in B_t \wedge b \leq_t b_{s+1}\}$. We define c_s as follows. For $d \leq_{t_s} b_s$ we set $c_s(|d|_{t_s}) = |d|_t$. This clearly gives $\text{range}(c_s) \subseteq \text{dom}(c_{s+1})$. This completes the construction.

Now we need to check that c_s is MOP and meets the condition ψ. If $|d|_{t_s} < |c|_{t_s}$ for $d, c <_{t_s} b_s$ then we have that $d <_{t_s} c$, so $d \leq_t c$ and $|d|_t \leq |c|_t$. Thus c_s preserves $\leq$. Since L is a strong η-s-representation, we have that $|d|_n \leq |d|_m$ for $n \leq m$, and so c_s is monotonic. If we combine this with the fact that there are s many points between relevant blocks in B_{t_s} we have that if $d_1 <_{t_s} \cdots <_{t_s} d_n \leq_{t_s} b_s$ but $d_1 =_t \cdots =_t d_n$ then we have $|d_i|_t + s \geq \sum_{i=1}^{n}(|d_i|_{t_s} + s)$. So we can conclude that c_n meets the condition ψ.

All that is left to check is that the limits exist and that they give us A. We have that $A_{n,m} = \{|d|_{t_{n+m}} : d \in B_{t_n}, d \leq b_n\}$. Since B_{t_n} is a finite set and each block in B_{t_n} only changes size finitely often, we have that the limit $A_{n,\omega}$ exists and $A_{n,\omega} \subseteq A$. On the other hand, every $d \in B_L$ is in some B_n, so the there is a stage s such that $t_s > n$ and then we have $|d|_{t_{s+1}} \in \text{dom}(c_{s+1})$. Thus $|d| \in A_{s+1,\omega}$. So we have that $A = \cup_n A_{n,\omega}$ and $(c_n)_n$ is a connected approximation of A as desired.

Now for the other direction. Suppose we have a connected approximation $(c_n)_n$ of A satisfying the conditions of the theorem. We construct an η-s-representation as follows. The main idea is that at stage s we will have a linear order L_s with successor relation and blocks B_s strictly ordered by size with s many points in between, and the sizes of blocks of B_s are the members of $\mathrm{dom}(c_s)$.

We define a computable function $H(L, c, m)$ that takes a finite linear order L with successor, a finite function c and a number m, and outputs a finite linear order D with successor extending L if it can. We assume that the blocks B_L are ordered by size the same way they are by $<_L$. We assume that $\{|b| : b \in B_L\} \subseteq \mathrm{dom}(c)$. We build D in steps as follows. If $c(|b|) = c(|d|)$ then we merge blocks b and d and all the points in between into one large block. This gives us a D_0 that differs from L only in the successor. We then go through each block b of D_0, and if d was a block of L and $d \subseteq b$ then we possibly add points to the end of b so that $|b| = c(|d|)$. If we have $|b| > c(|d|)$ already then H fails. If H does not fail then this gives us D_1. Now, for each $n \in \mathrm{range}(c) \setminus \{|b| : b \in B_{D_1}\}$, we add a new block of length n to D_1, keeping the ordering of blocks by size. This gives us a D_2. Finally, between each pair of adjacent blocks in D_2, we add points in a dense way so that there are exactly m many points between them. This is D. If one of the assumptions was wrong then H fails, otherwise it succeeds, and D is a linear order with blocks ordered by size the sizes of which are $\mathrm{range}(c)$, and there are exactly m many points between adjacent blocks.

We define our strong η-s-representation to be $L = \bigcup_s L_s$ where $L_0 = \emptyset$ and $L_{s+1} = H(L_s, c_s, s+1)$. From the definition of H and the fact that each c_n preserves $\leq$ and satisfies ψ we can see, using an induction argument, that H will always succeed, so the L_s are all well-defined. From the definition of H we can see that if two points are never part of the same block for some L_s then there is a point in between them. So we have that the successor relation on L is $S_L = \bigcup_n S_{L_n}$. So L is a computable linear order with c.e. successor relation. As the successor relation of a computable linear order is always co-c.e.

we have that S_L is computable.

By construction we have that $A_{n,m} = \{|b|_{n+m} : b \text{ is a block in } L_{n+1}\}$, and for every block b in L_n, $c_m(|b|_m) = |b|_{m+1}$. So we have that $A_{n,\omega} = \{|b|_L : b \text{ is a block in } L_n\}$ and L is an η-s-representation of A. As the blocks of L_n are ordered by increasing size so too are the blocks of L, so L is a strong η-s-representation of A. □

Note that if we replace $\psi(n)$ by the condition $\forall x \in \mathrm{dom}(c_n)[(\sum_{m \in c_n^{-1}(\{x\})} m + f(n) \leq x + f(n)]$ for any computable non-decreasing f with $\lim_n f(n) = \omega$ then a slight modification of the arguments above should still work and we get another characterization. The relativized version of the proof with $\mathbf{0}'$-computable connected approximation, does not necessarily build us a computable strong η-representation, so we do not have a characterization of the strongly η-representable sets.

2.4 The Many-One Degrees of η-Representable Sets

We know from Kach and Turetsky [21] that if $S \in \mathbf{SSILM}(\mathbb{Q})$ then S has a strong η-s-representation. The following is a condition on S under which the converse holds.

Theorem 2.4.1. *Suppose $g : \omega \to \omega$ is a computable increasing function. If a set A has a strong η-s-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in \mathbf{SSILM}(\mathbb{Q})$.*

Proof. The construction goes as follows. We use an enumeration of $L = \{x_i : i \in \omega\}$, and at stage s we look at the maximal blocks of L_s. We pick rationals to represent the blocks with the idea that $F(r)$ is the size of the block represented by r, but the block that r represents may change when blocks change. To keep track of what blocks rationals follow we will use a sequence of helper functions $h_s : \mathbb{Q} \to B_{t_s}$ with $\mathrm{dom}(h_s) = \{r : f(r,s) > 0\}$. Once we see a block b appear in L_s it can only grow, so it will remain a block in L_t for $t > s$. Like we did in the proof of Theorem 2.3.3 we will use $|b|_t$, $b =_t c$ and $b <_t c$ to denote the size and order of the blocks from L_s according to L_t. We let B_t be the set of blocks from L_t.

At stage 0 we start with $f(r,0) = 0$ for all $r \in \mathbb{Q}$ and $t_0 = 0$. At stage s let $b = \max(B_{t_s})$. Let $t_{s+1} = t$ be the least stage $t > t_s$ such that for each $c <_t d \leq_t b$ in B_t we have that $|c|_t < |d|_t$ and there are at least $g(|c|_t + |d|_t)$ many points between c and d in L_t, and furthermore for all n such that $g(n) \leq |b|_t + 1$ we have $|\{c \in B_t : c \leq_t b \wedge |c| < g(n)\}| \geq n$. As L is an η-s-representation of A there must be such a t.

Let $r_0 < \cdots < r_{n-1}$ be the domain of h_s; we begin defining h_{s+1} as follows. Let $h_{s+1}(r_0)$ be the smallest block $c_0 \in B_t$ such that $c_0 \leq_t h_s(r_0) \wedge |c_0|_t \geq f(r_0, s)$. Let $h_{s+1}(r_i)$ be the smallest block $c_i \in B_t$ such that $h_{s+1}(r_{i-1}) <_t c_i \leq_t h_s(r_i) \wedge |c_i|_t \geq f(r_i, s)$.

For each block $c \leq_t b$ that is not in range(h_{s+1}) we pick a rational r_c and set $h_{s+1}(r_c) = c$ so that h_{s+1} is order preserving and has image $\{c \in B : c \leq_t b\}$. We define

$$f(r, s+1) = \begin{cases} |h_{s+1}(r)|_t & r \in \mathrm{dom}(h_{s+1}) \\ 0 & \text{otherwise} \end{cases}$$

Now for the verification: first we need to show that the recursive definition of $h_{s+1}(r_i)$ actually works. Suppose it does not. Then let i be least such that we cannot find a block for r_i. $|h_s(r_i)|_t \geq |h_s(r_i)|_{t_s} \geq f(r_i, s)$ so if $h_s(r_i)$ does not work then there must be some smaller r_j with $h_{s+1}(r_j) = h_s(r_i)$. So $i > 0$. We have $h_{s+1}(r_{i-1}) \leq_t h_s(r_{i-1}) <_{t_s} h_s(r_i)$, so it must be that $h_s(r_{i-1})$ and $h_s(r_i)$ have merged. So $|h_s(r_i)|_t > g(|h_s(r_{i-1})|_{t_s} + |h_s(r_i)|_{t_s})$. So by our choice of t_s we have at least $|h_s(r_{i-1})|_{t_s} + |h_s(r_i)|_{t_s}$ many blocks before $h_s(r_i)$ and at least $|h_s(r_i)|_{t_s}$ have size at least $|h_s(r_{i-1})|_{t_s}$. But $i \leq |h_s(r_i)|_{t_s}$, so we would have chosen $h_{s+1}(r_{i-1})$ to be one of these, a contradiction.

From the definition of h_s we can see that $h_{s+1}(r) \leq_L h_s(r)$ for each r and s, so as the blocks of L are well ordered, $\lim_s h_s(r)$ exists. From the definition of f we have that it is limitwise monotonic and $F(r) = |\lim_s h_s(r)|_L$. So range$(F) \subseteq S$. If b is a block of L then after some stage t, all of b is in L_t as well as all smaller blocks. So at some stage s, $t_s > t$, so at stage $s+1$ we have an r such that $h_{s+1}(r) = b$ and for any $n > s$ we have $h_n(r) = b$ as the blocks in that part of the linear order no longer change.

So $S = \mathrm{range}(F)$ as desired. $\square$

Relativizing we get the following:

Corollary 2.4.2. *Suppose $g : \omega \to \omega$ is a $\mathbf{0}'$-computable increasing function. If a set A has a strong η-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in \mathbf{SSILM}^{0'}(\mathbb{Q})$.*

This means that for dense enough sets, the notions of Δ^0_2 strong η-s-representation, strong η-representation and support strictly increasing limitwise monotonic on $\mathbb{Q}$ all coincide.

Note that we cannot use Theorem 2.4.1 to give a characterization of $\mathbf{SSILM}(\mathbb{Q})$ as there are sparse sets in $\mathbf{SSILM}(\mathbb{Q})$. For instance consider the function $F(n) = n + \sum_{e \in \emptyset' \cap n} h(e)$ where $h(e)$ is the least s such that $\varphi_{e,s}(e)\downarrow$. Then as F cannot be computably bounded, $S = \text{range}(F)$ would not meet the condition $|S \cap g(n)| \geq n$ for all n for any computable g, but by definition it is clearly limitwise monotonic and increasing, so $S \in \mathbf{SSILM}(\mathbb{Q})$.

We can, however, use Theorem 2.4.1 to characterize the degrees of sets with computable strong η-representations.

Theorem 2.4.3. *If $\mathbf{a}$ is the m-degree of a set with a strong η-s-representation then there is $S \in \mathbf{a}$ such that $S \in \mathbf{SSILM}(\mathbb{Q})$*

To prove this we use the following lemma.

Lemma 2.4.4. *If A is a set with a strong η-s-representation then $A \oplus \omega$ also has a strong η-s-representation.*

Proof. Suppose that A is a set with a strong η-s-representation. Let $(c_n)_n$ be a computable MOP connected approximation of A satisfying condition ψ of Theorem 2.3.3. We build a connected approximation $(d_n)_n$ of $A \oplus \omega$ satisfying ψ as follows. The first idea is to use c_m with m much larger than n to build d_n. We want m to be large enough that when we see $c_m(x) = c_m(y)$ we can merge the corresponding numbers $2x, 2y \in \text{dom}(d_n)$ without violating ψ. The second idea is that when we see $c_m(x) > x$ without any merging, we shift the representative of x in $\text{dom}(d_n)$ to a lager number so that we can handle the case where the gaps between numbers shrink, i.e. when $c_m(y) - c_m(x) < y - x$ for $y > x$.

To start let $d_0 = \emptyset$ and $m_0 = 0$. We will ensure that $m_n > \sum_{x\in\mathrm{dom}(d_n)}(x+n)$, and if $2x \in \mathrm{range}(d_n)$ then $x \in \mathrm{range}(c_{m_n})$. Given d_n and m_n, let $N > m_n$ be the least number such that if $x = \max(A_{m_n,N+1})$ then $N > 2x(2x+n+1)$. Let $m_{n+1} = N$. This will ensure that $m_{n+1} > \sum_{x\in\mathrm{dom}(d_{n+1})}(x+n+1)$. There must be such an N as $(c_n)_n$ is a valid connected approximation, so eventually x will stabilize. Let $c = c_N \circ \cdots \circ c_{m_n+1}$.

Let $z \in \mathrm{dom}(c)$ be the least such that there is $y > z$, $c(z) = c(y)$. For $y \geq 2z \in \mathrm{range}(d_n)$ we define $d_{n+1}(y) = 2c(z)$. By assumption on m_n, this will not violate condition ψ. If there is no such z then set $z = \max(\mathrm{dom}(c)) + 1$. Now we define d_{n+1} on values smaller than $2z$.

Let $a_0, \ldots, a_{s-1}$ list the elements of $(\mathrm{dom}(c)\oplus\omega)\cap 2z-1$. Let $b_0, \ldots, b_{s-1}$ list the first s elements of $\mathrm{range}(c)\oplus\omega$. Note that $b_{s-1} < 2c(z)$ if $c(z)$ is defined. We define $d_{n+1}(a_i) = b_i$. This is definitely order preserving, and it is monotonic because c is injective and monotonic on $\mathrm{dom}(c) \cap z$. This completes the construction of $(d_n)_n$.

Verification: By construction we can see that $(d_n)_n$ is MOP and satisfies ψ; all that is left to check is that it is a connected approximation of $A \oplus \omega$. Let $x \in \mathrm{dom}(d_n)$ and let $2y$ be the least even number in $\mathrm{dom}(d_n) \setminus x$. Following the construction we can see that $d_n(x) \leq d_n(2y) \leq 2(c_{m_n} \circ \cdots \circ c_{m_{n-1}+1})(y)$. So if we repeat this then we can see that the limit of $x, d_n(x), d_{n+1}(d_n(x)), \ldots$ if it exists is less than the limit of $2y, 2c_{m_n}(y), 2c_{m_n+1}(c_{m_n}(y)), \ldots$, so by monotonicity it exists. Thus $(d_n)_n$ is a valid connected approximation.

Fix x. Let s be a stage at which $c_n \upharpoonright x = c_s \upharpoonright x$ for all $n \geq s$. Then by the construction we have that $d_n \upharpoonright 2x = d_s \upharpoonright 2x$ for all $n \geq s$ and $\mathrm{dom}(d_n) \cap 2x = (\mathrm{dom}(c_n) \cap x) \oplus x$. So $(d_n)_n$ is a connected approximation of $A \oplus \omega$.

$\square$

This allows us to characterize the m-degrees of sets with strong η-representations.

Corollary 2.4.5. *The following coincide.*

- *The m-degrees of sets with computable strong η-representations.*

- *The m-degrees of sets in* $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$.
- *The m-degrees of sets with* Δ^0_2 *strong* η*-s-representations.*

2.5 Open Questions

We can characterize the m-degrees of sets with strong η-representations, but we leave open the following.

Question 2.5.1. Is there a characterization of the sets with computable strong η-representations that is not in terms of linear orders?

A related question is in regards to the sets with Δ^0_2 strong η-s-representations.

Question 2.5.2. Is there a set with a Δ^0_2 strong η-s-representation, but no computable strong η-representation?

A negative answer to Question 2.5.2 would give us an answer to Question 2.5.1, using the connected approximation characterization of sets with computable strong η-s-representations.

Chapter 3

Strong minimal pairs in the enumeration degrees

3.1 Introduction

In this chapter we look at different types of minimal pairs, some of which can occur in the enumeration degrees and some of which cannot occur.

Definition 3.1.1. In an upper semilattice with least element $\mathbf{0}$ a pair $\mathbf{a}, \mathbf{b} > \mathbf{0}$ is a:

- *minimal pair* if $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.
- *strong minimal pair* if it is a minimal pair, and for all $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ we have $\mathbf{x} \vee \mathbf{b} = \mathbf{a} \vee \mathbf{b}$.
- *super minimal pair* if both $\mathbf{a}, \mathbf{b}$ and $\mathbf{b}, \mathbf{a}$ are strong minimal pairs.
- *strong super minimal pair* if it is a minimal pair, and for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ and $\mathbf{0} < \mathbf{y} \leq \mathbf{b}$ we have $\mathbf{x} \vee \mathbf{y} = \mathbf{a} \vee \mathbf{b}$.

Any pair of minimal degrees form a strong super minimal pair, so we know there are strong super minimal pairs in the Turing degrees. The question of whether these types exist is more interesting for structures with downwards density, like the enumeration degrees and the c.e. degrees. Cai, Liu, Liu, Peng and Yang [6] answer these questions for the c.e. degrees by proving that there are no strong minimal pairs in that structure. This chapter answers some of these questions for the enumeration degrees and the enumeration degrees below $\mathbf{0}'$.

In Section 3.2 we look at strong super minimal pairs. Using the Gutteridge operator and $\mathcal{K}$-pairs we give a proof that there are no strong super minimal pairs in the enumeration degrees. The Gutteridge operator was used by Gutteridge [16] to prove that the enumeration degrees are downwards dense. $\mathcal{K}$-pairs were used by Kalimullin [22] to prove that the jump on the enumeration degrees is definable. They have been used for other applications to the theory of the enumeration degrees [13], and in Section 3.3 we show how they can be used to give an example of a strong minimal pair. The example in Section 3.3 was conveyed to us by the anonymous referee for first submitted version of the paper this chapter is based on.

In Section 3.4 we give our own, direct construction of a strong minimal pair in the enumeration degrees. This uses a two stage forcing method to construct our sets. In Section 3.5 we modify this construction into a finite injury argument and are able to lower the complexity to construct a strong minimal pair A, B where A is Σ^0_2 and B is Π^0_2. The example from Section 3.3 has A is Π^0_2 and $B = \emptyset'$, so both sides of a strong minimal pair can be below $\mathbf{0}'$. It is an open question this can happen at the same time, i.e. if there are any strong minimal pairs in $\mathcal{D}_e(\leq \mathbf{0}')$. Also an open question is if there are any super minimal pairs in the enumeration degrees.

3.2 No strong super minimal pairs

We prove that there are no strong super minimal pairs in the enumeration degrees. This proof is similar to Gutteridge's proof of downwards density [16], and makes use of the Gutteridge operator Θ. Gutteridge's proof splits into two cases: one where $\mathbf{a}$ is Δ^0_2 and one where $\mathbf{a}$ is not Δ^0_2. Similarly our proof splits into two cases. For the first case we have the following lemma proven by Mariya Soskova.

Lemma 3.2.1 (M. Soskova). *If A is Δ_2^0 then A, B is not a strong minimal pair in $\mathcal{D}_e$ for any B.*

The proof relies on some results about Kalimullin pairs [22], defined below.

Definition 3.2.2. A and B are a Kalimullin pair ($\mathcal{K}$-pair) if there is a c.e. set $W \subseteq \omega^2$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is called *trivial* if one of A, B is c.e.

We use the following two facts about $\mathcal{K}$-pairs.

Theorem 3.2.3 (The minimal pair $\mathcal{K}$-property, Kalimullin [22]). *A, B are a $\mathcal{K}$-pair if and only if for all $X \subseteq \omega$, $A \oplus X$ and $B \oplus X$ form a minimal pair relative to X. i.e. $Y \leq_e A \oplus X, Y \leq_e B \oplus X \implies Y \leq_e X$.*

Theorem 3.2.4 (Kalimullin [22]). *Every nonzero Δ_2^0 degree computes a nontrivial $\mathcal{K}$-pair.*

Proof of Lemma 3.2.1. Suppose that A is Δ_2^0 and A, B form a minimal pair. Then by Theorem 3.2.4 let $X, Y \leq_e A$ be a nontrivial $\mathcal{K}$-pair. Then consider $X \oplus B$ and $Y \oplus B$. If $X \oplus B \equiv_e Y \oplus B \equiv_e A \oplus B$ then by Theorem 3.2.3 $A \leq_e B$ a contradiction. But by assumption X, Y are both non-c.e. and bounded by A, so A, B is not a strong minimal pair. □

For the second case of his proof Gutteridge constructed an operator Θ, now known as the Gutteridge operator. Gutteridge constructed Θ so that the following would hold:

$$\text{If } A \text{ is not } \Delta_2^0 \text{ then } \emptyset <_e \Theta(A) <_e A. \tag{3.1}$$

Our proof below relies on the particular form of Θ, not just the fact that (3.1) holds, so we remind the reader of this.

The construction of Θ uses a c.e. set B with the property that each column $B^{[k]} = \{x : \langle k, x \rangle \in B\}$ is finite and an initial segment of ω, that is $x + 1 \in B^{[k]} \implies x \in B^{[k]}$. We also have $B^{Int} = \{\langle k, x \rangle : \langle k, x + 1 \rangle \in B\}$ which is also c.e. Let $n_k = |B^{[k]}| - 1$. $\Theta(A)$ is defined to be the set $B^{Int} \cup \{\langle k, n_k \rangle : k \in A\}$.

From this we can see the following.

Lemma 3.2.5. *$\Theta(A \cup C) = \Theta(A) \cup \Theta(C)$.*

Proof.

$$\begin{aligned}\Theta(A \cup C) &= B^{Int} \cup \{\langle k, n_k\rangle : k \in A \cup C\} \\ &= B^{Int} \cup \{\langle k, n_k\rangle : k \in A\} \cup B^{Int} \cup \{\langle k, n_k\rangle : k \in C\} \\ &= \Theta(A) \cup \Theta(C)\end{aligned}$$

□

Using this we can prove the following lemma.

Lemma 3.2.6. *If A and C are not Δ^0_2 then there are X, Y such that $\emptyset <_e X \leq_e A$, $\emptyset <_e Y \leq_e C$, and $X \oplus Y <_e A \oplus C$.*

Proof. Take $X = \Theta(A \oplus \emptyset), Y = \Theta(\emptyset \oplus C)$. Since $A \oplus \emptyset \equiv_m A$, by (3.1) we have that $0 <_e X <_e A$ as desired. Similarly $0 <_e Y <_e C$. By Lemma 3.2.5 we have that $X \cup Y = \Theta(A \oplus C)$.

Next we show that $X \oplus Y \equiv_e X \cup Y$. It is clear that $X \cup Y \leq_e X \oplus Y$, so we need to consider the other direction. We have that

$$\begin{aligned}X \oplus Y =& \{2x : x \in X\} \cup \{2x+1 : x \in Y\} \\ =& B^{Int} \oplus B^{Int} \cup \{2\langle k, n_k\rangle : k \in A \oplus \emptyset\} \cup \{2\langle k, n_k\rangle + 1 : k \in \emptyset \oplus C\} \\ =& B^{Int} \oplus B^{Int} \cup \{2\langle k, n\rangle : \langle k, n\rangle \in X \cup Y, k \text{ is even}\} \cup \\ & \{2\langle k, n\rangle + 1 : \langle k, n\rangle \in X \cup Y, k \text{ is odd}\}\end{aligned}$$

From this we see that $X \oplus Y \leq_e X \cup Y$, as $B^{Int} \oplus B^{Int}$ is c.e. So, as $A \oplus C$ is not Δ^0_2, by (3.1) we have that $\emptyset <_e X \oplus Y \equiv_e \Theta(A \oplus C) <_e A \oplus C$. □

Putting both lemmas together we get the following theorem.

Theorem 3.2.7. *There are no strong super minimal pairs in the enumeration degrees.*

3.3 A strong minimal pair

In this section we use $\mathcal{K}$-pairs to show that there is a strong minimal pair in the enumeration degrees. We would like to thank the anonymous referee to an earlier version of this paper for pointing out this proof.

For this proof we need some more facts about $\mathcal{K}$-pairs.

Theorem 3.3.1 (The ideal $\mathcal{K}$-property, Kalimullin [22])**.** *For sets $A, B, C \subseteq \omega$ we have the following:*

1. *If A, B are a $\mathcal{K}$-pair and $C \leq_e B$ then A, C is a $\mathcal{K}$-pair.*
2. *If both A, B and A, C are $\mathcal{K}$-pairs then $A, B \oplus C$ is a $\mathcal{K}$-pair.*

We will only need 1 for our purposes.

Theorem 3.3.2 (The main $\mathcal{K}$-property, Kalimullin [22])**.** *If A, B are a nontrivial $\mathcal{K}$-pair then:*

1. *$A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.*
2. *$\overline{A} \leq_e B \oplus \emptyset'$ and $\overline{B} \leq_e A \oplus \emptyset'$.*

Now we look at a method of building $\mathcal{K}$-pairs. For a set X we define $L_X = \{\sigma \in 2^{<\omega} : \sigma \leq_{\text{lex}} X\}$. We have that $L_X \leq_e X$ since if $\sigma \leq_{\text{lex}} D \subseteq X$ then $\sigma \leq_{\text{lex}} X$. We define $R_X = \overline{L_X} = \{\sigma \in 2^{<\omega} : \sigma >_{\text{lex}} X\} \leq_e \overline{X}$. L_X, R_X form a $\mathcal{K}$-pair with witness $W = \{\langle\sigma, \tau\rangle : \sigma <_{\text{lex}} \tau\}$. Since $\rho \prec X \iff \exists \sigma \in L_X, \tau \in R_X[\rho \preceq \sigma, \tau]$ we have that $X \oplus \overline{X} \leq_e L_X \oplus R_X$. Now we have the tools needed to prove the existence of a strong minimal pair.

Theorem 3.3.3. *There is a strong minimal pair A, B in the enumeration degrees. Furthermore A can be Π^0_2 and B can be Π^0_1.*

Proof. Consider some non-low Δ^0_2 set Y (For example $Y = \emptyset'$). Consider $X = K_Y = \bigoplus_e \Psi_e(Y)$. We will prove that $A = R_X$ and $B = \emptyset'$ is a strong minimal pair. Since Y is

Δ^0_2 we have $L_X \leq_e K_Y \leq_e Y \leq_e \emptyset'$. Since Y is not low $Y' \nleq_e \emptyset'$ and $Y' = K_Y \oplus \overline{K_Y} \equiv_e L_X \oplus R_X$, we must have $R_X \nleq_e \emptyset'$. Consider any non-c.e. $C \leq_e A$. By the ideal $\mathcal{K}$-property we have that C, L_X is a $\mathcal{K}$-pair. So we have that $A = \overline{L_X} \leq_e C \oplus \emptyset'$ by the main $\mathcal{K}$-property since neither L_X nor C is c.e. So A, B is a strong minimal pair.

Since Y is Δ^0_2 we have that K_Y and L_X are Σ^0_2, so $A = \overline{L_X}$ is Π^0_2. $\emptyset' \equiv_e \overline{K_\emptyset}$ which is Π^0_1. □

Now we know there are strong minimal pairs, a question we can ask is how many such pairs are there? If A, B is a strong minimal pair and $C \subseteq \omega$ is such that $B \leq_e C$ and $A \nleq_e C$ then A, C is a strong minimal pair since $A \leq_e X \oplus B \leq_e X \oplus C$ for any X such that $\emptyset <_e X \leq_e A$. If $A \nleq_e B$ then there are $2^{\aleph_0}$ many $C \geq_e B$ such that $A \nleq_e C$ so the existence of one strong minimal pair tells us there are $2^{\aleph_0}$ many strong minimal pairs in the enumeration degrees.

If we restrict our attention to the left side A of a strong minimal pair A, B then we can observe that if $C \leq_e A$ is not c.e. then C, B is also a minimal pair and for any non-c.e. $X \leq_e C$ we have $C \leq_e A \leq X \oplus B$. So the existence of a strong minimal pair A, B tells us that there are at least $\aleph_0$ many degrees $\mathbf{a}$ such that $\mathbf{a}$ is the left side of a strong minimal pair.

In the proof of Theorem 3.3.3 we chose the set X so that R_X would be Π^0_2 and not below $\emptyset'$. If we take any X such that $X \nleq_e \emptyset'$ then either $L_X \nleq \emptyset'$ or $R_X \nleq \emptyset'$. Thus, by the same argument from Theorem 3.3.3, either $L_X, \emptyset'$ or $R_X, \emptyset'$ is a strong minimal pair. Since there are $2^{\aleph_0}$ many $X \subseteq \omega$ but only countably many of them are below $\emptyset'$ there are $2^{\aleph_0}$ many left sides of a strong minimal pair.

3.4 Forcing construction of a strong minimal pair

In this section we give a new proof of the existence of a strong minimal pair in the enumeration degrees. This proof is longer than the one given in Section 3.3, but it is a more direct construction and can be modified more easily.

Theorem 3.4.1. *There is a strong minimal pair A, B in the enumeration degrees.*

Proof. The first step is to consider the requirements. We have:

$$\mathcal{R}_e: \ \exists\Gamma[\Gamma(\Psi_e(A) \oplus B) = A] \vee \Psi_e(A) \text{ is c.e.}$$

and

$$\mathcal{N}_e: \ \Psi_e(B) \neq A$$

Satisfying $\mathcal{N}_e$ gives us that $A \not\leq_e B$ and satisfying $\mathcal{R}_e$ gives us that for all degrees x such that $\mathbf{0} < \mathbf{x} < \deg_e(A)$ we have $\mathbf{x} \vee \deg_e(B) = \deg_e(A) \vee \deg_e B$, so notably $\mathbf{x} \not< \deg_e(B)$. If $\mathbf{0} < \mathbf{y} \leq \deg_e(B)$ then $\mathbf{y} \not\leq \deg_e(A)$ as otherwise $\deg_e(A) \leq \mathbf{y} \vee \deg_e(B) = \deg_e(B)$ contradicting an $\mathcal{N}_e$ requirement. By downward density there is an $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} < \deg_e(A)$ so we will have that $B \notin \mathbf{0}$ and hence a strong minimal pair.

The Γ that we will use to satisfy $\mathcal{R}_e$ will have a very specific form and will in fact be chosen ahead of time. We define

$$\Gamma_e = \{\langle a, p\rangle : \exists v[D_p = D_v \oplus \{\langle e, a, v\rangle\}]\}$$

The intuitive idea is that we will enumerate $\langle e, a, v\rangle \in B$ to code the fact that $D_v \subseteq \Psi_e(A) \implies a \in A$. In other words, $B^{[e]}$ will look like an enumeration operator that computes A from $\Psi_e(A)$.

We will do two rounds of forcing to construct A and B. The first round will produce a pair $A(X), B(X)$ satisfying all $\mathcal{R}_e$ requirements for each $X \in 2^\omega$. Then we will force along 2^ω to find an X such that $A(X), B(X)$ satisfies all $\mathcal{N}_e$ requirements.

Definition 3.4.2. The forcing partial $\mathbb{P} = (P, \leq)$ we will use will be defined as follows. A condition $p \in P$ will consist of a disjoint pair of computable sets (A_p, C_p) with $A_p \cup C_p$ coinfinite. We say that $p \leq q$ if $A_p \supseteq A_q$ and $C_p \supseteq C_q$.

If $\mathcal{G}$ is a generic filter on $\mathbb{P}$ then we define $A_\mathcal{G} = \bigcup_{p\in G} A_p$. So we can think of p as determining a subset of $A_\mathcal{G}$ and a subset of $\overline{A_\mathcal{G}}$. The definition of $B_\mathcal{G}$ is more complex, and it will look at $p \notin \mathcal{G}$. We will give this definition later.

Definition 3.4.3. For $p \in P$ and $e, n \in \omega$ we say $p \Vdash n \in \Psi_e(A)$ if $n \in \Psi_e(A_p)$ and $p \Vdash n \notin \Psi_e(A)$ if $n \notin \Psi_e(\overline{C_p})$. We say n *is determined for* e *by* p and write $p \Vdash \Psi_e(A)(n)$ if either $p \Vdash n \in \Psi_e(A)$ or $p \Vdash n \notin \Psi_e(A)$. We say $p \Vdash \Psi_e(A)$ is c.e. if for all n we have $p \Vdash \Psi_e(A)(n)$.

It is clear from the definition that if $p \in \mathcal{G}$ and $p \Vdash n \in \Psi_e(A)$ then $n \in \Psi_e(A_{\mathcal{G}})$; similarly if $p \Vdash n \notin \Psi_e(A)$ then $n \notin \Psi_e(A_{\mathcal{G}})$. If $p \Vdash \Psi_e(A)$ is c.e. then, as each n is determined for e by p, we have $\Psi_e(A_{\mathcal{G}}) = \Psi_e(A_p) = \Psi_e(\overline{C_p})$ which is c.e. for any $\mathcal{G} \ni p$.

Lemma 3.4.4. *For every $p \in P, e \in \omega$ we have either*

1. *There is $q \leq p$ such that $q \Vdash \Psi_e(A)$ is c.e.*
2. *There is $n \in \omega$ and $F \subseteq_{\text{fin}} \overline{A_p \cup C_p}$ such that $(A_p \cup F, C_p) \Vdash n \in \Psi_e(A)$ and $(A_p, C_p \cup F) \Vdash n \notin \Psi_e(A)$.*

Proof. Suppose we are given $p \in P$ and case 2 fails. We will show that case 1 holds for $q = (A_q, C_p)$ where A_q is built as follows. We have requirements

$$\mathcal{P}_n : \ n \in \Psi_e(\overline{C_p}) \iff n \in \Psi_e(A_q)$$

along with the requirement that $A_q \cup C_p$ is coinfinite and A_q and C_p are disjoint. We build sequences $A_p = A_0 \subseteq A_1 \subseteq \ldots$ and $m_0 < m_1 < \ldots$ with $\{m_t : t \in \omega\}$ disjoint from all A_s. A requirement $\mathcal{P}_n$ is unmet at stage s if $n \notin \Psi_{e,s}(A_s)$ and a requirement needs attention at stage s if it is unmet and there is some pair $\langle n, u\rangle \in \Psi_{e,s}$ such that $D_u \subseteq \overline{C_p}$

We start with $m_0 = \max(A_p \cup C_p) + 1$ and $A_0 = A_p$. At stage s let $\mathcal{P}_n$ be the highest priority requirement that needs attention (if there is no such requirement then let $A_{s+1} = A_s, m_{s+1} = m_s + 1$). So there is a pair $\langle n, u\rangle \in \Psi_{e,s}$ such that $D_u \subseteq \overline{C_p}$. Wait until we see a possibly new pair $\langle n, v\rangle \in \Psi_e$ such that $D_v \subseteq \overline{C_p}$ and $\min(D_v \setminus A_s) > m_s$, then define $A_{s+1} = A_s \cup D_v$ and $m_{s+1} = \min(\overline{A_{s+1} \cup C_p} \setminus (m_s + 1))$. By assumption this search will always terminate eventually as otherwise $F = ((m_s + 1) \cup D_u) \setminus (A_p \cup C_p)$ will have $(A_p \cup F, C_p) \Vdash n \in \Psi_e(A)$ and $(A_p, C_p \cup F) \Vdash n \notin \Psi_e(A)$, a contradiction.

So the sequence $(m_s)_s$ is computable and increasing, and so the set $A_q = \overline{C_p} \setminus \{m_s : s \in \omega\}$ is computable and has that $A_q \cup C_p$ is coinfinite, A_q and C_p are disjoint and $A_p \subseteq A_q$. If a requirement $\mathcal{P}_n$ ever needs attention then it is met no more than n stages later and $n \in \Psi_e(A_q)$. On the other hand if $\mathcal{P}_n$ never needs attention then $n \notin \Psi_e(\overline{C_p})$. So every requirement is satisfied and $q \Vdash \Psi_e(A)$ is c.e. as desired. □

The key point of Lemma 3.4.4 is that if we cannot find a p that forces $\Psi_e(A)$ is c.e. and satisfy $\mathcal{R}_e$ that way, then we can always find n that is not determined for e. We will use this to satisfy $\mathcal{R}_e$ using Γ_e while maintaining the choice of whether $n \in \Psi_e(A)$ or not.

Next we will use this to build an embedding $H : 2^{<\omega} \to P$ and a function $S : 2^{<\omega} \to \{f :\subseteq \omega \to \omega : f \text{ is finite}\}$. The idea is that for $X \in 2^\omega$ we will have $A(X) = \bigcup_{\sigma \prec X} A_{H(\sigma)}$ and if $\Psi_e(A(X))$ is not c.e. then for all $\sigma \prec X$, $|\sigma| > e \implies [S(\sigma)(e) \in \Psi_e(A(X)) \leftrightarrow \sigma^\frown 1 \prec X]$.

Construction of H and S. We define H, S as follows. At each stage of the construction we will start considering a new $\mathcal{R}_e$ requirement. When we can force that $\Psi_e(A)$ is c.e. we will do so immediately. For other requirements, case 2 of Lemma 3.4.4 will always apply. These requirements will be considered active and will need to be handled at each step. To help us keep track which requirements are active for a given σ we use a function $Z : 2^{<\omega} \to \{F \subseteq_{\text{fin}} \omega\}$. We start with $H(\emptyset) = (\emptyset, \emptyset), Z(\emptyset) = \emptyset$.

Given a node σ and $H(\sigma), Z(\sigma)$ we ask if there is $p \leq H(\sigma)$ such that $p \Vdash \Psi_{|\sigma|}(A)$ is c.e. If yes, then we can satisfy $\mathcal{R}_{|\sigma|}$ by making sure we choose extensions of p for $H(\sigma^\frown j)$. Otherwise we redefine $Z(\sigma) := Z(\sigma) \cup \{|\sigma|\}$ so that $\mathcal{R}_{|\sigma|}$ is now active and set $p = H(\sigma)$.

Let $0 = e_0, \ldots, e_{k-1}$ list $Z(\sigma)$. For each $i < k$ define n_i, F_i to be a pair satisfying case 2 of Lemma 3.4.4 for p and e_i. By assumption of $e_i \in Z(\sigma)$ case 1 has failed so case 2 applies. Define $F = \bigcup_{i<k} F_i$, $S(\sigma)(e_i) = n_i$, $Z(\sigma^\frown j) = Z(\sigma)$.

Finally define $H(\sigma^\frown 1) = (A_p \cup F, C_p)$ and $H(\sigma^\frown 0) = (A_p, C_p \cup F)$. Clearly $p \geq H(\sigma^\frown 1), H(\sigma^\frown 0)$ and for each $i < k$, $H(\sigma^\frown 1) \Vdash n_i \in \Psi_{e_i}(A)$ and $H(\sigma^\frown 0) \Vdash n_i \notin \Psi_{e_i}(A)$.

End of Construction.

For $X \in 2^\omega$ we define $A(X) = \bigcup_{\sigma \prec X} A_{H(\sigma)}$ and

$$B(X) = \{\langle e, a, v\rangle : (\exists \sigma \geq_{\text{lex}} X\restriction|\sigma|)$$
$$[a \in A_{H(\sigma)} \wedge e \in \text{dom}(S(\sigma)) \wedge D_v = \{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}]\}$$

Let us try to understand the definition of $B(X)$. From the definition of Γ_e we want that if $\langle e, a, v\rangle \in B(X)$ then $D_v \subseteq \Psi_e(A(X)) \to a \in A(X)$. We also want that if $\Psi_e(A(X))$ is not c.e. then for all $a \in A(X)$ there exists a v such that $\langle e, a, v\rangle \in B(X) \wedge D_v \subseteq \Psi_e(A(X))$. If $\sigma \prec X$ then $A_{H(\sigma)} \subseteq A(X)$ and $\{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}]\} \subseteq \Psi_e(A(X))$ so that is where the conditions on v and a come from. The reason we need to add axioms for $\sigma \geq_{\text{lex}} X\restriction|\sigma|$ instead of just $\sigma \prec X$ is that the latter is too restrictive and will not allow us to meet the $\mathcal{N}_e$ requirements in the next stage.

