

Computability in Uncountable Binary Trees

By

Reese Johnston

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Abstract

Computability, while usually performed within the context of ω , may be extended to larger ordinals by means of α -recursion. In this thesis, I concentrate on the particular case of ω_1 -recursion, and study the differences in the behavior of Π_1^0 -classes between this case and the standard one.

Of particular interest are the Π_1^0 -classes corresponding to computable trees of countable width. Classically, it is well-known that the analog to König's Lemma - "every tree of countable width and uncountable height has an uncountable branch" - fails; I demonstrate that not only does it fail effectively, but that the failure is as drastic as possible. This is proven by showing that the ω_1 -Turing degrees of even isolated paths in computable trees of countable width are cofinal in the Δ_1^1 ω_1 -Turing degrees.

Finally, I consider questions of non-isolated paths, and demonstrate that the degrees realizable as isolated paths and the degrees realizable as non-isolated ones are very distinct; in particular, I show that there exists a computable tree of countable width so that every branch can only be realized in a tree with \aleph_2 branches.

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Chapter 1

Introduction

1.1 α -Recursion and Motivation

The results and discussion to follow are computability-theoretic in motivation and approach but set-theoretic in nature. The overarching theme in the study of α -recursion in general and in this work in particular is the question of “true finiteness” and its role in computability theory; that is to say, the motivating question is:

Question 1.1.1. *To what degree do the results of classical computability theory depend on genuine finiteness of the objects involved, rather than simply “smallness”?*

The natural approach to such a question, as in most mathematics, is to replace “finiteness” with another sense of “smallness” and examine the consequences. The most direct route is to simply replace the domain with a larger one; in this work, we will replace ω with ω_1 and “finite” with “countable”, leading to the following question.

Question 1.1.2. *To what degree do the results of classical computability theory hold in the setting of ω_1 -recursion?*

This is, of course, an ongoing project pursued by many mathematicians. In this thesis, we will concentrate on a single class of classical results.

Question 1.1.3. *To what degree do Π_1^0 -classes behave similarly between the settings of ω_1 -recursion and ω -recursion?*

In the standard setting, the behavior of Π_1^0 -classes varies depending on whether we consider Π_1^0 -classes in Cantor space or in Baire space; we will present a plausible analogue to this distinction in the uncountable setting, and demonstrate that the proposed analogues to Π_1^0 -classes in Cantor space behave very differently than in the classical setting.

1.2 Background

Arguably the field of α -recursion began with Kreisel and Sacks [12]; they studied the case in which α is taken to be the first admissible ordinal beyond ω . It was soon after expanded upon by Kripke [13], Platek [18], and Sacks [19], among many others. The fine structure theory developed by Jensen [8] provided the key techniques of the study.

In general, α -recursion is a complex subject; very little that holds at ω holds for all α . To take an extreme example, S.-D. Friedman [4] showed that when $\alpha = \aleph_{\omega_1}$ the α -Turing degrees are well-ordered by Turing reducibility above $\mathbf{0}'$. This result and others suggest that if a close analogue of ω is desired, α should be taken to be a regular cardinal. For the purposes of this thesis, we will consider the particular case of $\alpha = \omega_1$.

The following definitions are slight rephrasings of those originally set out by Kripke [13]; definitions specific to this work will be presented in the next section.

Definition 1.2.1. *A set is hereditarily countable if it is countable and all of its elements are hereditarily countable. Let HC be the collection of all hereditarily countable sets. Throughout this work, we will operate under the set-theoretic assumption that $L_{\omega_1} = HC$.*

A first-order formula φ is said to be Δ_0^0 if it is quantifier-free (for example, $x \in y$) or every quantifier is bounded (that is, of the form $\exists x \in y$ or $\forall x \in y$). A formula is Σ_1^0 if it is of the form $(\exists x)\varphi(x)$ for φ a Δ_0^0 formula. This extends to a general hierarchy: a

formula is Π_n^0 if it is the negation of a Σ_n^0 formula, and a formula is Σ_{n+1}^0 if it has the form $(\exists x)\varphi(x)$ for φ a Π_n^0 formula.

A subset $X \subseteq L_{\omega_1}$ is computably enumerable if there exists a Σ_1^0 formula φ and a parameter $c \in L_{\omega_1}$ so that for each $x \in L_{\omega_1}$, $x \in X$ iff $\varphi(x, c)$ is true.

X is computable if both X and $L_{\omega_1} \setminus X$ are computably enumerable. In general, a set is Σ_n^0 (resp. Π_n^0) if it is definable by a Σ_n^0 (Π_n^0) formula; it is Δ_n^0 if it is both Σ_n^0 and Π_n^0 . Thus X is computable iff X is Δ_1^0 .

There are many other definitions which are equivalent under the assumption that $L_{\omega_1} = HC$; perhaps the most compelling one defines a set to be computable if it is computable by a Turing machine with a tape of length ω_1 and permitted to run for any countable number of stages. In light of this, a useful intuition is that a set is *computable* if it is “intuitively” computable by a computer that can hold any hereditarily countable set in memory and is permitted to run for any countable length of time before halting.

Note that we now have two definitions of “computable”; the standard definition used in the setting of ω , and the newly defined one in the context of ω_1 . Henceforth, we will use *computable* to refer exclusively to the uncountable setting; on the occasions in which we need to refer to the more usual definition of computability, we will use phrases such as “ ω -computable” or “the standard setting”.

It is also useful to define a notion of *relative* computability: X is *computable from* Y if it is computable using information in Y . More precisely, if X and its complement can both be defined by Σ_1^0 formulas using Y as a parameter. If X is computable from Y and Y is also computable from X , we say that they are *Turing-equivalent* and we denote this $X \equiv_T Y$; the *Turing degree* of X , $\deg X$, is the set of all sets Turing-equivalent to X .

Definition 1.2.2. Given $X \subseteq \omega_1$, let X' denote the set of indices for Σ_1^0 formulas φ with parameters $c \in L_{\omega_1}$ so that $L_{\omega_1} \models \varphi(c)$. Let $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, and for a countable limit ordinal α let $X^{(\alpha)} = \{(\beta, \gamma) \mid \gamma \in X^{(\beta)}\}$. $X^{(\alpha)}$ for $\alpha \geq \omega_1$ can and will be

defined in a later section.

It will shortly become useful to extend the hierarchy noted above to higher-order levels.

Definition 1.2.3. $X \subseteq \omega_1$ is Σ_1^1 if there exists a first-order formula φ with a parameter $c \in L_{\omega_1}$ so that $x \in X \iff L_{\omega_1} \models (\exists Y)\varphi(Y, x, c)$, where Y ranges over subsets of L_{ω_1} .

X is Π_1^1 iff $\omega_1 \setminus X$ is Σ_1^1 , and X is Δ_1^1 if it is both Σ_1^1 and Π_1^1 .

When considering these higher-order quantifiers, another set-theoretic assumption is relevant: $L_{\omega_2} = H(\aleph_1)$, where $H(\aleph_1)$ is the collection of sets hereditarily of size $\leq \aleph_1$. For many of the results of this work, this assumption is not necessary; where it is, it will be noted as such.

One of the most important facts about ω_1 -recursion for our purposes is that there is a computable bijection between ω_1 and L_{ω_1} ; we will therefore interchange freely between the two.

Because we will be working within L , we will be able to make use of the canonical ordering on L , $<_L$. For a complete definition of this ordering, see [6]; for our purposes, the only important properties are the following.

- $<_L$ is a well-ordering.
- $<_L \upharpoonright L_{\omega_1}$ is (ω_1) -computable.

1.3 Notation and Conventions

Certain notation conventions we will preserve from the standard setting. For countable ordinals e , Φ_e or φ_e denotes the e th computable function; for $X \subseteq \omega_1$, Φ_e^X denotes the e th computable function with oracle X . A boldface lower-case letter $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ will denote an ω_1 -Turing degree unless otherwise noted.

By convention, we will use e, i, j for countable ordinals that represent indices of computable functions. n and m will be finite ordinals, and lower-case Greek letters $\alpha, \beta, \gamma, \dots$ will denote ordinals in general (usually countable). The lower-case Greek letters $\sigma, \tau, \rho, \theta$ will usually denote binary strings of countable length, and the symbol \frown will be used to denote concatenation of these. Upper-case Greek letters Γ, Δ, \dots will denote ω_1 -Turing functionals; Λ will usually denote a set of symbols. Upper-case letters (A, B, C, \dots) will be used for subsets of ω_1 or for requirements (as in a priority argument). T will almost exclusively refer to a binary tree (defined below).

$\Phi_{e,s}(x)$ denotes the e th computable function evaluated at x within the first s stages; the precise interpretation of this is not relevant, but one possible definition would be that $\Phi_{e,s}(x)$ is $\Phi_e(x)$ as evaluated over L_s rather than L_{ω_1} , in which case the computation is said to diverge if no witness to the relevant Σ_1^0 formulas exists in L_s .

By convention, when σ is a countable string, Φ_e^σ is evaluated only up to stage $|\sigma|$ and considered to diverge if it requires information past $|\sigma|$ from the oracle.

Throughout, we will use the ω_1 analogue of Church's thesis freely, and present constructions largely informally.

Chapter 2

Trees of Countable Width

2.1 Introduction

In this chapter, we will study a particular flavor of Π_1^0 class: those arising from trees that are *thin* in a particular sense. To begin, we must lay out a number of definitions that are the direct analogues of the corresponding notions in ω -computability.

Definition 2.1.1. $2^{<\omega_1}$ is the collection of all countable binary strings. For $\sigma, \tau \in 2^{<\omega_1}$, $\sigma \prec \tau$ if σ is a proper initial segment of τ ; likewise, $\sigma \prec X$ for $X \subseteq \omega_1$ if σ is a (proper) initial segment of the characteristic function of X . For $\sigma \in 2^{<\omega_1}$, $|\sigma|$ denotes its length: $\sup\{\alpha + 1 \mid \alpha \in \text{dom}(\sigma)\}$.

When σ is a string, and α is an ordinal, σ^α is the string produced by concatenating σ with itself α times; for example, 0^α is the sequence of length α consisting entirely of 0.

A binary tree is a subset of $2^{<\omega_1}$ that is downward-closed under \prec . If T is a binary tree, then a path through T is a set $X \subseteq \omega_1$ so that $\{\sigma \in 2^{<\omega_1} \mid \sigma \prec X\} \subseteq T$. $[T]$ denotes the set of paths through T . A path X is isolated if there is an initial segment σ of X so that X is the only path through T passing through σ . The unique path of a tree T is a path X so that $[T] = \{X\}$.

A binary tree is computable if it is computable as a subset of L_{ω_1} . A Π_1^0 -class is a

subset of $\mathcal{P}(\omega_1)$ that can be described as $[T]$ for some computable binary tree T .

In the countable setting, two distinct classes of trees are studied: finitely-branching trees, which are equivalent to binary trees; and infinitely-branching trees, equivalent to downward-closed subsets of $\omega^{<\omega}$. We could likewise define the latter in this setting, considering trees that are subsets of $\omega_1^{<\omega_1}$; however, it is evident that there is no difference between these and binary trees. The principle obstacle is that even in a binary tree, the set of elements of a particular length (e.g., length ω) may have size ω_1 , so an ω_1 -branching tree may be effectively “modeled” in a binary one. For this reason, we henceforth will not distinguish between binary trees and ω_1 -branching trees, and will use the word *tree* to refer to either or both of them.

However, most of the results in ω -computability regarding finitely-branching trees might be considered to depend less on the finiteness of the branching than on the finiteness of each level of the tree; while these are of course equivalent, their analogues in the setting of ω_1 are not. It seems natural to suppose, then, that the “correct” analogues of binary or finitely-branching trees in the countable setting might be trees of *countable width* in the uncountable setting.

Definition 2.1.2. *If T is a tree, T has countable width if $|T \cap 2^\alpha|$ is countable for every countable α .*

It should be noted that this is an imperfect analogue at best - for example, a finitely-branching tree in the countable setting is permitted to have levels of cardinality unbounded below ω , while a tree of countable width in our present setting must have levels of cardinality strictly bounded below ω_1 .

Before moving on to the principal definitions of this chapter, we prove a simple result that will be useful throughout the chapter.

Proposition 2.1.3. *There is a uniformly (partial) computable sequence $\langle T_i \rangle_{i < \omega_1}$ of subsets of $2^{<\omega_1}$ so that the following conditions hold:*

(i) T_i is a tree for every i , and

(ii) for every computable tree T , there is an i (uniformly in an index for T) so that $T = T_i$.

Proof. For any e , let T_e be the subset of $2^{<\omega_1}$ enumerated by Φ_e , except for the following:

(i) If $\Phi_{e,s}(\sigma) \downarrow = 1$ and an initial segment of σ has been excluded from T_e , then exclude σ from T_e ; and

(ii) If $\Phi_{e,s}(\sigma) \downarrow = 1$ and no initial segment of σ has been excluded from T_e , include all undecided initial segments of σ in T_e .

□

On the other hand, the natural extension of Prop. 2.1.3 to trees of countable width does not hold; that is, there is no uniformly computable sequence of trees of countable width which realizes every computable tree of countable width as above.

The primary topic of study in this document is the case of Π_1^0 -classes comprised of a single element.

Definition 2.1.4. A Π_1^0 singleton is the unique path in a computable tree. A thin Π_1^0 singleton is the unique path in a computable tree of countable width.

For ease of notation, we introduce some symbols to represent important classes of degrees.

Definition 2.1.5. Let $\mathcal{P} = \{\deg(D) : D \text{ is a } \Pi_1^0 \text{ singleton}\}$, and $\mathcal{P}_{\text{thin}} = \{\deg(D) : D \text{ is a thin } \Pi_1^0 \text{ singleton}\}$.

A degree in $\mathcal{P}_{\text{thin}}$ is called thin.

A degree \mathbf{d} is wide if no member of \mathbf{d} is contained in any thin Π_1^0 -class (of any cardinality).

An essential tool in the study of Π_1^0 -classes in the standard setting is König's Lemma, which for our purposes states the following:

Proposition 2.1.6. (*König's Lemma*) *If T is an infinite, finitely-branching tree, then T has an infinite branch.*

Among its most notable consequences is the following result:

Proposition 2.1.7. (*Folklore*) *If X is an isolated path of an ω -computable finitely-branching tree T , then X is also ω -computable.*

The analogous statement to König's Lemma in the uncountable setting might be formulated as “A countably-branching uncountable tree has an uncountable path”. However, the following example demonstrates that this fails:

Proposition 2.1.8. *There is a computable binary tree which is uncountable but has no uncountable path.*

Proof. $2^{\leq \omega}$, the set of binary strings of length $\leq \omega$, suffices. □

The triviality of this example suggests a re-formulation: “A countably-branching tree of height ω_1 has an uncountable branch”. This also has an immediate counterexample:

Proposition 2.1.9. *There is a computable binary tree with elements of every length but having no uncountable path.*

Proof. Note that there is a computable bijection $f : 2^\omega \rightarrow \omega_1$. Let $T = 2^{\leq \omega_1} \cup \{\sigma \frown 0^\alpha \mid \sigma \in 2^\omega \wedge \alpha \leq f(\sigma)\}$. Then T is the desired tree. □

The final, and most plausible, attempt at a rephrasing of König's Lemma comes from reinterpreting the original statement as a claim about *width* instead of *branching*. That is to say, we interpret the original Lemma as stating that “An infinite tree in which every level is finite has an infinite path”. This is a particularly compelling interpretation because

the proof of König's Lemma relies only on a pigeonhole principle argument. This suggests a new analogue to the Lemma in the uncountable case: *An uncountable tree of countable width has an uncountable path.* This notion was originally examined by Aronszajn as reported by Kurepa [15], leading to the following definition:

Definition 2.1.10. *An ω_1 -Aronszajn tree is a tree of countable width and height ω_1 with no uncountable path. Since κ -Aronszajn trees for $\kappa \neq \omega_1$ are not considered in this work, we will refer to an ω_1 -Aronszajn tree as simply an Aronszajn tree.*

That is to say, this latest analogue to König's Lemma states *There is no Aronszajn tree.* Before we demonstrate the falsity of this claim, we present another definition that will be useful:

Definition 2.1.11. *An Aronszajn tree T is well-pruned if every element of T has uncountably many extensions in T .*

Lemma 2.1.12. *There is a computable well-pruned Aronszajn tree with a computable uncountable antichain.*

Proof. The following is an effectivized version of the construction presented in Kunen's *Set Theory* [14].

We construct the tree T as a subtree of $\omega^{<\omega_1}$, ordered by extension. We define a sequence $\langle s_\alpha \rangle_{\alpha < \omega_1}$ of countable injections $s_\alpha : \alpha \rightarrow \omega$ so that for $\beta < \alpha$ the set $\{\gamma < \beta : s_\beta(\gamma) \neq s_\alpha(\gamma)\}$ is finite, and the range of s_α is coinfinite in ω . At the same time, we build a computable set of *constraints* $(\beta, n) \in \omega^1 \times \omega$ for β a limit ordinal; we say s_α *satisfies* the constraint if $\alpha \leq \beta$ or $s_\alpha(\beta) \neq n$, and otherwise it *violates* the constraint. We will ensure the following condition:

(*) If s_α violates the constraint (β, n) , then α is at most $\beta + \omega$.

The construction proceeds by induction on α , in three cases: either α is a successor, α takes the form $\beta + \omega$ for β a limit ordinal, or α is a limit of limit ordinals.

The desired antichain will be s_α for α in the second case.

Let s_0 be the empty function.

Let $\alpha = \beta + 1$. By construction, s_β violates at most one constraint (γ, n) . If so, let $s_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$ other than γ ; otherwise, let $s_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$. If s_α is not yet defined on γ , let $s_\alpha(\gamma)$ be the first m not already in the range of s_α other than n . Either way, let $s_\alpha(\beta)$ be the next m not already in the range of s_α .

Let $\alpha = \beta + \omega$ for β a limit ordinal. By construction, s_β violates at most one constraint (γ, n) ; if so, we define $s_\alpha^*(\delta) = s_\beta(\delta)$ for all $\delta < \beta$ other than γ . Otherwise, let $s_\alpha^*(\delta) = s_\beta(\delta)$ for all $\delta < \beta$. Let $\langle m_i \rangle_{i < \omega}$ enumerate the complement of the range of s_α^* not including n ; by construction, this is infinite. Let $\langle \delta_i \rangle_{i < \omega}$ enumerate the complement of the domain of s_α^* . Let $s_\alpha = s_\alpha^*$ on the domain of s_α^* , and let $s_\alpha(\delta_i) = m_{2i}$ for each i . Finally, place the constraint $(\beta, s_\alpha(\beta))$.

Finally, let α be a limit of limit ordinals. Let $\langle \alpha_i \rangle_{i < \omega}$ be the $<_L$ -least ω -sequence of limit ordinals cofinal in α . By construction, each s_{α_i} violates at most one constraint (γ_i, n_i) . Define a sequence of partial injections $s_{\alpha_i}^* : \alpha_i \rightarrow \omega$ inductively as follows:

(i) $s_{\alpha_0}^*(\delta) = s_{\alpha_0}(\delta)$ if $\delta \neq \gamma_0$.

(ii) $s_{\alpha_{i+1}}^*(\delta) = s_{\alpha_i}^*(\delta)$ for $\delta < \alpha_i$, and $s_{\alpha_{i+1}}^*(\delta) = s_{\alpha_{i+1}}(\delta)$ for $\delta \geq \alpha_i$ and $\delta \neq \gamma_{i+1}$ such that $s_{\alpha_{i+1}}(\delta)$ is not in the range of $s_{\alpha_i}^*$.

Let $s_\alpha^* = \bigcup s_{\alpha_i}^*$. Let β_i be the first $\alpha_i < \beta < \alpha_{i+1}$ so that $s_\alpha^*(\beta)$ is defined. Let $\langle \delta_i \rangle_{i < \omega}$ enumerate the β_i and the complement of the domain of s_α^* in α , let $\langle m_i \rangle_{i < \omega}$ enumerate $s_\alpha^*(\beta_i)$ and the complement of the range of s_α^* , and define s_α as follows:

(a) For β not in the sequence $\langle \delta_i \rangle_{i < \omega}$, let $s_\alpha(\beta) = s_\alpha^*(\beta)$.

(b) Let $s_\alpha(\delta_i) = m_{2j}$ for j least such that $m_{2j} \neq s_\alpha(\delta_k)$ for any $k < i$ and no constraint (δ_i, m_{2j}) is in place.

This completes the construction of the sequence of s_α . Let T be the subtree of $\omega^{<\omega_1}$ consisting of countable injections σ so that $\{\alpha < |\sigma| : \sigma(\alpha) \neq s_{|\sigma|}(\alpha)\}$ is finite, and let $C = \{s_{\beta+\omega} : \beta \text{ limit}\}$.

Claim 2.1.13. *If $\beta < \alpha$, then $\{\gamma < \beta : s_\beta(\gamma) \neq s_\alpha(\gamma)\}$ is finite.*

Proof. By induction on α . Suppose the claim holds for all $\delta < \alpha$.

If $\alpha = \delta + 1$, then s_α differs from s_δ on at most one point; if the claim holds for δ , then it holds for α as well.

If δ is a limit and $\alpha = \delta + \omega$, then s_α differs from s_δ on at most one point, so if $\beta \leq \delta$ then s_α differs from s_β at most finitely often. If $\delta < \beta < \alpha$, then $\beta = \delta + n$ for some n and s_β differs from s_δ at most finitely often. s_α differs from s_β at most finitely often before δ , and possibly everywhere after δ ; but this is still finite.

If α is a limit of limit ordinals, let $\langle \alpha_i \rangle_{i < \omega}$ be the cofinal sequence specified in the construction of s_α . Fix $\beta < \alpha$, and let i be least so that $\alpha_i > \beta$. By construction, $s_\alpha(\beta) = s_{\alpha_i}(\beta)$ except in one of three possible cases: (1) $(\beta, s_{\alpha_i}(\beta))$ is the one constraint violated by s_{α_i} ; (2) $s_{\alpha_i}(\beta)$ is in the range of $s_{\alpha_{i-1}^*}$; or (3) $\beta = \beta_{i-1}$. Cases (1) and (3) apply in at most one case each. Only finitely many β fall under case (2), because the range of $s_{\alpha_i}^*$ is a subset of the range of s_{α_i} , and $s_{\alpha_{i+1}}$ differs at most finitely from s_{α_i} . So s_α differs from each s_{α_i} at most finitely. Since each s_{α_i} differs only finitely from s_β for $\beta < \alpha_i$, s_α satisfies the claim as well. \square

T is therefore a tree of unbounded height; since it is a subtree of the tree of partial injections from ω_1 to ω , it cannot have an unbounded chain. It only remains to show that C is the desired antichain.

Claim 2.1.14. *C is an antichain.*

Proof. For each $s_\alpha \in C$, a constraint was placed during the construction of s_α of the form (β, n) , where $s_\alpha(\beta) = n$. By condition (*), all future $s_{\alpha'}$ satisfy this constraint, and so in

particular cannot extend s_α . □

□

As a consequence, the classical proof of Prop. 2.1.7 will not carry through to the uncountable setting; we will need to employ other tools in the effort to characterize the possible degrees of isolated paths, regardless of the type of tree we consider.

In fact, the only immediate upper bound on the complexity of isolated paths is the following:

Proposition 2.1.15. *The unique path in a computable tree (countable width or otherwise) is Δ_1^1 .*

Proof. Let T a computable tree, X the unique path through T . Then $\sigma \in 2^{<\omega_1}$ is an initial segment of X iff $(\exists Y) ((\forall n) Y \upharpoonright n \in T \wedge \sigma \prec Y)$, or equivalently iff $(\forall Y) ((\forall n) Y \upharpoonright n \in T \rightarrow \sigma \prec Y)$. □

The central result of this work is that, even with the additional constraint of countable width, this upper bound is optimal:

Theorem 2.1.16. *For every Δ_1^1 degree \mathbf{d} , there is (uniformly in a Δ_1^1 index for a representative of \mathbf{d}) a degree $\mathbf{c} \geq \mathbf{d}$ so that $\mathbf{c} \in \mathcal{P}_{\text{thin}}$.*

Theorem 2.1.16, which will be proven in Section 4, guarantees only a member of $\mathcal{P}_{\text{thin}}$ above each Δ_1^1 degree; we also present several theorems aimed at characterizing the degrees that are actually members of $\mathcal{P}_{\text{thin}}$.

Theorem 2.1.17. $\mathbf{0}^{(\alpha)} \in \mathcal{P}_{\text{thin}}$ for every hyperarithmetical ordinal α .

Theorem 2.1.18. $\text{deg } \mathcal{O} \in \mathcal{P}_{\text{thin}}$, where \mathcal{O} is an appropriate analogue of Kleene's \mathcal{O} .

In fact, Theorems 2.1.17 and 2.1.18 can be relativized, subject to a technical condition that will be presented in Section 3.

All of these results present a sharp contrast from the case of ω -recursion, as evidenced by the Low Basis Theorem of Jockusch and Soare [9]:

Theorem 2.1.19. (*Low Basis Theorem*)

(ω -computability) *Every Π_1^0 -class has an element of low Turing degree.*

In Section 2, we demonstrate that $\mathcal{P}_{\text{thin}} \subsetneq \mathcal{P}$, even restricted to relatively small degrees. In Section 3, we prove relativized forms of Theorems 2.1.17 and 2.1.18, and present the definitions required to express the theorems in full generality. In Section 4, we show Theorem 2.1.16 by introducing a large class of members of $\mathcal{P}_{\text{thin}}$ characterized by a combinatorial property. In Section 5, we examine the structure of $\mathcal{P}_{\text{thin}}$ in more detail, and concentrating on results relating to jump inversion and minimal degrees. In Section 6, we diverge from the theme of isolated paths to consider thin Π_1^0 -classes with more than one member; of principal interest will be the Π_1^0 -classes arising from computable Kurepa trees, which are thin but have cardinality \aleph_2 . Finally, in Section 7, we discuss the notable questions left open by the results of this chapter.

2.2 Separation

In light of the results of the previous section, it is reasonable to ask whether the property of countable width makes any difference at all in relation to isolated paths. The following result answers that question in the affirmative:

Theorem 2.2.1. *There is a Δ_2^0 Π_1^0 -singleton which is not the degree of any thin Π_1^0 singleton.*

To prove this result, we require the following lemma.

Lemma 2.2.2. *Suppose T is a tree and $f : 2^{<\omega} \rightarrow T$ is an embedding such that for all $x \in 2^\omega$, $\lim_{n < \omega} f(x \upharpoonright n) \in T$. Then T does not have countable width.*

Proof. Suppose T and f are as given in the hypothesis of the lemma. Since the domain of f is countable, the range of f is likewise countable. Since ω_1 is regular, there exists a countable ordinal α so that $|f(\sigma)| < \alpha$ for all $\sigma \in 2^{<\omega}$. Then for each $x \in 2^\omega$, $|\lim_{n < \omega} f(x \upharpoonright n)| = \lim_{n < \omega} |f(x \upharpoonright n)| \leq \alpha$. Since there are uncountably many such x , and each one yields a different element of T , we have that $T \cap 2^{\leq \alpha}$ is uncountable. But $T \cap 2^{\leq \alpha}$ has only countably many levels, so by the Pigeonhole Principle there is some $\beta < \alpha$ so that $T \cap 2^\beta$ is uncountable. Therefore T does not have countable width. \square

Proof. (Proof of Theorem 2.2.1)

Fix a recursive enumeration $\langle T_i \rangle_{i < \omega_1}$ of computable trees. Also fix a computable Aronszajn tree A .

We construct in stages a tree $T \subseteq 2^{<\omega_1}$ with a unique path P . T will be $\bigcup_s T_s$, where each T_s is itself a tree, $T \cap 2^{<s} = T_s \cap 2^{<s}$, and $T_t \subseteq T_s$ whenever $t \leq s$. P will be $\lim_s P_s$, where each P_s is a branch of T_s .

At any stage s , some strings in T_s may be designated as *Aronszajn roots*. While σ is an Aronszajn root, strings will automatically be added to each successive T_t in order to extend an embedding from A into the subset of T_t extending σ . The effect of this is to ensure that while σ is designated an Aronszajn root, there will always be an extension of σ that can be added to T_t , while at the same time refraining from constructing a path extending σ .

A string σ is considered *available* at stage s if $|\sigma| \geq s$ and every initial segment of σ is in T_s ; this means that σ is permitted to enter T_{s+1} .

The argument will be a variation on the infinite-injury priority argument frequently used in ω -computability. Let $\Lambda = \{0, 1\} \cup \{\langle 0, n \rangle : n < \omega\} \cup \{\langle 1, \alpha \rangle : \alpha < \omega_1\}$; this will be the set of *outcomes*. We will pursue the following requirements:

$$D_e : P \neq \Phi_e$$

$R_{e,i}$: If T_i is a tree of countable width, then either Φ_e^P is not total, $\Pi_e^P \leq_T \emptyset$, or $\Phi_e^P \notin [T_i]$.

Effectively order the D_e and $R_{e,i}$ requirements as $\langle Q_j \rangle_{j < \omega_1}$ in some manner so that each D_e and each $R_{e,i}$ appears exactly once as some Q_j . For each $j < \omega_1$, assign to each string in Λ^j a strategy pursuing requirement Q_j ; call the strategy assigned to σ $Q(\sigma)$. At any stage, a given strategy may or may not have been *initialized*; once initialized, it has an *outcome* taken from Λ .

$Q(\sigma)$ is *strictly higher-priority* than $Q(\tau)$ iff $\sigma \prec \tau$.

The *execution path at stage s* is the string $\eta(s) \in \Lambda^{\leq s}$ inductively defined as follows: for any $\alpha < \omega_1$, if $Q(\eta(s) \upharpoonright \alpha)$ has been initialized, then $\eta(s)(\alpha)$ is its outcome at stage s . Otherwise, $\eta(s)(\alpha)$ is not defined. A strategy is called *along the execution path* at stage s if it is $Q(\eta(s) \upharpoonright \alpha)$ for some $\alpha < |\eta(s)|$.

At any stage s , a strategy that has been initialized may *require attention*. The first strategy (that is, the one with least height) along the execution path that requires attention is *permitted to act*. If no strategy along the execution path requires attention, then $Q(\eta(s))$ is initialized.

Every requirement, once initialized, has an *anchor* in T ; it will be the case that whenever $\sigma \preceq \tau$, the anchor of $Q(\tau)$ extends that of $Q(\sigma)$.

Throughout the construction, we will maintain the following conditions:

(i) If Q is a strategy along the execution path at stage s , then P_s extends the anchor of Q .

(ii) If a strategy Q acts to add elements to the tree, then those elements extend the anchor of Q .

The D Requirements: When initialized at stage s , D_e sets its anchor to an available

extension σ of P_s , puts $\sigma \frown 0$ and $\sigma \frown 1$ into T_{s+1} , designates $\sigma \frown 1$ as an Aronszajn root, and sets $P_{s+1} = \sigma \frown 0$. D_e begins with outcome 0.

Once initialized, D_e *requires attention* at stage s if its outcome at stage s is 0 and $\Phi_{e,s}(|\sigma|) \downarrow = 0$. When permitted to act, D_e selects an available extension τ of $\sigma \frown 1$, sets $P_{s+1} = \tau$, and switches to outcome 1.

Observe that any fixed D_e strategy will eventually cease to act, because it will act at most once.

Claim 2.2.3. *If Q is a D_e strategy and lies along the execution path at unboundedly many stages, and if all higher-priority strategies eventually cease to act, then D_e is satisfied.*

Proof. Suppose otherwise. Then $\Phi_e = P$. Let σ be the anchor of Q . Then $\Phi_e(|\sigma|) = P(|\sigma|)$. Let s_0 be least so that $\Phi_{e,s_0}(|\sigma|) = P(|\sigma|)$. Let $s_1 > s_0$ be least so that all strategies of strictly higher priority than Q have ceased to act. If $\Phi_e(|\sigma|) \downarrow = 1$, then Q will never act at all, and will remain in outcome 0; therefore $P(|\sigma|) = 0$, contradicting our supposition. So $\Phi_e(|\sigma|) \downarrow = 0$. Let $s_2 > s_1$ be the least stage at which Q lies along the execution path. At this stage, Q requires attention, and since no strictly higher-priority strategy will act, Q must be the first strategy along the execution path which requires attention. Q is therefore permitted to act, and changes P so that $P(|\sigma|) = 1$ and hence $P \neq \Phi_e$. By construction, $P \upharpoonright (|\sigma| + 1)$ will not change again. \square

The R Requirements:

Definition 2.2.4. *An e -splitting tree is a partial map $f : 2^{<\omega} \rightarrow T$ with the following properties:*

(i) *For each $\sigma, \tau \in \text{dom}(f)$, if $\sigma \mid \tau$ then $\Phi_e^\sigma \mid \Phi_e^\tau$.*

(ii) *$f(\sigma) \prec f(\tau)$ iff $\sigma \prec \tau$.*

(iii) If $\sigma \in \text{dom}(f)$, then $\sigma \frown 0 \in \text{dom}(f)$ iff $\sigma \frown 1 \in \text{dom}(f)$.

An e -splitting tree f is complete if $\text{dom}(f) = 2^{<\omega}$. A leaf node of f is $\sigma \in \text{dom}(f)$ with no extensions in $\text{dom}(f)$. Note that an e -splitting tree is complete iff it has no leaf nodes.

When initialized, an $R_{e,i}$ strategy sets its anchor to be an available extension σ of P_s and includes σ and $\sigma \frown 0$ in T . It takes outcome $\langle 0, 0 \rangle$ and begins assembling an e -splitting tree $f_{e,i}$ by taking $f_{e,i}(\emptyset)$ to be σ .

At any stage after initialization, $R_{e,i}$ may be in either of two phases, *phase zero* and *phase one*.

While $f_{e,i}$ is not complete, $R_{e,i}$ remains in phase zero. During phase zero, $R_{e,i}$ is in some outcome of the form $\langle 0, n \rangle$ for $n \in \omega$. $R_{e,i}$ maintains a string τ_s so that τ_s is a leaf node of $f_{e,i}$. $R_{e,i}$ *requires attention* if there exists $t, u \leq s$ with $P_t, P_u \succcurlyeq f_{e,i}(\tau_s)$ and $\Phi_{e,s}^{P_t} \mid \Phi_{e,s}^{P_u}$. If $R_{e,i}$ is then permitted to act, it sets $f_{e,i}(\tau_s \frown 0) = P_t$ and $f_{e,i}(\tau_s \frown 1) = P_u$. It then takes τ_{s+1} to be the first string (in the standard ordering) that is still a leaf node of $f_{e,i}$, sets P_{s+1} to be some available extension of $f_{e,i}(\tau_{s+1})$, designates P_s an Aronszajn root, and takes on outcome $\langle 0, n + 1 \rangle$.

If at any stage there exists a $\tau \in 2^\omega$ so that $f_{e,i}(\tau \upharpoonright n)$ is defined for every $n \in \omega$, then include $\lim_{\rho \prec \tau} f_{e,i}(\rho)$ in T and designate it as an Aronszajn root. Note that this must occur regardless of whether the strategy is permitted to act or even along the execution path at all.

Once $f_{e,i}$ is complete, $R_{e,i}$ enters phase one, and takes outcome $\langle 1, 0 \rangle$. $f_{e,i}$ induces a map $g_{e,i} : 2^\omega \rightarrow 2^{<\omega_1}$, given by $g_{e,i}(\sigma) = \lim_{n < \omega} f_{e,i}(\sigma \upharpoonright n)$; $R_{e,i}$ takes P_{s+1} to be the first available extension of $g_{e,i}(\sigma_0)$, where σ_0 is the $<_L$ -least member of 2^ω , and designates P_s an Aronszajn root.

While in phase one, $R_{e,i}$ *requires attention* if it has not yet declared phase one *complete* and there is a $\tau \in 2^\omega$ such that $\Phi_{e,s}^{g_{e,i}(\tau)} \notin T_i$. If permitted to act, $R_{e,i}$ changes its outcome

to the outcome $\langle 1, \alpha \rangle$ with α the index of τ in the standard enumeration of 2^ω , sets P_{s+1} to the first available extension of $g_{e,i}(\tau)$, and designates P_s an Aronszajn root. Once this has occurred, $R_{e,i}$ declares that phase one is complete.

Claim 2.2.5. *For a fixed $R_{e,i}$ strategy η , if all strictly higher-priority requirements eventually cease to act, so does η .*

Proof. There are two cases: either η remains in phase zero indefinitely or it eventually enters phase one.

If η remains in phase zero indefinitely, then it may only act countably many times, because each action adds an element to the domain of the e -splitting tree maintained by η , and that domain must be a subset of $2^{<\omega}$. So at some countable stage, η will no longer act.

If η eventually enters phase one, then it performed a countably infinite number of actions in phase zero, and at most two actions in phase one (once to advance to phase one, and once when permitted to act). So η acts at most countably often and therefore eventually ceases to act. \square

Claim 2.2.6. *If η is an $R_{e,i}$ strategy and lies along the execution path at unboundedly many stages, and $P = \lim_s P_s$ exists, then η is satisfied.*

Proof. Note that, inductively by Claim 2.2.5, all strictly higher-priority strategies eventually cease to act. Let s_0 be a stage large enough that all strictly higher-priority strategies that will ever act have already finished doing so; note that since changing the outcome of a strategy requires an action, at this stage η will either always be along the execution path or will never be. Since η lies along the execution path unboundedly often, the former case must hold.

There are three possible cases.

- (a) η never proceeds to phase one (so remains in phase zero indefinitely).

(b) η proceeds to phase one, but never declares it completed.

(c) η eventually declares phase one complete.