Lemma 3.4.5. *If $X \in 2^\omega$ then the pair $A(X), B(X)$ satisfies $\mathcal{R}_e$ for each e.*

Proof. Case 1: $e \notin \text{dom}(S(\sigma))$ for any $\sigma \prec X$. Then by construction, for $\sigma = X\restriction(e+1)$ we have $H(\sigma) \Vdash \Psi_e(A)$ is c.e. So as $A_{H(\sigma)} \subseteq A(X) \subseteq \overline{C_{H(\sigma)}}$ we have $\Psi_e(A(X))$ is c.e.

Case 2: we assume $e \in \text{dom}(S(\sigma))$ for all $\sigma \prec X$ with $|\sigma| \geq e$. By induction along $\sigma \prec X$ we can see that $A_{H(\sigma)} \subseteq A(X) \subseteq \overline{C_{H(\sigma)}}$. So by construction we have that $A_{H(\sigma\frown 1)} \setminus A_{H(\sigma\frown 0)} \subseteq A(X) \iff S(\sigma)(e) \in \Psi_e(A(X))$, for all $\sigma \prec X$, $|\sigma| \geq e$. Given $\sigma \prec X$, $|\sigma| \geq e$, $a \in A_{H(\sigma)}$ and $D_v = \{S(\tau)(e) : \tau \prec \sigma, |\tau| \geq e, \sigma(|\tau|) = 1\}$ we have that $D_v \subseteq \Psi_e(A(X))$ and $\langle e, a, v\rangle \in B(X)$ so by definition of Γ_e, $a \in \Gamma_e(\Psi_e(A(X)) \oplus B(X))$. Therefore $A(X) \subseteq \Gamma_e(\Psi_e(A(X)) \oplus B(X))$.

On the other hand, if $a \in \Gamma_e(\Psi_e(A(X)) \oplus B(X))$ then by definition of Γ_e and $B(X)$, there is some $\sigma \geq_{\text{lex}} X\restriction|\sigma|$ such that $D_v = \{S(\tau)(e) : \tau \prec \sigma \wedge \sigma(|\tau|) = 1\}$, so we have $D_v \subseteq \Psi_e(H(X))$ and $a \in A_{H(\sigma)}$. If $\sigma >_{\text{lex}} X\restriction|\sigma|$ then let n be the first place that they differ. So $\sigma(n) = 1$ and hence $S(\sigma\restriction n)(e) \in D_v$ but $H(X\restriction(n+1)) \Vdash S(\sigma\restriction n)(e) \notin \Psi_e(A)$. So a was not put in $\Gamma_e(\Psi_e(H_e(X)) \oplus B(X))$ by $\langle e, a, v\rangle$, a contradiction. So $\sigma \prec X$ and hence $a \in A_{H(\sigma)} \subseteq A(X)$. Therefore $A(X) = \Gamma_e(\Psi_e(A(X)) \oplus B(X))$ □

Now all that is left is to diagonalize and satisfy all $\mathcal{N}_e$ requirements.

Construction of X. We pick a path, $X \in 2^\omega$, satisfying one $\mathcal{N}_e$ requirement at a time. We start with $\sigma_0 = \emptyset$. Suppose at stage $s+1$ we are given σ_s. Let $Y_s = \sigma_s\widehat{\ }1\widehat{\ }0^\omega$. To satisfy $\mathcal{N}_s$ ask if $A_{H(\sigma\widehat{\ }1)} \subseteq \Psi_s(B(Y_s))$. If yes then $\sigma_{s+1} = \sigma_s\widehat{\ }0$ otherwise $\sigma_{s+1} = \sigma_s\widehat{\ }1$. Let $X = \cup_s \sigma_s$.

End of Construction.

Lemma 3.4.6. *If X is defined as above then $A(X)$ and $B(X)$ satisfy $\mathcal{N}_e$ for each e.*

Proof. If $X \succ \sigma_s\widehat{\ }0$, then $A_{H(\sigma\widehat{\ }1)} \subseteq \Psi_s(B(Y_s))$ and by definition of B, $B(Y_s) \subseteq B(X)$ so $A_{H(\sigma\widehat{\ }1)} \subseteq \Psi_s(B(X))$ but $A_{H(\sigma\widehat{\ }1)} \not\subseteq A(X)$ as $\sigma\widehat{\ }0 \prec X$, so $\mathcal{N}_e$ is met. On the other hand if $X \succ \sigma_s\widehat{\ }1$, then $A_{H(\sigma\widehat{\ }1)} \not\subseteq \Psi_s(B(Y_s))$ and $B(X) \subseteq B(Y_s)$, so $A_{H(\sigma\widehat{\ }1)} \not\subseteq \Psi_s(B(X))$, but $A_{H(\sigma\widehat{\ }1)} \subseteq H_e(X)$. So $\mathcal{N}_e$ is satisfied. □

So $(A(X), B(X))$ satisfy all the requirements and form a strong minimal pair. □

An immediate corollary of this proof is that there are continuum many strong minimal pairs in the enumeration degrees.

Corollary 3.4.7. *For every set $Y \in 2^\omega$ there is a strong minimal pair A_Y, B_Y such that if $Y \neq Z$ then $A_Y \neq A_Z, B_Y \neq B_Z$.*

Proof. Fix Y. We build X as in the construction of X from the proof of Theorem 3.4.1 but on even stages $2s$ we set $X(2s) = Y(s)$ and on odd stages $2s+1$ we chose $X(2s+1)$ to satisfy $\mathcal{N}_s$ as before. □

It is also interesting to note the reduction $A \oplus B \leq_e \Psi_e(A) \oplus B$ is uniform in e with $\Gamma_e(\Psi_e(A) \oplus B) = \emptyset$ if $\Psi_e(A)$ is c.e. Furthermore B can enumerate the set $\{e : 0 <_e \Psi_e(A)\}$ by looking at which columns of B are nonempty. In this sense, we can think of A, B as being a uniformly strong minimal pair.

The forcing conditions are symmetric. By applying the same forcing steps to $\overline{A} = \bigcup_{p \in H(x)} C_p$ that we apply to A we can make it that both A, B and $\overline{A}, B$ are strong minimal pairs (we can also construct examples like this with the $\mathcal{K}$-pair construction). Note that $A \oplus \overline{A}, B$ will not be a strong minimal pair as $L_A, R_A \leq_e A \oplus \overline{A}$ and Lemma 3.2.1 says the left side of a strong minimal pair cannot bound a non trivial $\mathcal{K}$-pair.

If we wanted to modify the construction to get a super minimal pair we would quickly run into problems. The design of B is very precise and if we add some point $\langle e, a, v\rangle$ to B at some stage where we have ensured $A_s \subseteq A$, then it could be that already $D_v \subseteq \Psi_e(A_s)$. So we would have to ensure that $a \in A$, but then because we want $\Gamma_i(\Psi_i(B) \oplus A) = B$ we are in the reverse situation and may need to add things to B. This could go on indefinitely and end up making A and B cofinite or require us to add numbers to A or B that we have ensured are not in A or B and break a negative condition. We could try increasing C_p so that this case cannot happen, but the set $\{\langle e, a, v\rangle : p \Vdash D_v \subseteq \Psi_e(A)\}$ is not computable so we would be using a new partial order and Lemma 3.4.4 no longer holds.

3.5 The complexity of a strong minimal pair

Now we look at what oracle is needed to carry out the construction of the Section 3.4. To work out if case 1 of Lemma 3.4.4 can be applied for a given condition p and number e we ask if there exists $q \leq p$ such that $\Psi_e(A_q) = \Psi_e(\overline{C_q})$. Since P is the set of pairs of disjoint computable sets, we can encode it as a Π^0_2 set of natural numbers. Similarly asking if $\Psi_e(A_q) = \Psi_e(\overline{C_q})$ is a Π^0_2 question. So asking if case 1 of Lemma 3.4.4 can be applied is a Σ^0_3 question. Asking if a pair n, F witnesses case 2 of Lemma 3.4.4 holding is something $\mathbf{0}'$ can answer, so is not going to add to the complexity of the construction. Hence H and S are Δ^0_4.

$A, B \leq_T H \oplus S \oplus X$ so we need to work out the complexity of X. To construct X we ask questions of the form "is $A_{H(\sigma\frown 1)} \subseteq \Psi_e(B(\sigma\frown 1\frown 0^\omega))$?" which is $\Pi^0_2(H \oplus S)$. So X is Δ^0_6. When the answer was yes, B increased in size. Therefore A is Δ^0_6 and B is Π^0_5.

Clearly there are some minor modifications that would reduce the complexity. We make some more serious changes to get the following result.

Theorem 3.5.1. *There is a strong minimal pair A, B in the enumeration degrees such that A is Σ_2^0 and B is Π_2^0 and quasi-minimal.*

A set B is quasi-minimal if every function f with graph$(f) \leq_e B$ is a computable function. In other words the only degree below $\deg_e(B)$ that is the image of a Turing degree is $\mathbf{0}_e$.

Proof. This is a finite injury argument. The idea is that we run the construction using $0'$ as an oracle, but rather than building a whole tree we only build nodes along what we believe to be on the true path (on X). $0'$ will often be wrong about what the true path is, and this is where the injury comes in.

We use a restricted set of forcing conditions, $Q = \{p \in P : A_p, C_p \text{ are finite}\}$. In the proof of for Lemma 3.4.4 the q we build to meet case 1 was in fact infinite, so to ensure we satisfy $\mathcal{R}_e$ when case 2 does not apply we will make A enumeration 1-generic.

Definition 3.5.2 ([3])**.** A set A is *enumeration 1-generic* if for every W_e either there is $u \in W_e$ such that $D_u \subseteq A$ or there is $F \subseteq_{\text{fin}} \overline{A}$ such that for all $u \in W_e$, $D_u \cap F \neq \emptyset$.

For $q \in Q$ we say $q \Vdash \Psi_e(A)$ is c.e. if for all enumeration 1-generic $A \supseteq A_q$ with $\overline{A} \supseteq C_q$ we have that $\Psi_e(A) = \Psi_e(\overline{C_q})$. We have a new version of Lemma 3.4.4 that applies to Q.

Lemma 3.5.3. *For every $q \in Q, e \in \omega$ we have either*

1. *$q \Vdash \Psi_e(A)$ is c.e.*

2. *There is $n \in \omega$, $F \subseteq_{\text{fin}} \overline{A_q \cup C_q}$ such that $(A_q \cup F, C_q) \Vdash n \in \Psi_e(A)$ and $(A_q, C_q \cup F) \Vdash n \notin \Psi_e(A)$.*

Proof. Consider a pair $q \in Q, e \in \omega$, and suppose that case 2 does not hold. Let G be an enumeration 1-generic such that $A_q \subseteq G \subseteq \overline{C_q}$. Then suppose that there is n such that $n \in \Psi_e(\overline{C_q})$ but $n \notin \Psi_e(G)$. Then consider the c.e. set $W = \{u : \langle n, u\rangle \in \Psi_e\}$. Since G is enumeration 1-generic and there is no $u \in W$ such that $D_u \subseteq G$ we have that there is $E \subseteq_{\text{fin}} \overline{G}$ such that for all $u \in W$ we have $D_u \cap E \neq \emptyset$.

Pick $v \in W$ such that $D_v \subseteq \overline{C_q}$ (since $n \in \Psi_e(\overline{C_q})$ there must be some v). Now consider $F = (D_v \setminus A_q) \cup (E \setminus C_q)$. $D_v \subseteq A_q \cup F$ so $(A_q \cup F, C_q) \Vdash n \in \Psi_e(G)$. On the other hand $E \subseteq C_q \cup F$ so for each $u \in W$, $D_u \nsubseteq \overline{C_q \cup F}$, and thus $(A_q, C_q \cup F) \Vdash n \notin \Psi_e(G)$, a contradiction. So it must be that $\Psi_e(G) \supseteq \Psi_e(\overline{C_q})$. We already have $\Psi_e(G) \subseteq \Psi_e(\overline{C_q})$ as $G \subseteq \overline{C_q}$, so $\Psi_e(G) = \Psi_e(\overline{C_q})$. Since G was arbitrary, we have $q \Vdash \Psi_e(A)$ is c.e. □

The Γ_e we will use this time is a little different, only needing one witness from $\Psi_e(A)$:

$$\Gamma_e = \{\langle a, p\rangle : \exists m[D_p = \{m\} \oplus \{\langle e, a, m\rangle\}]\}$$

Since we are making B a Π^0_2 set we will start with all axioms in B and remove broken axioms as we go.

Construction of Σ^0_2 A and Π^0_2 B. At each stage of the construction we will have a tuple

$$(\sigma_s \in 2^{<\omega}, n_s = |\sigma_s|, H_s : n_s + 1 \to Q, (F_{n,s})_{n<n_s}, S_s :\subseteq \omega \times n_s \to \omega, B_s)$$

with $H = \lim_s H_s$, $S = \lim_s S_s$, $X = \lim_s \sigma_s$, $B = \bigcap B_s$, $F_n = \bigcup_s F_{n,s}$, $A = \bigcup_n A_{H(n)} = \bigcup_s A_{H_s(n_s)}$ and $\overline{A} = \bigcup_n C_{H(n)} = \bigcup\{F_n : X(n) = 0\}$. We will have $H_s(n+1) < H_s(n)$ and $F_{n,s} \subseteq A_{H_s(n+1)}$ if $\sigma(n)_s = 1$ and $C_{H_s(n)} = \bigcup\{F_k : \sigma_s(k) = 0, k < n\}$.

The requirements we will use are slightly different than in Section 3.4. We will break each $\mathcal{R}_e$ requirement into ω many requirements $\mathcal{R}_{e,n}$ for $n \geq e$.

$$\mathcal{R}_{e,n} :\ A_{H(n)} \cup F_n \subseteq \Gamma_e(\Psi_e(A_{H(n)} \cup F_n) \oplus B)) \subseteq \overline{C_{H(n)}}$$

This means that if every $\mathcal{R}_{e,n}$ requirement is satisfied (and X contains infinitely many 1's) then $\Gamma_e(\Psi_e(A) \oplus B) = A$. If some $\mathcal{R}_{e,n}$ cannot be satisfied then by Lemma 3.5.3 we will have $H(n) \Vdash \Psi(A)$ is c.e. The $\mathcal{N}_e$ requirements will not change.

$$\mathcal{N}_i :\ \Psi_i(B) \neq A$$

And we have new requirements to make sure that A is enumeration 1-generic.

$$\mathcal{E}_i : \ \exists u \in W_i[D_u \subseteq A_{H(i)}] \vee \forall u \in W_i[D_u \cap C_{H(i)} \neq \emptyset]$$

And requirements to make sure that B is quasi-minimal.

$$\mathcal{Q}_e : \ \Psi_e(B) \neq \text{graph}(f) \text{ for any non-computable } f$$

The priority of the requirements is given by $\mathcal{R}_{0,n} < \cdots < \mathcal{R}_{n,n} < \mathcal{N}_n < \mathcal{E}_n < \mathcal{Q}_n < \mathcal{R}_{0,n+1}$.

A requirement $\mathcal{R}_{e,n}$ requires attention at stage $s+1$ if it has not been satisfied, $\sigma_s(n) = 0$ and there are $m, u < s$ that show case 2 holds for $(A_{H_s(n_s)}, C_{H_s(n)})$ with m and D_u (note it may not hold for $(A_{H_s(n)}, C_{H_s(n)})$). $\emptyset'$ can answer this question.

We say that a requirement $\mathcal{N}_i$ requires attention at stage $s+1$ if it has not been initialized and $i < |\sigma_s|$. There are two cases as to how $\mathcal{N}_i$ will act depending on whether $F_{i,s} \nsubseteq \Psi_e(B_s)$ or $F_{i,s} \subseteq \Psi_e(D)$ for some finite $D \subseteq B_s$.

An $\mathcal{E}_i$ requirement needs attention at stage $s+1$ if $n_s = i$. It is only when satisfying these $\mathcal{E}_i$ requirements that we will increase n_s, so every $\mathcal{E}_i$ requirement will require attention at some stage.

We say that a requirement $\mathcal{Q}_i$ requires attention if it has not been initialized and $i < |\sigma_s|$ and there are x, y, z such that $z \neq y$ and $\langle x, y \rangle, \langle x, z \rangle \in \Psi_i(B_s)$.

Assume at stage s we have $(n_s, \sigma_s, H_s, (F_{n,s})_{n<n_s}, S_s, A_s B_s)$. At stage $s+1$ consider the highest priority requirement that requires attention. All lower priority requirements will be considered unsatisfied.

- *Case one: the requirement is $\mathcal{R}_{e,n}$.* By assumption we have $m, u < s+1$ that shows e is in case 2 for $(A_{H_s(n_s)}, C_{H_s(n)})$. We set
  - $\sigma_{s+1} = \sigma_s \restriction (n+1)$.
  - $n_{s+1} = n + 1$.

- $F_{n,s+1} = F_{n,s} \cup D_u \cup C_{H_s(n_s)} \setminus C_{H_s(n)}$.
- $F_{k,s+1} = F_{k,s}$ for $k < n$.
- $H_{s+1} = H_s\restriction(n+1) \cup \{(n+1, (A_{H_s(n_s)}, C_{H_s(n)} \cup F_{n,s+1}))\}$ if $\sigma(n) = 0$.
- $H_{s+1} = H_s\restriction(n+1) \cup \{(n+1, (A_{H_s(n_s)} \cup F_{n,s+1}, C_{H_s(n)}))\}$ if $\sigma(n) = 1$.
- $S_{s+1} = S_s\restriction(\omega \times n + 1) \cup ((e, n), m)$.

The reason we add all the extra elements to $F_{n,s+1}$ is because D_u may have contained some of them and we need to respect the axioms on modified parts of B. For each $a \in F_{n,s+1}$ and i, k such that $k = S_{s+1}(i, n)$ we ask if $\{v : \langle i, a, v\rangle \in B_s\} = \omega$. If yes then we define $\{v : \langle i, a, v\rangle \in B_{s+1}\} = \{k\}$. Intuitively this change to B means that $a \in \Gamma_i(\Psi_i(A) \oplus B)$ if and only if $k \in \Psi_i(A)$.

- *Case two: the requirement is $\mathcal{N}_i$ and $F_{i,s} \not\subseteq \Psi_i(B_s)$.* We have that $i < n_s$, $\sigma_s(i) = 0$. Since $F_{i,s} \not\subseteq \Psi_i(B_s)$ we will redefine $\sigma_{s+1}(i)$ to be 1 and add $F_{i,s}$ to A.
  - $\sigma_{s+1} = (\sigma_s\restriction i)^\frown 1$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s\restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup F_{i,s} \cup C_{H_s(n_s)} \setminus C_{H_s(i)}, C_{H_s(i)}))\}$.
  - $S_{s+1} = S_s\restriction(\omega \times n_{s+1})$ and $B_{s+1} = B_s$.
- *Case three: the requirement is $\mathcal{N}_i$ and $F_{i,s} \subseteq \Psi_i(B_s)$.* We have that $i < n_s$, $\sigma_s(i) = 0$ and there is finite $D \subseteq B$ such that $F_{i,s} \subseteq \Psi_i(D)$. We will add elements to A to ensure that D will remain a subset of B. Let $P = (\{a : \exists e, m[\langle e, a, m\rangle \in D]\} \cup C_{H_s(n_s)}) \setminus C_{H_s(i+1)}$. We set
  - $\sigma_{s+1} = (\sigma_s\restriction i + 1)$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s\restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup P, C_{H_s(i+1)}))\}$.

  - $S_{s+1} = S_s \restriction (\omega \times n_{s+1})$ and $B_{s+1} = B_s$.

- *Case four: the requirement is $\mathcal{Q}_i$ and $\langle x, y \rangle, \langle x, z \rangle \in \Psi_i(B_s)$.* We have that $i < n_s$ and a finite $D \subseteq B$ such that $\langle x, y \rangle, \langle x, z \rangle \in \Psi_i(D)$. We will add elements to A to ensure that D will remain a subset of B. Let $P = (\{a : \exists e, m[\langle e, a, m \rangle \in D]\} \cup C_{H_s(n_s)}) \setminus C_{H_s(i+1)}$. We set

  - $\sigma_{s+1} = (\sigma_s \restriction i + 1)$.
  - $n_{s+1} = i + 1$.
  - $F_{k,s+1} = F_{k,s}$ for $k \leq i$.
  - $H_{s+1} = H_s \restriction n_{s+1} \cup \{(n_{s+1}, (A_{H_s(n_s)} \cup P, C_{H_s(i+1)}))\}$.
  - $S_{s+1} = S_s \restriction (\omega \times n_{s+1})$ and $B_{s+1} = B_s$.

- *Case five: the requirement is $\mathcal{E}_i$.* Note that the other two cases do not increase n_s. Here is where we do so. Ask if there is $u \in W_i$ such that $p = (A_{H_s(n_s)} \cup D_u, C_{H_s(n_s)}) \in Q$. If not then set $p = H_s(n_s)$. Now take the least $m \in \overline{A_p \cup C_p}$ and set

  - $\sigma_{s+1} = \sigma_s {}^\frown 0$.
  - $n_{s+1} = n_s + 1$.
  - $F_{n_s,s+1} = \{m\}$.
  - $F_{k,s+1} = F_{k,s}$ for $k < n_s$.
  - $H_{s+1} = H_s \restriction n_s \cup \{(n_s, p), (n_{s+1}, (A_p \cup F_{n_s,s+1}, C_p))\}$.
  - $S_{s+1} = S_s$ and $B_{s+1} = B_s$.

End of Construction.

Now we move on to the verification.

Lemma 3.5.4. *A is Σ^0_s and B is Π^0_2.*

Proof. The construction only removes points from B so B is $\emptyset'$-co-c.e. At each stage we have that $A_{H_s(n_s)} \subseteq A_{H_{s+1}(n_{s+1})}$ so $\bigcup_s A_{H_s(n_s)}$ is $\emptyset'$-c.e. At each stage the only value of

H_s that changes is the final one, so for each n there is s such that $H(n) = H_s(n_s)$. Hence $A = \bigcup_n A_{H(n)} = \bigcup_s A_{H_s(n_s)}$ is Σ^0_2. □

Lemma 3.5.5. *A is enumeration 1-generic.*

Proof. Consider a requirement $\mathcal{E}_i$. Consider the last stage s where $n_s = i$. Then at stage $s+1$ we looked for $u \in W_i$ such that $D_u \cap C_{H_{s+1}(i)} = \emptyset$. If there was such a u then we set $D_u \subseteq A_{H_{s+1}(i)}$, and if not then, $D_u \cap C_{H_{s+1}(i)} \neq \emptyset$ for all $u \in W_i$. Since no higher priority requirements act after stage s we have $n_t \geq i$ for all $t > s$ and $H(i) = H_{s+1}(i)$. Thus $\mathcal{E}_i$ is satisfied. □

Lemma 3.5.6. *B is quasi-minimal.*

Proof. Consider a requirement $\mathcal{Q}_i$. Suppose that $\text{graph}(f) = \Psi_i(B)$. It is sufficient for us to show that f is computable. Let s be a stage such that $\mathcal{Q}_i$ is not injured at any stage $t \geq s$. We claim that $\text{graph}(f) = \Psi_i(B_s)$. Since $B \subseteq B_s$ we have $\text{graph}(f) \subseteq \Psi_i(B_s)$ so if $\text{graph}(f) \neq \Psi_i(B_s)$ then there are $x, y \neq z$ such that $\langle x, y\rangle, \langle x, z\rangle \in \Psi(B_s)$. This means that we would have acted with some finite $D \subseteq B_s$ and P at some stage $\leq s$ according to the strategy for $\mathcal{Q}_i$.

Since $D \nsubseteq B$ some $\mathcal{R}_{e,n}$ requirement removed an axiom $\langle e, a, m\rangle \in D$ from B at some stage $t > s$. Since $\mathcal{Q}_i$ is not injured after stage s we must have $n > i$ and the strategy for $\mathcal{R}_{e,n}$ put $a \in F_{n,t}$. So $a \notin A_{H_t(n)} \cup C_{H_t(n)} \supseteq P \cup C_{H_s(i+1)} \supseteq \{a : \exists e, m[\langle e, a, m\rangle \in D]\}$, a contradiction. □

Lemma 3.5.7. *A, B satisfies $\mathcal{R}_e$ and $\mathcal{N}_i$ for all $e, i \in \omega$.*

Proof. Consider an $\mathcal{N}_i$ requirement. Let s be the last stage where $\mathcal{N}_i$ is injured. So $F_i = F_{i,s}$ as only higher priority requirements can change $F_{i,s}$. If $F_i \subseteq \Psi_i(B)$ then $F_i \subseteq \Psi_i(B_t)$ for all t. So when $\mathcal{N}_i$ acted for the last time we must have set $\sigma_t(i) = 0$. Since no lower priority requirements will change $\sigma_t(i)$ we have $X(i) = 0$ and $F_i \subseteq \overline{A}$.

If $F_i \nsubseteq \Psi_e(B)$ then suppose that when $\mathcal{N}_i$ acted for the last time it set $\sigma_t(i) = 0$. At this stage we had $D \subseteq B_t$ so there must have been a later stage k where $D \nsubseteq B_k$.

If we remove an axiom $\langle e, a, m\rangle$ from B then we have $a \in F_{n,k}$ for some $n \leq n_k$. But because lower priority requirements do not remove elements from $H_k(i+1)$ we have that $a \notin A_{H_k(i+1)} \cup C_{H_k(i+1)} \supseteq P \cup C_{H_t(i+1)} \supseteq \{a : \exists e, m[\langle e, a, m\rangle \in D]\}$, a contradiction. So $F_i \subseteq A$. Hence each $\mathcal{N}_i$ is satisfied.

Consider an $\mathcal{R}_e$ requirement. We have two cases to deal with here. First, suppose that all $\mathcal{R}_{e,n}$ sub-requirements are satisfied. Now we show that $\Gamma_e(\Psi_e(A) \oplus B) = A$. Consider some $a \in A$. If $\{m : \langle e, a, m\rangle \in B\} = \omega$ then we must have $a \in \Gamma_e(\Psi_e(A) \oplus B$. If $\{m : \langle e, a, m\rangle \in B\} \neq \omega$ then when we removed the missing elements at some stage s, we ensured that there is m such that $\langle e, a, m\rangle \in B$ and $m, F_{k,s}$ satisfied case 2 of Lemma 3.5.3 for e and $(A_{H_s(n_s)}, C_{H_s(k)})$ and $a \in F_{k,s}$. Since $a \in A$ it must be that $A_{H_s(k)} \cup F_{k,s} \subseteq A$; this is because when we change $F_{k,s}$ at some later stage t we ensure that $F_{k,s}$ is a subset of one of $F_{n,t}, A_{H_t(n))}, C_{H_t(n)}$ for some n. So we have that $m \in \Psi_e(A)$ and hence $a \in \Gamma_e(\Psi_e(A) \oplus B)$.

Now consider $a \notin A$. Then there must be a least stage s with $a \in F_{k,s}$ for some k. Since $a \notin A$ there is some n such that $a \in F_n \subseteq \overline{A}$. Since $\mathcal{R}_{e,n}$ is satisfied and $X(n) = 0$ there is a stage t and m, u, n' such that $a \in D_u \subseteq F_n$ and m, D_u satisfied case 2 of Lemma 3.5.3 for e and $(A_{H_t(n_t)}, C_{H_t(n')})$. Note it is possible that t is smaller than the stabilizing stage of $F_{n,s}$ and that $n' \neq n$ but this does not matter. We would have used m, D_u to satisfy $\mathcal{R}_{e,n'}$ and have $\{m\} = \{v : \langle e, a, v\rangle \in B\}$. Since $a \notin A$ it must be that $C_{H_t(n')} \subseteq \overline{A}$ as every time we add a part of $C_{H_s(n)}$ to A we make sure $F_{n',s} \subseteq A$. Since $m \notin \Psi_e(\overline{C_{H_t(n')} \cup D_u})$ we have $m \notin \Psi_e(A)$ and so $a \notin \Gamma_e(\Psi_e(A) \oplus B)$.

Now suppose there is some $\mathcal{R}_{e,n}$ that is never satisfied. Then we argue in a similar vein to Lemma 3.5.3 that $\Psi_e(A) = \Psi_e(\overline{C_{H(n)}})$. Suppose not. Then there is $m \in \Psi_e(\overline{C_{H(n)}}) \setminus \Psi_e(A)$. Consider the c.e. set $\{u : D_u \subseteq \overline{C_{H(n)}}, \langle m, u\rangle \in W_e\}$. Since $m \notin \Psi_e(A)$ and A is enumeration-1-generic there must be finite $F \subseteq \overline{A} \setminus C_{H(n)}$ such that $m \in \Psi_e(A \cup F)$ but $m \notin \Psi_e(\overline{C_{H(n)} \cup F})$. But then we would have used m, F to satisfy $\mathcal{R}_{e,n}$ at some sufficiently large stage. □

This completes the proof. □

Is it possible for A or B to have lower complexity? Lemma 3.2.1 tells us that A cannot be Δ^0_2 or, in fact, be above any non c.e. Δ^0_2 set. So we have shown that Σ^0_2 is a strict lower bound on the complexity of A. As for B, we know from Theorem 3.3.3 that B can have complexity Π^0_1. B cannot have lower complexity because it cannot be c.e. We have shown that both sides of a strong minimal pair can be Σ^0_2 but we do not know if this can happen at the same time.

Question 3.5.8. Is there a strong minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$?

We leave open the questions about super minimal pairs.

Question 3.5.9. Is there a super minimal pair in the enumeration degrees?

Question 3.5.10. Is there a super minimal pair in $\mathcal{D}_e(\leq \mathbf{0}')$?

Before we can hope to find an algorithm that decides the two quantifier theory of $\mathcal{D}_e$ or $\mathcal{D}_e(\leq \mathbf{0}')$, we need be able to find to find answers to the questions above.

Chapter 4

Topological classification of classes of enumeration degrees

4.1 Introduction

In this chapter we look at some of the interactions between the enumeration degrees and topology. Kihara and Pauly [27] define degrees of points in arbitrary second-countable topological spaces, using the notion of a countably based space.

Definition 4.1.1. A cb_0 space $\mathcal{X}$ is a second countable T_0 space given with a listing of a basis $(\beta_e)_e$. Given a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ and a point $x \in X$ the coded neighborhood filter of x is $\mathrm{NBase}_{\mathcal{X}}(x) = \{e \in \omega : x \in \beta_e\}$. We define the degrees of a space $\mathcal{X}$ to be $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} \in \mathcal{D}_e : \exists x \in X[\mathrm{NBase}(x) \in \mathbf{a}]\}$.

Kihara and Pauly showed that $\mathcal{D}_e$ is the class of degrees of the ω-product of Sierpiński space $\mathbb{S}^\omega$, where $\mathbb{S} = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$. For a point $x \in \mathbb{S}^\omega$ we have that $\mathrm{NBase}(x) \equiv_e \{n : x(n) = 1\}$. $\mathbb{S}^\omega$ is a universal second-countable T_0 space. This can be seen by observing that the map $x \mapsto \mathrm{NBase}_{\mathcal{X}}(x)$ is a topological embedding.

From the definition, every cb_0 space $\mathcal{X}$ gives a class of enumeration degrees $\mathcal{D}_{\mathcal{X}}$. From the universality of $\mathbb{S}^\omega$ we have that every class $\mathcal{C}$ of enumeration degrees is $\mathcal{D}_{\mathcal{X}}$ for some cb_0 space $\mathcal{X}$, namely $\mathcal{X} = \{x \in \mathbb{S}^\omega : \deg(\mathrm{NBase}(x)) \in \mathcal{C}\}$. So the study of subclasses of the enumeration degrees is the study of cb_0 spaces.

In this chapter we answer several open questions of Kihara, Ng and Pauly [26], focusing on the topological separation axioms, and how they interact with classes of enumeration degrees. The separation axioms that we explore are as follows.

Definition 4.1.2. A topological space is considered

- T_0 (Kolmogorov) if for any $x \neq y$ there is an open set U such that either $x \in U, y \notin U$ or $x \notin U, y \in U$. In other words, points can be distinguished by the topology.
- T_1 (Fréchet) if for any $x \neq y$ there are open U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Equivalently if $\{x\}$ is closed for any x.
- T_2 (Hausdorff) if for any $x \neq y$ there are disjoint open U, V such that $x \in U, y \in V$.
- $T_{2.5}$ (Urysohn) if for any $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.
- *Submetrizable* if there is a coarser topology on the space that is metrizable. In other words, if $\mathcal{X} = (X, (\beta_e)_e)$ is submetrizable then there is a collection of $\mathcal{X}$-open sets $(\alpha_e)_e$ such that $(X, (\alpha_e)_e)$ is metrizable.

We have the following series of implications:

$$\text{metrizable} \implies \text{submetrizable} \implies T_{2.5} \implies T_2 \implies T_1 \implies T_0$$

It is well known that this hierarchy is strict for second countable spaces. Kihara, Ng and Pauly [26] prove that every enumeration degree is the degree of a point in a decidable, effectively submetrizable space, so the non-metrizable separation axioms do not give us new classes of degrees without additional computably assumptions. However we can use the separation axioms to classify classes of degrees.

Definition 4.1.3. Given a collection of cb_0 spaces $\mathcal{T}$ we say that a class $\mathcal{C}$ of enumeration degrees is $\mathcal{T}$ if there is some $\mathcal{X} \in \mathcal{T}$ such that $\mathcal{D}_{\mathcal{X}} = \mathcal{C}$.

We have the same implications of the separation axioms for classes of degree as we do for spaces, but because multiple different cb_0 spaces may give rise to the same class of degrees, it is not clear that these implications are strict.

Kihara, Ng and Pauly [26] gave some separations for this classification of classes of degrees. They showed that $\mathcal{D}_e$ is T_0 but not T_1, that cylinder-cototal degrees are T_1 but not T_2, and that $\mathcal{D}_{\mathbb{N}^\omega_{\mathrm{rp}}}$ is T_2 but not $T_{2.5}$. They did not show that $T_{2.5}$ and submetrizable are different notions for classes of degrees and asked as a question if there is a $T_{2.5}$ class that is not submetrizable. They also showed that the degrees of the Gandy-Harrington topology do not arise from any metrizable space, giving a separation between submetrizable and metrizable for classes of degrees.

Kihara, Ng and Pauly [26] suggested some candidates for classes that could be $T_{2.5}$ but not submetrizable. They introduced the Arens co-d-CEA degrees and the Roy halfgraph above degrees. Both classes arise from spaces that are $T_{2.5}$ but not submetrizable. Another candidate class introduced by Kihara, Ng and Pauly was the doubled co-d-CEA degrees. This class contains both the Arens co-d-CEA degrees and the Roy halfgraph degrees and arises from a space that is T_2 but not $T_{2.5}$. Kihara, Ng and Pauly asked if the doubled co-d-CEA degrees are a $T_{2.5}$ class or not.

In Section 4.5 we answer this question and show that the doubled co-d-CEA degrees are not $T_{2.5}$, giving a new example of a class that is T_2 but not $T_{2.5}$. As a result of this we have that the class of degrees that are Arens co-d-CEA or Roy halfgraph is a strict subset of the doubled co-d-CEA degrees. This separation is of interest in its own right because the doubled co-d-CEA degrees arise from a quasi-Polish space and the previous separation uses $\mathbb{N}^\omega_{\mathrm{rp}}$ which is not quasi-Polish.

In Section 4.6 we prove that the Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable answering the question of Kihara, Ng and Pauly about the distinction between $T_{2.5}$ and submetrizable. In the proof of these results we introduce a general method that could be used to get similar results. We also introduce the notion of a space being effectively submetrizable as part of the general method.

In Section 4.7 we look at the relationship between the Arens co-d-CEA degrees and the Roy halfgraph degrees and prove that neither class is contained in the other. This answers another question of Kihara, Ng and Pauly [26].

Kihara, Ng and Pauly [26] extend the notion of quasi-minimal to give a strong way in which a class of enumeration degrees can be not $\mathcal{T}$. An enumeration degree $\mathbf{a}$ is quasi-minimal if it is not above any nonzero total degrees. The following extends this idea to general classes.

Definition 4.1.4. For a cb_0 space $\mathcal{X}$ we say that a degree $\mathbf{a} \in \mathcal{D}_e$ is $\mathcal{X}$ quasi-minimal if $\mathbf{a} \notin \mathcal{D}_{\mathcal{X}}$ and for all $\mathbf{b} \in \mathcal{D}_{\mathcal{X}}$ if $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{b} = \mathbf{0}$.

For a class $\mathcal{C} \subseteq \mathcal{D}_e$ and a set of cb_0 spaces $\mathcal{T}$, we say that $\mathcal{C}$ is $\mathcal{T}$ quasi-minimal if for every $\mathcal{X} \in \mathcal{T}$ there is $\mathbf{a} \in \mathcal{C}$ such that $\mathbf{a}$ is $\mathcal{X}$ quasi-minimal.

Kihara, Ng and Pauly show that $\mathcal{D}_e$ is T_1-quasi-minimal and as a result any class of enumeration degrees that is downwards dense in the enumeration degrees (that is for, each $\mathbf{a} >_e \mathbf{0}$ there is a $\mathbf{b}$ in our class such that $\mathbf{0} <_e \mathbf{b} \leq_e \mathbf{a}$), like the class of semicomputable degrees (introduced by Jockusch [19] and proven to be downwards dense by Kihara, Ng and Pauly [26]) and the class of enumeration-1-generic degrees (introduced by Badillo and Harris [4] and proven to be downwards dense by Badillo, Liliana, Harris and Soskova [5]), is also T_1-quasi-minimal. They also prove that the telegraph-cototal degrees, a T_1 class containing the doubled co-d-CEA degrees, are quasi-minimal for countable disjoint unions of effective T_2 spaces. A question they ask is if there is a quasi-minimal separation of T_2 and $T_{2.5}$.

In section 4.3 we modify the proof that the cylinder-cototal degrees are not T_2 to show that they are T_2-quasi-minimal. In section 4.4 we answer the above question by modifying the proof that $\mathcal{D}_{\mathbb{N}^{\omega}_{\mathrm{rp}}}$ is not $T_{2.5}$ to show that it is $T_{2.5}$ quasi-minimal. An open question we ask is if there is a quasi-minimal separation of $T_{2.5}$ from submetrizable.

The continuous degrees are a subclass of the enumeration degrees, introduced by Miller [34], and come from computably represented metric spaces. They are a proper subclass of the cototal degrees and have some interesting interactions with the structure of the enumeration degrees [2]. Kihara and Pauly [27] show that their redefinition of the degrees of a space coincide with Miller's definition when the basis is given by the collection of balls centered at rational points with rational radii.

In Section 4.8 we look at metrizable cb_0 spaces that have different bases than the basis of rational radius balls centered at rational points. We show that there is a metrizable cb_0 space $\mathcal{X}$ such that $\mathcal{D}_{\mathcal{X}}$ contains all quasi-minimal doubled co-d-CEA degrees. As a result the doubled co-d-CEA degrees are not metrizable quasi-minimal. Hence neither are the Arens co-d-CEA or Roy halfgraph degrees. So a different $T_{2.5}$ space is needed if one wants to get a quasi-minimal separation of $T_{2.5}$ from submetrizable. In this section we also ask about the degrees of points in decidable, metrizable cb_0 spaces. We show that there is a decidable, metrizable cb_0 space whose degrees contain a quasi-minimal degree. Hence the class of degrees of points in decidable, metrizable cb_0 spaces is larger than the continuous degrees. We leave open the question of whether there is any degree that is not the degree of a point in a decidable, metrizable cb_0 space.

4.2 Preliminaries

In this section we go over some background notions related to cb_0 spaces that are needed for this chapter. More specific definitions, for instance the definition of a particular class of degrees, will be given in the relevant section. The content of this section gives some background and should apply to the whole chapter.

4.2.1 Represented spaces

Represented spaces are a way of defining notions of computability for arbitrary spaces, using the notions of computability on ω^ω. These are a key tool in the study of computable analysis [45].

Definition 4.2.1. A represented space $\mathcal{X}$ is a set X and a partial surjection $\delta :\subseteq \omega^\omega \to X$. Given a represented space $\mathcal{X} = (X, \delta)$ and a point $x \in X$ we say that a point $p \in \omega^\omega$ is a *δ-name* for x if $\delta(p) = x$. We define $\mathrm{Name}_{\mathcal{X}}(x) = \{p : \delta(p) = x\}$.

For an example one nonstandard representation of ω^ω is to say that p is a δ-name for $f \in \omega^\omega$ if $\mathrm{range}(p) = \{\langle n, m\rangle : f(n) \neq m\}$. In this example it is possible that a δ-name for a point f does not compute f, even though it describes f uniquely.

For arbitrary represented spaces $\mathcal{X}, \mathcal{Y}$ Kihara and Pauly [27] define a reducibility notion of points $\leq_{\mathbf{T}}$, by $x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y} \iff \forall \in \mathrm{Name}_{\mathcal{X}}(x) \exists q \in \mathrm{Name}_{\mathcal{Y}}(y)[p \leq_T q]$. Kihara and Pauly study the degree spectra of represented spaces in [27].

For cb_0 spaces $\mathcal{X} = (X, (\beta_e)_e)$ there is a natural representation δ given by $\delta(p) = x$ if p is an enumeration of $\mathrm{NBase}(x)$. This is well defined because $\mathcal{X}$ is a T_0 space: if $x \neq y \in \mathcal{X}$ then there is some open U such that $x \in U, y \notin U$ or $y \in U, x \notin U$, so there is an e such that $x \in \beta_e \subseteq U$ and $y \notin \beta_e$ or vice versa. Hence $\mathrm{NBase}(x) \neq \mathrm{NBase}(y)$. Kihara and Pauly [27] observe that when we use this representation of cb_0 spaces $\mathcal{X}, \mathcal{Y}$, we have $x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y}$ if and only if $\mathrm{NBase}_{\mathcal{X}}(x) \leq_e \mathrm{NBase}_{\mathcal{Y}}(y)$.

The example of a represented space above comes from a cb_0 space, the ω product of the cofinite topology, $(\omega_{\mathrm{cof}})^\omega$. Here a subbasis can be given as $\beta_{\langle n,m\rangle} = \{f : f(n) \neq m\}$ and the representation we used above was $\delta(p) = f$ if $\mathrm{range}(p) = \mathrm{NBase}_{(\omega_{\mathrm{cof}})^\omega}(f)$. For a cb_0 space we technically need a basis rather than a subbasis, however a subbasis can be turned to into a basis by taking finite intersections. Since $\{e : x \in \beta_e\} \equiv_e \{\sigma \in \omega^{<\omega} : x \in \beta_{\sigma(0)} \cap \cdots \cap \beta_{\sigma(|\sigma|-1)}\}$, using a subbasis rather than a basis will not change the degree of a point. In this thesis we will sometimes specify and work with a cb_0 space in terms of a subbasis rather than a basis.

Our remark above about a $(\omega_{\mathrm{cof}})^\omega$-name for f not necessarily computing f can be stated as saying that while $f : (\omega_{\mathrm{cof}})^\omega \leq_{\mathbf{T}} f : \omega^\omega$ there are f such that $f : \omega^\omega \not\leq_{\mathbf{T}} f : (\omega_{\mathrm{cof}})^\omega$. The degrees of $(\omega_{\mathrm{cof}})^\omega$ are the degrees of complements of graphs of total functions. This class is known as the graph cototal degrees [43, 1]. This class is known to be a proper subclass of the cototal degrees [1] and to contain non-total degrees.

As a tool in Sections 4.3, 4.4 and 4.5 we will make use of multi-representations [40]. The difference between a multi-representation and a single valued representation is that in a multi-representation $\delta :\subseteq \omega^\omega \rightrightarrows X$ is a multi function. So a point $p \in \omega^\omega$ may be a name for more than one distinct point in X.

4.2.2 Computability of spaces and functions

For represented spaces $\mathcal{X}, \mathcal{Y}$ and a partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ we say that a partial function $F :\subseteq \omega^\omega \to \omega^\omega$ is a *realizer*, if $f(\delta_\mathcal{X}(p)) = \delta_\mathcal{Y}(F(p))$ for every $p \in \mathrm{dom}(f \circ \delta_\mathcal{X})$. We say that a function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ is computable if it has a computable realizer.

For topological spaces there are several different notions of a computable representation. Since we are interested in the degrees of a space, we want a notion of computability that prevents coding non-computable information directly into the basis.

Definition 4.2.2. For a cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ we say $\mathcal{X}$ is *decidable* if the subset relation between positive Boolean combinations of $\emptyset$ and $(\beta_e)_e$ is computable. We say $\mathcal{X}$ is *strongly decidable* if the subset relation between positive Boolean combinations of $\emptyset$, $(\beta_e)_e$ and $(\overline{\beta_e})_e$ is computable.

Many natural spaces are strongly decidable, for instance $2^\omega, \omega^\omega, \mathbb{S}^\omega, [0,1]^\omega$ and $(\omega_{\mathrm{cof}})^\omega$. In section 4.6 we will introduce another notion of computability for spaces—that of being effectively submetrizable.

4.3 The cylinder-cototal degrees are T_2-quasi-minimal

The cylinder-cototal degrees were introduced by Kihara, Ng and Pauly [26]. They are defined to be the degrees of the cocylinder space $\omega^\omega_{\mathrm{co}} = (\omega^\omega, (\beta_e)_e)$ where $\beta_e = \{x \in \omega^\omega : \sigma_e \not\prec x\}$ for an effective enumeration $(\sigma_e)_e$ of $\omega^{<\omega}$. This is a coarser topology than the usual one on ω^ω because $\beta_e = \bigcup\{[\sigma] : \sigma \in \omega^{|\sigma_e|}, \sigma \neq \sigma_e\}$ is open under the usual topology. The space is T_1 but not T_2. Kihara, Ng and Pauly [26] prove that the cylinder-cototal degrees are a subclass of the graph cototal degrees by embedding $\omega^\omega_{\mathrm{co}}$ into $(\omega_{\mathrm{cof}})^\omega$, the space of the graph cototal degrees. One of the reasons the cylinder-cototal degrees are interesting is that they give us a separation of T_1 and T_2 for classes of degrees.

Theorem 4.3.1 (Kihara, Ng, and Pauly [26])**.** *The cylinder-cototal degrees are not T_2.*

By modifying the proof of the above theorem we are able to turn this into a quasi-minimal separation.

Theorem 4.3.2. *The cylinder-cototal degrees are T_2-quasi-minimal.*

To prove their result, Kihara, Ng and Pauly prove two important lemmas involving Hausdorff spaces and network representations of spaces. To state these lemmas we need to introduce the terminology used.

Definition 4.3.3. Given a topological space $\mathcal{X}$, a point $x \in X$ and a collection $\mathcal{N} \subseteq \mathcal{P}(X)$ have the following.

- $\mathcal{N}$ is a *network* at x if for each open $U \ni x$ there is $N \in \mathcal{N}$ such that $x \in N \subseteq U$.
- $\mathcal{N}$ is a *strict network* at x if it is a network at x and $x \in N$ for each $N \in \mathcal{N}$.
- $\mathcal{N}$ is a *cs-network* for $\mathcal{X}$ if for any convergent sequence $(x_n)_n$ and open $U \ni \lim_n x_n$ there is $N \in \mathcal{N}$ and $m \in \omega$ such that $\{x_n : n > m\} \subseteq N \subseteq U$.

Given a space $\mathcal{X}$ and a countable cs-network $\mathcal{N} \subseteq \mathcal{P}(X)$, the representation of $\mathcal{X}$ from $\mathcal{N}$ is $\delta_{\mathcal{N}}$ where $\delta_{\mathcal{N}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a strict network at x.

For some simple examples of cs-networks consider the following. If $\mathcal{X} = (X, (\beta_e)_e)$ is a cb_0 space then $(\beta_e)_e$ is a cs-network. If $\mathcal{X}$ is regular then $(\overline{\beta_e})_e$ is a cs-network.

A network representation does not necessarily give us a class of enumeration degrees like a cb_0 space does, but Kihara, Ng and Pauly [26] make the following observation that connects points in network representations with enumeration degrees.

Observation 4.3.4. *If $\mathcal{X} = (X, (\beta_e)_e)$ is a cb_0 space and $\mathcal{Y} = (Y, \mathcal{N})$ is a space with a countable cs-network, then $y : \mathcal{Y} \leq_{\mathbf{T}} x : \mathcal{X}$ if and only if there is $J \leq_e \text{NBase}_{\mathcal{X}}(x)$ such that $\{N_e : e \in J\}$ is a strict network at y.*

To prove Theorem 4.3.1 Kihara, Ng and Pauly [26] consider a different type of representation, they call the closure representation.

Definition 4.3.5. Given a space $\mathcal{X} = (X, \mathcal{N})$ with a countable cs-network $\mathcal{N}$, the *closure representation* of $\mathcal{X}$ is $\overline{\delta_{\mathcal{N}}}$ where $\overline{\delta_{\mathcal{N}}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a network at x and $x \in \overline{N_{p(e)}}$ for all $e \in \omega$. For a point $x \in \mathcal{X}$ we say that x is *nearly computable* if there is a computable $\overline{\delta_{\mathcal{N}}}$ name for x. For a space $\mathcal{Y}$ and point $y \in \mathcal{Y}$ we say that y is *nearly $\mathcal{X}$-quasi-minimal* if

$$\forall x \in \mathcal{X}[x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y} \implies x \text{ is nearly computable}]$$

The main difference between the closure representation and the network representation is that the closure representation includes more names for a point x. So being nearly computable is a weaker notion that being computable as a point x may have a computable $\overline{\delta_{\mathcal{N}}}$ name, but no computable $\delta_{\mathcal{N}}$ name.

In general, these representations are not single valued; that is, a name $p \in \omega^\omega$ may be a name for multiple distinct points. The following observation by Kihara, Ng and Pauly [26] gives a condition for the closure representation to be single valued.

Observation 4.3.6. *If $\mathcal{X}$ is a T_2 space and $\mathcal{N}$ is a cs-network for $\mathcal{X}$ then $\overline{\delta_{\mathcal{N}}}$ is a single valued representation of $\mathcal{X}$.*

From this observation we can conclude that if $\mathcal{X}$ is a T_2 space and $\mathcal{N}$ is a countable network for $\mathcal{X}$ then there are only countably many points with a computable name as each computable $p \in \omega^\omega$ represents at most one point and there are only countably many computable $p \in \omega^\omega$.

Recall that a function $f \in \omega^\omega$ is *A-computably dominated* if there is an A-computable function g such that $f(n) \leq g(n)$ for all n. Given a space $\mathcal{X} = (X, \mathcal{N})$ the *disjointness diagram* of $\mathcal{X}$ is the set $\{(e, i) : N_e \cap N_i = \emptyset\}$. Now we can state the main lemma used to prove Theorem 4.3.1.

Lemma 4.3.7 (Kihara, Ng, Pauly [26]). *Let $f \in \omega^\omega$ be a function that is not C'-computably dominated. Then for any second countable $\mathcal{X} = (X, \mathcal{N})$ with C-c.e. disjointness diagram we have that $f : \omega^\omega_{\text{co}}$ (viewing f as a point in $\omega^\omega_{\text{co}}$) is nearly $\mathcal{X}$-quasi-minimal.*

Since we are modifying Kihara, Ng and Pauly [26]'s proof of Lemma 4.3.1, we will give it here.

Proof of Theorem 4.3.1. Let $\mathcal{X} = (X, (\beta_e)_e)$ be a T_2 cb_0 space. Let C be an oracle such that the disjointness diagram of $(\beta_e)_e$ is C-c.e. By Lemma 4.3.7, if f is not C'-computably dominated, then $f : \omega^\omega_{\mathrm{co}}$ is nearly $\mathcal{X}$-quasi-minimal. Since X is T_2, there are only countably many points in $\mathcal{X}$ that are nearly C-computable by Observation 4.3.6. However, there are uncountably many functions which are not C'-computably dominated. Thus, one can choose a function which is not $\mathbf{T}$-equivalent to any nearly computable point in $\mathcal{X}$. □

The above proof is close to a quasi-minimal separation. If $f \in \omega^\omega_{\mathrm{co}}$ is not C'-computably dominated then there are only countably many degrees in $\mathcal{D}_\mathcal{X}$ that might be below $\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)$. We now use forcing to avoid computing any of these degrees.

Lemma 4.3.8. *Given a set A and a countable collection of non-c.e. sets $(C_i)_i$ there is a function f such that f is not A-computably dominated and for each i we have $C_i \not\leq_e \mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)$.*

Proof. We construct f in stages with $f = \cup f_s$.

At stage $s = 2n$ we consider the nth A-partial computable function φ^A_n. If $\varphi^A_n(|f_s|)\downarrow$ then set $f_{s+1} = f_s \cup \{(|f_s|, \varphi^A_n(|f_s|) + 1)\}$, otherwise $f_{s+1} = f_s$. At stage $s = 2\langle e, i\rangle + 1$ we ask if there is $\sigma \succ f_s$ such that $\Psi_e(\{\tau : \tau \perp \sigma\}) \nsubseteq C_i$ then take $f_{s+1} = \sigma$, otherwise set $f_{s+1} = f_s$.

The even stages give us that f is not A-computably dominated. So we now only need to show f satisfies the other condition. Suppose that $C_i = \Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f))$. Let $s = 2\langle e, i\rangle + 1$. If there was $\sigma \succ f_s$ such that $\Psi_e(\{\tau : \tau \perp \sigma\}) \nsubseteq X_i$ then we would have $f \succ \sigma$ for one such σ and $\Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)) \nsubseteq C_i$. So there is no such σ and thus $C_i = \Psi_e(\mathrm{NBase}_{\omega^\omega_{\mathrm{co}}}(f)) \subseteq \Psi_e(\{\tau : \tau \not\succeq f_s\}) \subseteq C_i$. Hence C_i is c.e., a contradiction. □

Using this we can modify the proof Theorem 4.3.1 to get a quasi-minimal separation.

Proof of Theorem 4.3.2. Let $\mathcal{X} = (X, (\beta_e)_e)$ be a T_2 cb_0 space. Let C be an oracle such that the disjointness diagram of $(\beta_e)_e$ is C-c.e. Let $(C_i)_i$ be a listing of sets whose degrees are those of the non-computable nearly C-computable points in $\mathcal{X}$. By Observation 4.3.6 we know that we can find such a listing. From Lemma 4.3.8 we can find an f that is not C'-computably dominated and has $C_i \not\leq_e \mathrm{NBase}_{\mathrm{co}}(f)$ for any i. By the proof of Theorem 4.3.1 we have that f is nearly $\mathcal{X}$-quasi-minimal. Since f is not $\emptyset'$-computably dominated we have that $\mathrm{NBase}_{\omega^{\omega}_{\mathrm{co}}}(f)$ is not c.e. and hence f is $\mathcal{X}$-quasi-minimal □

4.4 A $T_{2.5}$-quasi-minimal class

In this section we look at the relatively prime integer topology. This topology is defined as follows. Let $\mathbb{Z}_+$ be the set of positive integers. The basic open sets in this topology are $\{a + b\mathbb{Z} : \gcd(a, b) = 1\}$. We write $\mathbb{N}_{\mathrm{rp}} = (\mathbb{Z}_+, \{a + b\mathbb{Z} : \gcd(a, b) = 1\})$ for the cb$_0$ space. It is known that $\mathbb{N}_{\mathrm{rp}}$ is second countable, T_2 and not $T_{2.5}$ [44].

Proposition 4.4.1. *$\mathbb{N}_{\mathrm{rp}}$ is strongly decidable, and the sets $a + b\mathbb{Z}, \overline{a + b\mathbb{Z}}$ are uniformly computable.*

Proof. Given a finite collection of basic open sets $(a_0 + b_0\mathbb{Z}), \dots, (a_{n-1}, b_{n-1}\mathbb{Z})$ let $b_n = \prod_{i<n} b_i$. For each $k < n$ we can uniformly compute $c_{k,0} < \cdots < c_{k,n_k-1} < b_n$ such that $a_k + b_k\mathbb{Z} = \bigcup_{i<n_k} c_{k,i} + b_n\mathbb{Z}$. So we have that the subset relationships between Boolean combinations of $((a_0 + b_0\mathbb{Z}), \dots (a_{n-1}, b_{n-1}\mathbb{Z}), \emptyset)$ is the same as the subset relationships between Boolean combinations of $(\{c_{0,i} : i < n_0\}, \dots \{c_{n-1,i} : i < n_{n-1}\}, \emptyset)$ and is hence decidable uniformly in $a_0, \dots, a_{n-1}, b_0 \dots, b_{n-1}$. Hence the subset relationships between Boolean combinations of $(\{a + b\mathbb{Z} : \gcd(a, b) = 1\}, \emptyset)$ is decidable.

To see that $\mathbb{N}_{\mathrm{rp}}$ is strongly decidable, we first show that for any a, b, c, d (no relatively prime assumptions) $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) \neq \emptyset$ if and only if $a \equiv c \mod \gcd(b, d)$. If $a \equiv c \mod \gcd(b, d)$ then $a = c + k \gcd(b, d)$ for some k, so $a = c + k(nd - mb)$ for some m, n. So $a + kmb = c + knd \in (a + b\mathbb{Z}) \cap (c + d\mathbb{Z})$. On the other hand $a + bm \equiv a \mod \gcd(b, d)$ and $c + dn \equiv c \mod \gcd(b, d)$ so if $a \not\equiv c \mod \gcd(b, d)$ then $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) = \emptyset$.

So $(a+b\mathbb{Z}) \cap (c+d\mathbb{Z}) \neq \emptyset$ if and only if $(a+b\mathbb{Z}) \cap (c+\gcd(b, c)\mathbb{Z}) \neq \emptyset$. Let $F = \{0 \leq c < b : (\forall d \mid b)[\gcd(c, d) > 1 \vee (c+d\mathbb{Z}) \cap (a+b\mathbb{Z}) \neq \emptyset\}$. Then we have that $\overline{a + b\mathbb{Z}} = \bigcup_{c \in F} c+b\mathbb{Z}$.

So we get that $\overline{a+b\mathbb{Z}}$ is uniformly computable and using the same argument as we did to show $\mathbb{N}_{\text{rp}}$ is decidable we can show that $\mathbb{N}_{\text{rp}}$ is strongly decidable. □

Kihara, Ng and Pauly[26] proved that the notions of T_2 and $T_{2.5}$ are distinct for classes of enumeration degrees.

Theorem 4.4.2. *$\mathcal{D}_{(\mathbb{N}_{\text{rp}})^\omega}$ is not $T_{2.5}$.*

In this section we modify the proof of the above theorem show that there are T_2 classes of degrees that are $T_{2.5}$-quasi-minimal.

Theorem 4.4.3. *$\mathcal{D}_{(\mathbb{N}_{\text{rp}})^\omega}$ is $T_{2.5}$-quasi-minimal.*

The idea behind the proof of Theorem 4.4.2 is similar to that or Theorem 4.3.1, but instead of looking at the closure representation from Definition 4.3.5, Kihara, Ng and Pauly introduce a new representation.

Definition 4.4.4. Given a space $\mathcal{X} = (X, \mathcal{N})$ with countable cs-network, we define the representation $\widetilde{\delta_{\mathcal{N}}}$ where $\widetilde{\delta_{\mathcal{N}}}(p) = x$ if $\{N_{p(e)} : e \in \omega\}$ is a network at x and $\overline{N_{p(e)}} \cap \overline{N_{p(i)}} \neq \emptyset$ for all $e, i \in \omega$. For a point $x \in \mathcal{X}$ we say that x is *$\widetilde{*}$-nearly computable* if there is a computable $\widetilde{\delta_{\mathcal{N}}}$ name for x. For a space $\mathcal{Y}$ and point $y \in \mathcal{Y}$ we say that y is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal if

$$\forall x \in \mathcal{X}[x : \mathcal{X} \leq_{\mathbf{T}} y : \mathcal{Y} \implies x \text{ is } \widetilde{*}\text{-nearly computable}]$$

A $\overline{\delta_{\mathcal{N}}}$-name for a point x is a $\widetilde{\delta_{\mathcal{N}}}$-name for x, but a $\widetilde{\delta_{\mathcal{N}}}$-name for a point x may not be $\overline{\delta_{\mathcal{N}}}$-name. As we will see $\widetilde{\delta_{\mathcal{N}}}$ is not necessarily single valued on T_2 spaces, but Kihara, Ng and Pauly observed that it is single valued on $T_{2.5}$ spaces.

Observation 4.4.5. *If $\mathcal{X}$ is a $T_{2.5}$ space and $\mathcal{N}$ is a cs-network for $\mathcal{X}$ then $\widetilde{\delta_{\mathcal{N}}}$ is a single valued representation of $\mathcal{X}$.*

Rather than working directly with $\mathbb{N}_{\text{rp}}$ Kihara, Ng and Pauly use the fact that $\mathbb{N}_{\text{rp}}$ is countable and nowhere $T_{2.5}$ and prove results about an arbitrary countable nowhere $T_{2.5}$

space $\mathcal{H}$. We say $\mathcal{H}$ is nowhere $T_{2.5}$ if for all open $U, V \subseteq \mathcal{H}$ we have that $\overline{U} \cap \overline{V} \neq \emptyset$. It is known that $\mathbb{N}_{\mathrm{rp}}$ is nowhere $T_{2.5}$ [44]. If $\mathcal{H} = (\omega, (H_e)_e)$ is nowhere $T_{2.5}$ then a *witness for being nowhere* $T_{2.5}$ is a set $\Lambda \subseteq \omega^3$ such that $H_e, H_d \neq \emptyset$ then the set $\Lambda_{e,d} = \{n : (e,d,n) \in \Lambda\}$ is nonempty and $\Lambda_{e,d} \subseteq \overline{H_e} \cap \overline{H_d}$.

Lemma 4.4.6 (Kihara, Ng, Pauly). *Let $\mathcal{H} = (\omega, (H_e)_e)$ be a represented, second countable space with c.e. witness for being nowhere $T_{2.5}$ and let $x \in \omega^\omega$ be 1-C-generic. Then for any space $\mathcal{Y} = (Y, \mathcal{N})$ which is strongly decidable relative to C and $\mathcal{N}$ is a cs-network, we have $x : \mathcal{H}^\omega$ is $\widetilde{*}$-nearly $\mathcal{Y}$-quasi-minimal.*

In the case of $\mathbb{N}_{\mathrm{rp}}$ Proposition 4.4.1 tells us that there is a computable witness that $\mathbb{N}_{\mathrm{rp}}$ is nowhere $T_{2.5}$ namely $\Lambda = \{(n, \langle a,b\rangle, \langle c,d\rangle) : n \in \overline{a+b\mathbb{Z}} \cap \overline{c+d\mathbb{Z}}\}$. So the lemma can be applied here.

Proof of Theorem 4.4.2. Let $\mathcal{X} = (X, \mathcal{N})$ be a $T_{2.5}$ space. Let C be an oracle such that $\mathcal{X}$ is strongly decidable relative to C. By Lemma 4.4.6, for any 1-C-generic point $x \in \omega^\omega$ we have that $x : (\mathbb{N}_{\mathrm{rp}})^\omega$ is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal. By Observation 4.4.5 since $\mathcal{X}$ is a $T_{2.5}$-space, there are only countably many points in $\mathcal{X}$ that are $\widetilde{*}$-nearly computable. However, there are uncountably many points in ω^ω which are 1-C-generic. Thus, one can choose such a point which is not $\equiv_{\mathbf{T}}$-equivalent (in terms of $(\mathbb{N}_{\mathrm{rp}})^\omega$) to any $\widetilde{*}$-nearly computable points in $\mathcal{X}$. $\square$

In the above proof we use a counting argument to separate the two classes. Using forcing we can get a stronger result of $\mathcal{X}$-quasi-minimality.

Lemma 4.4.7. *Given a countable* cb_0 *space $\mathcal{H} = (\omega, (H_e)_e)$, a countable collection of non-c.e. sets $(X_i)_i$ and set C, there is a 1-C-generic set $x \in \omega^\omega$ such that $\mathbf{0} <_e \mathrm{NBase}_{\mathcal{H}^\omega}(x)$ and $X_i \not\leq_e \mathrm{graph}(x)$ for each i.*

Proof. We will use forcing to construct x in stages with $x = \bigcup_s x_s$. We fix $H_e \neq \omega, \emptyset$ and points $a \in H_e, b \notin H_e$.

At stage $s = 3n$ let W_n be the nth c.e. set. If $\langle |x_s|, e\rangle \in W_n$ then set $x_{s+1} = x_s{}^\frown b$ otherwise $x_{s+1} = x_s{}^\frown a$. This ensures that $W_n \neq \mathrm{NBase}_{\mathcal{H}^\omega}(x)$.

At stage $s = 3n+1$ let V_n be the nth C-c.e. subset of $\omega^{<\omega}$. If there is $\sigma \in V_n$ such that $x_s \prec \sigma$ then set $x_{s+1} = \sigma$ otherwise set $x_{s+1} = x_s$.

At stage $s = 3\langle e, i\rangle + 2$ let Ψ_e be the eth enumeration operator. Ask if there is a number $n \notin X_i$ and $\sigma \succ x_s$ such that $n \in \Psi_e(\text{graph}(\sigma))$. If yes then set $x_{s+1} = \sigma$, otherwise $x_{s+1} = x_s$.

At stages $3n$ we ensured that $\text{NBase}_{\mathcal{H}^\omega}(x)$ is not c.e. and at stages $3n+1$ we ensure that x is 1-C-generic. Now we need to show that $X_i \not\leq_e x$ for any i. Suppose that $X_i = \Psi_e(\text{graph}(x))$. Then let $s = 3\langle e, i\rangle + 2$. If there is an extension $\sigma \succ x_s$ such that $\Psi_e(\text{graph}(\sigma)) \not\subseteq X_i$ then we would have $x \succ \sigma$ for some such σ and $\Psi_e(\text{graph}(x)) \supseteq \Psi_e(\text{graph}(\sigma)) \not\subseteq X_i$, so it must be that $\Psi_e(\text{graph}(\sigma)) \subseteq X_i$ for all $\sigma \succ x_s$. So we have $X_i = \Psi_e(x) \subseteq \{n : \exists \sigma \succ x_s, n \in \Psi_e(\text{graph}(\sigma))\} \subseteq X_i$. So we have that X_i is c.e., a contradiction. □

Now we can replace the counting argument used in the proof of Theorem 4.4.2 to get the quasi-minimal separation.

Proof of Theorem 4.4.3. Let $\mathcal{X} = (X, (\beta_e)_e)$ be a $T_{2.5}$ space. Let C be an oracle such that $\mathcal{X}$ is strongly decidable relative to C. Let $(X_i)_i$ be a list of non-c.e. sets with enumeration degrees those of the non-computable $\widetilde{*}$-nearly computable points in $\mathcal{X}$. Since the open sets of $\mathbb{N}_{\text{rp}}$ are uniformly computable we have that $x : (\mathbb{N}_{\text{rp}})^\omega \leq_{\mathbf{T}} x : \omega^\omega$ for all x, so by Lemma 4.4.7 there is a 1-C-generic x such that $x : (\mathbb{N}_{\text{rp}})^\omega$ is non-computable and $z : \mathcal{X} \leq_{\mathbf{T}} x : (\mathbb{N}_{\text{rp}})^\omega$ means that $z : \mathcal{X}$ is computable or $z : \mathcal{X}$ is not $\widetilde{*}$-nearly-computable. But from Lemma 4.4.6 we have that $x : (\mathbb{N}_{\text{rp}})^\omega$ is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal, so $x : (\mathbb{N}_{\text{rp}})^\omega$ is $\mathcal{X}$-quasi-minimal. □

4.5 The doubled co-d-CEA degrees

In this section we show that the doubled co-d-CEA degrees are not $T_{2.5}$. Kihara, Ng and Pauly [26] introduced the doubled co-d-CEA degrees as the degrees of points in the product of the double origin topology $\mathcal{DO}^\omega$. Rather than working directly with this topology it is easier for us to work with the characterization in terms of sets that they came up with.

Definition 4.5.1. A set $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$ is doubled co-d-CEA if $(A \cup B)^c, P, N$ are Y-c.e. and A, B, P, N are disjoint. A degree is doubled co-d-CEA if it contains a doubled co-d-CEA set.

Since the doubled origin space is T_2 the doubled co-d-CEA degrees are a T_2 class. Question 4 of the open questions asked by Kihara, Ng and Pauly [26] is if these degrees are a proper T_2 class. We show that they are a proper T_2 class.

Theorem 4.5.2. *The doubled co-d-CEA degrees are not $T_{2.5}$.*

Proof. We will make use of the $\widetilde{*}$-name concept, Definition 4.4.4, from Section 4.4 again in this proof.

Consider some $T_{2.5}$ cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$. Let Y be such that $\mathcal{X}$ is strongly Y-decidable. We will build coinfinite Y-c.e. sets $P, N \subseteq C$ such that for any partition $A \sqcup B = C^c$ we have that the doubled co-d-CEA degree $Y \oplus Y^c \oplus (A \cup P) \oplus (B \oplus N)$ does not enumerate the name of any non $\widetilde{*}$-nearly-Y'-computable $x \in X$. By doing this we will have constructed a size continuum class of doubled co-d-CEA degrees, only countably many of which can contain the name of a point in $\mathcal{X}$. Hence we will have shown that $\mathcal{D}_{\mathcal{DO}^\omega} \nsubseteq \mathcal{D}_{\mathcal{X}}$.

Let $\mathcal{Q} = \{(C_q, P_q, N_q) : P_q \cap N_q = \emptyset \wedge P_q, N_q \subseteq C_q \subseteq_{\text{fin}} \omega\}$. For $p, q \in \mathcal{Q}$, $u \in \omega$ and $a \subseteq u$ we define the following:

- $q \preceq p$ if $C_q \supseteq C_p, P_q \supseteq P_p, N_q \supseteq N_p$.
- $q \preceq_u p$ (q extends p above u) if $q \preceq p$ and $C_q{\restriction}u = C_p{\restriction}u, P_q{\restriction}u = P_p{\restriction}u, N_q{\restriction}u = N_p{\restriction}u$.
- $a \lhd_u p$ (a is a p-compatible choice of $A{\restriction}u$) if $a \subseteq u \setminus C_p$.
- $p(a, u) = Y \oplus Y^c \oplus (a \cup P_p) \oplus (u \setminus (C_p \cup a) \cup N_p)$

Note that if $a \lhd_u p$ and $q \preceq_u p$ then $a \lhd_u q$ and $p(a, u) \subseteq q(a, v)$ for any $v \geq u$. This does not necessarily hold if $q \preceq p$.

We will build a Y-computable sequence $q_0 \succeq q_1 \succeq \dots$ and have $C = \bigcup_s C_{q_s}$, $P = \bigcup_s P_{q_s}$ and $N = \bigcup_s N_{q_s}$. This will ensure that C, P, N are Y-c.e. and $P \sqcup N \subseteq C$. The requirements $\mathcal{R}_e$ are that for any partition $A \sqcup B = C^c$ we have that if $\Psi_e(Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)) = \mathrm{NBase}(x)$ for some $x \in X$ then x is $\widetilde{*}$-nearly-Y'-computable.

The strategy for $\mathcal{R}_e$ works as follows. Each requirement has a restriction u and when it sets q_{s+1} it needs to ensure that $q_{s+1} \preceq_u q_s$. If at stage s the value of u is undefined then let $u = \max(C_{q_s}) + 2$. If at some later stage t a higher priority requirement acts and we have $q_t \not\preceq_u q_s$ then we consider $\mathcal{R}_e$ injured and u to be undefined. $\mathcal{R}_e$ needs to be able to handle any partition of $u \setminus C$ so for each $a \lhd_u q_s$ we create a new subrequirement $\mathcal{R}_e^a$. If $\mathcal{R}_e$ is injured then we remove these subrequirments. $\mathcal{R}_e^a$ is satisfied if $\mathcal{R}_e$ is satisfied for each partition $A \sqcup B = C^c$ with $A \restriction u = a$.

The idea of the strategy for $\mathcal{R}_e^a$ is as follows. We consider the potential point x which is named by Ψ_e of some partition extending a. We first wait until a stage when we see a way to force $x \in \beta_{n_0}$ and a separate way to force $x \in \beta_{n_1}$ for some n_0, n_1 with $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. We then put everything above u into C and injure lower priority requirements so that we always have the option to put x in β_{n_0} or β_{n_1} without changing other facts about x. The next step is to wait until we see a way to put $x \in \beta_m$ for some β_m disjoint from β_{n_i} for some i. We then put x in both β_m and β_{n_i}. If we get past the waiting step then we will be able to ensure that there is no potential point x and satisfy the requirement that way. If we are stuck at the waiting step forever then we will show that x is close to computable.

The details of the strategy for $\mathcal{R}_e^a$ use states w, c, d and work as follows.

- State w: we wait until a stage r when we see some $p_0, p_1 \preceq_u q_r$ with $n_0 \in \Psi_e(p_0(a, u))$ and $n_1 \in \Psi_e(p_1(a, u))$ such that $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. Then we set $q_{r+1} = (C_q \cup [u, r), P_{q_r}, N_{q_r})$ and injure all lower priority requirements.

- State c: we wait until a stage $v > r$ where we see some $q \preceq_r q_v$ with $m \in \Psi_e(q(a, u))$ such that $\beta_{n_0} \cap \beta_m = \emptyset$ or $\beta_{n_1} \cap \beta_m = \emptyset$. In the first case we set $q_{v+1} = (C_q, P_q \cup P_{p_0}, N_q \cup N_{p_0})$ and in the second case we set $q_{r+1} = (C_q, P_q \cup P_{p_1}, N_q \cup N_{p_1})$. All lower priority requirements are injured, along with $\mathcal{R}_e^b$ requirements that are in state

c. This requirement moves to state d.

- State d: the requirement is considered finished and cannot be injured by fellow $\mathcal{R}_e^b$ requirements.

This completes the construction of C, P, N. Now we move onto the verification.

Claim 4.5.2.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}_e^a$ requirements. Suppose that an $\mathcal{R}_e^a$ is never injured by higher priority requirements after stage s.

If $\mathcal{R}_e^a$ is in state c then either $\mathcal{R}_e^a$ is injured by an $\mathcal{R}_e^b$ requirement and moves back to state w or it acts once and moves to state d, after which it never acts again and can no longer be injured by other $\mathcal{R}_e^b$ requirements. So each $\mathcal{R}_e^a$ requirement acts finitely often from state c.

Since each $\mathcal{R}_e^b$ requirement can act only finitely often in state c, and there are only finitely many of these requirements, we can let $t > s$ be a stage after which all $\mathcal{R}_e^b$ will not act in state c. If $\mathcal{R}_e^a$ is in c or d then it will never again act. If $\mathcal{R}_e^a$ is in state w then $\mathcal{R}_e^a$ will not be injured at any later stage and will act at most once more to move into state c. □

Claim 4.5.2.2. *C^c is infinite.*

Proof. Let s be the last stage when $\mathcal{R}_e$ was injured. We have that if u is the restriction chosen by $\mathcal{R}_e$ at stage $t > s$ then $e \leq |C^c \restriction u|$. This follows from induction and the fact that $\max(C_{q_t}) + 1 < u$ means that $\max(C_{q_t}) + 1 \notin C_{q_j}$ for any $j \geq t$. □

Claim 4.5.2.3. *Each $\mathcal{R}_e$ is satisfied.*

Proof. Consider some partition $A \sqcup B = C^c$. Let $Q = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$ and fix e. We will show that $\mathcal{R}_e$ is satisfied for Q. Let s be the last stage when a subrequirement $\mathcal{R}_e^b$ changes its state. There is some subrequirement $\mathcal{R}_e^a$ such that $Q \restriction u = (q_s(a, u)) \restriction u$. Note that for any t we have $q_t(a,u) \subseteq Q$. Let l be the last state that $\mathcal{R}_e^a$ is in. We will look at the three cases.

- $\underline{l = d}$: when we entered state d at stage s we ensured that there were $m, n \in \Psi_e(Q)$ such that $\beta_m \cap \beta_n = \emptyset$. So $\Psi_e(Q)$ is not the $\mathcal{X}$-name of a point in $\mathcal{X}$.

- $\underline{l = c}$: Let $r > s$ be the stage when $\mathcal{R}_e^a$ moves to state c for the last time. Suppose that $\mathrm{NBase}(x) = (\Psi_e(Q))$ for some $x \in X$. Then since $\overline{\beta_{n_0}}$ and $\overline{\beta_{n_1}}$ are disjoint there must be $m \in \mathrm{NBase}(x)$ and $i \in 2$ such $\beta_m \cap \beta_{n_i} = \emptyset$. Since $m \in \Psi_e(Q)$ there are some finite $D \subseteq A \cup P, E \subseteq B \cup N$ and $t > s$ such that $m \in \Psi_{e,t}(Y \oplus Y^c \oplus D \oplus E)$. Consider $q = (C_{q_t} \cup D \setminus u \cup E \setminus u, P_{q_t} \cup D \setminus u, N_{q_t} \cup E \setminus u)$. We have that $m \in \Psi_{e,t}(q(a,u))$ and $q \preceq_r q_t$ since $[u, r) \subseteq C$. Thus at stage t we could have used q to enter state d, a contradiction.

- $\underline{l = w}$ Suppose that $\mathrm{NBase}(x) = \Psi_e(Q)$ for some $x \in X$. We will show that x is $\widetilde{*}$-nearly-Y'-computable. Consider the set

  $$J = \{n : n \in \Psi_{e,t}(Y \oplus Y^c \oplus (D \cup P) \oplus (E \cup N)) \qquad \text{for some } t > s, D \sqcup E \subseteq_{\text{fin}} C^c \text{ with } a = D{\restriction}u\}$$

  Note that $J \leq_e Y'$ because C, P, N are Y-c.e. We claim that J is a $\widetilde{\delta_\beta}$-name for x. Suppose not. Then since $J \supseteq \mathrm{NBase}(x)$ there must be some $n_0, n_1 \in J$ such that $\overline{\beta_{n_0}} \cap \overline{\beta_{n_1}} = \emptyset$. Since $n_i \in J$ there is $t_i > s, D \sqcup E \subseteq_{\text{fin}} C^c$ with $a = D{\restriction}u$ such that $n_i \in \Psi_{e,t_i}(Y \oplus Y^c \oplus (D \cup P_{q_t}) \oplus (E \cup N_{q_t}))$. Consider $p_i = (C_{q_t} \cup D \setminus u \cup E \setminus u, P_{q_t} \cup D \setminus u, N_{q_t} \cup E \setminus u)$. We have that $p_i \preceq_u q_t$ and $n_i \in \Psi_{e,t}(p_i(a,u))$. Then at stage $\max(t_0, t_1)$ we would have used p_0, p_1 to move to state c, a contradiction.