In case (a), η does not completely fill out its e -splitting tree f , so there must be some string τ that eventually becomes a leaf node of f and remains so for the rest of the construction. Let $s_1 > s_0$ be a stage sufficiently large that τ is a leaf node of f . Since τ remains a leaf node thereafter, η cannot act after stage s_1 . So η does not require attention after stage s_1 ; this means that for all $t, u > s_1$, it must be that $\Phi_e^{P_t}$ and $\Phi_e^{P_u}$ are always compatible. Then Φ_e^P is either computable or non-total.

In case (b), since η never completes phase one, it must be that either T_i is non-total or for every $\sigma \in 2^\omega$, $\Phi_e^{g_{e,i}(\sigma)}$ eventually enters T_i . In the latter case, $\Phi_e^{f_{e,i}(\cdot)} \cup \Phi_e^{g_{e,i}(\cdot)}$ is an embedding from $2^{\leq \omega}$ into T_i , so T_i is not a tree of countable width.

In case (c), the action η took to complete phase one ensures that there is an initial segment σ of P so that $\Phi_e^\sigma \notin T_i$. So Φ_e^P is either non-total or not a path through T_i ; in either case, the requirement is satisfied. \square

Limit stages: At any limit stage s , take $P_s = \lim_{t < s} P_t$ if that limit exists. The outcome of any strategy η at stage s is taken to be the limit of its outcomes at previous stages, provided that value stabilizes below s . If not, then η is an $R_{e,i}$ strategy that acted infinitely often in phase zero before stage s (because the D_e strategies act at most once, and $R_{e,i}$ strategies in phase one act at most twice). Then, according to the $R_{e,i}$ strategy, η meets the conditions to transition into phase one at stage s ; if η is the first such strategy along the execution path, permit it to do so.

If $\lim_{t < s} P_t$ does not exist, include all limit points of the $\{P_t\}$ sequence in T and designate them all as Aronszajn roots. By construction, some strategy along the execution path transitions to phase one at this stage, and specifies a new value of P_s independent

of the limit.

Verification:

Claim 2.2.7. $P = \lim_s P_s$ exists and is a path through T .

Proof. By construction, neither D strategies nor R strategies will revisit a previously abandoned outcome, so once a strategy leaves the execution path it will never be along the execution path again. $P_t \not\subseteq P_s$ for $t > s$ only if some strategy acted between stages s and t ; since every action precipitates a change of outcome, that means that the execution path at stage t is not an extension of the execution path at stage s . Furthermore, note that any strategy η only changes P at points above the length of its anchor, and that when a strategy is freshly initialized it takes the current value of P_s as its anchor. So once η changes its outcome at stage s , any further change to $P_t \upharpoonright s$ must be caused by η or by a strictly higher-priority strategy; since the strategy order is well-founded, and since each strategy acts only boundedly often, this means that $P_t \upharpoonright s$ eventually stabilizes as t goes to ω_1 . Because this holds for all s , $\lim_s P_s$ exists. Furthermore, since every P_s lies in T , P is a path through T . \square

Claim 2.2.8. P is the only path through T .

Proof. Let X be a path through T ; we will show that $X = P$.

Since X is a path, its initial segments cannot have been added solely on the behalf of Aronszajn roots (that is, only through additions to the tree not made by strategies). So unboundedly often a strategy must add an initial segment of X to the tree. But each strategy adds only countably many elements (since each addition involves an action, and no strategy acts unboundedly often) and therefore unboundedly many distinct strategies must contribute to X . So unboundedly many strategies are anchored along X , because every strategy adds only extensions of its anchor. But since abandoned strategies cannot

be revisited, this uncountable sequence of strategies must be the strategies along the true path (that is, strategies that are eventually always along the execution path). But these strategies are all anchored along P , so arbitrarily long initial segments of P are also initial segments of X . Therefore $X = P$. \square

Claim 2.2.9. *P is not computable.*

Proof. By Claim 2.2.3, every D_e requirement is eventually satisfied. So $\Phi_e \neq P$ for each e . \square

Claim 2.2.10. *P is not Turing-equivalent to the unique path through any computable tree of countable width.*

Proof. By Claim 2.2.6, every $R_{e,i}$ requirement is eventually satisfied. So for every e, i , if T_i is a computable tree of countable width and Φ_e^P is total and noncomputable, then $\Phi_e^P \notin [T_i]$. So P does not compute any noncomputable path through any computable tree of countable width. Since P is itself not computable, P cannot be equivalent to any path through a computable tree of countable width, regardless of uniqueness. \square

\square

2.3 State Systems

In light of Theorem 2.2.1, constructing a computable tree of countable width in which the unique path is of a particular degree is nontrivial. In order to determine which degrees can be represented by these unique paths, it is useful to develop some general machinery for constructing trees with various properties.

Definition 2.3.1. *A system of states is a partial order (\mathbb{S}, \leq) , with $\mathbb{S} \subseteq L_{\omega_1}$. The system is computable if both \mathbb{S} and \leq are computable.*

A filter for (\mathbb{S}, \leq) is a downward-closed set $X \subseteq \mathbb{S}$ so that the elements of X are pairwise comparable and there is no $x \in \mathbb{S}$ above every element of X .

A state function is an injective function $s : \omega_1 \rightarrow \mathbb{S}$ so that $s(t) \leq s(u)$ only if $t \leq u$ and the range of s in X is well-ordered. s is a X -true state function if $\{t < \omega_1 : s(t) \in X\}$ is a club in ω_1 and the range of s is unbounded in X .

Definition 2.3.2. If \mathbb{P} and \mathbb{Q} are partially ordered sets, then $f : \omega_1 \rightarrow \mathbb{P}$ is said to be smooth over $g : \omega_1 \rightarrow \mathbb{Q}$ if the following conditions hold:

(i) If $f(s) \leq f(t)$, then $g(s) \leq g(t)$.

(ii) If $r < s < t$ is such that $f(r) \leq f(t)$ and $g(s) \leq g(t)$, then $f(r) \leq f(s)$.

The second requirement in Definition 2.3.2 is a technical condition only of use in the development of the analogue of α -true relations introduced by Montalbán [17] based on the priority framework of Ash [1]. Nevertheless, we include it here in anticipation of its future usefulness and because it does not significantly complicate the proof.

Theorem 2.3.3 (State Theorem). *Let (\mathbb{S}, \leq) be a computable system of states, X a filter, and $\mathbf{s} : \omega_1 \rightarrow \mathbb{S}$ a computable X -true state function. Then there exists a computable tree T of countable width with a companion map f and unique path Y so that $Y \equiv_T X$ and f is smooth over \mathbf{s} . Furthermore, this holds with all possible uniformity.*

2.3.1 Proof of the Theorem

We construct T as $\bigcup_s T_s$, so that $T \cap 2^{<\alpha} = \bigcup_{\beta < \alpha} T_\beta \cap 2^{<\alpha}$ (so the first α levels are decided by stage α). We will suppress references to the particular levels of T , and instead refer simply to “adding σ to T ” or “removing σ from T ”.

As we build T , we will build f simultaneously; the path Y will be defined as the unique path which f visits on a closed and unbounded set.

Throughout the construction, some nodes will be designated *Aronszajn roots*; at most countably many of these will appear below any fixed height α , but uncountably many such designations may be present at any stage. While σ is designated an Aronszajn root, at every stage $\alpha > |\sigma|$, extensions of σ will be added to T to copy a fixed computable Aronszajn tree rooted at σ up to height α .

Definition 2.3.4. Anchoring the stage- s state at σ is to perform the following sequence of steps.

- (i) Include σ in T , if it isn't already present.
- (ii) Add an Aronszajn tree A_σ rooted at σ .
- (iii) Designate a computable uncountable antichain $Q_\sigma \subseteq A_\sigma$.
- (iv) Designate each member of Q_σ as an Aronszajn root.
- (v) Set $f(s) = \sigma$.

At the beginning of the construction, we anchor the stage-0 state at the root node λ . As a result, we will have defined an Aronszajn tree A_λ rooted at λ and a computable antichain $Q_\lambda \subseteq A_\lambda$, each member of which is designated as an Aronszajn root.

Junking the stage- s state is to perform the following sequence of steps.

- (i) Let σ be the first member of Q_λ that is still designated as an Aronszajn root.
- (ii) Remove the designation of σ as an Aronszajn root.
- (iii) Select an extension τ of σ of maximal length up to s .
- (iv) Anchor the stage- s state at τ .

Stage $s > 0$: There are several cases. Let $A = \{t < s : \mathbf{s}(t) \leq \mathbf{s}(s)\}$, so that A is the set of previous stages at which the state was compatible with the current state.

Case 1: Suppose that A is unbounded below s and $\lim_{t \in A} f(t)$ exists. Then anchor the stage- s state at $\lim_{t \in A} f(t)$.

Case 2: Suppose that A is unbounded below s , but $\lim_{t \in A} f(t)$ does not exist. Then junk the stage- s state.

Case 3: Suppose that A is bounded below s and has a maximal element t . Then let σ be the first member of $Q_{f(t)}$ that is still designated as an Aronszajn root. Remove the that designation, and select $\tau \in T$ extending σ of length s . Anchor the stage- s state at τ .

Case 4: Suppose that A is bounded below s but has no maximal element. Then junk the stage- s state.

Observe that if $f(s) \preceq f(t)$, then $\mathbf{s}(s) \leq \mathbf{s}(t)$.

Verification:

Claim 2.3.5. T has at least one path which computes X .

Proof. Let $C = \{t : \mathbf{s}(t) \in X\}$. By hypothesis, this is a club. Let $\alpha : \omega_1 \rightarrow C$ be the function enumerating C in order. We show by induction that for every t and every $u < t$, $f(\alpha(u)) \prec f(\alpha(t))$. The claim is trivial for $t = 0$.

Suppose $t = x + 1$ and the claim holds for x . At stage $\alpha(t)$, the construction proceeds as follows: $A = \{y < \alpha(t) : \mathbf{s}(y) \leq \mathbf{s}(\alpha(t))\}$ consists entirely of stages with state below $\mathbf{s}(\alpha(t)) \in X$. Since X is a filter and therefore downward closed, $\mathbf{s}(y) \in X$ for every $y \in A$. So A is the initial segment of C below $\alpha(t)$, and so by our hypothesis on t has a maximal element, namely $\alpha(x)$. Then the stage- $\alpha(t)$ state is anchored above $\alpha(x)$, and by construction $f(\alpha(x)) \prec f(\alpha(t))$.

Suppose t is a limit ordinal and the claim holds for every $x < t$. Again, $A = \{x < \alpha(t) : \mathbf{s}(x) \leq \mathbf{s}(\alpha(t))\}$ is the initial segment of C below $\alpha(t)$. Since C is a club and t is a limit, A

is unbounded below $\alpha(t)$. By the induction hypothesis, the $f(x)$ are comparable for every $x \in A$, so this stage falls into case 1. Then $f(\alpha(t)) = \lim_{x \in A} f(\alpha(x))$ is well-defined.

This completes the induction. Therefore $\bigcup_{t \in C} f(t)$ is a path Y through T . By the observation above, the only states anchored along Y are members of X , and these are unbounded in X ; so, to compute whether a state x is in X , Y need only wait for a state to be anchored along it that is either above or incomparable with x . \square

Claim 2.3.6. *T has at most one path.*

Proof. Let $Z \in [T]$. Let B be the set of stages at which the current state is anchored along Z . Observe that, by construction, B is unbounded and $\mathbf{s}(s) < \mathbf{s}(t)$ for every $s, t \in B$ with $s < t$.

Suppose that B is not closed. Let $t_0 < t_1 < \dots < t_i < \dots$ be an increasing sequence of members of B so that $t = \lim_i t_i \notin B$. At stage t , the state was not anchored at $\lim_t f(t_i)$; since $f(t_i) \prec Z$ for every i , this cannot be because we fell in case 2. Instead we must fall in case 3 or 4. In either case, $\lim_i f(t_i)$ is not added to T , so Z cannot be a path through T , contradicting our supposition.

Therefore B is a club in ω_1 . Let C be as specified in the previous claim. Since B and C are both clubs, they intersect unboundedly often. But then Z and Y intersect unboundedly often, so $Z = Y$. \square

Claim 2.3.7. *f is smooth over \mathbf{s} .*

Proof. Condition (i) in Definition 2.3.2 is clear from the construction. It remains to show condition (ii).

Suppose that $r < s < t$ are such that $f(r) \leq f(t)$ and $\mathbf{s}(s) \leq \mathbf{s}(t)$. We aim to show that $f(r) \leq f(s)$.

Suppose that the claim fails; so, $f(r) \not\leq f(s)$. Further suppose, without loss of generality, that s is least so that this is the case.

Observe that since Condition (i) holds, we have that $\mathbf{s}(r) \leq \mathbf{s}(t)$. Since \mathbb{S} is a tree, $\mathbf{s}(s)$ and $\mathbf{s}(r)$ are comparable; since \mathbf{s} is order-respecting, $\mathbf{s}(r) \leq \mathbf{s}(s)$.

Consider the construction at stage s . Let $A = \{u < s : \mathbf{s}(u) \leq \mathbf{s}(s)\}$. By the minimality of s we supposed above, if A is unbounded below s then $\lim_{u \in A} f(u)$ exists and extends $f(r)$. So the construction at this stage falls in either Case 1, Case 3, or Case 4.

Case 1: The stage- s state was anchored at $\lim_{u \in A} f(u) \succ f(r)$, so $f(s) \geq f(r)$, contradicting our assumption.

Case 3: Say A has maximal element u . By the minimality of s , $f(u) \geq f(r)$; by construction, $f(s) \geq f(u)$. This contradicts our assumption.

Case 4: The stage- s state is junked.

So the construction must fall into Case 4 at stage s . But since $f(t) \geq f(r)$, there must be a later stage at which this happens; let t_0 be least $> s$ so that $f(t_0) \geq f(r)$. By the same argument as above, we must be in case 4 again at stage t_0 . But junking anchors a state at a position incompatible with all previous states except the stage-0 state. Since $f(t_0)$ was the result of junking and $f(t_0) \geq f(r)$, it must be that $r = 0$. But the stage-0 state was anchored at the root element; $f(0) \leq f(u)$ for every u . This completes the contradiction. \square

2.3.2 Applications

Theorem 2.3.3 gives as largely straightforward consequences a number of lemmas that are useful to characterizing the members of $\mathcal{P}_{\text{thin}}$. In particular, unique paths of computable trees of countable width with companion maps are closed under the Turing jump and under joins that are uniformly computable in such a path. At the end of this section, we will relate this to the analogue of the hyperarithmetic hierarchy.

Lemma 2.3.8. *Let T be a computable tree of countable width with unique path X and companion map f . Then there is, uniformly in an index for T and f , a tree T' with a companion map g and unique path $Y \equiv_T X'$, with g smooth over f .*

Proof. Let \mathbb{S} be the subset of $2^{<\omega_1} \times 2^{<\omega_1}$ consisting of elements (σ, τ) with $|\sigma| = |\tau|$, where $(\sigma_1, \tau_1) \leq (\sigma_2, \tau_2)$ iff $\sigma_1 \preceq \sigma_2$ and $\tau_1 \preceq \tau_2$. Observe that (\mathbb{S}, \leq) is a system of states.

Let $F = \{(\sigma, \tau) \in \mathbb{S} : \sigma \prec X, \tau \prec X'\}$. Then F is a filter of \mathbb{S} .

Definition 2.3.9. *For $\sigma \in 2^{<\omega_1}$, σ' is the sequence τ of length $|\sigma|$ so that $\tau(e) = 1$ iff $\Phi_{e,t}^{\sigma}(e) \downarrow$ converges for some $t < |\sigma|$.*

Let $\mathbf{s}(s) = (f(s), f(s)')$. By the properties of the companion map, \mathbf{s} is immediately a state function.

Claim 2.3.10. *\mathbf{s} is F -true.*

Proof. Let $t_0 < t_1 < \dots$ be an increasing sequence so that $\mathbf{s}(t_i) \in F$ for each i . Let $t = \lim_i t_i$.

For each i , $f(t_i) \prec X$. By the properties of the companion map, $f(t) = \lim_i f(t_i) \prec X$. Let $\tau = f(t)'$. $\tau(e) = 1$ iff $\Phi_{e,t}^{f(t)}(e) \downarrow$ for some $t < |f(t)| = \lim_i |f(t_i)|$. Equivalently, there is an i with $t < |f(t_i)|$, or equivalently there is an i so that $f(t_i)'(e) = 1$. So $f(t)' = \lim_i f(t_i)'$. Therefore $\mathbf{s}(t) \in F$.

Therefore $\{t : \mathbf{s}(t) \in F\}$ is closed. It remains to show that it is unbounded.

For any α , let $h(\alpha)$ be the least β so that for all $e < \alpha$, $\Phi_e^X(e)$ converges by stage β or not at all. Observe that h is continuous, so it has a club C of fixed points. By the conditions on the companion map, the set $\{s : f(s) \prec X\}$ is a club D . $C \cap D$ is then an unbounded subset of ω_1 so that for every $s \in C \cap D$, $f(s) \prec X$ and $f(s)' \prec X'$, so $\mathbf{s}(s) \in F$. \square

By Theorem 2.3.3, then, there is a computable tree of countable width T' with a companion map f' and unique path Y so that $Y \equiv_T F$. Clearly $F \equiv_T X'$, so $Y \equiv_T X'$. \square

Lemma 2.3.11. *Let U be a computable tree of countable width with unique path X and companion map f . Let $F : U \rightarrow \omega_1$ be a computable function so that for every $\sigma \prec X$, $F(\sigma)$ is the index for a computable tree of countable width T_σ with unique path X_σ with companion map f_σ . Then there is a tree T with unique path Y and companion map g so that $Y \equiv_T X \oplus \bigoplus_{\sigma \prec X} X_\sigma$ and g is smooth over f and every f_σ .*

Proof. Let \mathbb{S} be the subset of $2^{<\omega_1} \times (2^{<\omega_1})^{<\omega_1}$ consisting of elements of the form $(\sigma, \langle \tau_i \rangle_{i < \alpha})$ so that $\alpha = |\sigma|$. Then \mathbb{S} is a system of states.

Let $G = \{(\sigma, \langle \tau_i \rangle_{i < |\sigma|}) : \sigma \prec X, \forall i \tau_i \prec X_{X \upharpoonright i}\}$. Then G is a filter.

Let $\mathbf{s}(s) = (f(s), \langle f_{f(s) \upharpoonright t} \rangle_{t < s})$, where $f_{f(s) \upharpoonright t}$ is evaluated at stage s . Then \mathbf{s} is a state function.

Claim 2.3.12. *\mathbf{s} is G -true.*

Proof. Let $C = \{s : \mathbf{s}(s) \in G\}$. We show that C is a club.