  So J is a $\widetilde{\delta_\beta}$-name for x and hence x is $\widetilde{*}$-nearly-Y'-computable.

$\square$

Since the requirements are satisfied, the construction works and we have a class of continuum many doubled co-d-CEA degrees $\mathcal{C}$ such that $\mathcal{C} \cap \mathcal{D}_{\mathcal{X}}$ is a countable set, and hence $\mathcal{D}_{\mathcal{DO}^\omega} \nsubseteq \mathcal{D}_{\mathcal{X}}$. Since $\mathcal{X}$ was an arbitrary $T_{2.5}$ cb_0 space we have that the doubled co-d-CEA degrees are not $T_{2.5}$. □

4.6 Separating $T_{2.5}$ classes from submetrizable classes

In this section we give our main result: there are $T_{2.5}$ classes of degrees that are not submetrizable. We do this for two example classes, the Arens co-d-CEA degrees and the Roy halfgraph above degrees. Kihara, Ng and Pauly [26] show that both these classes are $T_{2.5}$ as they arise from the decidable $T_{2.5}$ spaces $\mathcal{QA}^\omega$ and $\mathcal{QR}^\omega$, respectively. We will give formal definitions of these classes later in this section. The definitions of the spaces $\mathcal{QA}^\omega$ and $\mathcal{QR}^\omega$ can be found in [26]. For now we will go over the general method that is used to prove these separations.

4.6.1 General method

A submetrizable space arises by adding open sets to some underlying metric space, however a cb_0 submetrizable space does not tell us which sets are open under the metric. We would like an effective way to find a name for a point with respect to the underlying metric space from a name for a point with respect to the submetrizable space. We can do this with many natural examples, but it is not always possible. This motivates the following definition.

Definition 4.6.1. We call a submetrizable cb_0 space $\mathcal{X} = (X, (\beta_e)_e)$ *effectively submetrizable* if there is a continuous, injective function $f : \mathcal{X} \to [0,1]^\omega$ such that $\text{NBase}_{[0,1]^\omega}(f(x)) \leq_e \text{NBase}_{\mathcal{X}}(x)$.

If $\mathcal{X} = (X, (\beta_e)_e)$ is a computable metric space and $\mathcal{Y} = (X, (\beta_e)_e \sqcup (\alpha_e)_e)$ is submetrizable then $\mathcal{Y}$ is effectively submetrizable. If one looks at the examples of submetrizable cb_0 spaces in [26] they will see that these spaces are all effectively submetrizable, even the non-decidable cb_0 spaces like the Gandy-Harrington topology. Thus every enumeration degree is an $\mathcal{X}$-degree for some decidable, effectively submetrizable cb_0 space $\mathcal{X}$.

Since $[0,1]^\omega$ is universal for second countable metrizable spaces, for any submetrizable cb_0 space $\mathcal{X}$ there is an oracle Y such that $\mathcal{X}$ is Y-effectively submetrizable.

Miller [34] proved that there is no quasi-minimal continuous degree. So if x is a point in an effectively submetrizable cb_0 space $\mathcal{X}$ and f is a witness that $\mathcal{X}$ is effectively submetrizable, then $\mathrm{NBase}_{\mathcal{X}}(x)$ is quasi-minimal implies that $\mathrm{NBase}_{[0,1]^\omega}(f(x))$ is c.e. since we have $\mathrm{NBase}_{[0,1]^\omega}(f(x)) \leq_e \mathrm{NBase}_{\mathcal{X}}(x)$, so we can conclude that $\mathcal{X}$ has only countably many quasi-minimal degrees.

The above is one way of showing a class is not effectively submetrizable, but it does not help us in the case of the Arens co-d-CEA and Roy halfgraph degrees as we do not know if they contain uncountably many quasi-minimal degrees. By looking more closely at the total degrees below a continuous degree we come up with our method of separation.

Definition 4.6.2. A countable class $\mathcal{S} \subseteq 2^\omega$ is a *Scott set* if it is closed under join, Turing reducibility and for any $X \in \mathcal{S}$ and nonempty $\Pi^0_1(X)$ class G there is $Y \in \mathcal{S} \cap G$. The collection $\{\deg_T(X) : X \in \mathcal{S}\}$ is called a *Scott ideal.*

So every Scott ideal contains PA degrees. In fact, for any $Y \in \mathcal{S}$ we have that $\mathcal{S}$ contains a set that is PA relative to Y.

Theorem 4.6.3 (J. Miller [34])**.** *If* $\mathbf{v}$ *is a non-total continuous degree then the set* $\{\mathbf{b} <_e \mathbf{v} : \mathbf{b}$ *is total*$\}$ *is a Scott ideal. Notably, there is total* $\mathbf{b} <_e \mathbf{v}$ *such that* $\mathbf{b}$ *is a PA degree.*

Now we have the tools we need to prove the following

Lemma 4.6.4. *If* $\mathcal{C}$ *is an uncountable class of enumeration degrees and* $\mathcal{B}$ *is a countable class of non PA total degrees such that for any* $\mathbf{a} \in \mathcal{C}$ *we have* $\{\mathbf{b} \in \mathcal{D}_T : \mathbf{b} \leq_e \mathbf{a}\} \subseteq \mathcal{B}$*, then* $\mathcal{C} \nsubseteq \mathcal{D}_{\mathcal{X}}$ *for any effectively submetrizable* cb_0 *space* $\mathcal{X}$*.*

Proof. Take $\mathcal{C}$ as in the statement of the theorem. Let $\mathcal{X} = (X, (\beta_e)_e)$ be an effectively submetrizable cb_0 space with witness f. We will show that $\mathcal{C} \cap \mathcal{D}_{\mathcal{X}}$ is countable.

Consider some $x \in X$. Suppose that $\mathrm{NBase}_{\mathcal{X}}(x) \in \mathbf{a}$ for some $\mathbf{a} \in \mathcal{C}$. So we have that $\mathrm{NBase}_{[0,1]^\omega}(f(x)) \leq_e \mathbf{a}$. Since $\mathbf{a}$ does not bound any PA degrees we have that

$\text{NBase}_{[0,1]^\omega}(f(x))$ has total degree. So we have that that $\deg_e(\text{NBase}_{[0,1]^\omega}(f(x))) \in \mathcal{B}$. Since f is injective, $\text{NBase}_{[0,1]^\omega}(f(x))$ uniquely determines x and there are only countably many $x \in \mathcal{X}$ such that $\deg_e(\text{NBase}_{\mathcal{X}}(x)) \in \mathcal{C}$. So $\mathcal{C} \cap \mathcal{D}_{\mathcal{X}}$ is countable, and hence $\mathcal{C} \not\subseteq \mathcal{D}_{\mathcal{X}}$. □

Now we relativize to get the result for arbitrary submetrizable spaces.

Theorem 4.6.5. *Suppose that for each $Y \subseteq \omega$, we have an uncountable class of enumeration degrees $\mathcal{C}^Y$ and $\mathcal{B}^Y$ a countable class of non Y-PA total degrees such that for any $\mathbf{a} \in \mathcal{C}$ we have $Y \oplus Y^c \leq_e \mathbf{a}$ and $\{\mathbf{b} \in \mathcal{D}_T : \mathbf{b} \leq_e \mathbf{a}\} \subseteq \mathcal{B}^Y$. Then $\bigcup_Y \mathcal{C}^Y \not\subseteq \mathcal{D}_{\mathcal{X}}$ for any submetrizable* cb_0 *space $\mathcal{X}$.*

Proof. Let $\mathcal{X} = (X, (\beta_e)_e)$ be a submetrizable space. Let $f : X \to [0,1]^\omega$ be a continuous injection. Let $Y = \{\langle n, m\rangle : \beta_n \subseteq f^{-1}[\alpha_m]\}$ where $(\alpha_e)_e$ is the standard basis on $[0,1]^\omega$. So for any $x \in \mathcal{X}$ we have that $\text{NBase}_{[0,1]^\omega}(x) \leq_e Y \oplus Y^c \oplus \text{NBase}_{\mathcal{X}}(x)$. We will show that the $\mathcal{C}^Y \not\subseteq \mathcal{D}_{\mathcal{X}}$.

Consider some $x \in X$. Suppose that $\text{NBase}_{\mathcal{X}}(x) \in \mathbf{a}$ for some $\mathbf{a} \in \mathcal{C}^Y$. So we have that $Y \oplus Y^c \leq_e \text{NBase}_{\mathcal{X}}(x)$ and hence $\text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c \leq_e \mathbf{a}$. If $\text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ is not a total degree, then since it is a continuous degree and $Y \oplus Y^c \leq_e \text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$, by Theorem 4.6.3 there is some total degree $\mathbf{b} \leq \text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ such that $\mathbf{b}$ is PA relative to Y. So it must be that $\text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c$ is total and hence $\deg_e(\text{NBase}_{[0,1]^\omega}(x) \oplus Y \oplus Y^c) \in \mathcal{B}^Y$. Since $\text{NBase}_{[0,1]^\omega}(x)$ uniquely determines x and the downward closure of $\mathcal{B}^Y$ is countable there are only countably many $x \in \mathcal{X}$ such that $\deg_e(\text{NBase}_{\mathcal{X}}(x)) \in \mathcal{C}^Y$. So $\mathcal{C}^Y \cap \mathcal{D}_{\mathcal{X}}$ is countable and hence $\mathcal{C}^Y \not\subseteq \mathcal{D}_{\mathcal{X}}$. □

Now that we have a method to prove classes are not submetrizable, we can use it on some classes to get new separations.

4.6.2 Arens co-d-CEA degrees

Now we show that the Arena co-d-CEA degrees are not submetrizable. The Arens co-d-CEA degrees were introduced in [26] and are the degrees of points in a $T_{2.5}$ space. By proving that the Arens co-d-CEA degrees are not submetrizable we prove that the notions of $T_{2.5}$ and submetrizable are distinct for classes of enumeration degrees.

Definition 4.6.6. A degree is Arens co-d-CEA if it contains a set of the form:

$$Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$$

where $(A_0 \cup A_1)^c, N, P_0, P_1, M$ are Y-c.e. A_0, A_1, N are disjoint, $P_0, P_1, M \subseteq N$ are pairwise disjoint, and there is a partition $N_0 \sqcup N_1 = N$ such that N_0, N_1 are Y-c.e. and $P_0 \subseteq N_0, P_1 \subseteq N_1$.

When referring to an Arens co-d-CEA set (or subset) we will often use the notation $L \oplus R \oplus Z$ to keep the track of the different columns of $(A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$. By definition L, R, Z are disjoint and the $L \oplus R$ part is a doubled co-d-CEA set by itself. The Z part is of co-d-c.e. degree and keeps track of numbers that have been added to $(A_0 \cup A_1)^c$ but not to $P_0 \cup P_1$. In order for some number in $(A_0 \cup A_1)^c$ to not show up in Z it must appear in N. This means the number is in N_0 or N_1, so if the number is also in $P_0 \cup P_1$ then we know which of P_0, P_1 contains it. The set M gives us a way of adding numbers in N back into Z.

We will introduce some notation that will help us keep track of the different c.e. sets when constructing an Arens co-d-CEA degree, or class of degrees. Let

$$Q = \{(C^q, P^q = P_0^q \sqcup P_1^q, N^q = N_0^q \sqcup N_1^q, M^q) : P_i^q \subseteq N_i^q, M^q \subseteq N^q \subseteq C^q, M^q \cap P^q = \emptyset\}$$

Here C^q is meant to represent the c.e. set $(A_0 \cup A_1)^c$. For $p, q \in Q$, $u \in \omega$ and $a = a_0 \sqcup a_1 \subseteq u$ we define the following:

- $q \preceq p$ if $C^q \supseteq C^p, P_i^q \supseteq P_i^p, N_i^q \supseteq N_i^p, M^q \supseteq M^p$.
- $q \preceq_u p$ (q extends p above u) if $q \preceq p$ and $C^q \restriction u = C^p \restriction u, P^q \restriction u = P^p \restriction u, N^q \restriction u = N^p \restriction u, M^q \restriction u = M^p \restriction u$.
- $a \lhd_u q$ (a is a q-compatible choice of $A \restriction u$) if $a = u \setminus C^q$.

- $q(a) = (a_0 \cup P_0^q) \oplus (a_1 \cup P_1^q) \oplus (C^q \setminus N^q \cup M^q)$.
- q is considered *u-robust* if $C^q \setminus N^q \subseteq u$ and $C^q \setminus u$ is an interval.

Note that we can have $q \preceq p$ but $p(a) \nsubseteq q(a)$. This is why we need the notion of a condition being u-robust. Note that for a u-robust p if $a \lhd_u p$ and $q \preceq_u p$ then $a \lhd_u q$ and $p(a) \subseteq q(a)$. Note also that for every condition $p \in Q$ and $u \in \omega$ there is $q \preceq_u p$ such that $p(a) = q(a)$ and q is u-robust.

Theorem 4.6.7. *The Arens co-d-CEA degrees are not submetrizable.*

Proof. We use Lemma 4.6.5 and show there are c.e. sets $C, P = P_0 \sqcup P_1, N = N_0 \sqcup N_1, M$ such that C^c is infinite and for any partition $A_0 \sqcup A_1 = C^c$ we have that $X = (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M)$ has Arens co-d-CEA degree and if any $f \leq_e X$ is the graph of a total function, then $f \leq_T \mathbf{0}'$ and $\deg_T(f)$ is not PA.

Fix a nonempty Π^0_1 class G where each $x \in G$ has PA degree, and fix a computable tree T with $[T] = G$.

We will build a computable sequence $q_0 \succeq q_1 \succeq \dots$ of $q_i \in Q$ and have $C = \bigcup_s C^{q_s}$, $P = \bigcup_s P^{q_s}$, $N = \bigcup_s N_{q_s}$ and $M = \bigcup_s M^{q_s}$. This will ensure that C, P, N are c.e. and will produce an Arens co-d-CEA set for any partition of C^c. The requirements $\mathcal{R}_e$ are that for any partition $A_0 \sqcup A_1 = C^c$ we have that if $f = \Psi_e((A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M))$ is the graph of a total function then $f \leq_T \emptyset'$ and $f \notin G$.

The strategy for $\mathcal{R}_e$ works as follows. If at stage s, $\mathcal{R}_e$ is initialized then let $u_e = \max(C^{q_s}) + 2$. If at some later stage t we have $q_t \npreceq_{u_e} q_s$ then we consider $\mathcal{R}_e$ injured and will reinitialize it. For each $a \lhd_{u_e} q_s$ we create a new subrequirement $\mathcal{R}_e^a$. If $\mathcal{R}_e$ is injured then we remove these subrequirements. $\mathcal{R}_e^a$ is satisfied if $\mathcal{R}_e$ is satisfied for each partition $A_0 \sqcup A_1 = C^c$ with $A_i \restriction u = a_i$. We add an interval consisting of the $\mathcal{R}_e^a$ to the order of requirements placing the interval just below $\mathcal{R}_e$ in priority. Initially the restriction u_e^a given to each $\mathcal{R}_e^a$ is u_e but this may increase if $\mathcal{R}_e^e$ is injured by a higher priority $\mathcal{R}_e^b$. When $\mathcal{R}_e^a$ is injured, we set $u_e^a = \max(C^{q_s}) + 1$.

The strategy for each $\mathcal{R}_e^a$ has states g, w, c, n, d. Initially they are in state g. Let $u = u_e^a$, the actions for each state are as follows.

- <u>State g</u>: we start with $n = 0$ and $\sigma_0 = \emptyset$. If at some stage s we see some $x \in \omega$ and u-robust $q \preceq_u q_s$ such that $\langle n, x\rangle \in \Psi_{e,s}(q(a))$ then we set $q_{s+1} = q$ and injure all lower priority requirements. If $\sigma_n{}^\frown x \notin T$ then we go to state w otherwise we remain in state g and set $n = n+1, \sigma_{n+1} = \sigma_n{}^\frown x$.

- <u>State w</u>: we wait until at some stage s we see $m, x_0, x_1 \in \omega$ and a u-robust pair $r_0, r_1 \preceq_u q_s$ such that $\langle m, x_i\rangle \in \Psi_{e,s}(r_i(a))$ and $x_0 \neq x_1$. We set $q_{s+1} = (C^{q_s} \cup C^{r_0} \cup C^{r_1}, P^{q_s}, N^{q_s}, M^{q_s})$. Let $v = \max(C^{q_{s+1}}) + 1$. All lower priority requirements are injured and we enter state c.

- <u>State c</u>: we wait until we see a stage t such that for some v-robust $p \preceq_v q_t$ we have $\langle m, x_2\rangle \in \Psi_{e,t}(p(a))$ for some x_2. Pick i such that $x_i \neq x_2$. Set $q_{t+1} = (C^p, P^p, N^p \cup N^{r_i}, M^p)$ (N^{r_i} and N^p do not conflict because $p \preceq_v q_{s+1}$). Note that now we have $q_{t+1}(a)\restriction s \subseteq r_i(a), p(a)$. Set $o = \max(C^{q_{t+1}}) + 1$ and enter state n.

- <u>State n</u>: we wait until we see a stage ℓ such that for some o-robust $h \preceq_o q_t$ we have $\langle m, y\rangle \in \Psi_{e,t}(v(a))$ for some y. If $y \neq x_i$ then set $q_{\ell+1} = (C^h, P^h \cup P^{r_i}, N^h, M^h \cup M^{r_i})$ and move into state d. Otherwise $y \neq x_2$ and we set $q_{\ell+1} = (C^h, P^h, N^h, M^h \cup v \setminus (u \cup P^h))$. Then we move into state d.

- <u>State d</u>: in this state $\mathcal{R}^a_e$ is considered satisfied.

This completes the construction of C, P, N, M. Now we move onto the verification.

Claim 4.6.7.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}^a_e$ requirements.

Suppose that an $\mathcal{R}^a_e$ is never injured by higher priority requirements after stage s. The first case we need to deal with is if $\mathcal{R}^a_e$ remains in state g and injures lower priority requirements infinitely often. Each time it acts n increases, so we have that $f = \bigcup_n \sigma_n \in 2^\omega$ and f is computable. Since every $\sigma \prec f$ is in T we have that $f \in G$, but this is a contradiction as no PA degree is computable. So $\mathcal{R}^a_e$ acts only finitely often in state g.

In states w, c, n the requirement acts at most once, so there is a stage after which $\mathcal{R}_e^a$ stops injuring lower priority requirements. □

For each requirement $\mathcal{R}_e$ let u_e be the restriction that is given to $\mathcal{R}_e$ after the last time it is injured.

Claim 4.6.7.2. $C^c = \{u_e - 1 : e \in \omega\}$

Proof. Because the restriction u_e is defined to be $\max(C^{q_s}) + 2$ we have that $u_e - 1 \notin C^{q_s}$. Since lower priority requirements only work above u_e we have that $u_e - 1 \notin C$. On the other hand, whenever a subrequirement $\mathcal{R}_e^a$ acts it makes sure that $[u_e^a, \max(C^{q_s})] \subseteq C^{q_s}$. So we have that $[u_e^a, u_{e+1} - 1) \subseteq C$. □

Claim 4.6.7.3. *Each $\mathcal{R}_e$ is satisfied.*

Proof. Consider some partition $A \sqcup B = C^c$. Let $X = (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus (C \setminus N \cup M)$ and fix e. We will show that $\mathcal{R}_e$ is satisfied for X. Let s be the last stage where a subrequirement $\mathcal{R}_e^b$ changes its state. There is some subrequirement $\mathcal{R}_e^a$ such that $X{\upharpoonright}u_e^a = q_s(a){\upharpoonright}u_e^a$. Note that for any t we have $q_t(a) \subseteq X$. Let l be the last state that $\mathcal{R}_e^a$ is in and $u = u_e^a$. We will look at the four cases.

- $\underline{l = d}$: when we entered state d at stage t we ensured that $\langle m, y\rangle, \langle m, x_i\rangle \in \Psi_{e,t}(q_{t+1}(a)) \subseteq \Psi_e(X)$ with $y \neq x_i$, so $\Psi_e(X)$ is not the graph of a total function.

- $\underline{l = g}$: consider the last value n takes. We know from Claim 4.6.7.1 that $\mathcal{R}_e^a$ acts finitely often, so n is finite. Suppose that $n \in \operatorname{dom}(\Psi_e(X))$. Then there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $n \in \operatorname{dom}(\Psi_{e,t}(L \oplus R \oplus Z))$. Consider $q = (C^{q_t} \cup L \setminus u \cup R \setminus u \cup Z \setminus u, (P_0^{q_t} \cup L \setminus u) \sqcup (P_1^{q_t} \cup R \setminus u), N^{q_t} \cup L \setminus u \cup R \setminus u \cup Z \setminus u, M^{q_t} \cup Z \setminus u)$. We have that $q \prec_u q_t$ and $n \in \operatorname{dom}(\Psi_{e,t}(q(a)))$. So at stage t we would have set $q_{t+1} = q$, a contradiction. So $n \notin \operatorname{dom}(\Psi_e(X))$ and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = c}$: Since $\mathcal{R}_e^a$ is never injured after stage s we have a fixed collection m, r_0, r_1 such that $r_i \not\leq q_t$ for any t. Furthermore, because of the restriction imposed by $\mathcal{R}_e^a$ we have that for all $t > s$ we have $N^{r_i} \backslash u \subseteq C^{q_t} \backslash N^{q_t} \subseteq [u, v)$. Suppose that $m \in \text{dom}(\Psi_e(X))$. Then, like in the previous case, there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $m \in \text{dom}\Psi_{e,t}(L \oplus R \oplus Z)$. Let $z = \max(L \cup R \cup Z)$. Consider

$$q = (C^{q_t} \cup [v, z], (P_0^{q_t} \cup L \setminus v) \sqcup (P_1^{q_t} \cup R \setminus v), N^{q_t} \cup [v, z], M^{q_t} \cup Z \setminus v).$$

We have that $q \preceq_v q_t$ is v-robust and $m \in \text{dom}(\Psi_{e,t}(q(a)))$. Thus at stage t we could have used q to enter state n, a contradiction. So $m \notin \text{dom}(\Psi_e(X))$, and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = n}$: Since $\mathcal{R}_e^a$ is never injured after stage s we have a fixed collection m, r_i, p such that $r_i, p \not\preceq q_t$ for any t. Like before, we have that for all $t > s$ we have $N^p \backslash v \subseteq C^{q_t} \backslash N^{q_t} \subseteq [v, o)$. Suppose that $m \in \text{dom}(\Psi_e(X))$. Then, like in the previous case, there are some finite $L \oplus R \oplus Z \subseteq X$ and $t > s$ such that $m \in \text{dom}\Psi_{e,t}(L \oplus R \oplus Z)$. Let $z = \max(L \cup R \cup Z)$. Consider

$$q = (C^{q_t} \cup [o, z], (P_0^{q_t} \cup L \setminus v) \sqcup (P_1^{q_t} \cup R \setminus v), N^{q_t} \cup [o, z], M^{q_t} \cup Z \setminus v).$$

We have that $q \preceq_o q_t$ is o-robust and $m \in \text{dom}(\Psi_{e,t}(q(a)))$. Thus at stage t we could have used q to enter state n, a contradiction. So $m \notin \text{dom}(\Psi_e(X))$, and thus $\Psi_e(X)$ is not a total function.

- $\underline{l = w}$ Suppose that $f = \Psi_e(X)$ is a total function. We will show that $f \notin G$ and that $f \leq_T 0'$. Since we left state g there is some n such that $f \restriction n + 1 \notin T$, so $f \notin G$. To compute $f(m)$ from $\mathbf{0}'$ search for a stage $t > s$ and u-robust $r \preceq_u q_t$ such that $m \in \text{dom}(\Psi_{e,t}(r(a)))$ and $r \not\preceq q_k$ for all k. $\mathbf{0}'$ can carry out this search, and since $m \in \text{dom}(\Psi_e(X))$ the search will halt. We claim that $f(m) = \Psi_e(r(a))(m)$. Suppose not. Since $m \in \text{dom}(\Psi_e(X))$ there are some finite $L \oplus R \oplus Z \subseteq X$ and $k > t$ such that $m \in \text{dom}(\Psi_{e,k}(L \oplus R \oplus Z))$. Let $z = \max(L \cup R \cup Z)$. Consider

$$q = (C^{q_t} \cup [u, z], (P_0^{q_t} \cup L \setminus u) \sqcup (P_1^{q_t} \cup R \setminus u), N^{q_t} \cup [u, z], M^{q_t} \cup Z \setminus u).$$

We have that q is u-robust and $q \not\leq q_k$ for all k. But we also have $r \not\leq q_k$ for all k. So at some stage j we would have entered state c using robust extensions of q and r along with m, $f(m)$ and $\Psi_e(r(a))(m)$. This is a contradiction, so $f(m) = \Psi_e(r(a))(m)$ and hence $f \leq_T 0'$.

□

We can relativize this construction to get a relativizable class as in the statement of Theorem 4.6.5, and then apply the theorem to get that the Arens co-d-CEA degrees are not submetrizable. □

4.6.3 Roy halfgraph degrees

Now we give another example of a class that is $T_{2.5}$ but not submetrizable. The Roy halfgraph degrees were introduced in [26] and are defined as follows.

Definition 4.6.8. Define $\tilde{\omega} = \omega \cup \{-1, \infty\}$ For a function $f : \omega \to \tilde{\omega}$ we define

$$\begin{aligned}\mathrm{HalfGraph}(f) =& \{\langle n, m\rangle \in \omega : f(n) \in \omega \wedge f(n) = 2m\} \oplus \\ & \{\langle n, m\rangle \in \omega : f(n) \in \omega \wedge f(n) \geq 2m\}\end{aligned}$$

$$\mathrm{HalfGraph}^{+}(f) = \{\langle n, m\rangle \in \omega : f(n) \leq 2m\} \oplus \{\langle n, m\rangle \in \omega : f(n) \geq 2m\}$$

We say that f is Y-*computably dominated* if there is a Y-partial computable φ such that for all n we have $f(n) \in \omega \implies \varphi(n)\downarrow \geq f(n)$.

We say a degree $\mathbf{d}$ is *Roy halfgraph above* if it contains a set of the form

$$Y \oplus Y^c \oplus \mathrm{HalfGraph}^{+}(f)$$

where f is Y-computably dominated and $\mathrm{HalfGraph}(f)$ is Y c.e.

Kihara, Ng and Pauly [26] show that Roy halfgraph graph degrees are the degrees of the product space of Roy's lattice space $\mathcal{QR}^{\omega}$, a $T_{2.5}$ space which is not submetrizable. Now we prove that this class of degrees does not arise from any submetrizable cb_0 space.

Theorem 4.6.9. *The Roy halfgraph degrees are not submetrizable.*

Proof. Given a partial function $\psi :\subseteq \omega \to \omega$ we define a *Roy extension of* ψ to be a total function $f : \omega \to \tilde{\omega}$ such that $f^{-1}(\omega) = \mathrm{dom}(\psi)$.

We will build a partial function $\psi :\subseteq \omega \to \omega$ such that HalfGraph(ψ) is c.e., ψ is computably dominated, dom(ψ) is coinfinite and for any Roy extension f of ψ we have if $h \leq_e \mathrm{HalfGraph}^+(f)$ then $h \leq_T \mathbf{0}'$ and $\deg_T(h)$ is not PA.

Now we consider what enumeration operators on the halfgraph of a function look like. If we see $2\langle n, m\rangle \in \mathrm{HalfGraph}^+(f)$ then we know that $f(n) \leq 2m$ and if we see $2\langle n, m\rangle + 1 \in \mathrm{HalfGraph}^+(f)$ then we know that $f(n) \geq 2m$. So an enumeration of $\mathrm{HalfGraph}^+(f)$ can be viewed as a refinement of even ended intervals containing f. This is the idea behind the following notation.

Let I be the set of all closed intervals in $\tilde{\omega}$ with end points in $\{2n : n \in \omega\} \cup \{-1, \infty\}$. Let $(\alpha_v)_v$ be an effective listing of all functions $\alpha : \omega \to I$ such that $\alpha(n) = \tilde{\omega}$ for all but finitely many n. For $f : \omega \to \tilde{\omega}$ define $\Phi_e(f) = \{n : \exists\langle x, v\rangle \in W_e[\forall n(f(n) \in \alpha_v(n))]\}$. For any f we have that $X \leq_e \mathrm{HalfGraph}^+(f)$ if and only if $X = \Phi_e(f)$ for some e. For $f :\subseteq \omega \to \tilde{\omega}$, note that if $n \in \Phi_e(f)$ via α then there is $g : \omega \to \omega$ such that $f(m) = g(m)$ for all $m \in f^{-1}(\omega)$ and $n \in \Phi_e(g)$ via α.

We will use a finite injury construction to create partial functions ψ, φ such that HalfGraph(ψ) is c.e., φ is partial computable and φ dominates ψ.

We will build ψ and φ in stages and use the following notation. Let $Q = \{(q, p) : q, p :\subseteq \omega \to \omega \wedge \mathrm{dom}(q) = \mathrm{dom}(p) \subseteq_{\mathrm{fin}} \omega \wedge \forall n \in \mathrm{dom}(q)[q(n) \leq p(n)]\}$. For $(q_0, p_0), (q_1, p_1) \in Q$, $u \in \omega$ and $a : u \to \tilde{\omega}$ we define the following:

- $(q_0, p_0) \preceq (q_1, p_1)$ if $\mathrm{HalfGraph}(q_0) \supseteq \mathrm{HalfGraph}(q_1)$ and $p_0 \supseteq p_1$.

- $(q_0, p_0) \preceq_u (q_1, p_1)$ ((q_0, p_0) extends (q_1, p_1) above u) if $(q_0, p_0) \preceq (q_1, p_1)$, $q_0 \restriction u = q_1 \restriction u$ and $\mathrm{dom}(q_0) \setminus u$ is an interval.

- $a \lhd_u (q, p)$ if $a : u \to \tilde{\omega}$ and $\mathrm{HalfGraph}(a) = \mathrm{HalfGraph}(q \restriction u)$.

Note that if $a \lhd_u (q_1, p_1)$ and $(q_0, p_0) \preceq_u (q_1, p_1)$ then $a \lhd_u (q_0, p_0)$ and $\Phi_e(a \cup q_1) \subseteq$

$\Phi_e(a \cup q_0)$. Note that if $(q_0, p_0) \preceq (q_1, p_1)$ and $q_1(n) = 2m$ then $q_0(n) = 2m$, but if $q_1(n) = 2m+1$ then $q_0(n)$ can be any number in $[2m, p_1(n)]$.

For $\alpha : \omega \to I$ and $a \lhd_u (q,p)$ we say $a, (q,p) \Vdash_u f \in \alpha$ if for all n we have that if $\alpha(n) \neq \tilde{\omega}$ then $(a \cup q)(n) \in \alpha(n)$ and in addition if $n \geq u$ then $q(n) \in 2\mathbb{Z}$. We say $a, (q,p) \Vdash_u f \notin \alpha$ if for all $(q_0, p_0) \preceq_u (q,p)$ we have $a, (q_0, p_0) \nVdash_u f \in \alpha$. Note that because even values of q cannot change in extensions if $(q_0, p_0) \preceq_u (q_1, p_1)$ and $a, (q_1, p_1) \Vdash_u f \in \alpha$ then $a, (q_0, p_0) \Vdash_u f \in \alpha$.

We will build a computable sequence $(q_0, p_0) \succeq (q_1, p_1) \succeq \ldots$ and have $\psi = \lim_n q_n$ with $\varphi = \bigcup_n p_n$. This will ensure that HalfGraph(ψ) is c.e. and ψ is computably dominated by φ. Fix a Π^0_1 class G such that for each $x \in G$ $\deg_T(x)$ is PA and a computable tree T such that $[T] = G$. The requirements $\mathcal{R}_e$ are that for any Roy extension f of ψ we have that if $h = \Phi_e(f)$ is the graph of a total function then $h \leq_T \emptyset'$ and $h \notin G$.

The strategy for $\mathcal{R}_e$ works as follows. If at stage s $\mathcal{R}_e$ is initialized then let $u = \max(\text{dom}(q_s)) + 2$. If at some later stage t we have $(q_t, p_t) \npreceq_u (q_s, p_s)$ then we consider $\mathcal{R}_e$ injured and will reinitialize it. For each $a \lhd_u (q_s, p_s)$ we create a new subrequirement $\mathcal{R}_e^a$. We put an order on these subrequirements and add them to the list of all requirements as an interval just below $\mathcal{R}_e$ in priority. If $\mathcal{R}_e$ is injured then we remove these subrequirements. $\mathcal{R}_e^a$ is satisfied if $\mathcal{R}_e$ is satisfied for each Roy extension f of ψ with $a \subseteq f$.

Each $\mathcal{R}_e^a$ has states g, w, c, d and a bound u_a. Initially they are in state g and have $u_a = u$. Each time $\mathcal{R}_e^a$ is injured by a higher priority requirement we set $u_a = \max(\text{dom}(q_s)) + 1$ and return to state g. When an $\mathcal{R}_e^a$ sets q_s we will have that $[u_a, \max(\text{dom}(q_s))] \subseteq \text{dom}(q_s)$ so that lower priority $\mathcal{R}_e^b$ can work with a higher restriction than u_a without the need to split into more subrequirements. The actions for each state are as follows.

- State g: we start with $n = 0$ and $\sigma_0 = \emptyset$. If at some stage s we see some x, v such that $\langle\langle n, x\rangle, v\rangle \in W_{e,s}$ and we see $(q_{s+1}, p_{s+1}) \preceq_{u_a} (q_s, p_s)$ such that $a, (q_{s+1}, p_{s+1}) \Vdash_{u_a} f \in \alpha_v$ then we act and injure all lower priority requirements. If $\sigma_n{}^\frown x \notin T$ then we go to state w otherwise we remain in state g and set $n = n+1, \sigma_{n+1} = \sigma_n{}^\frown x$.

- State w: we wait until at some stage s we see $m, x_0, x_1, v_0, v_1 \in \omega$ such that

$\langle\langle m, x_i\rangle, v_i\rangle \in W_{e,s}$ and $a,(q_s,p_s) \nVdash_{u_a} f \notin \alpha_{v_i}$ for $i \in 2$. Chose a condition $(q,p) \preceq_{u_a} q_s$ such that for all $n \geq u_a$ the following hold.

1. If $\alpha_{v_0}(n) \cap \alpha_{v_1}(n) \neq \emptyset$ then $q(n) \in \alpha_{v_0}(n) \cap \alpha_{v_1}(n)$ and $q(n)$ is even.
2. If $\max(\alpha_{v_i}(n)) < \min(\alpha_{v_{1-i}}(n))$ then $q(n) = \max(\alpha_{v_i}(n)) + 1$ and $p(n) \geq \min(\alpha_{v_{1-i}})$

So we get that $q(n) \in \alpha_{v_0}(n) \iff q(n) \in \alpha_{v_1}(n)$. Set $(q_{s+1}, p_{s+1}) = (q,p)$. If $a,(q,p) \Vdash_{u_a} f \in \alpha_{v_0}, \alpha_{v_1}$ then move to state d. Otherwise move to state c. Let $r = \max(\mathrm{dom}(q) + 1$. Note that $a,(q_s,p_s) \nVdash_{u_a} f \notin \alpha_{v_i}$ but $a,(q,p) \Vdash_r f \notin \alpha_{v_i}$.

- State c: we wait until we at some stage t we see a pair $x, v \in \omega$ such that $\langle\langle m, x\rangle, v\rangle \in W_{e,t}$ and $a,(q_t,p_t) \nVdash_r f \notin \alpha_v$. Pick i such that $x_i \neq x$. Choose a condition $(q,p) \preceq_{u_a} (q_s,p_s)$ such that the following all hold.

  1. If $\alpha_{v_i}(n) \cap \alpha_v(n) \neq \emptyset$ then $q(n) \in \alpha_{v_i}(n) \cap \alpha_v(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_v(n)) < \min(\alpha_{v_i}(n))$ then $q(n) = \max(\alpha_v(n)) + 1$.

  We set $(q_{t+1}, p_{t+1}) = (q,p)$. If $a,(q,p) \Vdash_{u_a} f \in \alpha_v, \alpha_{v_i}$ then we will move to state d. Otherwise we will remain in state c. If we remain, then note that there is n such that $q(n) = \max(\alpha_v(n)) + 1$. Since $\max(\alpha_{v_{1-i}}) < q_t(n) \in \alpha_v(n)$ we now have $a,(q,p) \Vdash_{u_a} f \notin \alpha_{v_{1-i}}$. We redefine $x_{1-i} = x, v_{1-i} = v, r = \max(\mathrm{dom}(q) + 1$.

- State d: in this state $\mathcal{R}_e^a$ is considered satisfied.

This completes the construction of ψ and φ. Now we move onto the verification.

Claim 4.6.9.1. *Each requirement is injured only finitely often.*

Proof. If a requirement $\mathcal{R}_e$ is never injured after stage s then it acts only once more to split into the $\mathcal{R}_e^a$ requirements. Suppose that $\mathcal{R}_e^a$ is never injured after stage s. Then there is a stage after which the state of $\mathcal{R}_e^a$ remains the same. In states w and d it is clear $\mathcal{R}_e^a$ can act at most once. We now look at the other two states.

First, suppose that $\mathcal{R}_e^a$ remains in state g and injures lower priority requirements infinitely often. Each time it acts n increases, so we have that $h = \bigcup_n \sigma_n \in 2^\omega$ and h is computable. Since every $\sigma \prec h$ is in T we have that $h \in G$, but this is a contradiction as no PA degree is computable.

Second, suppose that $\mathcal{R}_e^a$ remains in state c and injures lower priority requirements infinitely often. Let $n_0 \dots n_{k-1}$ be the values where $q_s(n) \notin \alpha_{v_i}(n)$ for $i \in 2$. Each time $\mathcal{R}_e^a$ acts the collection $n_0 \dots n_{k-1}$ either stays the same or decreases. So there is a stage after which this collection is fixed. Consider one of these n. At each stage t when $\mathcal{R}_e^a$ acts we have that $q_t(n) \in \alpha_v(n)$ so $q_{t+1}(n) > q_t(n)$. This can only happen finitely often, a contradiction. □

Claim 4.6.9.2. $\mathrm{dom}(\psi)$ *is coinfinite.*

Proof. Let s be the last stage when $\mathcal{R}_e$ was injured. We have that if u is the restriction chosen by $\mathcal{R}_e$ at stage $t > s$, then $e = |\mathrm{dom}(\psi)^c \restriction u|$. This follows by induction and from the fact that $\max(\mathrm{dom}(q_t)) + 1 < u$ means that $\max(\mathrm{dom}(q_t)) + 1 \notin \mathrm{dom}(q_j)$ for any $j \geq t$. □

Claim 4.6.9.3. *Each* $\mathcal{R}_e$ *is satisfied.*

Proof. Consider some Roy extension f of ψ. Fix e. We will show that $\mathcal{R}_e$ is satisfied for f. There is some subrequirement $\mathcal{R}_e^a$ such that $f \restriction u_a = a \cup \psi \restriction u_a$. Note that for any t we have $\Phi_e(a \cup q_t) \subseteq \Phi_e(f)$. Let s be a stage such that $\mathcal{R}_e^a$ is never injured after this stage. Let l be the last state that $\mathcal{R}_e^a$ is in. We will look at the four cases.

- $\underline{l = d}$: when we entered state d at stage t we ensured that $\langle m, x_0\rangle, \langle m, x_1\rangle \in \Phi_{e,t}(a \cup q_{t+1}) \subseteq \Phi_e(f)$ for $x_0 \neq x_1$ so $\Phi_e(f)$ is not a function.

- $\underline{l = g}$: consider the last value n takes. We know from Claim 4.6.9.1 that $\mathcal{R}_e^a$ acts finitely often, so n is finite. Suppose that $n \in \mathrm{dom}(\Phi_e(f))$. Then there are some $x, v \in \omega$ and $t > s$ such that $\langle\langle n, x\rangle, v\rangle \in W_{e,t}$ and $f(m) \in \alpha_v(m)$ for all m. But then there is $(q, p) \preceq_{u_a} (q_t, p_t)$ such that $a \cup q = f \restriction \{n : \alpha_v(n) \neq \tilde{\omega}\}$. Define $q'(n) = q(n) - 1$

if $q(n)$ is odd and $n \geq u_a$, and $q'(n) = q(\alpha_v)$ otherwise. We have $(q', p) \preceq_{u_a} (q_t, p_t)$ and $a, (q', p) \Vdash f \in \alpha_v$. Thus the requirement would be able to act again using (q', p), a contradiction.

- $\underline{l = c}$: Let $t - 1$ be the last stage when $\mathcal{R}_e^a$ acts. So from stage t onward we have fixed m, r and $(q_k, p_k) \preceq_r (q_t, p_t)$ for all $k \geq t$. Suppose that $m \in \text{dom}(\Phi_e(f))$. Then like in the case above there is some stage k and $(q, p) \preceq_r (q_k, p_k)$ such that $m \in \text{dom}(\Phi_{e,k}(a \cup q))$. But then $\mathcal{R}_e^a$ would have acted again at stage k, a contradiction.

- $\underline{l = w}$: Suppose that $h = \Phi_e(f)$ is a total function. We will show that $h \notin G$ and that $h \leq_T 0'$. Since we left state g there is some n such that $h\restriction n + 1 \notin T$, so $h \notin G$. To compute $h(m)$ search for a stage $t > s$ and $(q, p) \preceq_{u_a} q_t$ such that $m \in \text{dom}(\Phi_{e,t}(a \cup q))$ and $\psi\restriction\text{dom}(q) = q$. $\mathbf{0}'$ can carry out this search since ψ is $\mathbf{0}'$ computable and $m \in \text{dom}(\Phi_e(f))$, so the search will halt. We claim that $f(m) = \Phi_e(a \cup q)$.

  Suppose not. Let α_v be the witness that $m \in \text{dom}(\Phi_{e,t}(a \cup q))$. Since $m \in \text{dom}(\Phi_e(f))$ there is $j \geq t$ and $(q_1, p_1) \preceq_{u_a} (q_j, p_j)$ such that $\langle m, f(m)\rangle \in \Phi_{e,j}(a \cup q_1)$ via some witness α_{v_1} and $\psi\restriction\text{dom}(q_1) = q_1$. So for all k we have that $(q_k, p_k) \nVdash f \notin \alpha_v$,$(q_k, p_k) \nVdash f \notin \alpha_{v_1}$. But we would have used α_v and α_{v_1} to enter state c, a contradiction. So $f(m) = \Phi_e(a \cup q)$ and hence $f \leq_T 0'$.

□

By meeting all $\mathcal{R}_e$ requirements we have ensured ψ has the desired properties. By relativizing this construction we can find classes of Roy halfgraph degrees $\mathcal{C}^Y$ as in the statement of Theorem 4.6.5. So the Roy halfgraph degrees are not submetrizable. □

4.7 Arens co-d-CEA degrees and Roy halfgraph degrees above

We have seen that the Arens co-d-CEA degrees and the Roy halfgraph above degrees are both examples of classes which are $T_{2.5}$ but not submetrizable. In this section we look at the relationship between these classes. Kihara, Ng and Pauly [26] showed that both of these classes contain the co-d-CEA degrees, and that the Roy halfgraph degrees are a subclass of the doubled co-d-CEA degrees. We show that the Arens co-d-CEA degrees are also a subclass of the doubled co-d-CEA degrees.

Proposition 4.7.1. *Every Arens co-d-CEA degree is doubled co-d-CEA*

Proof. Consider an Arens co-d-CEA set $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$. We have that $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1)$ is doubled co-d-CEA by definition. Consider the set $(A_0 \cup A_1 \cup N)^c \cup M = (A_0 \cup A_1)^c \cap N^c \cup M = (A_0 \cup A_1)^c \cap (N^c \cup M)$. Since $(A_0 \cup A_1)^c$ is Y-c.e. let f be a Y computable enumeration of $(A_0 \cup A_1)^c$. Then we have $Y \oplus Y^c \oplus (A_0 \cup A_1)^c \cap (N^c \cup M) \equiv_e Y \oplus Y^c \oplus \{n : f(n) \in N^c\} \cup \{n : f(n) \in M\}$. The right hand side is a co-d-CEA and hence doubled co-d-CEA. The join of two doubled co-d-CEA degrees is doubled co-d-CEA so we have that $Y \oplus Y^c \oplus (A_0 \cup P_0) \oplus (A_1 \cup P_1) \oplus ((A_0 \cup A_1 \cup N)^c \cup M)$ has doubled co-d-CEA degree. □

Now we know from Theorem 4.5.2 that these classes are both proper subclasses of the doubled co-d-CEA degrees and since the co-d-CEA degrees are a submetrizable class [26] we know that they both properly contain the co-d-CEA degrees. The next question to ask is if these two classes are distinct from each other. We now give separations that show neither class is contained in the other. The proofs below make use of some of the notation and ideas from the proofs of Theorems 4.6.7 and 4.6.9

Theorem 4.7.2. *There is a Roy halfgraph degree that is not an Arens co-d-CEA degree.*

Proof. Our proof is a finite injury construction. Since we are building a single Roy halfgraph degree we will use an extension of the partial order from 4.6.9. Let

$$Q = \{(q,p) : q : \omega \to \tilde{\omega} \wedge |q^{-1}[\omega \cup \{\infty\}]| < \omega \wedge p : q^{-1}[\omega] \to \omega \wedge \forall n \in \mathrm{dom}(p)[q(n) \leq p(n)]\}$$

The difference from before is that now q is total and its range is no longer restricted to ω. For $(q_0, p_0), (q_1, p_1) \in Q$ and $u \in \omega$ we define the following:

- $(q_0, p_0) \preceq (q_1, p_1)$ if $\text{HalfGraph}(q_0) \supseteq \text{HalfGraph}(q_1)$ and $p_0 \supseteq p_1$.
- $(q_0, p_0) \preceq_u (q_1, p_1)$ ((q_0, p_0) extends (q_1, p_1) above u) if $(q_0, p_0) \preceq (q_1, p_1)$, $q_0 \upharpoonright u = q_1 \upharpoonright u$.

We will again make use of the enumeration of sequences of even ended intervals $(\alpha_v)_v$ and the operators $(\Phi_e)_e$. This time we define $\Vdash_u$ a little differently. For $\alpha : \omega \to I$ we say $(q, p) \Vdash_u f \in \alpha$ if for all n we have $f(n) \in \alpha(n)$ and for all $n \geq u$ we have $f(n) \in 2\mathbb{Z}$ or $\alpha(n) = \tilde{\omega}$. We say $(q, p) \Vdash_u f \notin \alpha$ if for all $(q_0, p_0) \preceq_u (q, p)$ we have $(q_0, p_0) \nVdash_u f \in \alpha$. Note that because even values of q cannot change in extensions, if $(q_0, p_0) \preceq_u (q_1, p_1)$ and $(q_1, p_1) \Vdash_u f \in \alpha$ then $(q_0, p_0) \Vdash_u f \in \alpha$.

We will build a computable sequence $(q_0, p_0) \succeq (q_1, p_1) \succeq \ldots$ such that $f = \lim_s q_s$ is well defined. We will have that $\text{HalfGraph}^+(f)$ has Roy halfgraph degree. Note that if q_0 and q_1 differ only in that $q_0(n) = -1$ and $q_1(n) = \infty$ then for an appropriate p we have $(q_0, p) \preceq (q_1, p) \preceq (q_0, p)$, so $\preceq$ is not a partial order. The reason f will be well defined is that requirements put up restrictions, so for each u there is a large enough s such that $(q_s, p_s) \succeq_u (q_{s+1}, p_{s+1}) \succeq_u \ldots$ and so $f \upharpoonright u = q_s \upharpoonright u$.

The requirements will be $\mathcal{R}_{e,i,j,N,P,M,A}$ where $e, i, j \in \omega$, A, N, P, M are enumeration operators such that given some total set Y they produce Y-c.e. sets $A^Y, N^Y = N_0^Y \sqcup N_1^Y, P^Y = P_0^Y \sqcup P_1^Y, M^Y$ with $P_0 \subseteq N_0^Y \subseteq A^Y, P_1^Y \subseteq N_1^Y \subseteq A^Y$, and $M^Y \subseteq N_Y \setminus P^Y$. The intuition for $\mathcal{R}_{e,i,j,N,P,M,A}$ is that if $Y = \Phi_i(f)$ is the graph of a total function and $Y \oplus \Phi_e(f)$ is an Arens co-d-CEA set of the form $Y \oplus (A_0 \cup P_0^Y) \oplus (A_1 \cup P_1^Y) \oplus ((A_0 \cup A_1 \cup N^Y)^c \cup M^Y)$ for some $A_0 \sqcup A_1 = (A^Y)^c$ then we need to have $\Psi_j(Y \oplus \Phi_e(f)) \neq \text{HalfGraph}^+(f)$. So we say $\mathcal{R}_{e,i,j,N,P,M,A}$ is satisfied if one of the following conditions holds.

1. $Y = \Phi_i(X)$ is not the graph of a total function.
2. $\Phi_e(X) \neq (A_0 \cup P_0^Y) \oplus (A_1 \cup P_1^Y) \oplus ((A_0 \cup A_1 \cup N^Y)^c \cup M^Y)$ for any partition $A_0 \sqcup A_1 = (A^Y)^c$.
3. $\Psi_j(Y \oplus \Phi_e(f)) \neq \text{HalfGraph}^+(f)$.

Now we consider the strategies for $\mathcal{R}_{e,i,j,A,N,P,M}$. Each $\mathcal{R}_{e,i,j,A,N,P,M}$ will be given some restriction u by higher priority requirements and must ensure that $(q_{s+1}, p_{s+1}) \preceq_u (q_s, p_s)$ whenever it sets (q_{s+1}, p_s). Let $Y_s = \Phi_{i,s}(q_s)$. There are several strategies for $\mathcal{R}_{e,i,j,A,N,P,M}$. First we try to meet condition 2 directly. If we ever see some $(q,p) \preceq_u (q_s, p_s)$ such that for $L \oplus R \oplus Z = \Phi_{e,s}(q)$ we have L, R, Z are not disjoint or one of the pairs $(P_0^Y \cup M^Y, R), (P_1^Y \cup M^Y, L), (P^Y, Z)$ is not disjoint, then we set $(q_{s+1}, p_{s+1}) = (q,p)$ and injure lower priority requirements by forcing their restriction to be larger than the use of $\Phi_{e,s}, \Phi_{i,s}$. At each stage when $\mathcal{R}_{e,i,j,A,N,P,M}$ is active we will run this strategy and then move on to the next strategy.

For the second strategy we try to meet condition 1. We do this by running a simplified version of the strategy we used for Theorem 4.6.9 for $\mathcal{R}_i^a$ to try to make $\Psi_i(X)$ not the graph of a total function. This time we start in state w.

- <u>State w</u>: we wait until at some stage s we see $m, x_0, x_1, v_0, v_1 \in \omega$ such that $\langle\langle m, x_i\rangle, v_i\rangle \in W_{e,s}$ and $(q_s, p_s) \nVdash_u f \notin \alpha_{v_i}$ for $i \in 2$. Chose a condition $(q,p) \preceq_u q_s$ such that for all $n \geq u$ the following hold.

  1. If $\alpha_{v_0}(n) \cap \alpha_{v_1}(n) \neq \emptyset$ then $q(n) \in \alpha_{v_0}(n) \cap \alpha_{v_1}(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_{v_i}(n)) < \min(\alpha_{v_{1-i}}(n))$ then $q(n) = \max(\alpha_{v_i}(n)) + 1$ and $p(n) \geq \min(\alpha_{v_{1-i}})$.

  So we get that $q(n) \in \alpha_{v_0}(n) \iff q(n) \in \alpha_{v_1}(n)$. Set $(q_{s+1}, p_{s+1}) = (q,p)$. If $(q,p) \Vdash_u f \in \alpha_{v_0}, \alpha_{v_1}$ then move to state d. Otherwise move to state c. Let $r = \max(\mathrm{dom}(q) + 1$. Note that $(q_s, p_s) \nVdash_u f \notin \alpha_{v_i}$ but $(q,p) \Vdash_r f \notin \alpha_{v_i}$.

- <u>State c</u>: we wait until some stage t we see a pair $x, v \in \omega$ such that $\langle\langle m, x\rangle, v\rangle \in W_{e,t}$ and $(q_t, p_t) \nVdash_r f \notin \alpha_v$. Pick i such that $x_i \neq x$. Choose a condition $(q,p) \preceq_u (q_s, p_s)$ such that the following all hold.

  1. If $\alpha_{v_i}(n) \cap \alpha_v(n) \neq \emptyset$ then $q(n) \in \alpha_{v_i}(n) \cap \alpha_v(n)$ and $q(n)$ is even.
  2. If $\max(\alpha_v(n)) < \min(\alpha_{v_i}(n))$ then $q(n) = \max(\alpha_v(n)) + 1$.

We set $(q_{t+1}, p_{t+1}) = (q, p)$. If $a, (q, p) \Vdash_u f \in \alpha_v, \alpha_{v_i}$ then we will move to state d. Otherwise we will remain in state c. If we remain, then note that there is an n such that $q(n) = \max(\alpha_v(n)) + 1$. Since $\max(\alpha_{v_{1-i}}) < q_t(n) \in \alpha_v(n)$ we now have $(q, p) \Vdash_u f \notin \alpha_{v_{1-i}}$. We redefine $x_{1-i} = x, v_{1-i} = v, r = \max(\mathrm{dom}(q) + 1$.

- State d: in this state $\mathcal{R}_{e,i,j,A,N,P,M}$ is considered satisfied via case 1.

If the strategy finishes then we get that Y is multivalued, and thus not the graph of a function. If the strategy remains in state c then $m \notin \mathrm{dom}(Y)$ so Y is not total. In both these cases $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied.

If the strategy never leaves state w then it is possible that Y is the graph of a total function. However if this is the case then we have the following important observation: if Y is the graph of a total function and $(r, p) \not\leq_u (q_t, p_t)$ for all t then it must be that $\Phi_i(r) \subseteq Y$ as otherwise we would have used (r, p) to move into state c at some point. We make repeated use of this in the third strategy to ensure that Y is consistent at every stage when we act.

The third strategy tries to meet condition 3. Define $L_s \oplus R_s \oplus Z_s = \Phi_{e,s}(q_s)$ and $D_s = L_s \cup R_s \cup Z_s$. For $(q, p) \in Q$ and $v \in \omega$ we say that $(q, p) \to_v f(n) \leq 2m$ (implies $f(n) \leq 2m$) if $2\langle n, m\rangle \in \Psi_j(Y \oplus \Phi_e(q{\restriction}v)$ and $(q, p) \to_v f(n) \geq 2m$ if $2\langle n, m\rangle + 1 \in \Psi_j(Y \oplus \Phi_e(q{\restriction}v)$.

When $\mathcal{R}_{e,i,j,A,N,P,M}$ is initialized we pick a witness $x > u$ and start with $q_s(x) = -1$. The steps for this strategy are as follows.

1. If we ever see a stage s and $v \in \omega$ where we see $(q_s, p_s) \to_s f(x) \leq 0$ then we injure all lower priority requirements with restriction s and set $q_{s+1}(x) = \infty$. If we are waiting forever at this step then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 3.

2. Next we wait until we see a stage $t > s$ where $Y_s \subseteq Y_t$, $D_s \subseteq D_t \cup N^{Y_t}$ and $(q_t, p_t) \to_t f(x) \geq 2$. Then injure all lower priority requirements with restriction t and set $q_{t+1}(x) = -1$ and $D = D_s \cup D_t$.

Note that for any Arens co-d-CEA set $Y \oplus L \oplus R \oplus Z$ we have that $L \cup R \cup Z \cup N = \omega$. So if we wait forever to see $D_s \subseteq D_t \cup N^{Y_t}$ then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2.

If we have $L_s \oplus R_s \oplus Z_s \subseteq L_t \oplus R_t \oplus Z_t$ then the strategy is finished—this is because if $\mathcal{R}_{e,i,j,A,N,P,M}$ is not injured after stage t then we will have we have $(q_t, p_t) \to_t f(n) \leq 0 \wedge f(n) \geq 2$ meeting condition 3. Otherwise we move on to the next step.

3. We wait until a stage $l > t$ where $Y_t \subseteq Y_l$ and $L_l \oplus R_l \oplus Z_l$ looks like a subset of an Arens co-d-CEA set on D, more precisely, we must have $D \cap A^{Y_l} \cap (L_t \cup R_t) \subseteq P^{Y_l}$, $D \cap Z_t \subseteq A^{Y_l}$ and $D \cap N^{Y_l} \cap Z_t \subseteq M^{Y_l}$. Then set $q_{l+1}(x) = \infty$, and injure lower priority requirements with restriction l.

   For any Arens co-d-CEA set $Y \oplus L \oplus R \oplus Z$ we must have $(A_0 \cup A_1)^c \cap (L \cup R) \subseteq P$, $Z \subseteq (A_0 \cup A_1)^c$ and $N \cap Z \subseteq M$. So if we wait forever at this step, then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2.

4. We wait until a stage $r > l$ where we have $Y_l \subseteq Y_r$ and where $L_s \oplus R_s \oplus Z_s$ looks like an Arens co-d-CEA set on D. Then set $q_{r+1}(x) = -1$, and injure lower priority requirements with restriction r.

5. We repeat steps 4 and 3 until we have $l < r$ such that $A^{Y_l}, P^{Y_l}, N^{Y_l}, M^{Y_l}$ agree with $A^{Y_r}, P^{Y_r}, N^{Y_r}, M^{Y_r}$ on D. This must happen eventually as D is finite and $A^{Y_l}, P^{Y_l}, N^{Y_l}, M^{Y_l}$ can only increase each time we repeat these steps.

   Since the witness of $L_r \oplus R_r \oplus Z_r \subseteq \Phi_{e,t}(q_t)$ only uses finitely many axioms there is a large enough n such that for any axiom (z, α) used we have if $2n \in \alpha(x)$ then $\infty \in \alpha(x)$. Set $p_{r+1}(x) = 2n+1$, $q_{r+1}(x) = 1$. The next step will be repeated several times; we start with $i = 0$, $s_0 = l$ and $r_0 = r$.

6. We wait for a stage $k > r_i$ where $Y_{r_i} \subseteq Y_k$ and $L_k \oplus R_k \oplus Z_k$ looks like an Arens co-d-CEA set on $D_{s_i} \cup D_r$, more precisely $P_0^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq L_k$, $P_1^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq R_k$, $M^{Y_k} \cap (D_{s_i} \cup D_r) \subseteq Z_k$, $Z_k \cap (D_{s_i} \cup D_r) \subseteq A^{Y_k}$ and $D_{s_i} \cup D_r \subseteq D_k \cup N^{Y_k}$. Again,

if we are waiting forever at this step, then $\mathcal{R}_{e,i,j,A,N,P,M}$ is satisfied by condition 2 or condition 1.

We now have several cases to consider.

(a) If we have $L_{s_i} \oplus R_{s_i} \oplus Z_{s_i} \subseteq L_k \oplus R_k \oplus Z_k$ then $(q_k, p_k) \to_k f(x) \leq 2i$ but $q(k) > 2i$, so we injure all lower priority requirements with restriction k and set $q_{k+1} = q_k$. We have now met condition 3.

(b) If we have $L_{s_i} \nsubseteq L_k \cup N_0^{Y_k}$, $R_{s_i} \nsubseteq R_k \cup N_1^{Y_k}$ or $Z_{s_i} \nsubseteq Z_k \cup N^{Y_k}$, then set $q_{k+1}(x) = 2i$ and injure all lower priority requirements with restriction k. In this case we are meeting condition 2 directly with the first strategy, as either $(L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}) \cup (L_k \oplus R_k \oplus Z_k)$ is not the join of three disjoint sets or N^{Y_k} does not match with $L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}$.

(c) Otherwise set $i = i + 1$, $s_i = k$, $q_{k+1}(x) = 2i + 1$ and repeat this step.

Now we verify this part of the construction. Consider a requirement $\mathcal{R} = \mathcal{R}_{e,i,j,A,N,P,M}$. Let s be a stage after which $\mathcal{R}$ is never injured. We need to show that $\mathcal{R}$ acts only finitely often and is satisfied. If $\mathcal{R}$ acts at stage $t > s$ via the first strategy then it never acts again and has ensured that $\Phi_i(f) \oplus \Phi_e(f)$ is not an Arens co-d-CEA set with A, N, P, M, so $\mathcal{R}$ is satisfied via condition 2.

If at some point the second strategy acts then by the same sort of verification used in the proof of Theorem 4.6.9 we can see that $\mathcal{R}$ acts only finitely often and is satisfied via condition 1.

Now we must consider the third strategy. If we end up waiting forever at any of the steps then $\mathcal{R}$ acts only finitely often and by looking at each step and the current state of $q(x)$ one can see that $\mathcal{R}$ must be satisfied by one of the three conditions. Similarly if the strategy finishes then $\mathcal{R}$ never acts again and is satisfied directly.

Suppose towards a contradiction that $\mathcal{R}$ acts infinitely often. Then it must be that we are in case (6c) of step 6 for all values of $i \in \omega$. Since $Y_{s_i} \subseteq Y_{s_{i+1}}$ we have that $N^{Y_{s_i}} \subseteq N^{Y_{s_{i+1}}}$. Since case (6b) does not apply we have that $L_{s_0} \subseteq L_{s_1} \cup N_0^{Y_{s_1}} \subseteq L_{s_2} \cup N_0^{Y_{s_2}} \subseteq \ldots$

and $R_{s_0} \subseteq R_{s_1} \cup N_1^{Y_{s_1}} \subseteq R_{s_2} \cup N_1^{Y_{s_2}} \subseteq \dots$ Now consider when $i = n$. This means that $L_r \oplus R_r \oplus Z_r \subseteq L_{s_i} \oplus R_{s_i} \oplus Z_{s_i}$. However, since we finished repeating steps 3 and 4 we have that $Z_l \cap D = Z_r \cap D$ and since $L_r \oplus R_r \oplus Z_r$ and $L_l \oplus R_l \oplus Z_l$ disagree somewhere on D there must be $k \in L_r \cap R_l$ or $k \in L_l \cap R_r$. But then the first strategy would act, a contradiction. This also means that the largest i we can get is $i = n$ so $q_{s_i}(x)$ will not exceed $p_{s_i}(x)$.

So $\mathcal{R}$ acts only finitely often and is satisfied. So HalfGraph$^+(f)$ is not of Arens co-d-CEA degree.

□

Now we prove the other direction.

Theorem 4.7.3. *There is an Arens co-d-c.e. degree that is not a Roy halfgraph above degree.*

Proof. Note that this proof is very similar to the proof of Theorem 4.7.2 above and shares a lot of the same structure and ideas.

We will use a modified set of conditions from the set Q from the proof of Theorem 4.6.7 to construct a specific Arens co-d-c.e. set. Let $P = \{(a, q) : q \in Q, a \subseteq_{\text{fin}} C^q\}$. For $(a, q), (b, p) \in P$ we define the following new notions:

- $(a, q) \preceq (b, p)$ if $q \preceq p$.
- $(a, q) \preceq_u (b, p)$ if $q \preceq_u p$ and $a{\upharpoonright}u = b{\upharpoonright}u$.
- $q(a) = (a \cup P_0^q) \oplus ((C^q)^c \setminus a \cup P_0^q) \oplus (C^q \setminus N^q \cup M^q)$.

We will build a computable sequence $(a_0, q_0) \succeq (a_1, q_1) \succeq \dots$ and have $C = \bigcup_s C^{q_s}$, $P = \bigcup_s P^{q_s}$, $N = \bigcup_s N^{q_s}$,$M = \bigcup_s M^{q_s}$ and ensure that $A_0 = \lim_n a_s$ is a well defined. We will define $A_1 = C^c \setminus A_0$ so A_0, A_1 is partition of C^c. This will ensure that C, P, N, M are c.e. and that $X = L \oplus R \oplus Z := (A_0 \cup P_0) \oplus (A_1 \cup P_1) \cup (C \setminus N \cup M)$ is an Arens co-d-c.e. set.

The requirements are $\mathcal{R}_{e,i,j,H}$ where H is an c.e. operator. We think of $\Psi_i(X) \oplus \Psi_e(X)$ as being a Roy halfgraph set where $\Psi_i(X)$ is the total part Y and $\Psi_e(X)$ is $\text{HalfGraph}_+(f)$ for some $f : \omega \to \tilde{\omega}$ with $H^Y = \text{HalfGraph}(f)$. If this really is the case, then we want $\Psi_j(\Psi_i(X) \oplus \Psi_e(X)) \neq X$. We say that $\mathcal{R}_{e,i,j,H}$ is satisfied if one of the following holds.

1. $Y = \Psi_i(X)$ is not the graph of a total function.

2. There is no $f : \omega \to \tilde{\omega}$ such that $\Psi_e(X) = \text{HalfGraph}^+(f)$ and $H^Y = \text{HalfGraph}(f)$.

3. $X \neq \Psi_j(Y \oplus \Psi_e(X))$.

Fix a requirement $\mathcal{R} = \mathcal{R}_{e,i,j,H}$ and let u be the restriction given to $\mathcal{R}$ by higher priority requirements. We will now give a strategy for $\mathcal{R}$.

Like in the proof of Theorem 4.7.2, the first strategy for $\mathcal{R}$ is to attempt to make Y not the graph of a total function. Again, for this we use the strategy from the proof of Theorem 4.6.7. This time we only need states w, c, n, d and not g.

- <u>State w</u>: we wait until at some stage s we see $m, x_0, x_1 \in \omega$ and a pair $(b_0, r_0), (b_1, r_1) \preceq_u (a_s, q_s)$ such that $\langle m, x_k \rangle \in \Psi_{i,s}(r_k(b_k))$ and $x_0 \neq x_1$. Let v bound the use of $\langle m, x_k \rangle \in \Psi_{i,s}(r_k(b_k))$. Without loss of generality we can assume $[u, v] \subseteq C^{r_k} \subseteq [0, v]$ as we can find some $(b, r) \preceq_u (a_s, q_s)$ with this property that has $r(b) \restriction v = r_k(b_r) \restriction v$ by putting to $[u, v] \setminus C^{r_k}$ into P^r. Note this means we are assuming $b_0 = b_1 = a_s \restriction u$. We set $q_{s+1} = (C^{q_s} \cup C^{r_0} \cup C^{r_1}, P^{q_s}, N^{q_s}, M^{q_s})$ and $a_{s+1} = a_s \setminus C^{q_{s+1}}$. All lower priority requirements are injured with restriction v and we enter state c.

- <u>State c</u>: we wait until we see a stage t such that for some $(b, p) \preceq_s (a_t, q_t)$ we have $\langle m, x_2 \rangle \in \Psi_{i,t}(p(b))$ for some x_2. Pick k such that $x_k \neq x_2$. Set $q_{t+1} = (C^p, P^p, N^p \cup N^{r_k}, M^p)$ (we know N^{r_k} and N^p do not conflict because $p \preceq_v q_{s+1}$ and v bounds N^{r_k}) and set $a_{t+1} = b$. Note that now we have $q_{t+1}(b) \restriction v \subseteq r_k(b), p(b)$. Let o bound $C^{q_{t+1}}$ and the use of $\langle m, x_2 \rangle \in \Psi_{i,t}(p(b))$. We injure all lower priority requirements with restriction o and enter state n.

- <u>State n</u>: we wait until we see a stage ℓ such that for some $(a, h) \preceq_o (a_\ell, q_\ell)$ we have $\langle m, y \rangle \in \Psi_{i,\ell}(h(a))$ for some y. If $y \neq x_k$ then set $q_{\ell+1} = (C^h, P^h \cup P^{r_k}, N^h, M^h \cup$

M^{r_k}). Otherwise $y \neq x_2$ and we set $q_{\ell+1} = (C^h, P^h, N^h, M^h \cup v \setminus (u \cup P^h))$. In either case all lower priority requirements are injured with the use of $\langle m, y \rangle \in \Psi_{i,\ell}(v(a))$ and we enter state d.

- State d: in this state $\mathcal{R}$ is considered satisfied.

If this strategy ends in state d then we have ensured that Y is multivalued. If it ends in state c or n then we have that $m \notin \operatorname{dom}(Y)$. In either case $\mathcal{R}$ is satisfied via condition 1.

If we remain forever in state w then it might be Y is the graph of a total function. However if this is the case then we again have the following important observation: if Y is the graph of a total function and $(a, q) \not\leq_u (a_t, q_t)$ for all t then it must be that $\Psi_i(q(a)) \subseteq Y$ as otherwise we would have used (a, q) to move into state c at some point. We make repeated use of this in the third strategy to ensure that Y is consistent at every stage when we act. While this first strategy is waiting in state w we will enact the second strategy.

Before we describe the second strategy we introduce some notation. We define $Y_s = \Psi_i(q_s(a_s))$ and $H_s = H^{Y_s}$. Since it is easier to think of a Roy halfgraph set in terms of the function f rather than the formal definition we will think of elements of $\Psi_e(q(a))$ as putting restrictions on f. We say that $q(a) \to f(n) \geq 2m$ if $2\langle n, m \rangle \in \Psi_e(q(a))$ and $q(a) \to f(n) \leq 2m$ if $2\langle n, m \rangle + 1 \in \Psi_e(q(a))$. For an interval $[x, y] \in I$ we define $q(a) \to f(n) \in [x, y]$ if $q(a) \to x \leq f(n) \leq y$ (I is the set of even ended intervals from the proof of Theorem 4.6.9). For a sequence of intervals $\alpha : \omega \to I$ we say $q(a) \to f \in \alpha$ if for all n we have $q(a) \to f(n) \in \alpha(n)$. In a similar way we define $H_s \to f(n) \geq 2m$ and $H_s \to f(n) = 2m$ (we cannot define $H_s \to f(n) \leq 2m$ as the HalfGraph(f) does not give upper bounds on $f(n)$ unless $f(n)$ is even).

We also use notation for implications in the other direction. We say $Y_s, \alpha \to x \in L$ if $\{x\} \oplus \emptyset \oplus \emptyset \subseteq \Psi_j(Y_s \oplus (\{\langle n, m \rangle : 2m \leq \min(\alpha(n))\} \oplus \{\langle n, m \rangle : 2m \geq \max(\alpha(n))\}))$. Similarly we define $Y_s, \alpha \to x \in R$ and $Y_s, \alpha \to x \in Z$.

The steps of the second strategy for $\mathcal{R}$ are as follows:

1. Pick some $x \notin C^{q_s} \cup u$ and set $a_{s+1} = a_s \cup \{x\}, q_{s+1} = q_s$.

2. Wait for a stage s where we see some α_0 such that $Y_s, \alpha_0 \to x \in L$ and $q_s(a_s) \to f \in \alpha_0$. Set $a_{s+1} = a_s \setminus \{x\}, q_{s+1} = q_s$. Injure all lower priority requirements with the use of $Y_s, \alpha_0 \to x \in L$ and $q_s(a_s) \to f \in \alpha_0$.

3. Like in the previous step wait for a stage t such that $Y_s \subseteq Y_t$ and we see some α_1 such that $Y_t, \alpha_1 \to x \in R$ and $q_t(a_t) \to f \in \alpha_1$. Set $a_{s+1} = a_s \cup \{x\}, q_{t+1} = q_t$. Injure all lower priority requirements with the use of $Y_t, \alpha_1 \to x \in R$ and $q_t(a_t) \to f \in \alpha_1$.

4. Let $k = \max\{m : \alpha_i(m) \neq \tilde{\omega}\} + 1$. Now we wait until a stage $s_1 > t$ such that $Y_t \subseteq Y_{s_1}$ and H_{s_1}, f are well behaved up to k. More precisely, for each $n < k$ we want the following:

   - $H_{s_1} \to f(n) = 2m \iff q_{s_1}(a_{s_1}) \to f(n) = 2m$.
   - $H_{s_1} \to f(n) \geq 2m \wedge H_{s_1} \nrightarrow f(n) \geq 2m + 2 \iff q_{s_1}(a_{s_1}) \to 2m \leq f(n) \leq 2m + 2$.
   - $H_{s_1} \nrightarrow f(n) \geq 0 \implies q_{s_1}(a_{s_1}) \to f(n) \leq 0 \vee q_{s_1}(a_{s_1}) \to f(n) \geq \max(8, \min(\alpha_0(n)), \min(\alpha_1(n)))$.

   We use 8 here because the interval $[0, 8]$ can be divided into 4 even ended sub intervals. Set $a_{s_1+1} = a_{s_1} \setminus \{x\}, q_{s_1+1} = q_{s_1}$. Injure all lower priority requirements with the use that witnesses H_{s_1}, f are well behaved.

5. Again search for a $t_1 > s_1$ such that $Y_{s_1} \subseteq Y_{t_1}$ and H_{t_1}, f are well behaved up to k. Set $a_{t_1+1} = a^{t_1} \cup \{x\}, q_{t_1+1} = q_{t_1}$. Injure all lower priority requirements with the use that witnesses H_{t_1}, f are well behaved.

6. Repeat the previous two steps until we have $s_i < t_i$ such that H_{s_i} and H_{t_i} agree up to k. Since k is finite we only have to do this finitely many times. If we have that $q_{t_i}(a_{t_i}) \to f \in \alpha_0 \cap \alpha_1$ then the strategy is finished and we have satisfied condition 3.

   If this is not the case, then there is some $n < k$ such that $H_{s_i}, H_{t_i} \nrightarrow f(n) \geq 0$ and we have $q_{s_i}(a_{s_i}) \to f(n) \leq 0$ and $q_{t_i}(a_{t_i}) \to f(n) \geq 8$ or vica versa. Now set

$a^{t_i+1} = a^{t_i} \setminus \{x\}, C^{q_{t_i+1}} = C^{q_{t_i}} \cup \{x\}$ and leave the rest of q_{t_i+1} unchanged from q_{t_i}. Injure all lower priority requirements.

7. Wait until a stage $r > t_i$ when we see $Y_{t_i} \subseteq Y_r$ and β such that $Y_r, \beta \to x \in Z$ and $q_r(a_r) \to f \in \beta$. Furthermore we want that f and H_r are well behaved upto $\max\{m : \beta(m) \neq \tilde{\omega}\} + 1$.

   Since H_r, f are well behaved on k we have that $q_r(a_r) \to f(n) \geq 4$ or $q_r(a_r) \to f(n) \leq 4$ for the n from the previous step. We will assume that $q_r(a_r) \to f(n) \geq 4$ and $q_{s_i}(a_{s_i}) \to f(n) \leq 0$ and give the steps for this case. The cases where $q_r(a_r) \to f(n) \leq 4$ or $q_{t_i}(a_{t_i}) \to f(n) \leq 0$ are similar.

   In this case we add x to $N_0^{q_{r+1}}$, injure lower priority requirements and proceed to the next step.

8. We now wait for a stage $v > r$ where we see one of $q_v(a_v) \to f(n) \geq 2$ or $q_v(a_v) \to f(n) \leq 2$. If $q_v(a_v) \to f(n) \geq 2$ then we add x to $P_0^{q_{v+1}}$. If $q_v(a_v) \to f(n) \leq 2$ then we add x to $M^{q_{v+1}}$. In either case we have ensured $q_v(a_v) \to f(n) \in \emptyset$ so we have met condition 2.

If the strategy finishes, then as explained we meet either 2 or 3. If the strategy waits forever at one of the steps then $\mathcal{R}$ is also met. If we are waiting because we never see a $t > s$ where $Y_s \subseteq Y_t$ then either Y is not total or we have an opportunity to use the tools from the proof of Theorem 4.6.7. Either way we meet condition 1.

If we are waiting forever at a step for some other reason then one of the other conditions will be met. In the case of steps 2 and 3 we meet condition 3 and in the case of steps 4,5, 6 and 8 we meet condition 2. For step 7 will be condition 3 if we never see $Y_r, \beta \to x \in Z$ and condition 2 if we never see that H_r, f are well behaved.

$\square$

4.8 Metrizable classes and degrees

4.8.1 The doubled co-d-c.e. degrees

In Section 4.5.2 we showed that the doubled co-d-CEA degrees are not $T_{2.5}$ and in Section 4.6 we show that the Arens co-d-CEA degrees and the Roy halfgraph degrees are both not submetrizable. A natural question to ask is if these results can be improved to quasi-minimal separations. In this section we give a negative answer to that question. This result and others come from the following theorem.

Theorem 4.8.1. *For each Y there is a metrizable* cb_0 *space $\mathcal{X}_Y$ such that*

$$\mathcal{D}_{\mathcal{X}_Y} = \{\mathbf{a} : \mathbf{a} \textit{ is doubled co-d-c.e. in } Y\}$$

In fact $\mathcal{X}_Y$ is homeomorphic to $\omega \times 2^\omega$.

Proof. Fix a total set Y and Y-c.e. sets C, P, N such that $P \cap N = \emptyset$ and $P, N \subseteq C$. Let $X = \{f : C^c \cup P \cup N \to 2 : f[P] = \{0\}, f[N] = \{1\}\}$. We give a subbasis $(\beta_n)_n$ of $\mathcal{X}$ as:

- $f \in \beta_{3n}$ if $f(n) = 0$,
- $f \in \beta_{3n+1}$ if $f(n) = 1$ and
- $f \in \beta_{3n+2}$ if $n \in Y$.

If C^c is infinite then $\mathcal{X} \cong 2^\omega$ otherwise X is finite. We have that $\mathcal{D}_{\mathcal{X}} = \{\mathbf{a} : \exists A \sqcup B = C^c[Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N) \in \mathbf{a}]\}$. By taking the disjoint union over all Y-c.e. sets C, P, N where C^c is infinite we get the desired $\mathcal{X}_Y$. $\square$

Corollary 4.8.2. *There is a second countable metric space $\mathcal{X}$ such that for all computably submetrizable spaces $\mathcal{Y}$ we have $\mathcal{D}_{\mathcal{X}} \not\subseteq \mathcal{D}_{\mathcal{Y}}$.*

Corollary 4.8.3. *There is a second countable metric space $\mathcal{X}$ such that the quasi-minimal degrees in $\mathcal{D}_{\mathcal{X}}$ are the exactly the quasi-minimal doubled co-d-CEA degrees.*

Corollary 4.8.2 tells us that a lot of complexity can be coded into a non-computable basis of a metrizable space, but it does not tell much about submetrizable classes that are not metrizable. We know there are effectively submetrizable classes that are not metrizable, for instance the co-d-CEA degrees. This follows from the fact that there are quasi-minimal co-d-CEA degrees relative to any oracle. However, since the co-d-CEA degrees are contained in the doubled co-d-CEA degrees, Corollary 4.8.3 tells us that this is not a quasi-minimal separation. In fact every effectively submetrizable space $\mathcal{X}$ can have only countably many quasi-minimal degrees since points representing these degrees get mapped to computable points under the continuous injection $f : \mathcal{X} \to [0,1]^\omega$. Because it is possible to encode any countable set of degrees into the basis of a metric space we have the following result.