Let $t_0 < t_1 < \dots$ be an ω -sequence of elements of C with limit t . Let $\mathbf{s}(t) = (\sigma_t, \langle \tau_i^t \rangle_{i < \alpha_t})$. $\sigma_t = f(t) = \lim_i f(t_i) \prec X$, by the properties of the companion map. Fix $i < \alpha_t$, and let $j < \omega$ be such that $t_j > i$. Then $\tau_i^t = f_{X \upharpoonright i}(t) = \lim_k f_{X \upharpoonright i}(t_k)$. For $k > j$, $f_{X \upharpoonright i}(t_k) \prec X_{X \upharpoonright i}$, so $\tau_i^t \prec X_{X \upharpoonright i}$. So $\mathbf{s}(t) \in G$.

C is therefore closed; it remains to show that C is unbounded. $f(t) \prec X$ on a club D . For each α , note that $f_{X \upharpoonright \alpha}(t) \prec X_{X \upharpoonright \alpha}$ on a club C_α . Let $C_{<\alpha} = \bigcap_{\beta < \alpha} C_\beta$. Let $h(\alpha)$ be the least member of $D \cap C_{<\alpha}$. h is continuous, so has a club E of fixed points. $E \subseteq C$, so C is unbounded. □

The lemma follows by Theorem 2.3.3. □

Developing the hyperarithmetic hierarchy in the context of ω_1 -recursion is somewhat complicated by the fact that well-ordering is a Π_1^0 property; as a result, \emptyset' computes more ordinals than \emptyset does. The following definition is due to Greenberg and Turetsky in notes that are to the author's knowledge presently unpublished.

Definition 2.3.13. We define \mathcal{O} as the smallest set satisfying the following conditions, while simultaneously defining sets H_a for $a \in \mathcal{O}$, a rank function $|\cdot| : \mathcal{O} \rightarrow ORD$, and an ordering $<_{\mathcal{O}}$ on \mathcal{O} .

(i) $0 \in \mathcal{O}$, $H_0 = \emptyset$, and $|0| = 0$.

(ii) If $a \in \mathcal{O}$, then $b = \langle \text{succ}, a \rangle \in \mathcal{O}$. $|b| = |a| + 1$, $H_b = H'_a$, and $c <_{\mathcal{O}} b$ iff $c <_{\mathcal{O}} a$ or $c = a$.

(iii) If $a_i \in \mathcal{O}$ and $a_i <_{\mathcal{O}} a_{i+1}$ for $i < \omega$, then $b = \langle \omega, \langle a_i \rangle_{i < \omega} \rangle \in \mathcal{O}$. $H_b = \bigoplus_{i < \omega} H_{a_i}$ and $|b| = \sup_{i < \omega} |a_i|$. $c <_{\mathcal{O}} b$ iff there exists an i so that $c <_{\mathcal{O}} a_i$.

(iv) Suppose $a \in \mathcal{O}$, $\{e\}^{H_a}$ is total, $\{e\}^{H_a}(i) \in \mathcal{O}$ and $\{e\}^{H_a}(i) <_{\mathcal{O}} \{e\}^{H_a}(j)$ for all $i < j < \omega_1$. Then $b = \langle \omega_1, a, e \rangle \in \mathcal{O}$. $|b| = \sup_{i < \omega_1} |\{e\}^{H_a}(i)|$, and $H_b = \bigoplus_{i < \omega_1} H_{\{e\}^{H_a}(i)}$. $c <_{\mathcal{O}} b$ iff there is an $i < \omega_1$ with $c <_{\mathcal{O}} \{e\}^{H_a}(i)$.

A hyperarithmetical ordinal is an ordinal α so that there is $a \in \mathcal{O}$ with $|a| = \alpha$.

Proposition 2.3.14. (Greenberg, Turetsky) If $a, b \in \mathcal{O}$ and $|a| = |b|$, then $H_a \equiv_T H_b$. Thus $\mathbf{0}^{(\alpha)}$ can be defined as the degree of H_a where $a \in \mathcal{O}$ and $|a| = \alpha$.

The above can be relativized in a straightforward manner to any $X \subseteq \omega_1$, yielding a definition of \mathcal{O}^X , an ordinal hyperarithmetical in X and the degree $X^{(\alpha)}$ for any such ordinal α .

Corollary 2.3.15. (Theorem 2.1.17) Let X be the unique path in a computable tree of countable width T with a companion map f , and let α be an ordinal hyperarithmetical in X . Then, uniformly in indices for T and f and an X -notation for α , there is a computable tree \tilde{T} of countable width with a companion map \tilde{f} so that the unique path Y of \tilde{T} is equivalent to $X^{(\alpha)}$.

Proof. By effective induction on α , using Lemma 2.3.8 for successor steps and Lemma 2.3.11 for limit steps. □

Proposition 2.3.16. (Theorem 2.1.18) *Let T be a computable tree of countable width with unique path X and companion map f . Then there exists, uniformly in an index for T and f , a computable tree $T_{\mathcal{O}}$ with unique path $X_{\mathcal{O}}$ and companion map $f_{\mathcal{O}}$ so that $X_{\mathcal{O}} \equiv_T \mathcal{O}^X$.*

Proof. Let \mathbb{S} be the collection of tuples $\langle \sigma, \langle H_i \rangle_{\sigma(i)=1}, \langle D_i \rangle_{i < |\sigma|} \rangle$, where each H_i is a countable binary string and each D_i a function with domain a countable tree. Order \mathbb{S} in the natural way.

Note that, by Lemmas 2.3.8 and 2.3.11 we have a uniform map $a \rightarrow (T_a, f_a)$ so that if $a \in \mathcal{O}$, T_a is a tree of countable width with companion map f_a and unique path equivalent to H_a .

Define a map $\mathbf{s} : \omega_1 \rightarrow \mathbb{S}$ so that $\mathbf{s}(s) = \langle \sigma, \langle H_i \rangle_{\sigma(i)=1}, \langle D_i \rangle_{i < |\sigma|} \rangle$ as follows.

Let H_i (for each $i < |\sigma|$) be $f_i(s)$, evaluated at stage s . Define D_i by induction as follows:

- (i) $D_i(\lambda) = i$.
- (ii) If $D_i(\tau) = \langle \text{succ}, a \rangle$, then $D_i(\tau \frown 0) = a$.
- (iii) If $D_i(\tau) = \langle \omega, \langle a_j \rangle_{j < \omega} \rangle$, then $D_i(\tau \frown n) = a_n$ for each $n < \omega$.
- (iv) If $D_i(\tau) = \langle \omega_1, a, e \rangle$, and $a < s$, then $D_i(\tau \frown 0) = a$, and for each $\beta < s$, if $\Phi_{e,s}^{H_a}(\beta) \downarrow = c$ then $D_i(\tau \frown \beta) = c$.

Then σ is the string of length s so that $\sigma(i) = 1$ iff D_i is well-founded.

Let G be the set of elements $\langle \sigma, \langle G_i \rangle_{\sigma(i)=1}, \langle D_i \rangle_{i < |\sigma|} \rangle$ so that $\sigma \prec \mathcal{O}$, $G_i \prec H_i$ for each i , and D_i is defined as above for each i . Note that G is clearly downward-closed, and its elements are pairwise compatible, so G is a filter in \mathbb{S} .

Claim 2.3.17. \mathbf{s} is a state function.

Proof. The only condition that is not trivial is that the range of \mathbf{s} be well-founded. But this is clear from the fact that the first component has strictly increasing length; any descending sequence of elements of the range of \mathbf{s} would correspond to a descending sequence of lengths of the first component, and hence to a descending sequence of ordinals. \square

Claim 2.3.18. \mathbf{s} is G -true.

Proof. We show first that the set $S = \{t : \mathbf{s}(t) \in G\}$ contains a club, then that it is closed.

For $a \in \mathcal{O}$, $f_a(t) \prec X_a$ on a club C_a , by the definition of the companion map. Let $C_{<a} = \bigcap_{b < a, b \in \mathcal{O}} C_b$, and let $C^+ = \{s : s \in C_{<s}\}$. Note that C^+ is closed and unbounded, and that at stages in C^+ all of the approximations to H -sets used by \mathbf{s} are correct.

For $a \notin \mathcal{O}$, there is a first stage in C^+ at which this is recognized, either because D_a includes an element of non-notation form or because an infinite branch has appeared in D_a . Observe that if this behaviour is observed at a stage in C^+ , then a is not in \mathcal{O} , because all of the H -sets corresponding to true notations that were used in computing D_a at this stage were correct. Let $w(a)$ be this first stage. Let $C = \{s : s \in C^+ \wedge (\forall t < s) w(t) < s\}$. This is also a closed and unbounded subset of ω_1 , and at these stages t , $\mathbf{s}(t) \in G$.

To show that S is closed, let $t_0 < t_1 < \dots \in S$, with $t = \lim_i t_i$. At all of these stages, all of the approximations to H -sets are correct; in other words, $f_a(t_i) \prec X_a$ for $a \in \mathcal{O}$, $a < t_i$. By the conditions on the companion map, $f_a(t) = \lim_i f_a(t_i) \prec X_a$. Furthermore, at each of these stages, the first component is correct; so for each $a < t$ with $a \notin \mathcal{O}$, there is a $t_i > w(a)$. So $t \in C$ as defined above, and therefore $C \in S$. \square

We then obtain the desired tree by Theorem 2.3.3. \square

It is evident that similar approaches can push this hierarchy of thin degrees much further; combining the results of this section, for example, we can define a new \mathcal{O} -like notation system \mathcal{N} by adding to the closure conditions of \mathcal{O} the additional condition

(v) Suppose $a \in \mathcal{N}$. Then $b = \langle \mathcal{O}, a \rangle \in \mathcal{N}$. $H_b = \mathcal{O}^{H_a}$, and $|b|$ is the least ordinal not hyperarithmetic in H_a . $c <_{\mathcal{N}} b$ iff $c <_{\mathcal{N}} d$ for some $d \in \mathcal{N}$ constructed from H_a using only conditions (i)-(iv).

By a proof similar to Prop. 2.3.16, we can show that the degrees of paths of computable trees of countable width with companion maps are (uniformly) closed under the map $X \rightarrow \mathcal{N}^X$; and we can repeat this process further. But since this process can evidently be continued indefinitely, we must look elsewhere for an upper bound on the complexity of the thin degrees. In the next chapter, we will take a less constructive approach to building this degrees, culminating in the main theorem of this document.

2.4 Club Approximation

In the previous chapter, the presence of a companion map is critical to the construction of paths of higher and higher complexity; however, it is straightforward to construct an example in which a companion map is not possible. However, in a certain sense the path constructed in this fashion still has a “good” approximating function; it simply doesn’t obey the precise constraints placed on a companion map. In order to understand these approximating functions and their role in determining whether or not a path of a given complexity exists, we present in this chapter several extensions of the notion of a companion map, and use them to construct unique-path representations of a much larger class of degrees.

The following definition is well-known but presented here for completeness.

Definition 2.4.1. *In an ordered set A , a club is a set $C \subseteq A$ that is closed and unbounded, in that:*

- *If $a_0 < a_1 < a_2 < \dots$ is an increasing sequence in C that has an upper bound in A , then the least upper bound is in C ; and*

- For all $a \in A$, there is a $c \in C$ with $c > a$.

For our purposes, all clubs will be clubs in ω_1 with the usual ordering.

Definition 2.4.2. A function $f : A \rightarrow B$, with A and B ordered sets, is order-respecting if there exists no $x, y \in A$ with $x < y$ and $f(x) > f(y)$.

A set X is weakly club-approximable if there exists an order-respecting computable function $f : \omega_1 \rightarrow 2^{<\omega_1}$ so that $\{s : f(s) \preceq X\}$ contains a club.

X is strongly club-approximable if there exists an order-respecting computable function $f : \omega_1 \rightarrow 2^{<\omega_1}$ so that $\{s : f(s) \preceq X\}$ is a club.

X is continuously club-approximable if there exists a continuous order-respecting computable function $f : \omega_1 \rightarrow 2^{<\omega_1}$ so that $\{s : f(s) \preceq X\}$ is a club.

If X is weakly (strongly, continuously) club-approximable, then the computable function f witnessing that classification is a weak (resp. strong, continuous) club-approximation to X .

Observe that, in principle, these three notions differ widely in complexity. The most striking example is that the property “ A is a club” is a Π_2^0 property (A is unbounded and every countable increasing sequence in A has a limit in A) while the property “ A contains a club” is a strictly Σ_1^1 property [?]. It is reasonable to suppose, then, that the weakly club-approximable sets would comprise a much wider class than the strongly club-approximable sets, which in turn one would expect to be much more extensive than the continuously club-approximable sets.

The following is a straightforward consequence of the definitions:

Proposition 2.4.3. *The unique path of any tree with a companion map is strongly club-approximable, and the unique path of any tree with a continuous companion map is continuously club-approximable.*

Theorem 2.4.4. *Let X be a strongly club-approximable set. Then $\deg X \in \mathcal{P}_{\text{thin}}$.*

Proof. Let $\mathbb{S} = 2^{<\omega_1}$ with the usual ordering; then \mathbb{S} is a system of states, and X is a filter for \mathbb{S} .

Fix a strong club approximation f to X . Let $\mathbf{s}(t) = f(t)$. Then \mathbf{s} is X -true, by the definition of the companion map. The theorem follows from Theorem 2.3.3. \square

However, the same cannot be said of weakly club-approximable sets:

Proposition 2.4.5. *The weakly club-approximable sets are closed downwards under Turing reduction.*

Proof. Let f be a weak club approximation for X , and let $Y = \Phi_e^X$. $C = \{s : f(s) \prec X\}$ contains a club, by hypothesis. For each α , let $t(\alpha)$ be the least stage t so that $\Phi_{e,t}^X \upharpoonright \alpha \downarrow$; $t(\alpha)$ is then a continuous function of α , and hence has a club D of fixed points.

Let $g(s) = \Phi_{e,s}^{f(s)}$. For $s \in C \cap D$, $g(s) \prec Y$, so g is a weak club approximation to Y . \square

A similar result holds of strongly club-approximable sets, though as will be demonstrated later, the strongly club-approximable sets cannot be downward-closed under Turing reduction.

Theorem 2.4.6. *Let X be a strongly club-approximable set, and let $Y \equiv_T X$. Then Y is strongly club-approximable, uniformly in an index for a strong club-approximation to X and the reductions between Y and X .*

Proof. Fix f a strong club-approximation to X , and let Γ and Δ be Turing functionals so that $\Gamma^X = Y$ and $\Delta^Y = X$.

Let C be the set of stages s so that the following conditions hold:

- (i) $\Gamma_s^{f(s)} \upharpoonright s \downarrow$,
- (ii) $\Delta_s^{\Gamma^{f(s)}} \upharpoonright s \downarrow$, and
- (iii) $\Delta_s^{\Gamma^{f(s)}} \upharpoonright s = f(s)$.

Let $\langle s_\alpha \rangle_{\alpha < \omega_1}$ enumerate the members of C in increasing order, and let $g(t) = \Gamma^{f(s_t)} \upharpoonright s_t$.

Claim 2.4.7. *C is unbounded.*

Proof. Let C_0 be the set of stages s with $f(s) \prec X$; by hypothesis, C_0 is a club. For each s , let $h(s)$ be the least stage t at which $\Gamma_t^X \upharpoonright s \downarrow$. h is continuous, so has a club C_1 of fixed points. The elements of $C_0 \cap C_1$ satisfy conditions (i)-(iii), and therefore form a club inside C . \square

Claim 2.4.8. *g is a strong club approximation to Y .*

Proof. $g(t) \prec Y$ unboundedly often; for t with $s_t \in C_0 \cap C_1$, $g(t) \prec Y$.

Let $t_0 < t_1 < \dots$ be such that $g(t_i) \prec Y$, and let $t = \lim_i t_i$. For each i , $\Delta^{g(t_i)} = f(s_{t_i})$ by construction of C . Since $g(t_i) \prec Y$ for each i , $f(s_{t_i}) \prec X$. Since f is a strong club approximation to f , $u = \lim_i s_{t_i}$ is such that $f(u) \prec X$. $u \in C$, so $u = \lim_i s_{t_i}$, and $g(u) \prec Y$. \square

\square

As a result, we can make the following definition:

Definition 2.4.9. *A Turing degree \mathbf{a} is strongly club-approximable if either of the following equivalent conditions hold:*

- (i) \mathbf{a} contains a strongly club-approximable set, or
- (ii) every member of \mathbf{a} is strongly club-approximable.

Likewise, a Turing degree is weakly club-approximable if either it contains a weakly club-approximable set or, equivalently, it consists entirely of weakly club-approximable sets.

Based on these results and the main theorem of Chapter 2, we can conclude that the weakly club-approximable and strongly club-approximable degrees are different classes of degrees:

Proposition 2.4.10. *There is a Δ_2^0 Π_1^0 -singleton that is weakly club-approximable but not strongly club-approximable.*

Proof. By Theorem 2.2.1, there is a Δ_2^0 set X that is a Π_1^0 singleton but not a thin Π_1^0 singleton. By Theorem 2.4.4, this set cannot be strongly club-approximable. By Proposition 2.4.3 and Lemma 2.3.8, \emptyset is weakly club-approximable. By Proposition 2.4.5, so is X . \square

Given that every example presented thus far of a degree that cannot be represented as the unique path in a computable tree of countable width has also failed to be strongly club-approximable, it is reasonable to ask whether the strongly club-approximable degrees are exactly the degrees of such paths. The following result answers that question in the negative, even for relatively low levels of complexity.

Proposition 2.4.11. *There is a Δ_2^0 degree $\mathbf{d} \in \mathcal{P}_{\text{thin}}$ that is not strongly club-approximable.*

Proof. By Theorem 2.4.6, it suffices to construct a computable tree of countable width T with unique path X so that X is not a strongly club-approximable set.

We construct T in stages as $\bigcup T_s$, so that level s of the tree will be decided by stage $s + 1$. X will be the limit of stage- s guesses X_s . At any stage, there will be at most countably many elements of T_s designated as Aronszajn roots; extensions will be added to the tree above these nodes as necessary so that, in the limit, an Aronszajn tree will be built above each one. This ensures that while σ is designated an Aronszajn root, there will always be an extension of σ sufficiently long that none of its extensions have yet been decided.

We prove the claim by bounded injury, satisfying the following requirements, with the natural priority order:

R_e : Φ_e is not a strong club approximation to X .

At any stage s , the highest-priority requirement that has been initialized and requires attention is permitted to act. If no initialized requirement requires attention, then the first uninitialized requirement is initialized.

When initialized at stage s , R_e fixes $\sigma_e = X_s$, adds $\sigma_e \frown 0$ and $\sigma_e \frown 1$ to T_{s+1} , designates $\sigma_e \frown 1$ as an Aronszajn root, and sets $X_{s+1} = \sigma_e \frown 0$. R_e declares itself as *in phase zero*, and sets $\tau_e = \sigma_e \frown 1$.

While R_e is in phase zero, R_e *requires attention* if $\Phi_{e,s}(t) \downarrow = \tau$ for some new $t \leq s$ with $|\tau| > |\sigma_e|$ and $\tau \preceq X_s$. When permitted to act, R_e sets X_{s+1} to a free extension of τ_e , sets τ_e to X_s , declares X_s an Aronszajn root, and deinitializes all lower-priority requirements.