Proposition 4.8.4. *There is no effectively submetrizable class of degrees $\mathcal{C}$ that is metrizable quasi-minimal.*

However, if we drop the effective requirement then this becomes an open question.

Question 4.8.5. Is the a submetrizable class of degrees $\mathcal{C}$ that is metrizable quasi-minimal?

Corollary 4.8.3 means that there is no hope to separate the classes of degrees of a $T_{2.5}$ space from the classes of degrees of an arbitrary second countable submetrizable space using the notion of $\mathcal{T}$ quasi-minimal with the given examples of $T_{2.5}$ spaces. If there is such a separation we will need to look at new spaces.

Question 4.8.6. Is there a (decidable) $T_{2.5}$ class of degrees that is submetrizable quasi-minimal?

While the doubled co-d-CEA degrees are not metrizable quasi-minimal it may still be possible to get quasi-minimal separations by adding a computability constraint. In this vein, we have the following questions.

Question 4.8.7. For every decidable $T_{2.5}$ space $\mathcal{X}$ is there a doubled co-d-CEA degree that is $\mathcal{X}$ quasi-minimal?

Question 4.8.8. For every effectively submetrizable space $\mathcal{X}$ is there an Arens co-d-CEA degree or Roy halfgraph degree that is $\mathcal{X}$ quasi-minimal?

Since an effectively submetrizable space can have at most countably many quasi-minimal degrees and any countable collection of enumeration degrees can be encoded into an effectively submetrizable space the previous question is equivalent to the following.

Question 4.8.9. Are there uncountably many quasi-minimal Arens co-d-CEA degrees or quasi-minimal Roy halfgraph degrees?

Now we explore the spaces from Theorem 4.8.1 a little more. By combining them all together in the right way we can get a new cb_0 space that represents all doubled co-d-CEA degrees.

Definition 4.8.10. We define the doubled co-d-CEA space ($\mathcal{DCD}$) as follows. For each triple of c.e. functionals C, P, N with $P^Y, N^Y \subseteq C^Y, P^Y \cap N^Y = \emptyset$ define the set $\mathcal{X}_{C,P,N} = \{(Y, f) : Y \in 2^\omega, f : (C^Y)^c \cup P^Y \cup N^Y, f[P^Y] = 1, f[N^Y] = 0\}$. The subbasis $(\beta_e)_e$ of $\mathcal{X}_{C,P,N}$ is coded by pairs $\langle \sigma, m\rangle$, $\sigma \in 2^{<\omega}, m \in \omega$ with $\beta_{\langle \sigma, 2n\rangle} = \{(Y, f) : \sigma \prec Y, (n, 0) \in f\}$ and $\beta_{\langle \sigma, 2n+1\rangle} = \{(Y, f) : \sigma \prec Y, (n, 1) \in f\}$.

Let $\Gamma_e = (\Gamma_{e,2}, \Gamma_{e,1}, \Gamma_{e,0})$ be a effective listing of all valid triples of functionals. We define $\mathcal{DCD}$ to be $\bigsqcup_e \mathcal{X}_{\Gamma_e}$. The subbasis of $\mathcal{DCD}$ is given by $\beta_{\langle e,\sigma,m\rangle}$ where $\beta_{\langle e,\sigma,m\rangle}$ is the open set $\beta_{\langle \sigma,m\rangle}$ in X_{Γ_e}.

Theorem 4.8.11. *$\mathcal{D}_{\mathcal{DCD}}$ is the class of doubled co-d-CEA degrees.*

Proof. Consider a point $(Y, f) \in \mathcal{X}_{\Gamma_e}$. Let $(C, P, N) = \Gamma_e$. We have that

$$\mathrm{NBase}_{\mathcal{DCD}}(Y, f) = \{\langle e, \sigma, 2n\rangle : \sigma \prec Y, f(n) = 0\} \cup \{\langle e, \sigma, 2n + 1\rangle : \sigma \prec Y, f(n) = 1\}$$

Let $A = f^{-1}[\{1\}] \setminus C^Y, B = f^{-1}[\{0\}] \setminus C^Y$. Let $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$. So we have that X is doubled co-d-CEA and $\mathrm{NBase}_{\mathcal{DCD}}(Y, f) \equiv_e X$.

Now consider some doubled co-d-CEA set $X = Y \oplus Y^c \oplus (A \cup P) \oplus (B \cup N)$. Let e be such that $\Gamma_{e,2}(Y) = (A \cup B)^c, \Gamma_{e,1}(Y) = P, \Gamma_{e,0}(Y) = N$. Let $f : A \cup B \cup P \cup N \to 2$ be given by $f(n) = 1$ if $n \in A \cup P$, $f(n) = 0$ if $n \in B \cup N$. Then we have that $(Y, f) \in \mathcal{X}_{\Gamma_e}$. Then just like above we have that $\mathrm{NBase}_{\mathcal{DCD}}(Y, f) \equiv_e X$. $\square$

So we have a different space that represents the doubled co-d-CEA degrees. It is less natural than the double origin topology, but more explicitly represents these degrees.

Theorem 4.8.12. *$\mathcal{DCD}$ is $T_2 \setminus T_{2.5}$.*

Proof. Since $\mathcal{DCD}$ gives us the doubled co-d-CEA degrees we know that it cannot be $T_{2.5}$. So now we need to show that the space is T_2.

Fix an e and consider $\mathcal{X}_{C,P,N}$ for $(C, P, N) = \Gamma_e$. Consider two distinct points $(Y_0, f_0), (Y_1, f_1) \in \mathcal{X}_{C,P,N}$. If $Y_0 \neq Y_1$ then there is $\sigma_0 \prec Y_0$ and $\sigma_1 \prec Y_1$ such that $\sigma_0 \perp \sigma_1$. Consider the open sets $V_0 = \bigcup_m \beta_{\langle \sigma_0, m \rangle}$ and $V_1 = \bigcup_m \beta_{\langle \sigma_1, m \rangle}$. Since $\sigma_i \prec Y_i$ we have $Y_i \in V_i$. For any $(Y, f) \in V_0$, $\sigma_0 \prec Y$ so $\sigma_1 \not\prec Y$. Hence $(Y, f) \notin V_1$, so V_0 and V_1 are disjoint.

Now suppose that $Y_0 = Y_1$. So $f_0 \neq f_1$ and there is n such that $f_0(n) \neq f_1(n)$. So we have $(Y_i, f_i) \in \beta_{\langle \langle\rangle, 2n+f_i(n) \rangle}$. If $(Y, f) \in \beta_{\langle \langle\rangle, 2n+f_0(n) \rangle}$ then $f(n) = f_0(n) \neq f_1(n)$ so $(Y, f) \notin \beta_{\langle \langle\rangle, 2n+f_1(n) \rangle}$. Hence $\beta_{\langle \langle\rangle, 2n+f_0(n) \rangle}$ and $\beta_{\langle \langle\rangle, 2n+f_0(n) \rangle}$ are disjoint.

$\square$

4.8.2 Decidable, metrizable degrees

We know that any enumeration degree can be realized in a decidable, effectively submetrizable cb_0 space. A natural question to ask is the following.

Question 4.8.13. What is the class of degrees $\mathbf{a}$ such that $\mathbf{a} \in \mathcal{D}_{\mathcal{X}}$ for some decidable, metrizable cb_0 space $\mathcal{X}$?

We know that this class includes all continuous degrees, since $[0, 1]^\omega$ is decidable with the usual basis. Theorem 4.8.14 below shows that this class contains a quasi-minimal, and hence non continuous, degree. So this class is strictly larger than the class of the continuous degrees. It remains open whether there are any enumeration degrees that do not belong to this class.

Theorem 4.8.14. *There is a decidable, metrizable* cb_0 *space* $\mathcal{X}$ *such that* $\mathcal{D}_X$ *contains a quasi-minimal degree.*

Proof. The metric space $\mathcal{X} = (X, d)$ we will construct will be $X = \omega \times \omega \cup \{\infty\}$ with the metric given by $d((a,n),(b,m)) = 2^{-\min(n,m)}$ if $(a,n) \neq (a,m)$ and $d((a,n),\infty) = 2^{-n}$. So $\omega \times \omega$ has the discrete topology and ∞ is the limit of all sequences where the second coordinate is increasing. The basis we will use is given by $\beta_{2\langle a,n\rangle} = \{(a,n)\}$ and

$$\beta_{2n+1} = \{(a,m) : m \geq n \wedge (\forall k \leq n)(p_k \nmid a \vee n \notin B_k \vee m \in B_n)\} \cup \{\infty : B_n \text{ is cofinite}\}$$

where $(p_n)_n$ is the sequence of primes and $(B_n)_n$ are uniformly computable sets that we will build by finite injury. There will be infinitely many n such that B_n is cofinite, hence for each n there is $m > n$ such that $\infty \in \beta_{2m+1} \subseteq B(\infty, 2^{-n})$. So $(\beta_n)_n$ is a basis of $\mathcal{X}$.

To ensure that $(X, (\beta_e)_e)$ is decidable we will ensure that $n \in B_n \subseteq \omega \setminus n$ and we will ensure for all $m < n$ if $n \in B_m$ then $B_n \subseteq B_m$. This means that if $n \in B_m$ then $\beta_{2n+1} \subseteq \beta_{2m+1}$. To show that the $\subseteq$ relation on positive Boolean combinations is computable it is enough to look at questions of the form $\bigcap_{i<k} \beta_{e_i} \subseteq \bigcup_{j<k'} \beta_{d_j}$. If some e_i is even then, since the B_n are uniformly computable we can answer the question computably. So we can assume that all e_i are odd. Since $|\bigcap_{i<k} \beta_{e_i}|$ is either 0 or ω we can assume all d_j are odd. Let $e = \min\{e_i : i < k\}$. Let $e_i = 2r_i + 1$ and $d_j = 2v_j + 1$. If there is i, j such that $r_i \in B_{v_j}$ then it is true that $\bigcap_{i<k} \beta_{e_i} \subseteq \bigcup_{j<k'} \beta_{d_j}$ as $\beta_{e_i} \subseteq \beta_{d_j}$. If this is not the case then let $r = \max\{r_i : i < k\}$ and consider $p = \prod_{j:v_j \leq r} p_j$. Since $r \notin B_{v_j}$ for any j we have that $(p, r) \notin \beta_{d_j}$. On the other hand since $r_i \notin B_{v_j}$ for any i, j we have that $(p, r) \in \beta_{e_i}$ for each $i < k$.

Now we move on to the construction of $(B_n)_n$. We define $A_s = \{n : s \in B_n\}$, which means that $B_n = \{s : n \in A_s\}$. We will build A_s in stages and have $A = \lim_s A_s$. Note that $\emptyset \oplus A = \mathrm{NBase}(\infty)$ so we want to ensure that A is quasi-minimal. The requirements are $\mathcal{N}_e : A \neq W_e$ and $\mathcal{R}_e : \Psi_e(A)$ is the graph of a total function $\implies \Psi_e(A) \leq_e \emptyset$. Each requirement will be given a restriction $u \leq s$ by higher priority requirements and will not be allowed to change the value of $A{\upharpoonright}u$.

The strategy for an $\mathcal{N}_e$ requirement is as follows. If $\mathcal{N}_e$ is initialized at stage s then we will use s as a witness and give lower priority requirements restriction $s+1$ and will not change A. By definition we must have $s \in A_s$. We wait until a stage $t > s$ where we see $s \in W_e$. Then we set $A_{t+1} = (A_s \setminus \{s\}) \cup \{t+1\}$ and give all lower priority requirements restriction t. The strategy is finished after this step.

The strategy for an $\mathcal{R}_e$ requirement is as follows. If $\mathcal{R}_e$ is initialized at stage s then we give lower priority requirements restriction s. Now we wait until a stage t where we see a pair $\langle x, y\rangle, \langle x, z\rangle \in \Psi_{e,t}(A_s \cup [s,t])$ with $y \neq z$. When we see such a pair we set $A_{t+1} = A_s \cup [s, t+1]$ and give all lower priority requirements restriction t. The strategy is now finished. Since $A{\restriction}s$ has not changed between stages s and t defining A_{t+1} as above will not violate the requirement that $B_n \subseteq B_m$ if $n \in B_m$.

The $\mathcal{N}_e$ requirement ensures that $A \neq W_e$. The $\mathcal{R}_e$ requirement ensures that if $\Psi_e(A)$ is the graph of a total function then $\Psi_e(A) = \Psi_e(A_s \cup [s, \infty))$ for some s and is hence computable. So A has quasi-minimal degree and hence is $\mathcal{D}_{(X,(\beta_e)_e)} = \{\mathbf{0}, \deg_e(A)\}$ contains a quasi-minimal degree. $\square$

Chapter 5

E-pointed trees

5.1 Introduction

In this chapter we look at e-pointed trees. The work in this chapter was done in collaboration with Jun Le Goh, Joseph Miller and Mariya Soskova and can be found in [14].

E-pointed trees were studied by McCarthy [32] and have been used in computable model theory [35].

Definition 5.1.1. A tree T is *e-pointed* if for every path $P \in [T]$ we have that $T \leq_e P$. We say T is *uniformly* e-pointed if there is a single enumeration operator Ψ_e such that for all paths $P \in [T]$ we have $T = \Psi_e(P)$.

McCarthy [32] studied e-pointed trees on 2^ω and characterized their enumeration degrees as the cototal degrees. In this chapter we focus on e-pointed trees on ω^ω, henceforth referred to as *Baire* e-pointed trees. It turns out these have an interesting relationship with hyperenumeration reducibility (introduced by Sanchis [39]). We show in Section 5.3 that they characterize the hypercototal degrees. In Chapter 6 we make use of them to prove that Selman's theorem does not hold in the hyperenumeration degrees.

McCarthy [32] showed that in the case of Cantor e-pointed trees, allowing the trees to have dead ends does not change enumeration degrees represented by these trees. In the case of Baire e-pointed trees, we prove in Section 5.4 that the class of enumeration degrees of Baire e-pointed trees without dead ends is a strictly smaller class than the class of the degrees of Baire e-pointed trees with dead ends.

Also of interest to us in this chapter are introenumerable sets:

Definition 5.1.2. A set A is *introenumerable* if for all infinite $S \subseteq A$, $A \leq_e S$. A set A is *uniformly* introenumerable if there is an enumeration operator, Ψ_e, such that $A = \Psi_e(S)$ for all infinite $S \subseteq A$.

We study these in Section 5.5. It turns out that the class of introenumerable degrees, lies strictly between the cototal degrees and the degrees of Baire e-pointed trees without dead ends.

We end this chapter with a brief overview of the topological classification of these classes, using the notions from Chapter 4.

5.2 Hyperenumeration reducibility

Since this is the first chapter to make use of hyperenumeration reducibility we will use this section to discuss some concepts that will be useful for the rest of this thesis.

We defined enumeration reducibility in terms of operators $(\Psi_e)_e$. Using Definition 1.6.1 we can define hyperenumeration operators $(\Gamma_e)_e$.

Definition 5.2.1. For the eth c.e. set W_e we define the hyperenumeration operators Γ_e by $n \in \Gamma_e(A) \iff \forall f \in \omega^\omega \exists \sigma \preceq f, u \in \omega[\langle n, \sigma, u\rangle \in W_e \wedge D_u \subseteq A]$.

Now we examine the relationship between Γ_e and Ψ_e. Both use the same set W_e in their definition. Consider the tree S_e^A defined by $n\widehat{\ }\sigma \notin S_e^A \iff \exists \tau \preceq \sigma, u \leq |\sigma|[\langle n, \tau, u\rangle \in W_{e,|\sigma|} \wedge D_u \subseteq A]\}$. From the definition of Γ_e, we have that $n \in \Gamma_e(A)$ if and only if S_e^A does not have an infinite path starting with n. We have that $S_e^A \leq_e \overline{A}$ and $\overline{S_e^A} \leq_e A$.

While the notation is a little different, Sanchis [39] used a similar idea when proving the existence of a non-hypertotal degree. The form of S_e^A inspires us to come up with the notion of a hyperenumeration of a set.

Definition 5.2.2. We say that a tree S is a *hyperenumeration* of a set B if $B = \{n : \forall f \in [S](f(0) \neq n\}$.

From this we have that $B \leq_{he} \overline{S}$ via the same operator for every hyperenumeration S of B. By coding a set X into a layer of S_e^X, we have that for every X such that B is Π^1_1 in X, there is a hyperenumeration S of B such that $S \equiv_T X$. So the hyperenumerations of B characterize the hypertotal degrees above $\deg_{he}(B)$ much like how the enumerations of B characterize the total e-degrees above the $\deg_e(B)$. Recall the definition of hypertotal and hypercototal:

Definition 5.2.3. We say that a set A is *hypertotal* if $\overline{A} \leq_{he} A$. We say that A is *hypercototal* if $A \leq_{he} \overline{A}$. A degree is *hypertotal* (*hypercototal*) if it contains a hypertotal (hypercototal) set.

Sanchis proved some other results about hyperenumeration reducibility that we will use in this thesis:

Lemma 5.2.4 (Sanchis [39]). *For sets $A, B \subseteq \omega$ we have the following:*

1. *If there is a Π^1_1 set V such that $n \in A \iff \forall f \in \omega^\omega \exists \sigma \preceq f, u \in \omega[m^\frown\langle n, \sigma, u\rangle \in V \wedge D_u \subseteq B]$ then $A \leq_{he} B$.*
2. *If $A \leq_e B$ then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$.*

5.3 Baire e-pointed trees with dead ends

In this section we give a characterization of the enumeration degrees of e-pointed trees with dead ends on ω^ω as precisely the hypercototal degrees.

Theorem 5.3.1 (Goh, Jacobsen-Grocott, Miller and Soskova [14]). *For a degree $\mathbf{a} \in \mathcal{D}_e$ the following are equivalent.*

1. $\mathbf{a}$ *contains a Baire e-pointed tree (with dead ends).*
2. $\mathbf{a}$ *contains a uniformly Baire e-pointed tree (with dead ends).*

3. **a** *contains a hypercototal set.*

4. Every $A \in \mathbf{a}$ *is hypercototal.*

Proof. It is clear that $2 \implies 1$ and $4 \implies 3$.

$3 \implies 2$: consider a set $A \leq_{he} \overline{A}$ via some hyperenumeration operator Γ_e. This gives us a hyperenumeration $S_e^{\overline{A}}$ of A with $S_e^{\overline{A}} \leq_e A$. We use this to define a tree $T = \{\bigoplus_{n<m} i_n\widehat{\ }\sigma_n : \forall n < m[(i_n = 0 \wedge n\widehat{\ }\sigma_n \in S_e^{\overline{A}}) \vee (i_n = 1 \wedge n \in A)]\}$. Essentially a string $\sigma \in T$ breaks up into a finite join of strings $\oplus_{n<m} i_n\widehat{\ }\sigma_n$ where $\sigma(\langle n, k\rangle) = (i_n\widehat{\ }\sigma_n)(k)$. So for a path $p \in [T]$ we have $p = \bigoplus_n i_n\widehat{\ }p_n$. If $i_n = 0$ then we have $n\widehat{\ }p_n \in S_e^{\overline{A}}$ so $n \notin A$. If $i_n = 1$ then $n \in A$, so we have that $A \leq_e p$ uniformly. We have that $T \leq_e S_e^{\overline{A}} \oplus A \leq_e A$ so by composing enumeration operators we can see that T is uniformly Baire e-pointed and that $T \leq_e A$. We have that $n \in A$ if and only if there is $\sigma \in T$ such that $\sigma(\langle n, 0\rangle) = i_n = 1$ and hence $A \leq_e T$.

$1 \implies 4$: This proof follows the equivalent one by McCarthy [32] for e-pointed trees on Cantor space. Consider a Baire e-pointed tree T. We will build $S \subseteq T$ such that every path in S uniformly enumerates T and $S \leq_e T$. We build S in stages, by attempting to build a path in T that does not enumerate T. This process will fail, at which point we will have found our S.

We start with $S_0 = T$. At stage s we ask if there is a $\sigma \in S_s$ such that $\Psi_s(\sigma) \nsubseteq T$ and σ can be extended to an infinite path in S_s. If yes, then we define $S_{s+1} = \{\tau \in S_s : \tau \nmid \sigma\}$. If no, then we ask if there is $\sigma \in T$ such that the tree $\{\tau \in S_s : \sigma \notin \Psi_s(\tau)\}$ is illfounded. If yes then we take this tree as S_{s+1}. If no, then every path in S_s uniformly enumerates T with witness Ψ_s.

If this process does not stop, then we have a path $p = \bigcap_s S_s$ such that $\Psi_e(p) \neq T$ for e and thus T is not uniformly Baire e-pointed.

Since at each stage of the construction we have $S_{s+1} \leq_e S_s$ we get that $S \leq_e T$ as desired. Now we show how to use this tree to prove that every set in $\deg_e(T)$ is hypercototal. Suppose that $T \equiv_e A$. Then $S \leq_e A$ so by Lemma 5.2.4 we have that $\overline{S} \leq_{he} \overline{A}$. Observe that $\sigma \in T \iff \forall f \in \omega^\omega \exists \tau \preceq f[\tau \notin S \vee \sigma \in \Psi(\tau)]$ where Ψ is the witness that every path in S uniformly enumerates T. This shows that $T \leq_{he} \overline{S}$ and hence $A \leq_e T \leq_{he} \overline{S} \leq_{he} \overline{A}$. □

5.4 Baire e-pointed trees without dead ends

In this section consider e-pointed trees without dead ends. Notably it is this type of tree that Montalbán uses in [35]. It is also this kind of tree that we use in Chapter 6 to disprove Selman's theorem for hyperenumeration reducibility.

We observe that the uniformly Baire e-pointed tree we constructed in the proof of Theorem 5.3.1 contained many dead ends. This is unavoidable as the following shows:

Theorem 5.4.1 (Goh, Jacobsen-Grocott, Miller and Soskova [14]). *There is a Δ^0_3 set A that is not enumeration equivalent to any Baire e-pointed tree without dead ends.*

Proof. We build A in initial segments with $A = \bigcup_s a_s^+$ for $a_s \in 2^{<\omega}$ (here $a^+ = \{n : a(n) = 1\}$). For each enumeration operator Ψ_e we build a sequence $(p^e_n)_n$ such that if $\Psi_e(A)$ is a tree without dead ends then $p^e = \bigcup_n p^e_n$ is a path in that tree and for all $i \in \omega$, $\Psi_i(p^e) \neq A$. So either $\Psi_e(A)$ is not e-pointed or it is not equivalent to A. As part of ensuring this, we will also build an increasing sequence of forbidden sets $(F^e_n)_n$ with $\bigcup_n F^e_n \subsetneq A$.

The construction starts with $a_0 = p^e_0 = F^e_0 = \emptyset$.

- At sage $s = \langle e, 0\rangle$ we ask if there is a string $a \succeq a_s$ and string $p \in \Psi_e(a^+)$ such that for all $b \succeq a$ we have that p is a dead end in $\Psi_e(b^+)$. If yes then we take $a_{s+1} = a$ and we don't need to build p^e. If no, then we set $a_{s+1} = a_s$ and we will build p^e at future stages; to do this we will use the forbidden sets F^e_n.

- At stage $s = \langle e, n+1\rangle$ we build p^e_{n+1} and F^e_{n+1}. Given a_s, p^e_n and F^e_n we ask if there is $a \succeq a_s{}^\frown 0$ and $p \in \Psi_e(a^+ \setminus F^e_n)$ such that $|a_s| \in \Psi_n(p)$ and $p \succeq p^e_n$. If yes, then we set $p^e_{n+1} = p$, $F^e_{n+1} = F^e_n$ and $a_{s+1} = a$ and move to the next stage. If no, then we set $p^e_{n+1} = p^e_n$,$F^e_{n+1} = F^e_n \cup \{|a_s|\}$ and $a_{s+1} = a_s{}^\frown 1$.

Now we verify that the construction works. Suppose that towards a contradiction that A is equivalent to some Baire e-pointed tree without dead ends. Then there is some e

such that $\Psi_e(A)$ is this tree. If at stage $s = \langle e, 0\rangle$ we were able to find the extension a and finite path p that we were searching for, then we have a contradiction, as p must have some extension $p' \in \Psi_e(A)$, but then $p' \in \Psi_e(a_t)$ for some t. So we can conclude that the construction of p^e and F^e took place.

In a similar way we can see that we would always be able to find extensions of p^e_n in $\Psi_e(a_s^+ \setminus F^e_n)$. So (as there are infinitely many operators Ψ_n with $d \in \Psi_n(p) \iff |p| \geq n$) we can see that p^e must be an infinite sequence.

Since p^e is infinite, it can enumerate $\Psi_e(A)$, and $\Psi_e(A) \geq_e A$ by our assumption, so there must be n such that $\Psi_n(p^e) = A$. Now consider the question we asked at stage $s = \langle e, n\rangle$. If there was such an a and p, then $p \preceq p^e$ and $a \preceq A$ but $|a_s| \in \Psi_n(p^e) \setminus A$, a contradiction. If there was no such a and p, then $|a_s| \in A \cap F^e_{n+1}$. So $|a_s| \in \Psi_n(p^e)$, hence there is $m > n, t > s$ such that $|a_s| \in \Psi_n(p^e_m)$ and $p^e_m \in \Psi_e(a_t^+ \setminus F^e_m)$. But then if we take $a = a_s{}^\frown 0{}^\frown 1^{|a_t|}$ we get $p^e_m \in \Psi_n(a^+ \setminus F^e_n)$, a contradiction of there being no such a and p.

Hence we can conclude that $\Psi_n(p) \neq A$ for any n, and so A is not equivalent to any Baire e-pointed tree without dead ends. To see that A is Δ^0_3 we observe that each question we ask in the construction consists of only two quantifiers, so could be answered by $\mathbf{0}''$.

$\square$

Since every Δ^0_3 set is a Π^1_1 set, and hence hyperenumeration equivalent to $\emptyset$, we get the following:

Corollary 5.4.2. *There is a hypercototal enumeration degree that does not contain any Baire e-pointed trees without dead ends.*

This distinguishes the Baire e-pointed trees with dead ends from those without. In the question of Baire e-pointed trees without dead ends it is an open question if requiring uniformity has an effect on the degrees.

Question 5.4.3. Is there a Baire e-pointed tree without dead ends that is not equivalent to any uniformly Baire e-pointed tree without dead ends.

5.5 Introenumerable sets

In this section we look at introenumerable sets and their relationship to e-pointed trees on 2^ω and ω^ω.

Proposition 5.5.1 (Goh, Jacobsen-Grocott, Miller and Soskova [14])**.** *Every cototal enumeration degree contains a uniformly introenumerable set.*

Proof. If $\mathbf{a}$ is a cototal degree then $\mathbf{a}$ contains a uniformly Cantor e-pointed tree T without dead ends [32]. Let Ψ be such that $\Psi(p) = T$ for all infinite $p \in [T]$. Consider an infinite $S \subseteq T$. Consider the set $A = \{\sigma : \exists \tau \in S[\sigma \in \Psi(\tau)]\}$. It is clear that $A \leq_e S$. By König's Lemma, we know that there must some path $p \in [T]$ such that S contains arbitrarily large initial segments of p. This means that $T \subseteq A$ as $T = \Psi(p)$. On the other hand, since T has no dead ends and $\Psi(p) = T$ for all $p \in T$ it must be that $A \subseteq T$. So $A = T$.

Since the definition of A is uniform, T is uniformily introenumerable. □

For the case of Baire e-pointed trees we have the following.

Proposition 5.5.2 (Goh, Jacobsen-Grocott, Miller and Soskova [14])**.** *For a set A*

- *If A is introenumerable then $A \equiv_e T$ for some Baire e-pointed tree T without dead ends.*
- *If A is uniformly introenumerable then $A \equiv_e T$ for some uniformly Baire e-pointed tree T without dead ends.*

Proof. Fix A. Consider the tree $T = \{\sigma \in \omega^{<\omega} : \text{range}(\sigma) \subseteq A \wedge \sigma \text{ is strictly increasing}\}$. So $A \equiv_e T$ and if A is infinite, then T does not have any dead ends. For any path $p \in [T]$ we have range(p) is an infinite subset of A. So if A is (uniformly) introenumerable then every path in T (uniformly) enumerates A and hence T is a (uniformly) Baire e-pointed tree without dead ends. □

This completes the known implications regarding introenumerable degrees. Now we move on to showing that these implications are strict.

Theorem 5.5.3 (Goh, Jacobsen-Grocott, Miller and Soskova [14]). *There is a uniformly introenumerable set that is not of cototal enumeration degree.*

Before we begin the construction of this set we will specify the operator via which it will be uniformly introenumerable.

Lemma 5.5.4. *There is a computable $f\colon \mathcal{P}_{\text{fin}}(\omega) \to \mathcal{P}_{\text{fin}}(\omega)$*

1. *$f(\emptyset) = \emptyset$.*
2. *For each finite set $D \subseteq \omega$ and function $g : \mathcal{P}(D) \to \mathcal{P}_{\text{fin}}(\omega)$ there are infinitely many $x \in \omega$ such that $(\forall E \subseteq D)\ f(E \cup \{x\}) = g(E)$.*

Proof. Take an effective enumeration $\{(D_n, g_n, k_n)\}_n$ of tuples with $D_n \subseteq_{\text{fin}} \omega$, $g_n : \mathcal{P}(D_n) \to \mathcal{P}_{\text{fin}}(\omega)$ and $k_n \in \omega$. We define f by recursion.

$f(\emptyset) = \emptyset$. Next assume that $n > \max D$ for some $D \in \text{dom}(f)$ We let

$$f(D \cup \{x\}) = \begin{cases} g_n(D) & D \subseteq D_n, \\ \emptyset & \text{otherwise} \end{cases}$$

It is clear that 1 holds. For 2 fix a D and $g : \mathcal{P}(D) \to \mathcal{P}_{\text{fin}}(\omega)$. Observe that there are infinitely many n such that $D_n = D$ and $g_n = g$. If $n > \max(D)$ then $f(E \cup \{x\}) = g(E)$ for each $E \subseteq D$. □

To define our witness enumeration operator Ψ we take f as in Lemma 5.5.4 and define

$$\Psi(A) = \bigcup_{D \subseteq_{\text{fin}} A} f(D).$$

Now that we have our operator, the next step is to define the forcing partial order we use in this construction. For this we will use an order like structure $\mathcal{N} = \omega \cdot 2 \cup \{\propto\}$ with the ordering $0 < 1 < \cdots < \propto < \propto < \omega < \omega + 1 \ldots$. The idea with $\propto$ is that it represents an arbitrarily large finite number that has not yet been specified, hence we want that $\propto < \propto$. A condition p is a tuple $(G^p, B_0^p, \ldots, B_k^p, L^p)$ that satisfies the following properties:

1. $G^p \sqcup B_0^p \sqcup \cdots \sqcup B_k^p$ is a partition of some finite $A \subseteq \omega$.

2. $L^p : A \times \mathcal{P}(A) \to \mathcal{N}$.

3. For all $D \subseteq A, m \in \omega$, we have $L(m, D) = 0 \iff m \in \Psi(D)$.

4. If $E \subsetneq D \subseteq A$ and $m \in A$, then either $L^p(m, D) < L^p(m, E)$ or $L^p(m, D) = L^p(m, E) = 0$.

5. If $L(m, D) = \propto$, then for some $j \leq k$ we have $D \subseteq G \cup \bigcup_{i>j} B_i^p$ and $m \in B_j^p$.

6. If we have $D \subseteq G \cup \bigcup_{i>j} B_i^p$ and $m \in B_j^p$ then $L(m, D) \geq \propto$.

7. If $L(m, E \cup D) < \omega$ and $\forall n \in E(L(n, D) < \omega)$, then $L(m, D) < \omega$.

A condition p *extends* q, if $G^p \supseteq G^q, L^p \supseteq L^q$ and $B_0^p, \ldots, B_k^p \succeq B_0^q, \ldots, B_{k'}^q$.

The idea here is that G represents the numbers that will become part of our introenumerable set and $\bigcup_{i \leq k} B_i$ represents numbers that will not become part of our introenumerable set. The labeling function $L^p(n, D)$ tells us how many numbers we have to add to D before Ψ enumerates n. In our construction it is important to have the ability to add B_k^p to G and end up with a valid condition. This is why the labeling extends to the B_i^p. However we cannot allow Ψ to enumerate anything in B_i^p, hence the need to add $\propto$ to $\mathcal{N}$. Note that by 3 and 6 we have that $G \supseteq \Psi(G)$.

For any generic filter $\mathcal{G}$ we define $I^{\mathcal{G}} = \bigcup_{p \in \mathcal{G}} G^p$. We will show that for any sufficiently generic $\mathcal{G}$, $I^{\mathcal{G}}$ is uniformly introenumerable and not of cototal degree. Since $\Psi(\emptyset) = \emptyset$ we observe that there is at least one condition, $(\emptyset, \emptyset, \emptyset)$. Now we give two ways of extending conditions.

Definition 5.5.5. For a condition p and number $n \in \omega$ we define $q = p[+n]$ as follows:

- $G^q = G^p \cup \{n\}$.

- $B_i^q = B_i^p$ for $i \leq k$.

- $$L^q(m,D) = \begin{cases} \omega + |A_{k+1}| - |D| & n = m \\ L^p(m,D) & n \notin D \cup \{m\} \\ \propto & n \in D \wedge \exists j[m \in B^q_j \wedge D \subseteq A_j] \\ 0 & \text{otherwise} \end{cases}$$

  where $A_j = G^q \cup \bigcup_{i>j} B^q_i$.

Lemma 5.5.6. *For each p there are infinitely many n such that $p[+n]$ is a valid condition and $p[+n] < p$.*

Proof. We use the same meaning for A_j as above. For $m \in A$ and $D \subseteq A$, we say that m is *left of* D if there is some j such that $D \subseteq A_j$ and $m \in B_j$. Using this we define the function $g : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$g(D) = \{m \in A \colon m \text{ is } \textit{not} \text{ left of } D\}$$

for any $D \subseteq A$. By part 2 of Lemma 5.5.4 we know that there are infinitely many $n \notin A$ such that $(\forall D \subseteq A)\ g(D \cup \{x\}) = f(D)$. We now argue that $p[+n]$ is well defined.

It is clear from the definition that $p[+n]$ satisfies 1 and 2. 3 to 7 all hold for p so if they fail for $p[+n]$, then n must be involved. So we will look at those cases.

By our choice of g and n we have that $L(n,D) \neq 0$ and $n \notin g(D) = \Psi(D \cup \{n\})$ so 3 does not fail for $p[+n]$ when n is on the left. If $n \in D$ and $m \in A \setminus \{n\}$ then $L(m,D) = 0$ unless m is left of $D \setminus \{n\}$ and by definition of g that is the only time when $m \notin f(D)$ and so $m \notin \Psi(D)$. So 3 holds for $p[+n]$.

To prove 4 for $p[+n]$ we consider some $E \subsetneq D \subseteq A$ and $m \in A$. If $m = n$, then $L(n,D) < L(n,E)$ as $|D| > |E|$. If $n \in D$ and m is to the left of D then $L(m,D) = \propto$. Since m is also to the left of E we have that $L(m,E) \geq \propto$ whether or not $n \in E$.

For 5 and 6 we observe that $L(n,D) \geq \omega$, so they both hold when $m = n$. When $n \in D$ and $n \neq m$, the only time we have $L(m,D) = \propto$ is exactly when m is to the left of D so both 5 and 6 hold.

The last part to check is 7. Suppose that $L(m, E \cup D) < \omega$ and for each $i \in E$, $L(i, D) < \omega$. We know that $L(n, F) \geq \omega$ for all $F \subseteq A$, so $m \neq n$ and $n \notin E$. If $n \notin D$ then 7 holds, since it holds for p. If $n \in D$ then as $m \neq n$ we have $L(m, D) \in \{0, \propto\}$ so $L(m, D) < \omega$.

The fact that $p[+n], p[-1] < p$ follows from their definition. $\square$

From this we can see that our forcing partial order is not empty and that every condition has an extension that increases the set G. This allows us to conclude that for any sufficiently generic $\mathcal{G}$, $I_{\mathcal{G}}$ is infinite. Furthermore, we can prove the following.

Lemma 5.5.7. *If $\mathcal{G}$ is sufficiently generic, then $I_{\mathcal{G}}$ is uniformly introenumerable.*

Proof. As pointed out, $I_{\mathcal{G}}$ is infinite because the set of conditions when $|G^p| > n$ is a dense set for each n.

Fix $n \in I_{\mathcal{G}}$ and consider some infinite $S \subseteq I_{\mathcal{G}}$. We will prove that $n \in \Psi(S)$. Let p be a condition such that $n \in G^p$ and consider $\alpha = L^p(n, \emptyset)$. Let $i \in 2, j \in \omega$ be such that $\alpha = \omega \cdot i + j$. Let $q \leq p$ be such that $q \in \mathcal{G}$ and $|G^q \cap S| > j$. Since S is infinite and $\mathcal{G}$ is a filter there must be such a q. Let $\beta = L^q(n, G^q \cap S)$. By 4 it must be that $\beta + j < \alpha$ so $\beta < \omega$. Now we let $r \leq q$ be such that $r \in \mathcal{G}$ and $|G^r \cap S \setminus G^q| > \beta$. Now by 4 it must be that $L^r(n, G^r \cap S) = 0$ so by 3 we have that $n \in \Psi(G^r \cap S) \subseteq \Psi(S)$.

We have shown that $I_{\mathcal{G}} \subseteq \Psi(S)$ for any infinite $S \subseteq I_{\mathcal{G}}$. Now we prove the other direction. If $D \subseteq_{\text{fin}} I_{\mathcal{G}}$, then there is $p \in \mathcal{G}$ such that $D \subseteq G^p$. If $n \in \Psi(D)$ then by 3 $L^p(n, D) = 0$ so $n \in G^p \subseteq I_{\mathcal{G}}$. So $\Psi(S) \subseteq \Psi(I_{\mathcal{G}}) \subseteq I_{\mathcal{G}}$ for any $S \subseteq I_{\mathcal{G}}$. $\square$

We have given a tool for extending conditions by adding numbers to G, now we give a tool that can be used to extend a condition by adding to the sequences of B_i's.

Definition 5.5.8. For conditions $q \leq p$ we define $r = \text{fin}(q, p)$ to be the condition:

- $G^r = G^p \cup \{n \in G^q : L^q(n, G^p) < \omega\}$.
- $B_i^r = B_i^p$ for $i \leq k$.
- $B_{k+1}^r = G^q \setminus G^r$.

- $L^r = L^q$

Lemma 5.5.9. *If $q \leq p$ then* $\operatorname{fin}(q,p)$ *is a valid extension of p.*

Proof. Fix $q \leq p$ and let $r = \operatorname{fin}(q,p)$. By definition and the fact that q was a valid condition, it is clear that 1–4 and 7 all hold. Since we have only increased the sequences of B_i's 5 will hold. For 6 the only time this might fail is if $m \in B^r_{k+1}$ and $D \subseteq G^r$. In this case, since $m \notin G^r$ there is some $L^r(m, G^p) \geq \omega$. On the other hand, since $D \subseteq G^r$ for each $n \in D \setminus G^p$ we have that $L^r(n, G^p) < \omega$. So by 7 it cannot be that $L^r(m, D \cup G^p) < \omega$. So $L^r(m, D) \geq \omega \geq \propto$.

The fact that $r \leq p$ should be clear from the definition. □

The final tool we will need before we can prove Theorem 5.5.3 is that that we can move numbers from the B_i's to G and still have a valid condition.