At a limit stage, if R_e was permitted to act unboundedly often, then include both limits of the X_s assignments made by R_e in T , declare both as Aronszajn roots. R_e then requires attention, and when permitted to act it *enters phase one* and sets X_s to some free extension of the limit point above $\sigma_e \frown 0$ and deinitializes all lower-priority requirements. Let u_e be the current stage.

While R_e is in phase one, R_e *requires attention* if $\Phi_{e,s}(u_e) \downarrow = \tau$ with $|\tau| > |\sigma_e|$ and $\tau \preceq X_s$. When permitted to act, R_e sets X_{s+1} to some available extension of $\sigma_e \frown 1$, deinitializes all lower-priority requirements, and declares itself satisfied.

This completes the construction. Observe that, unless reinitialized by a higher-priority requirement, every strategy acts at most $\omega + 2$ times; ω -many times in phase zero, and at most twice in phase one.

Claim 2.4.12. *For each α , $\lim_s X_s \upharpoonright \alpha$ exists and is in T .*

Proof. Suppose otherwise. Then there is some $\beta < \alpha$ so that $X_s(\beta)$ changes uncountably often. Each change to X_s must be the result of an action on the part of some requirement. Since each requirement acts at most $\omega + 2$ times once its predecessors are done acting, it must be that an uncountable sequence of requirements change $X_s(\beta)$. But each requirement R_e causes changes only above $|\sigma_e|$; since X_s is increasing in length and newly

initialized requirements take $\sigma_e = X_s$, only countably many requirements can cause a change at position β . \square

In fact, it is clear from the above argument that $X_s(\beta)$ changes at most $(\omega + 2)^\beta$ -many times (where the exponentiation here is ordinal exponentiation, not cardinal), so the limit set X is in fact ω_1 -c.e.; this is not, however, important for the result we intend to prove.

Claim 2.4.13. *X is not strongly club-approximable.*

Proof. Suppose that Φ_e is a strong club-approximation to X . Let s_0 be a stage large enough that R_i has finished acting by stage s_0 for all $i < e$, and suppose that R_e is not already satisfied. Note that it will be the case that $\Phi_e(s) \preceq X_s$ for unboundedly many s , because the initial segments of the X_s must stabilize; R_e will therefore get uncountably many opportunities to act. So R_e acts infinitely many times in phase zero, and proceeds to phase one at stage s_1 . $\Phi_e(s) \prec X$ unboundedly often for $s < s_1$, and Φ_e is a strong club approximation to X ; therefore $\Phi_e(s_1) \prec X$. But when this occurs, R_e changes X_s to a value incompatible with $\Phi_e(s_1)$, and no strategy is able to change it back. This is a contradiction. \square

Claim 2.4.14. *X is the only path through T .*

Proof. Let $Y \in [T]$. Y is a path, so Y is not entirely contained in an Aronszajn tree. But the only strings that are not contained in Aronszajn trees are those that are initial segments of X_s uncountably often. Since the initial segments of the X_s stabilize, this can only happen for strings that are initial segments of X . Then every initial segment of Y is an initial segment of X , so $Y = X$. \square

\square

On the other hand, the weakly club-approximable degrees are very extensive:

Theorem 2.4.15. *A set X is Δ_1^1 if and only if it is weakly club-approximable. Furthermore, this holds with all possible uniformity.*

Proof. Suppose X is weakly club-approximable, with weak club approximation f . Then $\alpha \in X$ iff $\exists C$ a club such that $\alpha \in f(\beta)$ for all $\alpha < \beta \in C$, iff $\forall C$ if C is a club such that $\forall \beta, \gamma \in C (\beta < \gamma \rightarrow f(\beta) \prec f(\gamma))$, then $\forall \alpha < \beta \in C \alpha \in f(\beta)$. These are, respectively, a Σ_1^1 and a Π_1^1 definition of X , so X is Δ_1^1 .

Suppose instead that X is Δ_1^1 , and φ and ψ are first-order formulas of $\mathcal{L}_{\omega_1\omega}$ so that $\alpha \in X$ iff $\exists Y \varphi(\alpha, Y)$ iff $\forall Z \psi(\alpha, Z)$. Define a function $f : \omega_1 \rightarrow 2^{<\omega_1}$ as follows:

- (1) Suppose that there exists $\beta > \alpha$ such that the following hold:
 - (i) $L_\beta \models V = L_{\omega_2}$,
 - (ii) $\alpha = (\omega_1)^{L_\beta}$, and
 - (iii) $L_\beta \models \forall \gamma \exists Y \varphi(\gamma, Y) \iff \forall Z \psi(\gamma, Z)$.

Then $f(\alpha) = \sigma \in 2^{<\alpha}$, where for $\gamma < \alpha$, $\sigma(\gamma) = 1$ iff $L_\beta \models \exists Y \varphi(\gamma, Y)$. (Note that this is independent of the choice of the witnessing β .)

- (2) Otherwise, $f(\alpha) = 0^\alpha$.

Let $M_0 \subset M_1 \subset \dots$ be an increasing continuous ω_1 -sequence of countable elementary substructures of L_{ω_2} , and let \overline{M}_i be the transitive collapse of M_i . $\overline{M}_i \models ZF^- + V = L$, so $\overline{M}_i = L_{\gamma_i}$ for some countable ordinal γ_i . Let $\alpha_i = (\omega_1)^{L_{\gamma_i}}$; by the continuity of the original sequence, these α_i form a club. And $f(\alpha_i)$ is always a correct initial segment of X . \square

As a result of Theorem 2.4.15, the main theorem Theorem 2.1.16 can be proven by concentrating on weak club-approximations rather than on Δ_1^1 definitions of sets.

We now proceed to prove Theorem 2.1.16, for which we will need the following lemma:

Lemma 2.4.16 (Folklore). *Fix $\langle C_\alpha \rangle_{\alpha < \omega_1}$ a sequence of clubs. Let D be the diagonal intersection of the C_α ; that is, $\alpha \in D$ iff $\alpha \in \bigcap_{\beta < \alpha} C_\beta$. Then D is a club.*

Proof. D is unbounded: Fix a countable ordinal δ . Let $\alpha_0 = \delta$. For each β , let α_β be the first member of $\bigcap_{\gamma < \beta} C_\gamma$ greater than α_γ for every $\gamma < \beta$. Observe that this sequence is continuous: for a limit ordinal β , every club C_γ for $\gamma < \beta$ contains a tail segment of the sequence of α_γ for $\gamma < \beta$, so every C_γ includes the supremum of that sequence, which will be α_β . The sequence therefore has a fixed point $\alpha = \alpha_\alpha$. This α is a member of D greater than δ .

D is closed: Let $\alpha_0 < \alpha_1 < \dots$ an ω -sequence in D with limit α . Each C_β for $\beta < \alpha$ contains a tail segment of this sequence, so all of the C_β include α . Therefore $\alpha \in D$. \square

Theorem 2.4.17. (*Theorem 2.1.16*)

For every Δ_1^1 degree \mathbf{d} , there is (uniformly in a Δ_1^1 index for a representative of \mathbf{d}) a degree $\mathbf{c} \geq \mathbf{d}$ so that \mathbf{c} is the degree of the unique path in a computable tree of countable width.

Proof. We will use Theorems 2.4.15 and 2.4.4, and show instead that every weakly club-approximable set is computable from a strongly club-approximable set.

Let Y be Δ_1^1 . Then by Theorem 2.4.15, X is weakly club-approximable. Let f be a weak club approximation to X . The property “ C is a club so that for all $\alpha \in C$, $f(\alpha) \prec X$ ” is an arithmetic one; the $<_L$ -least witness is therefore Δ_1^1 (definable by statements of the form “there exists a well-founded model of $V = L_{\omega_2} \dots$ ” and “for every well-founded model of $V = L_{\omega_2} \dots$ ”). Let $C(X)$ be that $<_L$ -least club. Note that $C(X) \geq_T X$, because $C(X)$ filters out all of the incorrect “guesses” from f .

Define $C^\alpha(X)$ inductively as follows for countable ordinals α :

- (i) $C^0(X) = C(X)$,
- (ii) $C^{\alpha+1}(X) = C^\alpha(X) \cap C(C^\alpha(X))$, and
- (iii) For limit α , $C^\alpha(X) = \bigcap_{\beta < \alpha} C^\beta(X)$.

Note that for each α , $C^\alpha(X)$ is a set of stages on which the approximations to X and every $C^\beta(X)$ for $\beta < \alpha$ are simultaneously correct.

Without loss of generality, suppose $0 \in C(X)$.

Let f_β be the weak club approximation to $C^\beta(X)$. Note that by the uniformity of Theorem 2.4.15, and the uniformity in the selection of $C(\cdot)$, the sequence of indices for the f_β is computable.

Let $X^* = \bigoplus_{\alpha < \omega_1} C^\alpha(X)$; for the time being, consider X^* as a member of $2^{\omega_1 \times \omega_1}$. We construct a computable function g from ω_1 to countably-supported members of $2^{\omega_1 \times \omega_1}$ as follows:

(1) Suppose that the $f_\beta(\alpha)$ is closed and unbounded below α for every $\beta \leq \alpha$. Then let $g(\alpha)$ be the function $h : \alpha \times \alpha \rightarrow 2$ given by $h(\beta, \gamma) = f_\beta(\alpha)(\gamma)$.

(2) Otherwise, let $g(\alpha)$ be the function $h : \alpha \times \alpha \rightarrow 2$ given by $h(\beta, \gamma) = 0$.

Fix a bijective pairing function $\langle \cdot, \cdot \rangle : \omega_1 \times \omega_1 \rightarrow \omega_1$ so that $\langle 0, 0 \rangle = 0$. Let $F : \omega_1 \rightarrow 2^{<\omega_1}$ be defined as follows:

(1) If $\langle \beta, \gamma \rangle < \alpha$ for all $\beta, \gamma < \alpha$, then $F(\alpha) = \sigma \in 2^{<\alpha}$, where $\sigma(\langle \beta, \gamma \rangle) = g(\alpha)(\beta, \gamma)$.

(2) Otherwise, $F(\alpha) = 0^\alpha$.

Identify X^* with its image under $\langle \cdot, \cdot \rangle$.

Claim 2.4.18. *F is a strong club approximation to X^* .*

Proof. Let $A = \{\alpha \mid F(\alpha) \prec X^*\}$. We aim to show that A is closed and unbounded.

Let E be the set of α so that $\langle \beta, \gamma \rangle < \alpha$ for all $\beta, \gamma < \alpha$; note that E is a club.

For each α , let $C^\alpha(X)'$ be the set of limit points of $C^\alpha(X)$; note that $C^\alpha(X)'$ is also a club. Let D be the diagonal intersection of $C^\alpha(X)'$; that is, $\beta \in D$ iff $\beta \in C^\alpha(X)'$ for all $\alpha < \beta$. Then D is a club. Note that if $\beta \in D$ then $\beta \in C^\beta(X)'$, because the $C^\alpha(X)'$ are nested.

Let $\alpha \in D \cap E$. Then α falls under case (1) in the definition of F , so $F(\alpha)$ is the image of $g(\alpha)$. Since $\alpha \in D$, α also falls under case (1) in the definition of g . So $g(\alpha)$ is given by $h(\beta, \gamma) = f_\beta(\alpha)(\gamma)$. But $\alpha \in C^{\beta+1}(X)'$ for each $\beta < \alpha$; so $\alpha \in C^{\beta+1}(X)$; so $f_\beta(\alpha) \prec C^\beta(X)$. Therefore $g(\alpha)$ is a “correct square” of X^* , and hence $F(\alpha)$ is a correct initial segment of X^* .

Therefore $D \cap E \subseteq A$, so A contains a club; in particular, A is unbounded. It remains to show that A is closed.

Suppose $\alpha_0 < \alpha_1 < \dots$ is an increasing ω -sequence in A with limit α . $F(\alpha_i) \prec X^*$ for each i . Note that by the simplifying assumptions $0 \in C(X)$ and $\langle 0, 0 \rangle = 0$, the first bit of X^* is 1; so every α_i must fall under case (1) in the construction of F . It's then clear that α likewise falls under case (1).

Likewise, every α_i must fall under case (1) in the definition of g . Fix $\beta < \alpha$. By hypothesis, α_i is a limit point of $C^\beta(X)$ for all sufficiently large i ; so $\alpha_i \in C^\beta(X)$ for all sufficiently large i . Since the $C^\beta(X)$ are nested, $\alpha_i \in C^\beta(X)$ for all i . $C^\alpha(X) = \bigcap_{\beta < \alpha} C^\beta(X)$, so α is a limit point of $C^\alpha(X)$. So α also falls under case (1).

Since $g(\alpha_i)$ is correct for each i , $f_\beta(\alpha_i) \prec C^\beta(X)$ for each $\beta < \alpha$ and each sufficiently large i . But $\alpha \in C^{\beta+1}(X)$ for each $\beta < \alpha$, so $f_\beta(\alpha) \prec C^\beta$. Thus $g(\alpha)$ is also correct, so $F(\alpha) \prec X^*$ and $\alpha \in A$. □

Therefore X^* is a strongly club-approximable set computing X . By Theorem 2.4.4, X^* is Turing-equivalent to a thin Π_1^0 singleton; so X is computable from a thin Π_1^0 singleton. □

2.5 Structure of $\mathcal{P}_{\text{thin}}$

2.5.1 Jump Inversion

Theorem 2.5.1. *Let $A \subseteq \omega_1$ be the unique path of a computable tree of countable width T_A with a companion map f_A . Let T be a computable tree of countable width with unique path $X \geq_T A'$ and companion map f . Then, uniformly in an index for T , f , and the reductions from X to A' and A , there is a computable tree of countable width T^{-1} with unique path $Y \geq_T A$ and companion map g so that $Y' \equiv_T Y \oplus A' \equiv_T X$.*

Proof. For clarity, we prove the claim when $A = \emptyset$, and discuss relativization at the end.

We build T^{-1} in stages as $\bigcup_s T_s$, so that level s will be decided by the end of stage s . Simultaneously, at stage s $g(s)$ will be decided; we also build a Δ_2^0 partial map $h : T^{-1} \rightarrow T$ so that $\bigcup_\alpha h(Y \upharpoonright \alpha) = X$.

We perform the construction on a tree of strategies. Let $\Lambda = \omega_1 + 1$ with the natural ordering; this is the collection of outcomes for any strategy. The outcomes corresponding to countable ordinals are designated *coding outcomes*; the outcome corresponding to ω_1 is designated the *convergence outcome*. To each node of the tree $\Lambda^{<\omega_1}$ we assign a strategy η corresponding to the requirement $R_{|\eta|}$, defined below:

R_e : Either $\Phi_e^Y(e) \downarrow$, or there is an initial segment σ of Y such that $h(\sigma) = X \upharpoonright e$.

Strategies at nodes higher in the tree have lower priority.

At any stage s , the *execution path* $\delta(s)$ is the string in $\Lambda^{<\omega_1}$ of length s so that $\delta(s)(t)$ is the outcome at stage s of the strategy assigned to $\delta(s) \upharpoonright t$.

During stage s , if any strategy along the execution path requires attention, then the highest-priority strategy along the execution path that requires attention is permitted to act. If no strategy along the execution path requires attention, then the first uninitialized strategy along the execution path is initialized.

At any stage, some strategies may be *cancelled*. A cancelled strategy does not act or require attention, and no strategy of strictly lower priority than a cancelled strategy may be initialized. If the execution path at stage s passes through a cancelled strategy, and no higher-priority requirement requires attention or is uninitialized, then set $g(s+1) = g(s) \frown 0$.

At every stage, if $g(s+1)$ is not compatible with $g(s)$, then $g(s)$ is designated an Aronszajn root.

Strategy for R_e :

Let η be a strategy pursuing requirement R_e . When η is initialized at stage s , set $\sigma_\eta = g(s)$. Include an Aronszajn tree A_η rooted at $g(s)$, with a computable uncountable antichain $\langle q_i^\eta \rangle_{i < \omega_1}$. Designate every q_i^η as an Aronszajn root, set $g(s+1) = q_0^\eta$, set $h(q_0^\eta) = f(s) \upharpoonright e$, and set η to outcome 0.

If η ever enters its convergence outcome, then it never requires attention from that point forward.

If η is in a coding outcome, and there is a $\tau \in T_s$, $\tau \succ \sigma_\eta$, so that $\Phi_{e,s}^\tau(e) \downarrow$, then η requires attention. If permitted to act, η switches to its convergence outcome and sets $g(s+1)$ to some extension of τ with length at least s .

If η is in coding outcome α and $f(s) \upharpoonright e \neq h(q_\alpha^\eta)$, then η requires attention. If there is an outcome β which η has previously visited so that $h(q_\beta^\eta) = f(s) \upharpoonright e$, then η returns to outcome β and sets $g(s+1)$ to a string consistent with the outcomes of all strategies along the new execution path. If no such β exists, then let β be the first coding outcome that η has not yet visited. Set $h(q_\beta^\eta) = f(s) \upharpoonright e$ and $g(s+1)$ to some extension of q_β^η of sufficient length, and switch η to outcome β .

Limit Stages:

Let s be a limit stage. Let U be the set of limit points of $g(t)$ for $t < s$. Observe that,

since each corresponds to a particular configuration of outcomes, each corresponds to a particular limit point of $f(t)$ for $t < s$. Only countably many of these are in T ; include only the corresponding members of U in T^{-1} . Cancel all limit strategies corresponding to elements of U not included this way.

Verification:

Claim 2.5.2. *There exists $P \in \Lambda^{\omega_1}$ so that $\{s : \delta(s) \prec P\}$ is a club.*

Proof. Let C be the set of stages at which $f(s) \prec X$; by hypothesis, this is a club. We inductively define P and a sequence $\langle s_\alpha \rangle_\alpha$ as follows: Let η be the strategy assigned to the node $P \upharpoonright \alpha$. If η ever enters its convergence outcome, then $P \upharpoonright (\alpha + 1) = P \upharpoonright \alpha \frown \omega_1$ and s_α is the first stage after $\sup_{\beta < \alpha} s_\beta$ at which η is in its convergence outcome. Otherwise, η eventually assigns a correct initial segment of X to some outcome γ . Then $P \upharpoonright (\alpha + 1) = P \upharpoonright \alpha \frown \gamma$, and s_α is the first stage after $\sup_{\beta < \alpha} s_\beta$ at which η enters outcome γ .

For limit α , $P \upharpoonright \alpha = \bigcup_{\beta < \alpha} P \upharpoonright \beta$ and $s_\alpha = \sup_{\beta < \alpha} s_\beta$. The map $\alpha \rightarrow s_\alpha$ is then continuous, so its set of fixed points forms a club.