Definition 5.5.10. For a condition $q = (G^q, B^q_0, \ldots, B^q_\ell, L^q)$ and $k \leq \ell$ we define $r = \operatorname{merge}(q,k)$ as follows:

- $G^r = G^q \cup \bigcup_{i>k} B^q_i$.
- $B^r_i = B^p_i$ for $i \leq k$.
-

$$L^r(m,D) = \begin{cases} L^q(m,D) & L^q(m,D) \neq \propto \vee \exists j \leq k [m \in B^r_j \wedge D \subseteq A_j] \\ x + 1 + |A_{k+1}| - |D| & \text{otherwise} \end{cases}$$

where $x = \max\{L^q(n,D) : L^q(n,D) \in \omega\}$ and $A_j = G^r \cup \bigcup_{i>j} B^r_i$.

Lemma 5.5.11. *For all conditions q we have that* $\operatorname{merge}(q,k)$ *is a valid condition.*

Proof. We start by showing that $r = \operatorname{merge}(q,1)$ is a valid condition. 1 and 2 hold because they held for q. For 3 observe that $L^r(m,D) = 0 \iff L^q(m,D) = 0$.

Fix m, D and $E \subsetneq D$. If $L^r(m,D) = L^q(m,D)$ and $L^r(m,E) = L^q(m,E)$ then 4 holds. If $L^r(m,D) \neq L^q(m,D)$ then $L^q(m,D) = \propto$ and $L^r(m,D) = x + 1 + |A_{k+1}| - |D| <$

$\min(x+1+|A_{k+1}|-|E|,\propto) \leq L^r(m,E)$. If $L^r(m,E) \neq L^q(m,E)$ then $L^q(m,E) =\propto$ and m is not to the left of E. Hence m is not to the left of D and $L^q(m,D) <\propto$ means $L^r(m,D) \leq x+1+|A_{k+1}|-|D| < x+1+|A_{k+1}|-|E|$. So 4 holds.

If $L^r(m,D) =\propto$ then there is $j \leq k$ such that $m \in B^r_j \wedge D \subseteq A_j$, so 5 holds. On the other hand if there is $j \leq k$ such that $m \in B^r_j \wedge D \subseteq A_j$, then $j \leq k'$ so $L^r(n,D) = L^q(n,D) \geq\propto$. Hence 6 holds. For 7 we observe that $L^r(n,D) < \omega \iff L^q(n,D) < \omega$ and 6 holds for q. So we can conclude that r is a valid condition. The result for large k comes from applying merge to r.

□

Now we are ready to prove Theorem 5.5.3.

Proof of Theorem 5.5.3. We will prove that $I_{\mathcal{G}} \not\leq_e \overline{K_{I_{\mathcal{G}}}}$ for any sufficiently generic $\mathcal{G}$. Consider some condition p_0 and an enumeration operator Ψ_e. We will build an $r \leq p_0$ such that $r \Vdash I_{\mathcal{G}} \neq \Psi_e(\overline{K_{I_{\mathcal{G}}}})$. In this way we will have shown that the set $\{p : p \Vdash I_{\mathcal{G}}$ is not cototal$\}$ is dense.

By Lemma 5.5.6, we know that there is n such that $p_0[+n]$ is well defined. If $p_0[+n] \nVdash n \in \Psi_e(\overline{K_{I_{\mathcal{G}}}})$, then we can take $r \leq p_0[+n]$ such that $r \Vdash n \notin \Psi_e(\overline{K_{I_{\mathcal{G}}}})$. So $r \Vdash n \in I_{\mathcal{G}} \setminus \Psi_e(\overline{K_{I_{\mathcal{G}}}})$.

So now we will assume that $p_0[+n] \Vdash x \in \Psi_e(\overline{K_{I_{\mathcal{G}}}})$. This means that there must be some $p \leq p_0[+n]$ and axiom $(n,u) \in \Psi_e$ such that $p \Vdash D_u \subseteq \overline{K_{I_{\mathcal{G}}}}$.

Now we take $q_0 = \text{fin}(p, p_0)$. Observe that since $L^{p_0[+n]}(n, G^{p_0}) \geq \omega$, $n \notin G^q$. If $q_0 \nVdash n \notin \Psi_e(\overline{K_{I_{\mathcal{G}}}})$, then we can take any $r \leq q_0$ such that $r \Vdash n \in \Psi_e(\overline{K_{I_{\mathcal{G}}}})$. Then $r \Vdash n \in \Psi_e(\overline{K_{I_{\mathcal{G}}}}) \setminus I_{\mathcal{G}}$ as desired.

Suppose that $q_0 \Vdash n \notin \Psi_e(\overline{K_{I_{\mathcal{G}}}})$. Now we work toward getting a contradiction. Since $q_0 \Vdash n \notin \Psi_e(\overline{K_{I_{\mathcal{G}}}})$ there is some extension $q \leq q_0$ and $m \in D_u$ such that $q \Vdash m \in K_{I_{\mathcal{G}}}$. Furthermore we can ask that $m \in K_{G^q}$.

Suppose that $q = (G^q, B^q_0, \dots, B^q_\ell, L^q)$ and $p = (G^p, B^p_0, \dots, B^p_k, L^p)$. We now claim that $r = \text{merge}(q,k)$ is an extension of p. By definition of fin we have that $L^p = L^{q_0} \subseteq L^q$, $B^p_0, \dots, B^p_k \preceq B^q_0, \dots, B^q_\ell$. $G^p = G^{q_0} \cup B^q_{k+1}$. So we get that $B^p_0, \dots, B^p_k = B^r_0, \dots, B^r_k$ and

$G^p \subseteq G^r$. Recall that $L^r(m,D) \neq L^q(m,D)$ only if $L^q(m,D) = \propto$ and m is not to the left of D for condition r. This can only happen when $m \in \bigcup_{k<j\leq \ell} B_j^q$, so $(m,D) \notin \mathrm{dom}(L^p)$. Thus we can conclude that $r \leq p$.

However, we have that $p \Vdash D_u \subseteq \overline{K_{I_\mathcal{G}}}$, so $r \Vdash D_u \subseteq \overline{K_{I_\mathcal{G}}}$, but $G^r \supseteq G^q$ so $m \in K_{G^r}$ and thus $r \Vdash D_u \nsubseteq \overline{K_{I_\mathcal{G}}}$, a contradiction. □

Theorem 5.5.12 (Goh, Jacobsen-Grocott, Miller and Soskova [14])**.** *There is a uniformly Baire e-pointed tree that is not enumeration equivalent to any introenumerable set.*

Rather than repeat the proof of this theorem from [14] we will instead make use of the forcing used in Chapter 6 to prove a stronger separation.

Theorem 5.5.13. *There is a uniformly Baire e-pointed tree that is not hyperenumeration equivalent to any introenumerable set.*

We will leave the proof of this until Chapter 6.

This completes the known separations. The following we leave as an open question.

Question 5.5.14. Is there an introenumerable set that is not equivalent to any uniformly introenumerable set?

5.6 Topological classification

In this section we consider the topological classification of the classes we have studied in this chapter. We have proven that they all contain the cototal degrees, and hence the cylinder-cototal degrees, so from Theorem 4.3.2 we know that they are all T_2-quasi-minimal. Now we explain how they are all T_1 classes.

Theorem 5.6.1. *There is a T_1 cb_0-space $\mathcal{X}$ such that the hypercototal degrees are $\mathcal{D}_\mathcal{X}$.*

Proof. We start by considering the space $\omega^{\leq\omega} = \omega^\omega \cup \omega^{<\omega}$ with basis $(\beta_\sigma)_{\sigma\in\omega^{<\omega}}$ where $\beta_\sigma = \{\tau \in \omega^{\leq\omega} : \sigma \preceq \tau\}$. Observe that β_σ is the smallest open set containing σ. Consider some closed $F \subseteq \omega^{\leq\omega}$. F is closed under initial segments as every open set containing some σ contains all extensions of σ. If $\sigma_0 \prec \sigma \cdots \subseteq F$ then the usual limit $f = \lim_n \sigma_n$ is also the limit of this sequence in $\omega^{\leq\omega}$ so the closed set F is uniquely determined by the tree $F \cap \omega^{<\omega}$.

Now we consider a new cb$_0$-space $\mathcal{X} = (X, (\alpha_\sigma)_{\sigma\in\omega^{<\omega}})$ where $X = \{F \subseteq \omega^{\leq\omega} : F \text{ is closed}\}$ and $\alpha_\sigma = \{F \in X : \beta_\sigma \cap F \neq \emptyset\}$.

Now we define the hypercototal space $\mathcal{HCT}$ as the disjoint union of $\mathcal{HCT}_e$ where $\mathcal{HCT}_e = \{F \in X : \forall f \in F \cap \omega^\omega[F \cap \omega^{<\omega} = \Psi_e(f)]\}$ is the set of uniformly Baire e-pointed trees via Ψ_e. From this definition it should be clear that $\text{NBase}_{\mathcal{HCT}}(e, F) \equiv_e \{\sigma \in F \cap \omega^{<\omega}\}$, so $\mathcal{D}_{\mathcal{HCT}}$ is class of hypercototal degrees.

To see that $\mathcal{HCT}$ is a T_1 space it is enough to show that $\mathcal{HCT}_e$ is T_1 for each e. Consider $F_0 \neq F_1 \in \mathcal{HCT}_e$. Since these are distinct uniformly Baire e-pointed trees via the same operator there must not be infinite paths $f \in F_0 \cap F_1$ as $\Psi_e(f)$ is a single tree. So pick two infinite paths $f_0 \in F_0 \setminus F_1$ and $f_1 \in F_1 \setminus F_0$. Since $f_0 \notin F_1$, there is finite $\sigma_0 \prec f_0$ such that $\sigma_0 \notin F_1$. Similarly we can find σ_1. So we have now have $F_0 \in \alpha_{\sigma_0} \setminus \alpha_{\sigma_1}$ and $F_1 \in \alpha_{\sigma_1} \setminus \alpha_{\sigma_1}$. Thus $\mathcal{HCT}_e$ is T_1. □

Worth observing here is that $\mathcal{HCT}$ as we defined it above is not a decidable cb$_0$-space. We can improve this by observing that there is a fixed e such that if T is uniformly Baire e-pointed, then T is equivalent to a uniformly Baire e-pointed tree via Ψ_e (for example the enumeration operator we use in our construction in Chapter 6 has this property). If $\mathcal{HCT}_e$ has the property that every finite tree extends to a point in $\mathcal{HCT}_e$ (as is the case with the operator from Chapter 6) then $\mathcal{HCT}_e$ is strongly decidable, as it is a dense subset of $\mathcal{X}$.

Another point to note is that this space is not a G_δ space (a space where every closed set is the countable intersection of open sets). Kihara, Ng and Pauly [26] characterized the degrees of effectively G_δ spaces as precisely the cototal degrees. Ever G_δ cb$_0$-space is effectively G_δ relative to some oracle, but relativizing Theorem 5.5.3 shows that even on an enumeration cone, there are non-cototal uniformly introenumerable degrees.

Other general questions one might ask this class would be its size in terms of measure or category. Sanchis' [39] construction of a non hypertotal degree can be made symmetric to construct a set that is neither hypertotal or hypercototal. This shows that these classes are meager. In terms of measure recent work by Ang Li [31] shows that these classes are null sets.

Chapter 6

The hyperenumeration degrees

In Chapter 5 we introduced hyperenumeration reducibility and used it to characterize the degrees of Baire e-pointed trees with dead ends. In this chapter we investigate hyperenumeration reducibility in more depth and try to lift some results about enumeration reducibility to this context. In particular we are interested in Selman's theorem and downwards density.

In Section 6.2 we look at Selman's theorem. The work done in this section was started at the Dagstuhl Seminar on Descriptive Set Theory and Computable Topology in 2021. In particular, the connection to e-pointed trees, and Corollary 6.2.8 were discovered during that workshop.

As it turns out, Selman's theorem fails for hyperenumeration reducibility.

Corollary 6.2.8. *There are sets A, B such that $B \not\leq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.*

This follows from the fact that there is a uniformly Baire e-pointed tree without dead ends that is not hypertotal.

Theorem 6.2.1. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

Most of Section 6.2 is dedicated to proving this theorem. At the end of the section we make use of the forcing notions developed to prove Theorem 5.5.13 from Chapter 5.

In section 6.3 we look at downwards density. It turns out that the hyperenumeration degrees are downwards dense and Gutteridge's [16] proof does lift to the hyperenumeration context. We explore some issues that can arise when attempting priority constructions, and how these issues can be overcome for particular constructions like downwards density.

Since Selman's theorem fails for hyperenumeration reducibility it is natural to ask if this is the correct hyperarithmetic analogue of enumeration reducibility. In Section 6.4 we look at some other readabilities that could be considered analogues of enumeration reducibility.

6.1 Preliminaries

We will give a brief overview of some of the tools of higher computability theory that we will use in this chapter. A more in depth introduction to higher computability can be found in Sacks' book [38]. For an introduction to the hyperenumeration degrees, see Section 5.2 of Chapter 5.

Some basic points of notation. We use n, m, i, j, k for natural numbers. We use α, β, γ for ordinals. We use $\sigma, \tau, \rho, \upsilon, x, y, z$ to represent strings of natural numbers. $\langle\sigma\rangle$ corresponds to the Gödel number of the string σ. We use T and S to refer to trees in $\omega^{<\omega}$.

6.1.1 Admissible sets and higher computability theory

The usual definition of a Π^1_1 set of natural numbers is a set of the form $m \in X \iff \forall f \in \omega^\omega \exists n[R(f, n, m)]$ where R is a computable relation. However admissibility gives us another definition in terms of $L_{\omega_1^{CK}}$ that is useful.

Definition 6.1.1. A set M is *admissible* is it is transitive, closed under union, pairing and Cartesian product as well as satisfying the following two properties:

Δ_1-comprehension: for every Δ_1 definable class $A \subseteq M$ and set $a \in M$ the set $A \cap a \in M$.

Σ_1-collection: for every Σ_1 definable class relation $R \subseteq M^2$ and set $a \in M$ such that $a \subseteq \text{dom}(R)$ there is $b \in M$ such that $a = R^{-1}[b]$.

The smallest admissible set is HF, the collection of hereditarily finite sets. Looking at the Δ_1 and Σ_1 subsets of HF is one notion of computability. We have that the Δ_1 subsets of HF are computable sets and the Σ_1 subsets of HF are the c.e. sets. We generalize this to an arbitrary admissible set M by calling a set $A \subseteq M$ M-computable if it is a Δ_1 subset of M and M-c.e. if it is a Σ_1 subset of M.

The smallest admissible set containing ω is $L_{\omega_1^{CK}}$. We have that the $L_{\omega_1^{CK}}$-c.e. subsets of ω are precisely the Π^1_1 sets. This means that the $L_{\omega_1^{CK}}$-computable subsets of ω sets are the hyperarithmetic sets. Note that Δ_1-comprehension means that the hyperarithmetic sets are precisely the sets in $\mathcal{P}(\omega) \cap L_{\omega_1^{CK}}$.

These results about Π^1_1 and hyperarithmetic sets can be relativized for some set X. We define L_X to be the smallest admissible set containing X. We have that $A \subseteq \omega$ is Π^1_1 in X if and only if it is L_X-c.e. and hyperarithmetic in X if and only if $A \in L_X$. Note that while we have $\text{ORD}^{L_X} = \omega_1^X$ and $L_{\omega_1^X} \subseteq L_X$ it is only sometimes the case that $L_X = L_{\omega_1^X}$.

6.1.2 Some facts about trees

We will deal a lot with trees in this chapter so it is useful to have operations on trees. For a tree $S \subseteq \omega^{<\omega}$ and string x we define $\text{Ext}(S, x)$ to be the tree of extensions of x. $\{y : x^\frown y \in S\}$. A relation on trees that we will use is $\preceq$. We say $T \preceq S$ if S is an end extension of T. That is, $T \subseteq S$ and for all $\sigma \in S$ the longest initial segment of σ that is in T is a leaf in T.

Now we define $\text{rank}(S)$ for a well founded tree S using transfinite recursion. We define $\text{rank}(\emptyset) = 0$. Given a tree S we define $\text{rank}(S) = \sup_{i \in S} \text{rank}(\text{Ext}(S, i)) + 1$.

As it turns out, this function rank is in fact $L_{\omega_1^{CK}}$-partial computable, i.e. its graph is $L_{\omega_1^{CK}}$-c.e. To help the reader feel more familiar with computability on $L_{\omega_1^{CK}}$ we include a sketch of the proof of this fact.

For a tree $T \in L_{\omega_1^{CK}}$ and function $f \in L_{\omega_1^{CK}}$ we say that f is a rank function on T if dom$(f) = T$, range$(f) \subseteq \omega_1^{CK}$, for each leaf $x \in T$ we have $f(x) = 1$ and for each non leaf $y \in T$ we have that $f(y) = \sup_{y^\frown i \in T} f(y^\frown i) + 1$. Since the quantifies are all bounded it is $L_{\omega_1^{CK}}$-computable to check if f is a rank function on T. If f is a rank function on T then f is unique and $f(\emptyset) = \text{rank}(T)$. So we can define rank by rank$(T) = \alpha$ if there is a rank function f on T such that $f(\emptyset) = \alpha$ or $\alpha = 0$ and $T = \emptyset$. So we now have a Σ_1 definition of rank. The only problem is that its domain may not consist of all well founded trees $T \in L_{\omega_1^{CK}}$.

To prove that the domain of rank is all well founded trees in $L_{\omega_1^{CK}}$ we use induction on the true rank of T. Suppose all trees of rank less than T are in the domain of rank. Then for each $i \in T$ there is a rank function f_i for Ext(T, i). Since the map, $S \mapsto f$ where f is the rank function on S, is $L_{\omega_1^{CK}}$-c.e, Σ_1-collection tells us that the map $i \mapsto f_i$ is in $L_{\omega_1^{CK}}$. So we can build a rank function $f \in L_{\omega_1^{CK}}$ on T by $f(i^\frown x) = f_i(x)$ and $f(\emptyset) = \sup_{i \in T} f_i(\emptyset) + 1$.

One nice result of this is that if a tree $T \in L_{\omega_1^{CK}}$ is well founded, then it has rank $< \omega_1^{CK}$ and the set of all well founded trees in $L_{\omega_1^{CK}}$ is $L_{\omega_1^{CK}}$-c.e. This could also be seen by observing that trees in $L_{\omega_1^{CK}}$ are Δ_0 definable and so hyperarithmetic.

6.2 A uniformly e-pointed tree in ω^ω without dead ends that is not of hyper total degree

In this section we prove the following theorem.

Theorem 6.2.1. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

6.2.1 The forcing partial order

To build this we will need a new set of forcing conditions similar to those used in the construction of a uniformly e-pointed tree without dead ends that is not of introenumerable degree. So let $\{T_\sigma : \sigma \in \omega^{<\omega}\}$ be an effective listing of all finite trees in $\omega^{<\omega}$ where for each $\sigma \in \omega^{<\omega}$ the sequence $T_{\sigma\frown 0}, T_{\sigma\frown 1}, \ldots$ lists each finite tree that contains T_σ infinitely often. We will need a labeling that is allowed to use any ordinal below ω_1^{CK}. From now on a condition is some $p = (T^p, L^p : T^p \times T^p \to \omega_1^{CK}) \in L_{\omega_1^{CK}}$ where the following hold:

1. $T^p \subseteq \omega^{<\omega}$ is a well founded tree.

2. For each $\sigma \in T^p$ we have that $T_\sigma \subseteq T^p$.

3. $L^p(\sigma, \tau) = 0$ if and only if $\sigma \in T_\tau$.

4. If $\rho \prec \tau$ then $L^p(\sigma, \tau) = 0$ or $L^p(\sigma, \tau) < L^p(\sigma, \rho)$.

5. For each $\tau \in T^p$ and $n < \omega$ the set $\{\sigma : L^p(\sigma, \tau) \leq n\}$ is finite.

For two conditions p and q we say $p \leq q$ if $T^q \preceq T^p$ and $L^q \subseteq L^p$. For a filter $\mathcal{G}$ we define $T^{\mathcal{G}} = \bigcup_{p\in\mathcal{G}} T^p$. The fact that we must have $T^q \preceq T^p$ means that if $p \in \mathcal{G}$, σ is not a leaf in T^p and $\sigma\frown i \notin T^p$ then $\sigma\frown i \notin T^{\mathcal{G}}$. So we have a way of forcing strings into the complement of $T^{\mathcal{G}}$.

Proposition 6.2.2. *The set of conditions is $L_{\omega_1^{CK}}$-c.e. and the relation $\leq$ on conditions is $L_{\omega_1^{CK}}$-computable.*

Proof. Properties 2—5 are all straightforwardly Δ_1 conditions. To check if a tree T is well founded we ask if there is a rank function $f \in L$ such that $f(\sigma) = \text{rank}(\text{Ext}(T, \sigma))$, so a Σ_1 question. So property 1 is a Σ_1 condition. Hence the set of valid conditions is $L_{\omega_1^{CK}}$-c.e.

$q \leq p$ is clearly Δ_0 so $\leq$ is an $L_{\omega_1^{CK}}$-computable relation with $L_{\omega_1^{CK}}$-c.e. domain. □

Proposition 6.2.3. *For a condition p we have $L^p(\sigma, \tau) \geq \text{rank}(\{\rho : \tau\frown\rho \in T^p, \sigma \notin T_{\tau\frown\rho}\})$ for all $\sigma, \tau \in T^p$.*

Proof. We will use induction on $L^p(\sigma, \tau)$. Base case, $L^p(\sigma, \tau) = 0$. Then $\sigma \in T_\tau$ so $\{\rho : \tau\frown\rho \in T^p, \sigma \notin T_{\tau\frown\rho}\} = \emptyset$ and $\text{rank}(\emptyset) = 0$. Now suppose the proposition holds for all $\beta < \alpha$ and $L^p(\sigma, \tau) = \alpha$. Then we have for each $\tau\frown i \in T^p$ we have $L^p(\sigma, \tau\frown i) \geq \text{rank}(\{\rho : \tau\frown i\frown\rho \in T^p, \sigma \notin T_{\tau\frown i\frown\rho}\})$ by induction hypothesis. By property 4 and definition of rank we have $L^p(\sigma, \tau) \geq \sup_{\tau\frown i\in T^p} L^p(\sigma, \tau\frown i) + 1 \geq \sup_{\tau\frown i\in T^p} \text{rank}(\{\rho : \tau\frown i\frown\rho \in T^p, \sigma \notin T_{\tau\frown i\frown\rho}\}) + 1 = \text{rank}(\{\rho : \tau\frown\rho \in T^p, \sigma \notin T_{\tau\frown\rho}\})$. □

In order for this forcing notion to have nontrivial generics we need a way to extend conditions. Fix a condition p. Let $A \subseteq \omega^{<\omega}$ be a set such that for all $\sigma^\frown i \in A$ we have $\sigma \in T^p$ and $\{\tau : L^p(\tau, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} \subseteq T^P \cup A$. For such an A we can define $q = p[A]$ by $T^q = T^p \cup A$ and L^q given by

$$L^q(\sigma, \tau) = \begin{cases} L^p(\sigma, \tau) & \sigma, \tau \in T^p \\ \langle\sigma\rangle + \text{rank}(\text{Ext}(T^q, \tau)) & \sigma \in A, \sigma \notin T_\tau \\ 0 & \sigma \in T_\tau \\ L^p(\sigma, \rho) - 1 & \rho^\frown i = \tau \in A, \sigma \notin T_\tau, L^p(\sigma, \rho) < \omega \\ \langle\sigma\rangle & \text{otherwise} \end{cases}$$

Lemma 6.2.4. *If A meets the requirement of the definition then $p[A]$ is a valid condition. If we also have that $T^p \preceq T^p \cup A$ then $p[A] \leq p$.*

Proof. We show that $q = p[A]$ is well defined. Our requirement for A ensures that 1 and 2 hold. For 3—5, since $L^p = L^q \restriction T^p \times T^p$ the only way we can run into a problem is when considering $\sigma^\frown i \in A$. If $\rho \prec \tau \in T^q$ then by definition $L^q(\sigma^\frown i, \rho) > L^q(\sigma^\frown i, \tau)$. If $\rho \prec \sigma^\frown i$ then $L^p(\tau, \rho) \geq L^p(\tau, \sigma)$. If $0 < L^p(\tau, \sigma) < \omega$ then $L^q(\tau, \sigma^\frown i) = L^p(\tau, \sigma) - 1 < L^p(\tau, \sigma)$. If $L^p(\tau, \sigma) \geq \omega$ then $L^q(\tau, \sigma^\frown i) < \omega$. So 4 holds.

Fix n and τ and consider the set $\{\rho : L^q(\rho, \tau) \leq n\}$. If $\tau \in T^p$ then we have added at most n many elements to the set, so it is still finite. If $\tau = \sigma^\frown i \in A$ and ρ is in this set then either ρ belongs to the finite set $\{\rho : L^p(\rho, \sigma) \leq n + 1\}$ or $\langle\rho\rangle \leq n$. So there are only finitely many ρ that can be in $\{\rho : L^q(\rho, \tau) \leq n\}$. So 5 holds.

Now consider the set $\{\rho : L^q(\rho, \tau) = 0\}$. If $\tau \in T^p$ then $L^q(\sigma, \tau) \geq 1$ for each $\sigma \in A$, so we have $\{\rho : L^q(\rho, \tau) = 0\} = \{\rho : L^p(\rho, \tau) = 0\} = T_\tau$. If $\tau \in A$ then by definition of L^q we have $\rho \in T_\tau$ if and only if $L^q(\rho, \tau) = 0$. So 3 holds.

Since $L^p \subseteq L^q$ if $T^p \preceq T^p \cup A = T^q$ then $p[A] \leq p$. □

Corollary 6.2.5. *If $\mathcal{G}$ is a sufficiently generic filter then $T^{\mathcal{G}}$ is a uniformly e-pointed tree with no dead ends.*

Proof. First we show that for each condition p and $\sigma \in T^p$ the set $\{q \leq p : \sigma \text{ is not a dead end}\}$ is dense below p. If σ is a dead end in T^p then enumeration of $(T_\sigma)_{\sigma \in \omega^\omega}$ gives us an i such that $T_{\sigma^\frown i} = \{\rho : L^p(\rho, \sigma) \leq 1\}$. Thus we can take $p[\{\sigma^\frown i\}] < p$ where σ is no longer a dead end. So $T^{\mathcal{G}}$ does not have any dead ends.

To show $T^{\mathcal{G}}$ is uniformly e-pointed consider some path $P \in [T^{\mathcal{G}}]$. We will show that $T^G = \bigcup_{\sigma \prec P} T_\sigma$. If $\sigma \in T^{\mathcal{G}}$ then $\sigma \in T^p$ for some $p \in G$. So by property 2 we have that $T_\sigma \subseteq T^p \subseteq T^{\mathcal{G}}$. On the other hand if $\sigma \in T^{\mathcal{G}}$ then consider a sequence $p_0 > p_1 > \cdots \subseteq G$ with $P \restriction n \in T^{p_n}$. Now consider the sequence $(L^{p_n}(\sigma, P \restriction n))_n$. Since $L^{p_n} \subseteq L^{p_{n+1}}$ property 4 means that this is a decreasing sequence. Since ω_1^{CK} is a well order there is n such that $L^{p_n}(\sigma, P \restriction n) = 0$. So we have that $\sigma \in T_{P \restriction n}$. Hence $T^G = \bigcup_{\sigma \prec P} T_\sigma$. □

6.2.2 The forcing relation

Now that we have a forcing partial order and some useful operations on conditions, we will talk about forcing with conditions. We define $S_e^p \subseteq \omega^{<\omega}$ to be the tree where $x \notin S_e^p \iff \exists y \prec x[y \in \Psi_e(T^p)]$. For a filter $\mathcal{G}$ we define $S_e^{\mathcal{G}} \bigcap_{p \in \mathcal{G}} S_e^p$. So $x \notin S_e^{\mathcal{G}} \iff \exists y \prec x[y \in \Psi_e(T^{\mathcal{G}})]$. By definition of Γ_e we have that $\Gamma_e(T^{\mathcal{G}}) = \{n : \text{Ext}(S_e^{\mathcal{G}}, n) \text{ is well founded}\}$.

We define $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$ if $\text{rank}(\text{Ext}(S_e^p, x)) \leq \alpha$. From this definition it is clear that if $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$ then for any $\mathcal{G} \ni p$ we have that $\text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$. We now work towards proving the opposite direction.

Lemma 6.2.6. *Fix a condition p. Suppose that for each $i \in \omega, r \leq p$ there is $q \leq r$ such that $q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) \leq \beta$ for some $\beta < \omega_1^{CK}$ then there is $\hat{p} \leq p$ and $\alpha < \omega_1^{CK}$ such that $\hat{p} \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$.*

Proof. The function $(q, e) \mapsto S_e^q$ is $L_{\omega_1^{CK}}$-partial computable so by composition, the map $(q, e, x) \mapsto \text{rank}(\text{Ext}(S_e^q, x))$ is also $L_{\omega_1^{CK}}$-partial computable. So the set

$$C = \{(i, r, q, \beta) : q \leq r \wedge \text{rank}(\text{Ext}(S_e^q x^\frown i)) \leq \beta\}$$

is $L_{\omega_1^{CK}}$-c.e.

For each i we will define a condition r_i as follows. For each leaf $\sigma \in T^p$ let k_σ be the ith number such that $T_{\sigma^\frown k_\sigma} = \{\tau : L^p(\tau, \sigma) \leq 1\}$. Now we define $A_i = \{\sigma^\frown k_\sigma : \sigma \text{ is a leaf in } T^p\}$ and define $r_i = p[A_i]$. The definition of r_i only involves computable operations so the map $i \mapsto r_i$ is $L_{\omega_1^{CK}}$-computable and since $\omega \in L_{\omega_1^{CK}}$ the set $\{(i, r_i)\} \in L_{\omega_1^{CK}}$ by Σ_1-collection. Using Σ_1-collection again, this time with the set C, we get that there is a function $f \in L_{\omega_1^{CK}}$ such that $f(i) = (q_i, \beta_i)$ for some $q_i \leq r_i$ and β_i such that $q_i \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) \leq \beta_i$. Let $\alpha = \sup_i\{\beta_i : i \in \omega\}$. Since $f \in L_{\omega_1^{CK}}$, $\alpha < \omega_1^{CK}$.

To build $\hat{p}$ let $T^{\hat{p}} = \bigcup_{i\in\omega} T^{q_i}$. Since $f \in L_{\omega_1^{CK}}$ we have that $T^{\hat{p}} \in L_{\omega_1^{CK}}$. $T^{\hat{p}}$ will satisfy property 1 because the sets $T^{q_i} \setminus T^p$ are disjoint and so $T^{\hat{p}}$ is well founded.

We define $L^{\hat{p}}$ using the following tools. For $\tau \in T^{\hat{p}}$ let τ_p be the longest initial segment of τ that is in T^p. For $\sigma, \tau \in T^{\hat{p}}$ let $\text{rank}(\sigma, \tau) = \text{rank}(\{\rho : \tau^\frown\rho \in T^{\hat{p}}, \sigma \notin T_{\tau^\frown\sigma}\})$. Note that both of these operations are $L_{\omega_1^{CK}}$-computable. Define

$$L^{\hat{p}}(\sigma, \tau) = \begin{cases} L^p(\sigma, \tau) & \sigma, \tau \in T^p \\ 0 & \sigma \in T_\tau \\ L^p(\sigma, \tau_p) - |\tau| + |\tau_p| & \sigma \in T^p \setminus T_\tau, \tau \notin T^p, L^p(\sigma, \tau_p) < \omega \\ \langle\sigma\rangle + \text{rank}(\sigma, \tau) & \text{otherwise} \end{cases}$$

Now we prove that $\hat{p}$ is a valid condition. Since it is built in an effective way out of $L_{\omega_1^{CK}}$-computable functions $L^{\hat{p}}$ is $L_{\omega_1^{CK}}$-computable. Since $\text{dom}(L^{\hat{p}}) \in L_{\omega_1^{CK}}$ we have that $L^{\hat{p}} \in L_{\omega_1^{CK}}$. So we have that $\hat{p} \in L_{\omega_1^{CK}}$.

Now we show that $\hat{p}$ has the properties of a condition. Property 2 is straightforward. Property 3 follows from the definition of $L^{\hat{p}}$ and the fact that it held for each q_i.

For property 4 consider $\sigma, \rho \prec \tau$, and suppose that $L^{\hat{p}}(\sigma, \rho) > 0$. We look at several cases.

- $\sigma, \tau \in T^p$. Then $\rho \in T^p$ so by 4 for p we have $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau) < L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.
- $\sigma \in T_\tau$. Then $L^{\hat{p}}(\sigma, \tau) = 0 < L^{\hat{p}}(\sigma, \rho)$.

- $\sigma \in T^p \setminus T_\tau, \tau \notin T^p, L^p(\sigma, \tau_p) < \omega$. We have two subcases: if $\rho \notin T^p$ then $\tau_p = \rho_p$ so $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau_p) - |\tau| + |\tau_p| < L^p(\sigma, \tau_p) - |\rho| + |\rho_p| = L^{\hat{p}}(\sigma, \rho)$. If $\rho \in T^p$ then $\rho \preceq \tau_p$ so $L^{\hat{p}}(\sigma, \tau) = L^p(\sigma, \tau_p) - |\tau| + |\tau_p| < L^p(\sigma, \tau_p) \leq L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.

- Otherwise $L^{\hat{p}}(\sigma, \tau) = \langle\sigma\rangle + \text{rank}(\sigma, \tau)$. If $\rho \notin T^p$ or $\sigma \notin T^p$ then $L^{\hat{p}}(\sigma, \rho) = \langle\sigma\rangle + \text{rank}(\sigma, \rho) > \langle\sigma\rangle + \text{rank}(\sigma, \tau)$ as $\rho \prec \tau$. If $\rho, \sigma \in T^p$ then consider i such that $\tau \in T^{q_i}$. By Proposition 6.2.3 we have that $\text{rank}(\sigma, \tau) \leq L^{q_i}(\sigma, \tau) < L^{q_i}(\sigma, \rho) = L^p(\sigma, \rho) = L^{\hat{p}}(\sigma, \rho)$.

For property 5 fix τ and n. Suppose that $L^{\hat{p}}(\sigma, \tau) \leq n$. Then one of the following is true: $L^p(\sigma, \tau) \leq n$ or $\sigma \in T_\tau$ or $L^p(\sigma, \tau_p) - |\tau| + |\tau_p| \leq n$ or $\langle\sigma\rangle + \text{rank}(\sigma, \tau) \leq n$. So σ is a member of the finite set $\{\sigma : L^p(\sigma, \tau) \leq n\} \cup T_\tau \cup \{\sigma : L^p(\sigma, \tau_p) \leq n + |\tau|\} \cup \{\sigma : \langle\sigma\rangle \leq n\}$. Hence the set $\{\sigma : L^{\hat{p}}(\sigma, \tau) \leq n\}$ is finite.

So we have shown that $\hat{p}$ is a valid condition. Since $T^p \preceq T^{\hat{p}}$ and $L^p \subseteq L^{\hat{p}}$ we have $\hat{p} \leq p$. Consider $\text{Ext}(S_e^{\hat{p}}, x)$. By definition of $T^{\hat{p}}$ we have that $S_e^{\hat{p}} \subseteq S_e^{q_i}$ for each $i \in \omega$, so $\text{rank}(\text{Ext}(S_e^{\hat{p}}, x^\frown i)) \leq \text{rank}(\text{Ext}(S_e^{q_i}, x^\frown i)) < \alpha$. Thus we have $\text{rank}(\text{Ext}(S_e^{\hat{p}}, x)) \leq \alpha$ as desired. □

Now we use this lemma to show that if a condition p cannot be extended to some q that forces $S_e^{\mathcal{G}}$ to have computable rank then p in fact forces $S_e^{\mathcal{G}}$ to be ill founded. We say $p \Vdash \text{Ext}(S_e^{\mathcal{G}}, x)$ is ill founded if for all sufficiently generic filters $\mathcal{G} \ni p$ we have that $\text{Ext}(S_e^{\mathcal{G}}, x)$ contains an infinite path.

Lemma 6.2.7. *If for all $q \leq p$ and $\alpha < \omega_1^{CK}$ we have $q \nVdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$ then $p \Vdash \text{Ext}(S_e^{\mathcal{G}}, x)$ is ill founded.*

Proof. We define $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ if $\forall q \leq p, \alpha < \omega_1^{CK}[q \nVdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha]$. To prove this lemma, we first prove the simpler statement: if $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then there is $q \leq p, i \in \omega$ such that $q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$.

Suppose this statement fails for some p and x. Then $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$, so we have that $\forall q \leq p, \alpha < \omega_1^{CK}[q \nVdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha]$, and there is no $q \leq p, i \in \omega$ such that $q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$ so we have $\forall q \leq p, i \in \omega[\exists r \leq q, \alpha < \omega_1^{CK}(r \Vdash$

$(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) \leq \alpha)]$. So by Lemma 6.2.6 there is $\hat{p} \leq p$ such that $\hat{p} \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) \leq \alpha$. This contradicts the fact that $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$, so the statement holds.

Now we use this to prove the lemma. Since the set $\{q : q \leq p\} \supseteq \{q : q \leq r\}$ for $r \leq p$ we have that if $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then $r \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ for all $r \leq p$. So if $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$ then the set $\{q \leq p : \exists i \in \omega[q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty]\}$ is dense above p. So if $p \in \mathcal{G}$ for some sufficiently generic $\mathcal{G}$ then there is $q \in \mathcal{G}$ and $i \in \omega$ such that $q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x^\frown i)) = \infty$. By repeating this argument we can build a sequence $X \in \omega^\omega$ such that for all $y \prec X$ there is $q \in \mathcal{G}$ such that $q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, y)) = \infty$. We have that $X \in S_e^{\mathcal{G}}$ as otherwise there would be some $r \in \mathcal{G}$ and $y \prec X$ such that $r \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, y)) = 0$, a contradiction of $\mathcal{G}$ being a filter and $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, x)) = \infty$. $\square$

Now we have all the tools needed to prove the main result of this section.

Theorem 6.2.1. *There is a uniformly e-pointed tree $T^{\mathcal{G}} \subseteq \omega^{<\omega}$ with no dead ends such that $T^{\mathcal{G}}$ is not hypertotal.*

Proof. We show that for a sufficiently generic $\mathcal{G}$ we have that $T^{\mathcal{G}}$ is not hypertotal. We say $p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$ if there is $\sigma \in T^p$ and $\alpha < \omega_1^{CK}$ such that $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)) \leq \alpha$, or if there is $\sigma \notin T^p$ such that the initial segment of σ in T^p is not a leaf and $p \Vdash \text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded. To show that $T^{\mathcal{G}}$ is not hypertotal it is enough for us to show that the sets $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ are dense for each e. To see this consider the two cases. If $p \in \mathcal{G}$ and there is $\sigma \in T^p$ and $\alpha < \omega_1^{CK}$ such that $p \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)) \leq \alpha$ then we have that $\text{Ext}(S_e^p, \langle\sigma\rangle)$ is well founded and so $\text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle) \subseteq \text{Ext}(S_e^p, \langle\sigma\rangle)$ is also well founded so $\sigma \in T^{\mathcal{G}} \cap \Gamma_e(T^{\mathcal{G}})$. On the other hand if there is $\sigma \notin T^p$ such that the initial segment of σ in T^p is not a leaf and $p \Vdash \text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded, then by definition $p \in \mathcal{G}$ means that $\text{Ext}(S_e^{\mathcal{G}}, \langle\sigma\rangle)$ is ill founded, so $\sigma \notin \Gamma_e(T^{\mathcal{G}})$. Since the initial segment of σ in T^p is not a leaf, no $q \leq p$ has $\sigma \in T^q$ so $\sigma \notin T^{\mathcal{G}}$.