Let $C' = \{\alpha \in C : \alpha = s_\alpha\}$. Then C' is a club, and for $t \in C'$ $\delta(t) \prec P$. It is evident by construction that whenever $\delta(t) \prec P$, $t \in C'$. □

Claim 2.5.3. *There is a club C of stages so that for $s < t \in C$, $g(s) \prec g(t)$. Then $Y = \bigcup_{s \in C} g(s)$ is a path through T^{-1} .*

Proof. The club C is precisely the club of stages s so that $\delta(s) \prec P$. □

Claim 2.5.4. *Y is the only path through T^{-1} .*

Proof. Let Z be a path through T^{-1} . Z does not fall permanently inside an Aronszajn tree, so unboundedly often $g(s)$ is an initial segment of Z . Therefore, unboundedly many strategies η have σ_η along Z . But it is clear by construction that this can only happen if these strategies are assigned to compatible strings; in other words, Z corresponds to a

path through the tree of strategies that is visited unboundedly often. Call this path P' . But then there are unboundedly many coding outcomes along P' , which together must code a path through T . The only such path is X , so the coding outcomes of P' agree with the coding outcomes of P . Then by induction, the convergence outcomes must also match; so $P' = P$, and hence $Z = Y$. \square

Claim 2.5.5. $Y' \geq_T X$.

Proof. Y' clearly computes h , because h is Δ_2^0 . The domain of h is unbounded along Y . So Y' computes $h[Y] = X$. \square

Claim 2.5.6. $Y \oplus \emptyset' \geq_T Y'$.

Proof. Once a strategy η is initialized, \emptyset' is sufficient to determine whether η will enter its convergence outcome, which occurs if and only if $\Phi_e^Y(e) \downarrow$. Y is sufficient to determine where σ_η falls. \square

As a consequence, $X \equiv_T Y' \equiv_T Y \oplus \emptyset'$. \square

To relativize, we may *interleave* T^{-1} with T_A , taking a new tree T_A^{-1} to be $\{\sigma \oplus \tau \mid \sigma \in T^{-1} \wedge \tau \in T_A \wedge |\sigma| = |\tau|\}$, and similarly interleave g and f_A to form a corresponding companion map.

2.5.2 Gaps

Proposition 2.5.7. *If $X \not\leq_T \emptyset$ is hyperimmune-free, then X is not strongly club-approximable, and is therefore not equivalent to the unique path in any computable tree with a companion map.*

Proof. Suppose that X is both hyperimmune-free and strongly club-approximable. Let f be a computable function witnessing that X is strongly club-approximable; that is, $C = \{s : f(s) \prec X\}$ is a club. Clearly, $C \leq_T X$. Let $g \leq_T X$ be the function enumerating the

elements of C in order. Because C is a club, g is continuous. Since X is hyperimmune-free, there is a computable function $h \geq g$. Without loss of generality, h is also continuous.

Let x be a fixed point of h with $x > \omega$. $x = h(x) \geq g(x) \geq x$, so this is also a fixed point of x . Then $x \cap C$ is a club in x of order type x .

Claim 2.5.8. *Let $\alpha < \omega_1$, $\alpha > \omega$, and suppose that A_1 and A_2 are closed and unbounded subsets of α with order type α . Then $A_1 \cap A_2$ is an unbounded subset of α .*

Proof. Let g_1 and g_2 enumerate the members of A_1 and A_2 respectively.

$g_1 \circ g_2$ is a continuous function. Its fixed points are in both A_1 and A_2 , and there are clearly unboundedly many below α . \square

As a result of the claim, there is only one string σ of length x so that the set $\{s < x : f(s) \prec \sigma\}$ is a club of order type x , and that σ is $X \upharpoonright x$. So X is computable. \square

Definition 2.5.9. *A degree is thin if it can be represented as a path through a computable tree of countable width. A degree is axial if it can be represented as the unique path through a computable tree of countable width.*

\mathbf{a} is thinimal above \mathbf{b} if $\mathbf{a} > \mathbf{b}$ and there are no thin degrees \mathbf{c} with $\mathbf{a} \geq \mathbf{c} > \mathbf{b}$.

Proposition 2.5.10. *For any degree \mathbf{a} , there exists a degree \mathbf{d} so that $\mathbf{a} < \mathbf{d} \leq \mathbf{a}'$ and the thin degrees computable from \mathbf{d} are exactly the thin degrees computable from \mathbf{a} .*

Proof. Fix a representative $A \in \mathbf{a}$; for example, the $<_L$ -least, if canonicity is desired. We will construct by forcing with oracle A' a set G so that $D = A \oplus G$ is of the desired degree.

Fix also a standard effective bijection taking ω_1^3 to ω_1 , denoted $\langle \cdot, \cdot, \cdot \rangle$.

Finally, fix a uniformly A -computable enumeration $\langle T_i \rangle$ of A -computable trees (countable width or otherwise).

Let $\sigma_0 = \langle \rangle$.

Stage s : If s is a limit, let $\sigma_s = \lim_{t < s} \sigma_t$.

Stage $s+1 = \langle e+1, a, b \rangle + 1$: Let $t = \langle e, a, b \rangle$. Take $\sigma_s = \sigma_t \frown i$, where $i \neq \{e\}^A(\lvert \sigma_t \rvert)$.

Stage $s + 1 = \langle 0, e, i \rangle + 1$: Let $t = \langle 0, e, i \rangle$. Search for one of the following:

(i) $\tau \succ \sigma_t$ and $x \in \omega_1$ such that $(\forall \rho \succ \tau)(\{e\}^{\rho \oplus A}(x) \uparrow)$;

(ii) $\tau \succ \sigma_t$ such that

$$(\forall x \in \omega_1)(\forall \rho_0, \rho_1 \succ \tau) [(\{e\}^{\rho_0 \oplus A}(x) \downarrow \wedge \{e\}^{\rho_1 \oplus A}(x) \downarrow) \rightarrow \{e\}^{\rho_0 \oplus A}(x) = \{e\}^{\rho_1 \oplus A}(x)]$$

(iii) $\tau \succ \sigma_t$ such that $\{e\}^{\tau \oplus A} \notin T_i$; or

(iv) an order-preserving map $f : 2^{<\omega} \rightarrow 2^{<\omega_1}$ so that $\{e\}^{f(\cdot) \oplus A}$ is an order-preserving map $2^{<\omega} \rightarrow T_i$.

In cases (i)-(iii), take $\sigma_{s+1} = \tau$. In case (iv), note that $\{e\}^{f(\cdot) \oplus A}$ induces an injection $F : 2^\omega \rightarrow 2^{<\omega_1}$ with bounded range. If the range of F is entirely contained in T_i , then T_i does not have countable width, and we take $\sigma_{s+1} = \sigma_s \smallfrown 0$. Otherwise, let $\rho \in 2^\omega$ be such that $F(\rho) \notin T_i$ and take $\sigma_{s+1} = \lim f(\rho)$.

Verification: It is clear that $A' \geq_T G$, so $A' \geq_T A \oplus G \geq_T A$.

Suppose that $\{e\}^{G \oplus A}$ is a path through T_i . This pair was considered at stage $s + 1 = \langle 0, e, i \rangle + 1$. It is easy to see that one of the four cases was encountered. If the construction fell into case (i), then $\{e\}^{G \oplus A}$ is not total. In case (ii), $\{e\}^{G \oplus A}$ is A -computable. In case (iii), $\{e\}^{G \oplus A}$ does not remain in T_i ; and in case (iv), either $\{e\}^{G \oplus A}$ does not remain in T_i or T_i does not have countable width.

Finally, by the other class of stages, G is not computable from A , so $G \oplus A >_T A$. \square

Proposition 2.5.11. *Let J be a one-place operation on Turing degrees so that if $\mathbf{a} \geq J(\mathbf{0})$, then there exists $\mathbf{b} \not\leq J(\mathbf{0})$ such that $J(\mathbf{b}) = \mathbf{a}$.*

Then there exists a degree \mathbf{d} so that the thin degrees computable from $J(\mathbf{d})$ are exactly the thin degrees computable from \mathbf{d} .

Proof. By Prop 2.5.10, there is a degree \mathbf{a} with $J(\mathbf{0}) < \mathbf{a} \leq J(\mathbf{0})'$ so that the thin degrees computable from \mathbf{a} are exactly the thin degrees computable from $J(\mathbf{0})$. By the conditions on J , there exists a degree $\mathbf{d} \not\leq J(\mathbf{0})$ so that $J(\mathbf{d}) = \mathbf{a}$.

Suppose that there were a thin degree \mathbf{c} with $\mathbf{d} \leq \mathbf{c} \leq \mathbf{a}$. By the choice of \mathbf{a} , $\mathbf{c} \leq J(\mathbf{0})$. But then $\mathbf{d} \leq J(\mathbf{0})$, contradicting the choice of \mathbf{d} .

Thus \mathbf{d} is the desired degree. □

2.5.3 Types of Degrees

Theorem 2.5.12. *There is a computable tree T of countable width with exactly one path X , so that X has minimal Turing degree.*

Proof. Let A be a fixed computable Aronszajn tree.

Let $\Lambda = \{S\} \cup \omega_1$. Let $\mathcal{T} = \Lambda^{<\omega_1}$. To each string $\sigma \in \mathcal{T}$ we attach a *worker* η_σ . At any stage s , each worker which has been initialized has an outcome in Λ ; let $\delta(s)$ be the maximal string so that $\delta(s)(i+1)$ is the stage- s outcome of $\eta_{\delta(s) \upharpoonright i}$.

Each worker maintains one or more *scaffolds*, each one corresponding to an outcome of that worker. The *active* scaffold at stage s is the one corresponding to the worker's outcome at stage s .

Definition 2.5.13. *A scaffold is a partial function $f : 2^{<\omega_1} \rightarrow 2^{<\omega_1}$ together with a set $S \subseteq 2^{<\omega_1}$ such that the following conditions hold.*

- (i) $\text{dom}(f) \subseteq S$;
- (ii) For $\sigma, \tau \in \text{dom}(f)$, $\sigma \preceq \tau$ iff $\sigma \preceq \tau$;
- (iii) If $\sigma \frown i \in \text{dom}(f)$, then $f(\sigma \frown i)(|f(\sigma)|) = i$; and
- (iv) If $\sigma_0 \prec \sigma_1 \prec \dots$ is a sequence in $\text{dom}(f)$ and $\sigma = \lim_i \sigma_i \in \text{dom}(f)$, then $f(\sigma) = \lim_i f(\sigma_i)$.

We denote the scaffold (S, f) .

For scaffolds (S_0, f_0) and (S_1, f_1) , say (S_0, f_0) is a subscaffold of (S_1, f_1) if $\text{ran}(f_0) \subseteq \text{ran}(f_1)$. Observe that there is no restriction on the relationship between S_0 and S_1 .

It will be the case that every scaffold maintained by η_σ is a subscaffold of η_τ for each $\tau \prec \sigma$. The root worker η_\emptyset maintains only one scaffold, and the corresponding function is the identity; this scaffold will form the final tree T .

At any stage s , a worker η_σ may issue a *request* for any $\theta \in 2^{<\omega_1}$ every initial segment of which is currently in the range of the active scaffold of every η_τ for $\tau \prec \sigma$. Provided that $|\theta| \geq s$, this request will be *honored* by every such η_τ : η_τ will add an assignment to its active scaffold (possibly adding to the domain of the scaffold) so that θ will be in the range, and then add to the domain of the scaffold so that the preimage of θ has a copy of A above it.

Successor workers pursue two sorts of requirements:

D_e : $\Phi_e \neq X$

M_e : One of the following holds:

(i) $(\exists x)\Phi_e^X(x) \uparrow$,

(ii) $\Phi_e^X \leq_T \emptyset$, or

(iii) $\Phi_e^X \geq_T X$.

Divide the countable successor ordinals into two computable unbounded sets \mathcal{D} and \mathcal{M} . Workers η_σ for $|\sigma|$ the e th member of \mathcal{D} pursue the diagonalization requirement D_e ; workers η_σ for $|\sigma|$ the e th member of \mathcal{M} pursue the minimality requirement M_e . Workers η_\emptyset and η_σ for σ of limit length behave differently and pursue no requirement.

For ease of notation, when σ is a string of successor length, denote by σ^- its immediate predecessor.

For any worker η_σ , let (S^σ, f^σ) be the currently active scaffold.

Strategy for Limit Workers: $\eta_\langle \rangle$ does nothing except respond to requests; we therefore concentrate on η_σ for $|\sigma|$ of limit length.

When initialized at stage s , η_σ takes $S^\sigma = A$ and sets $f^\sigma(\langle \rangle)$ to be the limit of $f^\tau(\langle \rangle)$ for $\tau \prec \sigma$. It enters outcome 0, and remains there permanently.

At later stages, η_σ works to define f^σ on progressively higher and higher levels of S^σ . Limit levels are straightforward: η_σ simply defines the function as necessary to preserve continuity, requesting any strings not already present in the ranges of the previous scaffolds. At successor levels, suppose η_σ has defined $f^\sigma(\tau)$ and needs to define $f^\sigma(\tau \frown i)$. For each $\rho \prec \sigma$, let $\theta(\rho)$ be the element in the domain of f^ρ so that $f^\rho(\theta(\rho)) = f^\sigma(\tau)$. Let $f^\sigma(\tau \frown i) = \lim f^\rho(\theta(\rho) \frown i)$, requesting the string if necessary. This limit exists and has the property that every f^ρ has a sequence cofinal along it; so the request is a valid one.

Strategy for Diagonalization Requirements: Let η_σ be a successor worker pursuing requirement D_e . When initialized at stage s , η_σ looks for θ_0 and θ_1 in the range of f^{σ^-} with length at least s differing at some position $\alpha < s$. η_σ sets $S^\sigma = A$, puts $f^\sigma(\langle \rangle) = \theta_0$, and enters outcome 0.

If at some stage $t > s$, $\Phi_{e,t}(\alpha) \downarrow = \theta_0(\alpha)$, then η_σ requires attention. If it receives attention, it deletes its existing scaffold, sets $S^\sigma = A$ and $f^\sigma(\langle \rangle) = \theta_1$, and enters outcome 1.

At any successor stage, η_σ considers each leaf node τ of $\text{dom}(f^\sigma)$. For each such τ , by induction there exists $\theta \in \text{dom}(f^{\sigma^-})$ with $f^{\sigma^-}(\theta) = f^\sigma(\tau)$; if $\theta \frown 0, \theta \frown 1 \in \text{dom}(f^{\sigma^-})$, then η_σ takes $f^\sigma(\tau \frown i) = f^{\sigma^-}(\theta \frown i)$ for each i . Otherwise, it simply waits until the necessary assignments have been made.

At a limit stage, η_σ considers each limit node $\tau \in S^\sigma$ so that f^σ has been defined on every initial segment of τ . f^σ induces an obvious choice of $f^\sigma(\tau)$; if this string is not already in the range of f^{σ^-} , then η_σ requests it. Regardless, it takes this as the assignment

of $f^\sigma(\tau)$.

Strategy for Minimality Requirements: Let η_σ be a successor worker pursuing requirement M_e . When initialized at stage s , η_σ sets $S^\sigma = A$ and puts $f^\sigma(\langle \rangle)$ some element of $\text{ran}(f^{\sigma^-})$ with length at least s (if none exists, η_σ waits until it does). This scaffold is designated the *splitting scaffold*; it will be the active scaffold whenever η_σ is in outcome S . η_σ begins in outcome S .

η_σ will maintain the condition that if (S, f) is its splitting scaffold and $\tau_0, \tau_1 \in S$ are incomparable, then $\Phi_e^{f(\tau_0)}$ and $\Phi_e^{f(\tau_1)}$ are incomparable. Call this condition (*).

While still in outcome S , η_σ attempts to extend f^σ while preserving (*). At a limit stage, it simply fills in any limit nodes in S^σ , requesting whatever is necessary. At a successor stage, η_σ will attempt to extend each leaf node of $\text{dom}(f^\sigma)$ in both directions. For each leaf $\tau \in \text{dom}(f^\sigma)$, if there exist $\rho_0, \rho_1 \in \text{ran}(f^{\sigma^-})$ with length at least s and $x < s$ so that $\rho_i \succ f^\sigma(\tau) \frown i$ and $\Phi_e^{\rho_0}(x) \not\downarrow \Phi_e^{\rho_1}(x) \downarrow$, then η_σ takes $f^\sigma(\tau \frown i) = \rho_i$. If none exists, then let $\epsilon = f^\sigma(\tau)$. η_σ switches to a new ordinal outcome and associates with it a new scaffold as follows: $f^\sigma(\langle \rangle) = \theta$, where θ is the first extension of $\epsilon \frown 0$ in the range of f^{σ^-} (as usual, if none exists, wait until it does); S^σ is the translation of the part of S^{σ^-} extending that point.

While in an ordinal outcome, η_σ copies the scaffold of η_{σ^-} , just like the diagonalization workers do.

If a splitting pair is found within this new scaffold (that is, $\rho_0, \rho_1 \in S^\sigma$ with $\Phi_e^{\rho_0}$ and $\Phi_e^{\rho_1}$ incomparable) then η_σ again switches to a new ordinal outcome and replaces the current scaffold with a new scaffold, defined the same way as before but rooting at an extension of $\epsilon \frown 1$.

If a splitting pair is found within this scaffold as well, then between the two sets of splitting pairs is a pair appropriate for extending the splitting scaffold at τ ; η_σ returns to the S outcome, re-activating the splitting scaffold, and adds those elements.

To ensure that the search for strings of sufficient length is successful (that is, that higher-priority scaffolds are not unexpectedly short) at every stage we allow the active workers to act *in order*, beginning from the root. Responses to requests are immediate, but propagate in the same manner.

This completes the construction.

Claim 2.5.14. *For any α , only countably many requests are made for strings of length $< \alpha$, and none are requested past stage α .*

Proof. Any worker initialized past stage α will, by construction, place its root at a string of length at least α . Again by construction, no worker ever makes requests that do not extend its root; therefore, only the workers initialized before stage α will ever request strings of length less than α .

Let η be such a worker, and suppose that it requests a limit string θ at stage s . In every case, this request is made as soon as the initial segments are available; that is, it must be that there was a sequence $s_0 < s_1 < \dots$ cofinal in s at which η assigned strings in the domain of its scaffold to initial segments of θ . But by construction a worker can only make stage- t assignments of length at least t ; so these initial segments of θ have lengths at least s_0, s_1, \dots . Thus θ has length at least s .

Clearly no worker requests uncountably many strings in a single stage; therefore no individual worker requests uncountably many strings of length $< \alpha$. Since only countably many workers can request these strings at all, only countably many such requests are made. □

Claim 2.5.15. *The tree T is computable and of countable width.*

Proof. This is an immediate consequence of Claim 2.5.14. □

Claim 2.5.16. *All requests are honored.*

Proof. Again, this is an immediate consequence of Claim 2.5.14; any request made at stage s for a string with length at least s is honored, and by the claim no requests not satisfying this condition are made. \square

Claim 2.5.17. *The set of strings in \mathcal{T} that lie along the execution path unboundedly often forms a path.*

Proof. Observe that limit workers (including the root worker) never change outcomes; diagonalization workers change outcome at most once; and while minimalization workers may change outcome unboundedly often, the only outcome they may visit unboundedly often is S . So every worker visits exactly one outcome unboundedly often; the inductive argument is straightforward. \square

We call this the *true path*, and the workers along this path *true workers*; their outcomes that lie along the true path are *true outcomes*.

Note that the set of strings $f^\sigma(\langle \rangle)$ evaluated during stages at which η_σ has its true outcome exactly defines a path through the tree T ; we call this path X .

Claim 2.5.18. *X is the only path through T .*

Proof. Suppose for contradiction that $Y \neq X$ is a path through T . Y cannot be “native” - that is, it must be obtained through uncountably many requests, because otherwise it would be part of an Aronszajn tree.

No worker makes requests incompatible with its root; it therefore cannot be that the requests for initial segments of Y are unbounded along the true path. There is therefore a minimal worker η so that only countably many requests for initial segments of Y are made above the true outcome of η . Note that η is not a limit worker or the root worker, because neither class of worker changes outcomes. Note also that η is not a diagonalization worker; at some stage, a worker pursuing a diagonalization requirement will enter its true outcome and never leave, so only countably many requests for anything can be made above the

other outcome. So η is a minimalization worker. Furthermore, η 's true outcome must be S - otherwise, the same argument holds as for the diagonalization case.

Since η 's true outcome is not one of the ordinal outcomes, it cannot be that uncountably many requests for initial segments of Y are made above only one of them; so it must be that an uncountable sequence of ordinal outcomes have at least one request each. The scaffolds for these outcomes must be rooted at points along Y ; but all of them come from points at which the attempt to split in the splitting outcome temporarily failed. So all of these initial segments of Y are in the splitting scaffold - which means uncountably many requests for initial segments of Y must have been made above the S outcome, contradicting our assumption on η . \square

Claim 2.5.19. *X is not computable.*

Proof. Suppose $\Phi_e = X$. Let η be the (unique) worker along the true path pursuing requirement D_e . When initialized, η selected some pair of strings and an α at which they differed. At some stage t , $\Phi_{e,t}(\alpha) \downarrow$; at that stage, η switched outcomes if necessary and directed construction along the string of the pair that did not agree with Φ_e at α . This is a contradiction. \square

Claim 2.5.20. *X has minimal Turing degree.*

Proof. Suppose that Φ_e^X is total but not computable. Let η be the (unique) worker along the true path pursuing requirement M_e . Observe that if the true outcome of η were an ordinal, Φ_e^X would be either partial or computable, because a split would never be found; $\Phi_e^X(x)$ could be computed by taking the first computation $\Phi_e^\tau(x)$ to converge for τ in the range of the scaffold of η . So the true outcome of η is S , and therefore in the limit η constructs a scaffold (U, f) so that if σ and τ are incomparable members of U then $\Phi_e^{f(\sigma)}$ and $\Phi_e^{f(\tau)}$ disagree somewhere. But then given Φ_e^X we can determine a unique branch Y of U so that $\Phi_e^{f(Y)} = \Phi_e^X$; this branch must be the preimage of Y . Thus $\Phi_e^X \geq_T X$.

So for every e we have that Φ_e^X is either partial, computable, or above X ; since we already have that $X >_T \emptyset$, X is of minimal Turing degree. \square

\square

2.6 Nonisolated Paths

2.6.1 General Results

Once one has examined the behavior of thin Π_1^0 -classes containing exactly one element, a natural follow-up question is what happens to thin Π_1^0 classes more generally.

The following basis theorem appeared in a more specialized form in the proof of Theorem 2.1.16, but is now more directly applicable.

Theorem 2.6.1. *Let P be an arithmetic property of sets so that $(\exists X)P(X)$. Then $(\exists X \in \Delta_1^1)P(X)$.*

Proof. Let X be $<_L$ -least such that $P(X)$ holds. Then

$$\sigma \prec X \iff (\exists Z)(Z \models ZF^- + V = L + \sigma \prec X \wedge Z \text{ is well-founded})$$

and

$$\sigma \prec X \iff (\forall Z)((Z \models ZF^- + V = L + (\exists Y)P(Y) \wedge Z \text{ is well-founded}) \rightarrow Z \models \sigma \prec X)$$

These are, respectively, Σ_1^1 and Π_1^1 definitions of X ; X is hence Δ_1^1 . \square

2.6.2 Kurepa Trees

Throughout this section, we will operate under the additional assumption $L_{\omega_2} = H(\aleph_1)$.

Definition 2.6.2. (Kurepa [15])

A Kurepa tree is a tree T with countable width and $||T|| = \aleph_2$.

A perfect Kurepa tree is a Kurepa tree K so that $K(\sigma) = \{\tau \in K : \tau \succ \sigma\}$ is also Kurepa.

Related is the following definition, due to Jensen [?].

Definition 2.6.3. A \diamond^+ sequence (in ω_1) is a sequence $\{\mathcal{A}_\alpha\}_{\alpha < \omega_1}$ of sets so that \mathcal{A}_α is a countable set of subsets of α and, for any $X \subseteq \omega_1$, there is a club $C \subseteq \omega_1$ so that for all $\alpha \in C$, $X \upharpoonright \alpha$ and $C \upharpoonright \alpha$ are in \mathcal{A}_α .

The proof of the following result is an effectivization of the similar one given by Kunen [14]. We use the same notation, insofar as possible.

Lemma 2.6.4. *There is a computable \diamond^+ -sequence.*

Proof. Let $ZF - P$ be the collection of axioms of Zermelo-Frankel set theory without Choice and Power Set. For each $\alpha < \omega_1$, let α^* be least $> \alpha$ so that $L_{\alpha^*} \models ZF - P + |\alpha| = \aleph_0$. Let $\mathcal{A}_\alpha = L(\alpha^*) \cap \mathcal{P}(\alpha)$. This is certainly a computable sequence; we show that it is also a \diamond^+ sequence.

Let $A \subseteq \omega_1$. By the supposition $L_{\omega_2} = H(\aleph_1)$, $A \in L_{\omega_2}$. For $\sigma < \omega_1$, let $M_{A,\sigma} = \mathfrak{H}(L_{\omega_2}, \{A\} \cup \sigma)$ (that is, the set of elements of L_{ω_2} definable using parameters from $\{A\} \cup \sigma$). Then $M_{A,\sigma}$ is a countable elementary substructure of L_{ω_2} including A and the ordinals $< \sigma$. Let $C = \{\sigma < \omega_1 : M_{A,\sigma} \cap \omega_1 = \sigma\}$. C is clearly a club in ω_1 .

Let $\alpha \in C$. The transitive collapse of $M_{A,\alpha}$ is some L_γ ; since $\alpha = (\omega_1)^{L_\gamma}$, we have that $\alpha < \gamma < \alpha^*$. Then $A \cap \alpha$, which is preserved under the transitive collapse, is a member of L_{α^*} and hence of \mathcal{A}_α .

Since $L_{\alpha^*} \models ZF - P$, we can define $\hat{C} = \{\sigma < \alpha : \mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \sigma) \cap \alpha = \sigma\}$ in L_{α^*} . Since $\hat{C} \in L_{\alpha^*}$, it remains to show that $\hat{C} = C \cap \alpha$. To show that $C \cap \alpha \in \mathcal{A}_\alpha$, it suffices to show that for $\sigma < \alpha$, $\mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \sigma) \cap \alpha = \sigma$ iff $\mathfrak{H}(L_{\omega_2}, \{A\} \cup \sigma) \cap \omega_1 = \sigma$.

Let $\zeta < \omega_1$, $\zeta \in \mathfrak{H}(L_{\omega_2}, \{A\} \cup \sigma)$, and $\zeta \geq \sigma$. Then ζ is definable in L_{ω_2} from $\{A\} \cup \sigma$. $\sigma < \alpha \in C$, so $\zeta < \alpha$. Since $M_{A,\alpha}$ is an elementary submodel of L_{ω_2} , ζ is likewise definable in $M_{A,\alpha}$. Since the transitive collapse fixes ordinals $< \alpha$ and takes A to $A \cap \alpha$, ζ is also definable in L_γ using parameters in $\{A \cap \alpha\} \cup \sigma$, so $\mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \sigma) \cap \alpha \neq \sigma$. The converse is similar. \square

Lemma 2.6.5. *There is a computable perfect Kurepa tree.*

Proof. Fix a computable \diamond^+ -sequence $\langle A_\alpha \rangle_{\alpha \in \omega_1}$ so that each A_α is closed under finite difference, includes the empty set, and satisfies

$$(\forall \beta < \alpha, A \in A_\alpha, B \in B_\beta)(\exists C \in A_\alpha) (C \upharpoonright (\beta + 1) = B \frown 1 \wedge C \upharpoonright (\alpha \setminus (\beta + 1)) = A \upharpoonright (\alpha \setminus (\beta + 1)))$$

In other words, each A_α is closed under replacing initial segments with elements of A_β for $\beta < \alpha$.

Let K be the set consisting of the *null elements* $\mathbf{0}^\alpha$ for each α and all pairs $(A, C) \in 2^\alpha \times 2^\alpha$ such that the following holds:

- (i) $A, C \in A_\alpha$;
- (ii) For all $\beta \in C$, $A \upharpoonright \beta, C \upharpoonright \beta \in A_\beta$; and
- (iii) If $\langle \alpha_i \rangle_{i < \omega}$ is an increasing sequence in C that is bounded below α , then $\sup \alpha_i \in C$.

In the above conditions, we use the convention that for $C \in 2^\alpha$, $\beta \in C$ iff $C(\beta) = 1$.

Impose an ordering $<$ on K as follows:

- (a) $\mathbf{0}^\alpha < \mathbf{0}^\beta$ iff $\alpha < \beta$.
- (b) $\mathbf{0}^\alpha < (A, C)$ iff $\text{dom} A > \alpha$.

(c) $(A_0, C_0) < (A_1, C_1)$ iff $A_0 \prec A_1$ and $C_0 \prec C_1$.

$(K, <)$ is then obviously a computable tree of countable width, and is clearly Kurepa; it remains to show that it is perfect.

Fix $(A, C) \in K$, and let $\mathcal{A} \in 2^{\omega_1}$ so that $\mathcal{A} \succ A$. Because $\langle A_\alpha \rangle$ is a \diamond^+ -sequence, there exists a club \mathcal{C} so that for every $\alpha \in \mathcal{C}$, $\mathcal{A} \upharpoonright \alpha, C \upharpoonright \alpha \in A_\alpha$. Take \mathcal{C}^* such that $\mathcal{C}^* \upharpoonright (\alpha + 1) = C \frown 1$ and $\mathcal{C}^*(\beta) = C(\beta)$ for $\beta > \alpha + 1$. Then, by the closure property imposed on the \diamond^+ -sequence, \mathcal{C}^* has the same property as \mathcal{C} ; thus the initial segments of $(\mathcal{A}, \mathcal{C}^*)$ form a path in K extending (A, C) . Since \mathcal{A} was chosen to be an arbitrary extension of A , it is clear that there are ω_2 -many such paths. \square

Proposition 2.6.6. *There are \aleph_2 -many nonthin, nonwide degrees.*

Proof. There is a computable Kurepa tree K . K has countable width, so every path in K is nonwide. K is Kurepa, so K has ω_2 -many paths; but there are obviously only \aleph_1 -many thin degrees, so \aleph_2 -many of these paths are nonthin. \square

Definition 2.6.7. *A Kurepa degree is a degree \mathbf{d} so that*

(i) \mathbf{d} is thin, but

(ii) If T is a computable tree of countable width and $\mathbf{d} \cap [T] \neq \emptyset$, then $||T|| = \aleph_2$.

Proposition 2.6.8. *There exists a thin Π_1^0 -class consisting entirely of sets of Kurepa degree.*

The proposition is in fact a corollary of the following more powerful theorem:

Theorem 2.6.9. *Let K be a computable perfect Kurepa tree. Then there exists, uniformly in an index for K , a Δ_3^0 functional Γ and a computable Kurepa tree T such that the following holds:*

(i) For each $X \in [K]$, $\Gamma^X \in [T]$;

(ii) For each $Y \in [T]$, there is exactly one $X \in [K]$ so that $Y = \Gamma^X$; and

(iii) Every $Y \in [T]$ has Kurepa degree.

Proof. We will construct T in stages, so that $T = \bigcup_s \mathcal{T}_s$. At the same time, we will define at each stage a partial map $F_s : K \rightarrow T_s$, so that $F = \liminf_s F_s$ will be the desired Γ .

The tree of strategies:

This will be an argument by a modified form of unbounded injury. Let $\Lambda = \{\text{Kurepa}, \text{Halt}\} \cup \{\langle \text{Div}, \alpha \rangle \mid \alpha \in \omega_1\} \cup \{0, 1\}$, with the attached ordering $\text{Halt} < \text{Kurepa} < \langle \text{Div}, \alpha \rangle < \langle \text{Div}, \beta \rangle$ for every $\alpha < \beta$; 0 and 1 are not included in the ordering.

Let \mathfrak{S} be the subtree of $\Lambda^{<\omega_1}$ consisting of those σ with the following properties. For ease of notation, say an ordinal is *even* if it is of the form $\omega\beta + n$ for n even. Let $\text{even}(\sigma)$ denote the string τ so that $\tau(\omega\beta + n) = \sigma(\omega\beta + 2n)$ for all $\beta < \omega_1$ and $n < \omega$; in other words, $\text{even}(\sigma)$ is the string consisting of the even-indexed entries of σ .

(i) $\sigma(\alpha) \in \{0, 1\}$ iff n is even, and

(ii) $\text{even}(\sigma) \in K$.

To each node σ of \mathfrak{S} we attach a strategy η_σ . The strategy η_σ will attempt to satisfy the requirement at index $|\text{even}(\sigma)|$ in the priority order outlined below.

The requirements:

The requirements take the form

$R_{e,i,j}$: For some string $\sigma \in T$, one of the following holds:

(a) For every $X \in [T]$ with $\sigma \prec X$, $\Phi_e^X = Y \in [T_j]$ and $\Phi_i^Y = X$ (in which case T_j will have ω_2 -many branches); or

(b) For every $X \in [T]$ with $\sigma \prec X$, one of the following holds:

(i) Φ_e^X is not total;

- (ii) $\Phi_e^X \notin [T_j]$; or
- (iii) $\Phi_e^X = Y$ with $\Phi_i^Y \neq X$.

It should be clear that if we can ensure that (1) every path through T passes through the σ chosen by at least one $R_{e,i,j}$ strategy for each triple (e, i, j) , and (2) every strategy satisfies its chosen requirement, then the desired properties will hold of $[T]$. Accordingly, we call such a σ the *satisfaction point* of the corresponding requirement.

Fix a standard effective enumeration $\psi : \omega_1 \rightarrow \omega_1^3$ of triples. We then order the requirements $R_{\psi(0)} < R_{\psi(1)} < \dots$.

The preliminaries:

At any stage s , countably many strategies will have been *initialized*. If a strategy η_σ with $|\sigma|$ odd (an *odd-level strategy*) has been initialized, it will have an *outcome*, which will be a member of $\Lambda \setminus \{0, 1\}$.

The *execution tree at stage s* , δ_s , is the subtree of \mathfrak{S} consisting of those σ such that the following hold:

- (i) η_σ has been initialized by stage s ; and
- (ii) for every $\tau \prec \sigma$ of odd length, $\sigma \succ \tau \frown u$, where u is the stage- s outcome of η_τ .

The *true execution tree* δ is the subset of \mathfrak{S} consisting of those σ such that $\sigma \in \delta_s$ for uncountably many s . Note that δ must be a tree since every δ_s will be a tree.

Strategy for requirement $R_{e,i,j}$:

Let η_σ be a strategy assigned to requirement $R_{e,i,j}$. The behavior of η_σ varies depending on whether it is an even-level strategy or an odd-level strategy. Regardless, when initialized at stage s , η_σ is given a string $\tau \in T_s$ of length at least s ; this will be the *anchor point* of η_σ .

Even-Level Strategies:

Definition 2.6.10. Let $\text{Kurepa}(\sigma) = \{\alpha < |\text{even}(\sigma)| \mid \text{even}(\sigma)(\alpha) = \text{Kurepa}\}$; that is, this is the set of indices of strategies along σ that have the Kurepa outcome. A string θ is σ -coherent if the following holds for every $\alpha \in \text{Kurepa}(\sigma)$:

(i) $\Phi_e^\theta \upharpoonright |\theta| \downarrow$, and

(ii) $\Phi_i^\rho \upharpoonright |\theta| \downarrow = \theta$ where $\rho = \Phi_e^\theta$,

where $\psi(\alpha) = \langle e, i, j \rangle$.

Let η_σ be an even-level strategy assigned to $R_{e,i,j}$, and let τ be its anchor point. For convenience, we drop the subscript σ . Let $S = \{a \mid \sigma \frown a \in \mathfrak{G}\}$; η maintains a partial map $f : S \rightarrow 2^{<\omega_1}$ with range contained in \mathcal{T}_s . When initialized, η immediately builds Aronszajn trees rooted at $\tau \frown a$ for $a \in S$, and adds these to \mathcal{T}_s . η requires attention at stage s if $\text{dom} f = \emptyset$ and for every $a \in S$ there exists $\theta \succ \tau \frown a$ with $\theta \in \mathcal{T}_s$ so that θ is σ -coherent. If permitted to act, η takes $f(a)$ to be the $<_L$ -least length- s extension of θ in \mathcal{T}_s and initializes $\eta_{\sigma \frown a}$ with anchor point $f(a)$.

While the require θ have not yet been found, η_σ maintains *bookmark points* θ_a for each $a \in S$. Initially, $\theta_a = \tau \frown a$. At any stage s , if $\sigma \in \delta_s$ and η_σ does not require attention, then it adds $\theta_a \frown 0$ to \mathcal{T}_s for each a , moves each θ_a to $\theta_a \frown 0$, and adds an Aronszajn tree rooted at the new θ_a to \mathcal{T}_s .

Suppose σ leaves δ_s and reenters at some later stage t . Then σ reentered because some strategy along σ switched from a Div outcome to the Kurepa outcome. Let that strategy be η . When η returned to outcome Kurepa, it *posted* a triple $\langle k, e, i, x \rangle \in 2 \times \omega_1^3$. η_σ chooses its new bookmarks θ_a depending on the value of k as follows; let θ'_a be the previous value of θ_a .

$k = 0$: θ_a is the $<_L$ -least length- s available extension of θ'_a such that if $(\exists \theta_a \prec \theta \in \mathcal{T}_s) \Phi_e^\theta(x) \downarrow$, then $\Phi_e^{\theta_a}(x) \downarrow$.

$k = 1$: θ_a is the $<_L$ -least length- s available extension of θ'_a such that if $(\exists \theta_a \prec \theta \in \mathcal{T}_s) \Phi_i^{\Phi_e^\theta}(x) \downarrow$ then $\Phi_i^{\Phi_e^{\theta_a}}(x) \downarrow$.