Suppose towards a contradiction that $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is not dense. Let p be such that for all $q \leq p$ we have $q \nVdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$. Consider some leaf $\sigma \in T^p$ and let i, j be such that $T_{\sigma^\frown i} = T_{\sigma^\frown j} = \{\rho : L^p(\rho, \sigma) \leq 1\}$. Now consider $q = p[\{\sigma^\frown i\}]$; this is well defined by Lemma 6.2.4. By assumption on p we have that $q \nVdash \text{Ext}(S_e^{\mathcal{G}}, \langle \sigma^\frown j \rangle)$ is ill founded, so by Lemma 6.2.7 there is $r \leq q, \alpha < \omega_1^{CK}$ such that $r \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, \langle \sigma^\frown j \rangle)) \leq \alpha$. Now consider $r' = r[\{\sigma^\frown j\}]$. Since $\sigma^\frown i \in T^r$ we have $\{\rho : L^r(\rho, \sigma) \leq 1\} \subseteq T_{\sigma^\frown i} = T_{\sigma^\frown j}$ and thus the condition r' is a valid condition. Since $r \leq p$ and σ is a leaf in T^p we have that $r' \leq p$. But we have $S_e^r \supseteq S_e^{r'}$ so $r' \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, \langle \sigma^\frown j \rangle)) \leq \alpha$ a contradiction. So we have that the set $\{p : p \Vdash \overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})\}$ is dense.

So for sufficiently generic $\mathcal{G}$ we have that $T^{\mathcal{G}}$ is uniformly e-pointed without dead ends and for all e we have $\overline{T^{\mathcal{G}}} \neq \Gamma_e(T^{\mathcal{G}})$, and thus $\overline{T^{\mathcal{G}}} \nleq_{he} T^G$. □

This now allows us to conclude the following:

Corollary 6.2.8. *There are sets A, B such that $B \nleq_{he} A$ and for any X, if $A \leq_{he} X \oplus \overline{X}$ then $B \leq_{he} X \oplus \overline{X}$.*

Proof. We will have $A = T$ and $B = \overline{T}$ where T is a uniformly e-pointed tree with no dead ends that is not hypertotal. Suppose that T is Π_1^1 in X. Since T has no dead ends, there must be a path $P \in [T]$ such that $P \leq_h X$. So $T \leq_e P$ and by Lemma 5.2.4 we have $\overline{T} \leq_{he} \overline{P} \leq_h X$. So we get that $\overline{T} \leq_{he} X \oplus \overline{X}$. □

6.2.3 Relationship to introenumerable sets

As promised in Chapter 5 we now have the tools to prove the following theorem:

Theorem 5.5.13. *There is a uniformly Baire e-pointed tree that is not hyperenumeration equivalent to any introenumerable set.*

Proof. To do this we use the following lemma.

Lemma 6.2.9. *If $T^{\mathcal{G}}$ is sufficiently generic and $\Gamma_e(T^{\mathcal{G}})$ is infinite then there is $p \in \mathcal{G}$ such that $\Gamma_e(T^p)$ is infinite.*

Proof. Fix e and some condition p. For each n consider the set $D_{n,p} = \{q \leq p : \exists \alpha < \omega_1^{CK} m > n[q \Vdash \text{rank}(\text{Ext}(S_e^{\mathcal{G}}, m)) \leq \alpha]\}$. If for some n this is not dense above p then by Lemma 6.2.7 there is an extension of p that forces $\Gamma_e(T^{\mathcal{G}})$ to be finite.

So suppose that this set is dense above p for each n. Like in Lemma 6.2.6 we can define a sequence of extensions $r_i < p$ such that for any $q_j < r_j$ and $q_i < r_i$, $T^{q_j} \cap T^{q_i} = T^p$. Observe that the set D_{n,r_i} is uniformly $L_{\omega_1^{CK}}$-c.e. in i, n so by Σ_1-collection we have that there is a function $f \in L_{\omega_1^{CK}}$ such that $f(i) \in D_{i,r_i}$. Using the same merging of conditions that we did in Lemma 6.2.6 we can build $\hat{p} < p$ such that $\Gamma_e(T^{\hat{p}})$ is infinite.

We have now shown that the set $\{p : |\Gamma_e(T^p)| = \omega \vee p \Vdash |\Gamma_e(T^{\mathcal{G}})| < \omega\}$ is dense. Thus a sufficiently generic $\mathcal{G}$ satisfies this lemma. □

For a sufficiently generic $\mathcal{G}$ we now claim that $T^{\mathcal{G}}$ does not hyperenumerate any non-Π^1_1 introenumerable sets. Suppose that $\Gamma_e(T^{\mathcal{G}})$ is infinite. By Lemma 6.2.9 there is a $p \in \mathcal{G}$ such that $\Gamma_e(T^p)$ is infinite. If $\Gamma_e(T^{\mathcal{G}})$ is introenumerable then $\Gamma_e(T^{\mathcal{G}}) \leq_{he} \Gamma_e(T^p) \leq_{he} T^p \in L_{\omega_1^{CK}}$, so $\Gamma_e(T^{\mathcal{G}})$ is Π^1_1.

Since for any sufficiently generic $\mathcal{G}$, $T^{\mathcal{G}}$ is not a Π^1_1 set, $T^{\mathcal{G}}$ cannot be hyperenumeration equivalent to any introenumerable set below it. □

6.3 Downwards density

In this section we prove that the hyperenumeration degrees are downwards dense. The first part involves lifting the finite injury construction of the Gutteridge operator to a construction in $L_{\omega_1^{CK}}$.

6.3.1 The hyper Gutteridge operator

Gutteridge [16] proved the downwards density of the non-Δ^0_2 enumeration degrees using an operator Θ with the properties that if $\Psi_e(\Theta(A)) = A$ then A is c.e. and if $\Theta(A)$ is c.e. then A is Δ^0_2. Here we will take Gutteridge's construction and run it in $L_{\omega_1^{CK}}$ to produce a hyperenumeration operator Λ with similar properties. Thus we get the following result

Theorem 6.3.1. *If $A \subseteq \omega$ and $A \not\leq_{he} \overline{\mathcal{O}}$ then there is $C \subseteq \omega$ such that $\emptyset <_{he} C <_{he} A$.*

Proof. Recall the definition of Θ: there is a c.e. set $B = \bigoplus_{k \in \omega} n_k$ which is the join of ω many initial segments of ω. Θ is defined by $\Theta(A) = B \cup \{(k, n_k) : k \in A\}$. B is built using finite injury to ensure that if $\Psi_e(\Theta(A)) = A$ then A is c.e. If $\Theta(A)$ is c.e. then $A \leq_e \overline{B}$ and so A is Δ^0_2. Hence for any non-Δ^0_2 set A we have that $\emptyset <_e \Theta(A) <_e A$.

To ensure that if $\Psi_e(\Theta(A)) = A$ then A is c.e. B has the property that for any $D \subseteq n \geq e$ we have that $n \in \Psi_e(\Theta(D)) \iff n \in \Psi_e(\Theta(D \cup (\omega \setminus n)))$. So if $\Psi_e(\Theta(A)) = A$ then

$$D \preceq A \iff D{\upharpoonright}e \prec A{\upharpoonright}e \wedge \forall n \in D \setminus e[n \in \Psi_e(\Theta(D{\upharpoonright}n)]$$

We will use this idea to build Λ.

Before we start building Λ we need to set up some notation. For an $L_{\omega_1^{CK}}$-c.e. set A, given by formula $\exists y\varphi(x, y)$ where φ is Δ_0, we define $A_\alpha = \{x \in L_\alpha : L_\alpha \models \exists y\varphi(x, y)\}$. Since φ is Δ_0 we have that $A = \bigcup_{\alpha<\omega_1^{CK}} A_\alpha$. In this manner we can think of $L_{\omega_1^{CK}}$-c.e. sets as being enumerated over ordinal stages. Using this, for a set $B \in L_{\omega_1^{CK}}$ and ordinal $\alpha < \omega_1^{CK}$ we can define $\Gamma_{e,\alpha}(B) \in L_{\omega_1^{CK}}$ and get an $L_{\omega_1^{CK}}$-computable map $(B, e, \alpha) \mapsto \Gamma_{e,\alpha}(B)$. This is the hyperenumeration analogue of $\Psi_{e,s}(D)$. We can define $\Gamma_{e,\alpha}(B)$ more explicitly as $\Gamma_{e,\alpha}(B) = \{n : \text{rank}(S_{e,n}(B)) \leq \alpha\}$.

In the enumeration case, it is clear that $\Psi_e(W) = \bigcup_s \Psi_{e,s}(W_s)$ for a c.e. set W, but this is not so clear for an $L_{\omega_1^{CK}}$-c.e. set A. Γ_e is monotonic, so we have that $\bigcup_{\alpha\in\omega_1^{CK}} \Gamma_{e,\alpha}(A_\alpha) \subseteq \Gamma_e(A)$. The other direction is needed for our construction, so we will prove it here using the rank of nodes in $S_e(A)$.

Claim 6.3.1.1. $\bigcup_{\alpha\in\omega_1^{CK}} \Gamma_{e,\alpha}(A_\alpha) = \Gamma_e(A)$ *for any* $L_{\omega_1^{CK}}$*-c.e.* A.

Proof. Consider some node $x \in S_e(A)$ with ordinal rank. We will use induction on the rank of x and Σ^1_1 bounding to prove that there is $\alpha < \omega_1^{CK}$ such that x has rank $< \alpha$ in $S_e(A_\alpha)$. Base case: if x is a leaf then by definition of Γ_e there is a finite $D_u \subseteq A$ such that $(x, u) \in W_e$. There is $\alpha < \omega_1^{CK}$ such that $D_u \subseteq A_\alpha$ so x is a leaf in $S_e(A_\alpha)$.

For the inductive step, suppose by the inductive hypothesis that for each $i \in \omega$ there is a least $\alpha_i < \omega_1^{CK}$ such that $x^\frown i$ has rank $< \alpha_i$ in $S_e(A_{\alpha_i})$. Consider the map $i \mapsto \alpha_i$. This is $L_{\omega_1^{CK}}$-computable and hence by Σ_1-collection there is a $\beta < \omega_1^{CK}$ such that $\alpha_i < \beta$ for all i. So it must be that x has rank $\leq \beta$ in $S_e(A_\beta)$.

So we have that if $n \in \Gamma_e(A)$ then there is $\alpha < \omega_1^{CK}$ such that $\text{rank}(S_{e,n}(A_\alpha)) < \alpha$ hence we have that $n \in \Gamma_{e,\alpha}(A_\alpha)$. □

For our construction of Λ we will modify Gutteridge's proof. We will build an $L_{\omega_1^{CK}}$-c.e. set $B = \bigoplus_{k\in\omega} n_k$ and define $\Lambda(A) = B \cup \{(k, n_k) : k \in A\}$. We will build B using stages in ω_1^{CK} and satisfy the following requirements for $D \subseteq m \geq e$:

$$\mathcal{R}_{e,m,D} : m \in \Gamma_e(B \cup \{(k, n_k) : k \in D\}) \iff m \in \Gamma_e(B \cup \{(k, n_k) : k \in D \vee k \geq m\})$$

We chose an ordering of requirements so that $\mathcal{R}_{e,m,D}$ is higher priority than $\mathcal{R}_{i,m+1,E}$. Note that this means the priority of our requirements has order type ω.

Now we can move onto the construction. A requirement $\mathcal{R}_{e,n,D}$ requires attention at stage α if there is $n \notin \Gamma_{e,\alpha}(B_\alpha \cup \{(k, B_\alpha^{[k]}) : k \in D\})$ and there is $B \in L_\alpha$ such that $B_\alpha \subseteq B$ and $B_\alpha^{[k]} = B^{[k]}$ for $k < n$, B is the join of initial segments of ω and $n \in \Gamma_{e,\alpha}(B \cup \{(k, B^{[k]}) : k \in D\})$.

At stage α we consider the highest priority requirement that requires attention with some witness B. We then define $B_{\alpha+1} = B$. This completes the construction.

By the monotonicity of the Γ_e each requirement will need to act at most once, and that means that each column of B is finite. Now suppose that $\Gamma_e(\Lambda(A)) = A$ for some A and e. It is enough for us to show that A is $L_{\omega_1^{CK}}$-c.e. We claim that $A = \cup\{D : D{\restriction}e = A{\restriction}e \wedge \exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D{\restriction}n))]\}$. If D is such that $\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D{\restriction}n))]$ then by induction on $n \geq e$ we can see that $D \subseteq A$. So what we need to prove is that all $n \in A$ are contained in some such D. Fix $n \in A \setminus e$ and consider $D = A{\restriction}n$. We have $\Lambda(D \cup (\omega \setminus n))$ is $L_{\omega_1^{CK}}$-c.e. Since $n \in A$ by Claim 6.3.1.1 there is a stage $\alpha < \omega_1^{CK}$ such that $n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D \cup (\omega \setminus n)))$. So at stage α or earlier the requirement $R_{e,n,D}$ will have acted and we have $n \in \Gamma_{e,\alpha+1}(\Lambda_{\alpha+1}(D))$. We can assume by induction that D has the property $\exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D{\restriction}n))]$. We have now proven that $D \cup \{n\} = A{\restriction}n+1$ also has this property, thus by induction $A = \cup\{D : D{\restriction}e = A{\restriction}e \wedge \exists\alpha\forall n \in D \setminus e[n \in \Gamma_{e,\alpha}(\Lambda_\alpha(D{\restriction}n))]\}$. Hence A is $L_{\omega_1^{CK}}$-c.e. □

6.3.2 Downwards density below $\overline{\mathcal{O}}$

We have proven downwards density for most degrees in $\mathcal{D}_{he}$, but the proof may not work when a degree is below $\overline{\mathcal{O}}$. If we look at some of the proofs of downwards density for the degrees below $\mathbf{0}'_e$ and try to translate them, then we have a problem. They are finite injury constructions and rely on the following property of enumeration operators: $\Psi_e(A) = \bigcup_{D \subseteq_{\text{fin}} A} \Psi_e(D)$. This property does not hold for hyperenumeration operators. In fact there are many sets A and operators e such that $\Gamma_e(A) \neq \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$. For example, if A is the graph of a non-hyperarithmetic function and Γ_e is such that $0 \in \Gamma_e(B)$ if and only if B contains the graph of some function. None of the hyperarithmetic subsets of A will contain a function, but A does contain a function, so $0 \in \Gamma_e(A) \setminus \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$.

The reason we did not have this problem when adapting the Gutteridge operator was because of the special way that it was constructed. First, it is important to note that for Π^1_1 sets A we do have $\Gamma_e(A) = \bigcup_{H \subseteq_{\text{hyp}} A} \Gamma_e(H)$ by Claim 6.3.1.1. This property also holds for sets X of the form $X = \Lambda(A)$. To see this fix some Γ_e. For $n \geq e$ we have $n \in \Gamma_e(X) \iff n \in \Gamma_e(\Lambda(A \upharpoonright n))$. Here $\Lambda(A \upharpoonright n)$ is a Π^1_1 set, if $n \in \Gamma_e(X)$ then there is some hyperarithmetic $H \subseteq \Lambda(A \upharpoonright n) \subseteq X$ with $n \in \Gamma_e(H)$. The result for $n < e$ comes from some coding of indices of reductions.

We will make use of this idea in the proof of the following:

Theorem 6.3.2. *If A is not Π^1_1 and $\Lambda(A)$ is Π^1_1 then there are $X <_{he} A$ such that $X >_{he} \mathbf{0}$.*

Proof. Let $(A_s)_{s<\omega_1^{CK}}$ be an $L_{\omega_1^{CK}}$-computable approximation to A. We know there must be one since $\Lambda(A)$ is Π^1_1. We will build $L_{\omega_1^{CK}}$-c.e. operator Ψ, and define $X = \Psi(A)$. There are two types of requirements we need to satisfy

$$\mathcal{A}_e : X \neq V_e \text{ where } V_e \text{ is the } e\text{th } \Pi^1_1 \text{ set}$$

that will ensure that $\mathbf{0} <_{he} X$, and

$$\mathcal{B}_e : \Gamma_e(X) \neq A$$

that will ensure that $A \not\leq_{he} X$. The ordering of requirements is $\mathcal{A}_0 < \mathcal{B}_0 < \mathcal{A}_1 < \dots$. We will build Ψ and X in ω_1^{CK} many stages. We will want to be able to add infinitely much to columns of X so we will build X as a subset of $\omega_1^{CK^2}$. By fixing an $L_{\omega_1^{CK}}$-computable injection from ω_1^{CK} to ω (for instance the well founded part of a Harrison order) we can turn X into a subset of ω.

We will use an infinite injury construction here, putting the requirements on a tree of strategies. The outcome of node σ on the tree will be a set $\hat{A} \in L_{\omega_1^{CK}}$ that represents the strategy σ's guess at A at this stage of the construction. Strategies $\sigma\widehat{\ }\hat{A}$ and below will only add axioms of the form (n, H) to Ψ for sets $H \supseteq \hat{A}$. This way their work will not interfere with strategies who think $\hat{A} \nsubseteq A$. Each strategy σ can put up a restriction $u < \omega_1^{CK}$ and require that strategies to the right of them on the tree only put things in columns $\geq u$ of X. For σ we consider its restriction u to be the sup of all the restrictions put up by nodes to the left of or above σ. The ordering of outcomes is $\hat{A} < \hat{B}$ if $\hat{A} \supseteq \hat{B}$. This is only a partial order on sets, but we will see in the construction that we only use a limited collection of outcomes for each strategy, and this collection will be linearly ordered. We will argue that the leftmost path visited cofinally often is correct in its guesses about A and along this path all requirements are met. When a strategy has outcome $\hat{A}$ to the left of a previous outcome $\hat{B}$, it adds $(n, \hat{A})$ to Ψ for all n added by strategies below $\hat{B}$ to ensure that axioms added by strategies below $\hat{B}$ cannot interfere with the strategies below $\hat{A}$.

Strategies: The strategy for a node σ of requirement $\mathcal{B}_e$ is to find a witness m where $m \in A \backslash \Gamma_e(X)$. When this strategy is initialized at some stage s it is given outcome $\hat{A}$ from its parent node and restriction u, the sup of the restrictions put up by nodes to the left of σ. σ has one variable m_s that it keeps track of. When initialized, we start with $m_{s+1} = 0$.

At a limit stage s, we define $m_s = \liminf_{t<s} m_t$ if that exists, otherwise $m_s = 0$. We also have $X_s = \Psi_s(\hat{A})$ which is σ's guess at X at stage s. In the verification, we will prove that X_s eventually agrees with X on the first u many columns, that $m = \lim_{s<\omega_1^{CK}} m_s$ exists, and that $A(m) \neq \Gamma_e(X)(m)$.

Given m_s and X_s at stage s, the strategy asks if $m_s \notin \Gamma_{e,s}(X_s)$ and there is some $H \in L_s$ such that $H \subseteq (\omega_1^{CK} \setminus u) \times \omega_1^{CK}$ and $m_s \in \Gamma_{e,s}(X_s \cup H)$. If yes, then we put $m_s \in X$: we add axioms $(\langle \alpha, \beta \rangle, \hat{A})$ to Ψ_{s+1} for each $\langle \alpha, \beta \rangle \in X_s \cup H$ with $\alpha \geq u$. This will injure all lower priority $\mathcal{A}$ requirements. If no and $m_s \notin (A_s \cup \hat{A}) \triangle \Gamma_{e,s}(X_s)$, then we need to pick a new witness: set m_{s+1} to be the least m in $(A_s \cup \hat{A}) \triangle \Gamma_{e,s}(X_s)$. Otherwise $m_{s+1} = m_s$. No matter what, the outcome of σ is always $\hat{A}$, and σ does not put up any restriction.

The strategy for a node τ of requirement $\mathcal{A}_e$ is as follows. We will try to build an $L_{\omega_1^{CK}}$-c.e. aproximation $(P_s)_s$ to A by encoding parts of A into column u of X. The aproximation to A will eventually fail, as A is not $L_{\omega_1^{CK}}$-c.e., and we will use this point of difference to ensure that so that $X \neq V_e$. To this end we will build a sequence of coding points $(n_\beta, m_\beta)_{\beta<\alpha_s}$. The only changes we will make to this sequence are to add a new element to the sequence or remove the last element (if there is one). So at limit stages s we can have $\alpha_s = \liminf_{t<s} \alpha_t$ and that will ensure that all the coding points are well defined at stage s. The idea with the coding points is that for all but the top one we have ensured that $\langle u, n_\beta \rangle \in V_e$ and that $\langle u, n_\beta \rangle \in X$ only if $m_\beta \in A$. Our aproximation P_s will be $\hat{A} \cup \{m_\beta : \langle u, n_\beta \rangle \in V_{e,s}\}$. Since $V_{e,s}$ is increasing, P_s is increasing as long as we do not remove coding points (n_β, m_β) after putting $m_\beta \in P$.

We are trying to make $X \neq V_e$, so when we notice $m_\beta \notin A_s$ it appears that we have succeeded and do not need to do anything. The problem with this is that once we have $\alpha_s > \omega$ there are infinitely many m_β so we may see $m_\beta \notin A_t$ for a different β at each stage $t > s$ but have $m_\beta \in A$ for all $\beta < \alpha_s$. We could avoid this problem if we knew that A_s stabilized on hyperarithmetic sets, but we only know that it stabilizes on finite sets. This is enough for us, but we will have to keep track of the β where $m_\beta \notin A_s$, and sometimes our outcome will need to include numbers that are not in A_s.

To this end, τ will keep track of a finite sequence of victories $\hat{\beta}_s = \beta_0 > \beta_1 > \cdots > \beta_{k-1}$ with the property that $m_{\beta_0} < \cdots < m_{\beta_{k-1}}$ are numbers we thought were out of A at the previous stage. When τ is initialized we start with $\hat{\beta} = \emptyset$ and at limit stages s we define $\hat{\beta}_s$ to be the longest sequence $\hat{\beta}$ such that $\hat{\beta}_i = \lim_{t<s} \hat{\beta}_{t,i}$. At a stage s the requirement updates the victories as the first step. We consider β such that m_β is the smallest $m \in \overline{A_s}$ with the property for all $i < k$, $m_{\beta_i} \leq m \implies \beta_i > \beta$. If there is such a β then we add it to our sequence of victories for $\hat{\beta}_{s+1}$ and remove all victories $\beta_i < \beta$. Next we remove invalid victories: if there is any $i < k$ such that $m_{\beta_i} \in A_s$ then we remove that victory and all victories for $j \geq i$. The reason we also remove larger victories is to deal with the fact that A_s only stabilizes on finite sets. In the verification we will prove that an initial segment of $\hat{\beta}_s$ stabilizes, and to ensure that that initial segment is β_s cofinally often, we need to remove smaller β_i whenever we see a change.

If the sequence of victories has become empty, then we think $P_s \subseteq A$ so we need to consider adding a coding point. If α_s is a successor, then we consider the highest coding point (n, m). If $\langle u, n\rangle \in X \setminus V_e$ then we can satisfy the $\mathcal{A}_e$ without any victories. So if $\langle u, n\rangle \notin V_{e,s}$ and $m \notin A_s$, then we add the axiom $(\langle u, n\rangle, P_s)$ to Ψ_{s+1} to keep $\langle u, n\rangle \in X$. This will, however, invalidate the coding point, meaning we will have to remove it and try again if we ever see $\langle u, n\rangle$ enter V_e. If $\langle u, n\rangle \in V_e$ then it is time to add a new coding point. If (n, m) has been invalidated, then we remove it from the sequence of coding points first. If (n, m) has not been invalidated then we add m to P_{s+1}. To pick a new coding point we choose n_{α_s} to be the least unused number in column u and m_{α_s} to be the least member of $A_s \setminus P_s$ if there is one. We then add the axiom $(\langle u, n_{\alpha_s}\rangle, P_s \cup \{m_{\alpha_s}\})$ to Ψ_{s+1}. If $A_s \subseteq P_s$ then we cannot add a new coding point yet.

If there are no victories and α_s is a limit or 0, then we proceed to add a new coding point as above.

Finally we come to defining the outcome and restriction of τ. We always impose restriction $u + 1$ on lower priority requirements so they cannot interfere with our coding points in column u. To define the outcome, we use the sequence of victories. If there are no victories this means we have outcome P_s since it looks like $P_s \subseteq A$. If there are victories, then we consider the least victory β_{k-1}. Since it looks like $m_{\beta_{k-1}} \notin A$ but $m_\beta \in A$ for all $\beta < \beta_{k-1}$ we give outcome $\hat{A} \cup \{m_\beta : \beta < \beta_{k-1}\}$. Note that, as promised above, the collection of outcomes is linearly ordered.

Verification: We will use induction to argue that for each node σ on the true path, the following hold:

1. There is a left most outcome $\hat{A}$ that is visited cofinally often.

2. $\hat{A} \subseteq A$.

3. For all $\hat{B} < \hat{A}$ that were outcomes of σ we have $\hat{B} \nsubseteq A$.

4. $\Psi(\hat{A})^{[u]} = \Psi(A)^{[u]}$.

5. σ stops adding axioms to Ψ after some stage.

6. The requirement for σ is satisfied.

We start with the case where σ is a strategy for a $\mathcal{B}_e$ requirement. Let s be a stage after which no node to the left of σ is visited and no node above σ adds axioms to Ψ. In this case σ only has one outcome $\hat{A}$, and since it only adds axioms using $\hat{A}$, 1. through 4. hold. So we just need to check that the requirement was met and stops adding axioms. Consider the set $W = \bigcup_{s<\omega_1^{CK}} \Gamma_{e,s}(X_s) = \lim_{s<\omega_1^{CK}} \Gamma_{e,s}(X_s)$. Since W is $L_{\omega_1^{CK}}$-c.e. there is some least $m \in W \triangle A$. So we have that $m = \lim_{s<\omega_1^{CK}} m_s$, and once this limit settles down σ puts axioms into Ψ at most once more, so 5. is satisfied.

Now to show the requirement is met. We have two cases. Case 1: suppose that $m \in W$. Then $m \notin A$ and $m \in \Gamma_e(X)$ since $X_s \subseteq \Psi_s(\hat{A}) \subseteq \Psi(A) = X$, so the requirement is satisfied.

Case 2: suppose that $m \in A$. Then consider the set $X^* = \Psi(\hat{A}) \cup \{n : \exists H, \tau \prec \sigma[\tau \text{ put } (n, H) \in \Psi]\}$. Since any number put into X by a strategy to the right of σ is put into X with a subset of $\hat{A}$ when we next visit σ, and because requirements to the left of σ only add axioms (n, H) for $H \nsubseteq A$ by 3., we have that $X \subseteq X^*$ and $X^{[v]} = X^{*[v]}$ for all $v < u$. It is clear that X^* is $L_{\omega_1^{CK}}$-c.e. so if $m \in \Gamma_e(X^*)$ then by Claim 6.3.1.1 there is $t < \omega_1^{CK}$ and hyperarithmetic $H \subseteq X^*$ such that $m \in \Gamma_{e,t}(H)$. Since X and X^* agree on the first u many columns we have that $m \in W$ as σ will have acted at some stage $\geq t$ to ensure this. But $m \in A$, a contradiction, so $m \notin \Gamma_e(X^*) \supseteq \Gamma_e(X)$. So the requirement is satisfied.

Now we consider the case where σ is a strategy for an $\mathcal{A}_e$ requirement. First we will argue that we stop adding coding locations after some stage. Consider the set $P = \bigcup_s P_s$. This is an $L_{\omega_1^{CK}}$-c.e. set, so there is some least $m \in P \triangle A$. Consider some stage s such that $A \upharpoonright m + 1 = A_t \upharpoonright m + 1 = P_s \upharpoonright m + 1 \triangle \{m\}$ for all $t > s$. If $m \in P$ then after stage s we will always have $m = m_{\beta_0}$ as the first victory, so σ will stop growing P and will not add any more coding locations. If $m \notin P$ then after stage s whenever we chose a new coding location (n', m') we will have $m' = m$. Since m never leaves A_t this location will never be invalidated, so, since $m \notin P$, it must be that $\langle u, n' \rangle \notin V_e$ so we stop adding coding locations. In either case 5. is satisfied.

Next we argue that the sequence of victories stabilizes on an initial segment. If $P \subseteq A$ then this initial segment will be the empty set. Otherwise, observe that $P_s = \{m_\beta : \beta < \alpha_s\}$ (with the last element excluded if it has not been added). Consider the sequence $(\beta_i)_{i<k}$ defined by taking β_i is the least β such that m_β is the least element of $\{m_\beta : \beta < \beta_{i-1}\} \setminus A$ if this set is nonempty. Since α_s is well founded this sequence must be finite and have some length k. Consider a stage s after which $A_t \upharpoonright m_{\beta_{k-1}} + 1$ and P_t have stabilized. At all stages $t > s$ where σ is visited we must have $\beta_0 > \cdots > \beta_k$ as a proper initial segment of the sequence of victories as these are all true victories and no other victories could be added for $m_\beta < m_{\beta_{k-1}}$ after stage s. So after stage s the outcome of σ will always be a subset of $\hat{A} := \hat{B} \cup \{m_\beta : \beta < \beta_{k-1}\}$ where $\hat{B}$ was the outcome of the parent of σ. We now claim that $\hat{A}$ will satisfy 1. through 3.

If the outcome is ever $\hat{C} < \hat{A}$ then it must be that $m_{\beta_{k-1}} \in \hat{C}$ so 3. is satisfied and after stage s we never have any outcome left of $\hat{A}$. Since we could not extend our sequence $(\beta+i)_{i<k}$ it must be that $\hat{A} \subseteq A$ so 2. is satisfied. This also means that any victory β added to the end of our sequence of victories must have $m_\beta \in A$. This means that β will eventually be removed from our sequence of victories when we see $m_\beta \in A_t$ for some t. We remove a victory, we also remove all victories for $m > m_\beta$, so there will be cofinally many stages where the sequence of victories is just $(\beta_i)_{i<k}$. Hence the outcome of σ will cofinally often be $\hat{A}$, satisfying 1. To see 4. recall that when coding location (n, m) was added at stage t we used $P_t \cup \{m\}$ to put it in X. So if it was added before the coding location $(n_{\beta_{k-1}}, m_{\beta_{k-1}})$ was added then $\langle u, n\rangle \in \Psi_e(\hat{A})$, and if it was added after, then $\langle u, n\rangle \notin \Psi(A)$.

To see that the requirement for σ is satisfied, we need to look at two cases. First, if the sequence of victories was empty this meant that $P \subseteq A$ and we stopped adding coding locations because the top location (n, m) had $\langle u, n\rangle \notin V_e$. If this location was never invalidated then, $m \in A$ and $\langle u, n\rangle \in \Psi(A)$. If it was invalidated, then we added the axiom $(\langle u, n\rangle, P)$ to Ψ so $\langle u, n\rangle \in \Psi(A)$. Second, if the sequence of victories was not empty then $m_{\beta_0} \notin A$, so $\langle u, n_{\beta_0}\rangle \notin \Psi(A)$ but $m_{\beta_0} \in P$ so $\langle u, n_{\beta_0}\rangle \in V_e$.

This completes the induction. Note that condition 1. ensures that there is a true path. Since each requirement on the true path is satisfied we have that X is not $L_{\omega_1^{CK}}$-c.e. and $A \not\leq_{he} X$. The fact that $X \leq_{he} A$ follows from Proposition 6.4.4, which is proved in the next section.

$\square$

6.4 Other reducibilities

We now look at some other reducibilities that are different from $\leq_{he}$ but could be considered notions of hyperenumeration reducibility. We show most of these reducibilities $\leq_*$ share some of the properties of $\leq_{he}$, like extending enumeration reducibility and having $A \leq_* B \oplus \overline{B} \iff A$ is Π^1_1 in B. The reducibility to consider is the notion of relatively Π^1_1.

Definition 6.4.1. We say that A is *relatively* Π^1_1 *in* B, $A \leq_{\Pi^1_1} B$, if whenever B is Π^1_1 in X we have that A is Π^1_1 in X.

We say that A is *uniformly relatively* Π^1_1 *in* B, $A \leq_{u\Pi^1_1} B$, if there is a computable f such that if $B = \Gamma_e(X \oplus \overline{X})$ then $A = \Gamma_{f(e)}(X \oplus \overline{X})$

We used hyperenumeration operators to define $\leq_{u\Pi^1_1}$, but it could equivalently be defined by saying there is Turing operator that turns hyperenumerations of B into hyperenumerations of A or that there is a computable function that turns Π^1_1 formulas for B into Π^1_1 formulas for A.

The fact that composition of hyperenumeration operators is uniform means that $A \leq_{he} B \implies A \leq_{u\Pi^1_1} B$ and by definition we have $A \leq_{u\Pi^1_1} B \implies A \leq_{\Pi^1_1} B$. It is natural to ask if these implications are strict. From Theorem 6.2.1 we can see that $\leq_{he}$ is different from $\leq_{\Pi^1_1}$ because by definition each relatively Π^1_1 degree is uniquely determined by the total degrees above it. A closer look at the proof of Corollary 6.2.8 show us that $\overline{T} \leq_{u\Pi^1_1} T$ for any uniformly e-pointed tree without dead ends, hence $\leq_{he}$ and $\leq_{u\Pi^1_1}$ are different. The remaining possible separation is an open question.

Question 6.4.2. Are there sets A and B such that $A \leq_{\Pi^1_1} B$ and $A \not\leq_{u\Pi^1_1} B$.

A negative answer to the above questions could be seen as a proof of Selman's theorem for $\leq_{u\Pi^1_1}$. One approach to try to answer this question is to see if one can transform Selman's original proof to this context.

On the other hand, we observe that the uniformity of an e-pointed tree T without dead ends is important for the proof of $\overline{T} \leq_{u\Pi^1_1} T$. Perhaps there is a sufficiently generic non-uniformly e-pointed tree T without dead ends such that $\overline{T} \not\leq_{u\Pi^1_1} T$. Such a result would be interesting because it would show that there is no notion hyperenumeration operators for relatively Π^1_1.

Another way that may be natural to define hyperenumeration reducibility is by changing the nature of the set W in the usual definition of enumeration reducibility.

Definition 6.4.3. We say that A is *continuously higher enumeration reducible* to B, $A \leq_{che} B$ if there is a Π^1_1 set W such that $n \in A \iff \exists u[\langle n, u\rangle \in W \wedge D_u \subseteq B]$.

We say that A is ω_1^{CK}-*enumeration reducible* to B, $A \leq_{\omega_1^{CK}} B$, if there is an $L_{\omega_1^{CK}}$-c.e. set W such that $n \in A \iff \exists H[(n,H) \in W \wedge H \subseteq B]$

Both these reducibilities can be thought of as relativizations of enumeration reducibility to $L_{\omega_1^{CK}}$. In the case of continuously higher enumeration operators these are, like enumeration operators, continuous functions $\S^\omega$, hence the name. This reducibility could be thought of as an enumeration analogue of continuously higher Turing reducibility.

Both of these reducibilities imply hyperenumeration reducibility. For $\leq_{che}$ this follows from the fact (Sanchis [39]) that we can replace the c.e. set in the definition of hyperenumeration reducibility with a Π^1_1 set. It takes a bit more work for ω_1^{CK}-enumeration reducibility.

Proposition 6.4.4. *If $A \leq_{\omega_1^{CK}} B$ then $A \leq_{he} B$.*

Proof. Suppose W is a $L_{\omega_1^{CK}}$-c.e. set of pairs such that $n \in A \iff \exists H[(n,H) \in W \wedge H \subseteq B]$. Since W is $L_{\omega_1^{CK}}$-c.e. there is an $L_{\omega_1^{CK}}$-computable injection $f : \omega_1^{CK} \to W$. Consider the set

$$V = \{\langle n, \sigma\widehat{\ }k, u\rangle : \exists H, i, e[(n,H) \in W, D_u = H \restriction k, |\sigma| = \langle i, e\rangle, e \in \mathcal{O}, H = \Psi_i(\emptyset^{(e)})]\}$$

Since W and $\mathcal{O}$ are $L_{\omega_1^{CK}}$-c.e., V is also $L_{\omega_1^{CK}}$-c.e. and hence Π^1_1. Now all that is needed is to check that $A \leq_{he} B$ via V. If $H \subseteq B$ and $(n,H) \in W$ then V will put $n \in A$ as every path of length $\langle i, e\rangle + 1$ will be removed for $H = \Psi_i(\emptyset^{(e)})$. If there is no $H \subseteq B$ such that $(n,H) \in W$ then we can build a path f as follows:

$$f(\langle i, e\rangle) = \begin{cases} 0 & e \notin \mathcal{O} \vee (n, \Psi_i(\emptyset^{(e)})) \notin W \\ \text{least } k \text{ such that } \Psi_i(\emptyset^{(e)}) \restriction k \not\subseteq B & \text{otherwise.} \end{cases}$$

Note: for all $\sigma \prec f$ we have that $\langle n, \sigma, u\rangle \notin V$ for any u with $D_u \subseteq B$. □

So both these reducibilities imply hyperenumeration reducibility. These implications are strict. In fact, if we consider some set X with $L_{\omega_1^{CK}} \in L_X$ then anything ω_1^{CK} enumeration reducible to $X \oplus \overline{X}$ will be hyperarithmetic in X and there are sets hyperenumeration reducible to $X \oplus \overline{X}$ that are not hyperarithmetic in X, for instance $\mathcal{O}^X$.

Since these are weaker than hyperenumeration reducibility, it can be that Selman's theorem holds for these reducibilities. For continuously higher enumeration reducibility we have a proof of Selman's theorem that uses the enumeration degrees.

Theorem 6.4.5. *The continuously higher enumeration degrees embed as the enumeration degrees above $\mathcal{O}$ via the map $X \mapsto \mathcal{O} \oplus X$.*

Proof. For one direction, suppose that $X \oplus \mathcal{O} \leq_e Y \oplus \mathcal{O}$. Then $X \leq_e Y \oplus \mathcal{O} \leq_{che} Y$.

For the other direction, suppose that $X =\leq_{che} Y$ via the Π^1_1 set W. Let f be an m-reduction of W to $\mathcal{O}$. We define a c.e. set $W_e = \{\langle n, u\rangle : D_u = D_v \oplus D_q \wedge D_q = \{2f(\langle n, v) + 1\}\}$. So we have that $n \in \Psi_e(Y \oplus \mathcal{O}) \iff \exists v[D_v \subseteq Y \wedge f(\wedge n, v\rangle) \in \mathcal{O}] \iff \exists v[D_v \subseteq Y \wedge \langle n, v\rangle \in W] \iff n \in X$. So $X \oplus \mathcal{O} \leq_e Y \oplus \mathcal{O}$.

To see that this embedding is onto, observe that every enumeration degree above $\mathcal{O}$ contains a set of the form $X \oplus \mathcal{O}$. □

To see how this gives us Selman's theorem recall that every enumeration degree **a** above $\mathcal{O}$ is uniquely determined by the class of total degrees above **a**. This means that every *che*-degree is uniquely determined by the class of degrees above it that map to a total enumeration degree. If an enumeration degree above $\mathcal{O}$ is total then it will contain a set of the form $X \oplus \overline{X} \oplus \mathcal{O}$ and be the image of a *che*-total degree.

Note that there are *che*-total degrees that get mapped to non-total *e*-degrees. For instance $\mathcal{O}$ is not total.

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