Odd-Level Strategies:

Let η_σ be an odd-level strategy assigned to $R_{e,i,j}$, and let τ be its anchor point. Again, for convenience, we drop the subscript σ . η will work directly to satisfy $R_{e,i,j}$.

At any stage s , if η is not in outcome Halt, then η *requires attention* if there exists $\rho \succ \tau$, $\rho \in \mathcal{T}_s$ with either $\Phi_e^\rho \notin T_j$ or $\Phi_e^\rho = \theta$ and $(\exists t < s) \Phi_i^\theta \upharpoonright t \downarrow \notin \mathcal{T}_s$. If permitted to act, η changes its anchor to a maximal extension of ρ on \mathcal{T}_s and switches to outcome Halt.

Suppose that $\sigma \in \delta_s$, η currently has outcome Kurepa, and there exists a string $\rho \succ \sigma$ such that the following holds:

- (i) $\rho \in \delta_s$,
- (ii) η_ρ was initialized *before* stage s with an anchor τ_ρ , and
- (iii) For some $a \in \{0, 1\}$, $\tau_\rho \frown a$ has no σ -coherent extension in \mathcal{T}_s .

Then η *requires attention* at stage s . If permitted to act, it re-anchors itself at a maximal extension θ of τ_ρ , switches to the first $\langle \text{Div}, \alpha \rangle$ outcome it has never used before. θ is not σ -coherent. Since η has been permitted to act, it must be that θ is σ^* -coherent for every $\sigma^* \prec \sigma$ of odd length; otherwise η_{σ^*} would have acted instead; so either $\Phi_e^\theta \upharpoonright |\theta| \uparrow$ or $\Phi_e^\theta = \chi$ and $\Phi_i^\chi \upharpoonright |\theta| \uparrow$. Let α be least so that $\Phi_e^\theta(\alpha) \uparrow$, and let β be least so that $\Phi_i^\chi(\beta) \uparrow$. Then η *asserts* one of the following.

- (a) If $\alpha < |\theta|$, then η asserts that for every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_e^{\theta'}(\alpha) \uparrow$.
- (b) Otherwise, η asserts that for every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_i^{\Phi_e^{\theta'}}(\beta) \uparrow$.

We say assertion (a) has been *violated* if there exists $\theta' \succ \theta$, $\theta' \in \mathcal{T}_s$, so that $\Phi_e^{\theta'}(\alpha) \downarrow$. Likewise, (b) has been *violated* if there exists θ' with the same conditions so that $\Phi_i^{\Phi_e^{\theta'}}(\beta) \downarrow$.

Suppose instead that $\sigma \in \delta_s$, η currently has outcome $\langle \text{Div}, \alpha \rangle$ for some α , and the assertion η made when it entered that outcome has been violated. Then η *requires attention*. If permitted to act, η returns to its original anchor point, returns to the Kurepa outcome, and *posts* the tuple $\langle k, e, i, x \rangle$, where $k = 0$ and $x = \alpha$ if η was in case (a) and $k = 1$ and $x = \beta$ otherwise.

If η_σ has outcome i , no strategy η_τ requires attention for $\tau \preceq \sigma$, and $\eta_{\sigma \frown i}$ has not yet been initialized, $\eta_{\sigma \frown i}$ is initialized with anchor the $<_L$ -least length- s extension of the anchor of η_σ .

Limit Stages:

Let s be a limit stage, and let η be an odd-level strategy. The outcome of η at stage s is taken to be the lim inf of its outcomes at previous stages. \mathcal{T}_s is taken to be $\bigcup_{t < s} \mathcal{T}_t \cup U$, where U is a (countable) collection of limit nodes consisting of exactly those strings which are limits of anchor points of strategies on δ_s along branches of δ_s with limit in \mathfrak{S} . Any limit-level strategy all of whose predecessors have been initialized and are in δ_s is then initialized, with anchor equal to the limit of the anchors of its predecessors.

Defining F :

At any stage s , take $F_s(\sigma)$ to be the anchor point of η_ρ , where ρ is the (unique) member of δ_s such that $\text{even}(\rho) = \sigma$.

Construction:

At any stage s , any strategy η_σ is permitted to act as long as (1) η_σ requires attention, and (2) η_τ does not require attention for any $\tau \prec \sigma$.

Verification:

Claim 2.6.11. *Let $\sigma \in \delta$ have even length. Then η_σ eventually initializes $\eta_{\sigma \frown a}$ for each a such that $\sigma \frown a \in \mathfrak{S}$.*

Proof. Suppose otherwise. Then η_σ extends its bookmarks θ_a unboundedly often; so for each a , the limit X_a of θ_a is a branch of length ω_1 . If η_σ never initializes its successors,

then at least one of the X_a must have no σ -coherent initial segment. Fix one such a , and let $X = X_a$.

For each pair (e, i) so that η_σ has a predecessor attached to an $R_{e,i,j}$ requirement and η_σ is above that predecessor's Kurepa outcome, suppose that Φ_e^X is total and that $\Phi_i^{\Phi_e^X} = X$. For any j, Y , let $f_j^Y(\alpha)$ be the least s so that $\Phi_j^{Y \upharpoonright s}(\beta) \downarrow$ for all $\beta < \alpha$. Note that, as long as Φ_e^Y is total, f_j^Y is continuous, and therefore has a club C_j^Y of fixed points. $C_e^X \cap C_i^{\Phi_e^X}$ is therefore a club.

Let $C = \bigcap \left(C_e^X \cap C_i^{\Phi_e^X} \right)$, where the intersection is taken over all pairs (e, i) meeting the aforementioned condition. Then C is a countable intersection of clubs, and hence is itself a club. Let $\alpha \in C$ be greater than the length of the anchor of η_σ . Then $X \upharpoonright \alpha$ is σ -coherent.

It must therefore be the case that for some such pair (e, i) , either Φ_e^X is not total or $\Phi_i^{\Phi_e^X} \neq X$. If the latter holds because there is some α with $\Phi_i^{\Phi_e^X} \upharpoonright \alpha \not\downarrow \neq X \upharpoonright \alpha$, then the first strategy corresponding to (e, i) should have used an initial segment of X to move to a Halt outcome. So either Φ_e^X is not total or $\Phi_i^{\Phi_e^X}$ is not total. Then there is an initial segment θ of X sufficiently long that this is realized; that is, for some x , either $\Phi_e^{\theta'}(x) \uparrow$ for every $\theta \prec \theta' \prec X$ or $\Phi_i^{\Phi_e^{\theta'}}(x) \uparrow$ for every $\theta \prec \theta' \prec X$.

At some stage, the first (e, i) strategy below σ will notice this θ and x , and switch to a divergence outcome. Call this strategy η_* . If σ is to be visited unboundedly often, then it must be that η_* eventually returns to its Kurepa outcome, because its assertion was violated by some extension of θ . But by construction, the bookmark η_σ now selects, which will be an initial segment of X , has the same convergence - contradicting our hypothesis on X . □

Claim 2.6.12. $F = \liminf_s F_s$ is a Δ_3^0 total embedding $K \rightarrow T$.

Proof. That F is Δ_3^0 is clear from definition.

To see that F is total, observe that it is sufficient to show that for every $\tau \in K$, some strategy η_σ for $\text{even}(\sigma) = \tau$ is eventually initialized. But this is evident by Claim 2.6.11.

Finally, to see that F is an embedding, observe that successor strategies always take incomparable anchors. \square

Claim 2.6.13. *If $\sigma \in \delta$ has odd length, $\sigma \smallfrown i \in \delta$ for exactly one i , and the following holds, with τ the liminf of the anchor points of η_σ .*

(a) *If $i = \text{Halt}$, then either $\Phi_e^\tau \notin T_j$ or $\Phi_i^{\Phi_e^\tau} \upharpoonright \tau$,*

(b) *If $i = \langle \text{Div}, \alpha \rangle$, then there is an x so that for every $\rho \succ \tau$, $\rho \in T$, either $\Phi_e^\rho(x) \uparrow$ or $\Phi_i^{\Phi_e^\rho}(x) \uparrow$, and*

(c) *If $i = \text{Kurepa}$, then $F \circ \Phi_e$ embeds the part of K above $\text{even}(\sigma)$ into T_j .*

Proof. By construction, there are three possible cases for the limit behavior of η_σ , for $\sigma \in \delta$.

- (1) η_σ eventually enters outcome Halt and does not leave.
- (2) η_σ eventually enters outcome $\langle \text{Div}, \alpha \rangle$ for some α and does not leave.
- (3) η_σ visits outcome Kurepa unboundedly often.

Observe that this list is complete, because whenever a $\langle \text{Div}, \alpha \rangle$ outcome is abandoned, η_σ returns to outcome Kurepa (however briefly).

In case (1), η_σ moved to outcome Halt because it encountered a string ρ with either $\Phi_e^\rho \notin T_j$ or $\Phi_i^{\Phi_e^\rho} \upharpoonright \rho$, and set $\tau = \rho$. There is no occurrence that could cause η_σ to give up this definition, so η_σ falls into case (a).

In case (2), when η_σ entered outcome $\langle \text{Div}, \alpha \rangle$ it made one of the following assertions, for some choice of θ , α , and β :

(i) For every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_e^{\theta'}(\alpha) \uparrow$.

(ii) For every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_i^{\Phi_e^{\theta'}}(\beta) \uparrow$.

It also set $\tau = \theta$. If η_σ never leaves this outcome, then whichever assertion it made was never violated; thus case (b) holds.

In case (3), observe that by Claim 2.6.11, every even-level strategy on δ eventually initializes its successors. Odd-level strategies have no particular constraints on initializing their successors, so odd-level strategies always initialize their successors. And limit-level strategies are initialized when their predecessors are all initialized. So sufficiently many strategies will be initialized that the part of K above $\text{even}(\sigma)$ embeds into the part of δ above σ . Note also that for any even-level $\tau \succ \sigma$, η_τ chooses the anchor points of its successors to be τ -coherent and hence σ -coherent; in particular, if θ is one of these anchor points, then $\Phi_e^\theta \upharpoonright |\theta| \downarrow$ and $\Phi_i^{\Phi_e^\theta} \upharpoonright |\theta| \downarrow = \theta$. This latter clause entails that $\Phi_e^{\theta_0} \upharpoonright \Phi_e^{\theta_1}$ whenever $\theta_0 \upharpoonright \theta_1$ are such strings, so $F \circ \Phi_e$ is an embedding. \square

Claim 2.6.14. *For each $\sigma \in K$, there is exactly one $\tau \in \delta$ with $\text{even}(\tau) = \sigma$.*

Proof. By Claim 2.6.13 and induction on $|\sigma|$. \square

Claim 2.6.15. *Every requirement is eventually satisfied, and every path through T eventually passes through a satisfaction point of every requirement.*

Proof. The latter part of the claim is clear from the construction: if X eventually avoids the anchor points of strategies that are visited unboundedly often, then X must be contained inside an Aronszajn tree built to bookmark the position of an inactive strategy; but then X cannot be a path. And the anchor point of a strategy that succeeds at satisfying its requirement will be a satisfaction point of that requirement.

By Claim 2.6.13, the requirements are satisfied. \square

Claim 2.6.16. *For any $X \in [K]$, $F[X]$ has Kurepa degree.*

Proof. Fix $X \in [K]$, U a computable tree of countable width, and $F[X] \equiv_T Y \in [U]$. Let e, i such that $\Phi_e^{F[X]} = Y$ and $\Phi_i^Y = F[X]$. Fix j so that $T_j = U$.

By Claim 2.6.15, $F[X]$ passes through a satisfaction point τ of the requirement $R_{e,i,j}$. One of the following holds:

- (a) For every $Z \succ \tau$ in $[T]$, there is a $W \in [T_j]$ so that $\Phi_e^Z = W$ and $\Phi_i^W = Z$;
- (b) $\Phi_e^{F[X]}$ is not total;
- (c) $\Phi_e^{F[X]} \notin [T_j]$; or
- (d) $\Phi_i^Y \neq F[X]$.

Cases (b) through (d) directly contradict our hypothesis, so it must be that τ falls under case (a). But F embeds the part of K above some initial segment of X into the part of T above τ , and K was a perfect Kurepa tree, so that part of K has ω_2 -many branches. Therefore there are ω_2 -many Z satisfying the hypothesis of (a), and hence $U = T_j$ must have ω_2 -many branches. \square

Claim 2.6.17. $F''[[K]] = [T]$.

Proof. This is evident from Claim 2.6.15 and the fact that every path in T must pass through a satisfaction point of every (and hence unboundedly many) requirement. \square

By Claims 2.6.12, 2.6.16, and 2.6.17, T and F have the desired properties. \square

It seems evident that the above construction relied very little on the nature of K , and in fact could be used to show the following instead.

Theorem 2.6.18. *Let K be a computable tree of countable width. Then there exists, uniformly in an index for K , a Δ_3^0 functional Γ and a computable tree T of countable width such that the following holds:*

(i) For each $X \in [K]$, $\Gamma^X \in [T]$;

(ii) For each $Y \in [T]$, there is exactly one $X \in [K]$ so that $Y = \Gamma^X$; and

(iii) For every $Y \in [T]$ and every computable tree U , if there exists $Y \equiv_T Z \in [U]$ then there exists a $\sigma \in K$ so that $K(\sigma) = \{\tau \in K : \tau \succ \sigma\}$ embeds in U . Furthermore, this embedding is Δ_3^0 uniformly in an index for U and the equivalence between Y and Z .

To take an example at random:

Corollary 2.6.19. *There exists a thin Π_1^0 class of degrees \mathbf{d} so that \mathbf{d} appears as a path in a computable tree T only if T has uncountably many Δ_3^0 branches.*

Proof. Take K to be any computable tree of countable width with uncountably many computable paths, and apply the theorem. \square

Theorem 2.6.9 can be combined with Theorem 2.5.12 to produce a Kurepa tree in which every path has minimal Turing degree.

Theorem 2.6.20. *Let K be a computable perfect Kurepa tree. Then there exists a computable tree \mathcal{T} of countable width together with a Δ_3^0 embedding $f : K \rightarrow \mathcal{T}$ (uniformly in an index for K) so that the following holds:*

- Every branch of \mathcal{T} is the f -image of a branch of K , and
- For each $X \in [K]$, $f[X] \in \mathcal{T}$ is of minimal Turing degree.

An immediate corollary is the following:

Corollary 2.6.21. *There is a thin Π_1^0 -class of size \aleph_2 consisting entirely of sets of minimal Turing degree.*

Proof. (Proof of Theorem)

We mimic the previous proofs involving Kurepa trees and minimal degrees, respectively. The argument will be by a modified form of unbounded injury, merged with a worker argument.

Let A be a fixed computable Aronszajn tree.

Let $\Lambda = \{S\} \cup \omega_1$. Let $\mathcal{T} = \Lambda^{<\omega_1}$. To each string $\sigma \in \mathcal{T}$ we attach a *worker* η_σ . At any stage s , each worker which has been initialized has one or two outcomes from Λ ; let $\delta(s)$ be the maximal partial embedding $2^{<\omega_1} \rightarrow \mathcal{T}$ so that $\delta(s)(\sigma \frown 0)$ and $\delta(s)(\sigma \frown 1)$ are the leftmost and rightmost immediate extensions of $\delta(s)(\sigma)$ by its stage- s outcomes.

Each worker maintains one or more *scaffolds*, each one corresponding to an outcome of that worker. The *active* scaffold at stage s is the one corresponding to the worker's outcome at stage s .

Definition 2.6.22. *A scaffold is a partial function $f : 2^{<\omega_1} \rightarrow 2^{<\omega_1}$ together with a set $S \subseteq 2^{<\omega_1}$ such that the following conditions hold.*

- (i) $\text{dom}(f) \subseteq S$;
- (ii) For $\sigma, \tau \in \text{dom}(f)$, $\sigma \preceq \tau$ iff $f(\sigma) \preceq f(\tau)$;
- (iii) If $\sigma \frown i \in \text{dom}(f)$, then $f(\sigma \frown i)(|f(\sigma)|) = i$; and
- (iv) If $\sigma_0 \prec \sigma_1 \prec \dots$ is a sequence in $\text{dom}(f)$ and $\sigma = \lim_i \sigma_i \in \text{dom}(f)$, then $f(\sigma) = \lim_i f(\sigma_i)$.

We denote the scaffold (S, f) .

For scaffolds (S_0, f_0) and (S_1, f_1) , say (S_0, f_0) is a subscaffold of (S_1, f_1) if $\text{ran}(f_0) \subseteq \text{ran}(f_1)$. Observe that there is no restriction on the relationship between S_0 and S_1 .

It will be the case that every scaffold maintained by η_σ is a subscaffold of η_τ for each $\tau \prec \sigma$. The root worker η_\emptyset maintains only one scaffold, and the corresponding function is the identity; this scaffold will form the final tree T .

At any stage s , a worker η_σ may issue a *request* for any $\theta \in 2^{<\omega_1}$ every initial segment of which is currently in the range of the active scaffold of every η_τ for $\tau \prec \sigma$. Provided that $|\theta| \geq s$, this request will be *honored* by every such η_τ : η_τ will add an assignment to its active scaffold (possibly adding to the domain of the scaffold) so that θ will be in the range, and then add to the domain of the scaffold so that the preimage of θ has a copy of A above it.

Successor workers pursue two sorts of requirements:

$$D_e: \Phi_e \neq X$$

M_e : One of the following holds:

- (i) $(\exists x)\Phi_e^X(x) \uparrow$,
- (ii) $\Phi_e^X \leq_T \emptyset$, or
- (iii) $\Phi_e^X \geq_T X$.

Divide the countable successor ordinals into three computable unbounded sets \mathcal{D} , \mathcal{M} , and \mathcal{K} . Workers η_σ for $|\sigma|$ the e th member of \mathcal{D} pursue the diagonalization requirement D_e ; workers η_σ for $|\sigma|$ the e th member of \mathcal{M} pursue the minimality requirement M_e . Workers η_σ for $|\sigma|$ the e th member of \mathcal{K} build an embedding from K into the tree. Workers η_\emptyset and η_σ for σ of limit length behave differently and pursue no requirement.

Given σ , let $k(\sigma)$ be the string obtained from σ by deleting all entries with indices not in \mathcal{K} ; that is, by considering only outcomes of \mathcal{K} workers.

For ease of notation, when σ is a string of successor length, denote by σ^- its immediate predecessor.

For any worker η_σ , let (S^σ, f^σ) be the currently active scaffold.

Strategy for Limit Workers: η_\emptyset does nothing except respond to requests; we therefore concentrate on η_σ for $|\sigma|$ of limit length.

A limit worker η_σ is initialized at stage s if and only if it isn't yet initialized, every direct predecessor is active, and $k(\sigma) \in K$.

When initialized at stage s , η_σ takes $S^\sigma = A$ and sets $f^\sigma(\langle \rangle)$ to be the limit of $f^\tau(\langle \rangle)$ for $\tau \prec \sigma$. It enters outcome 0, and remains there permanently.

At later stages, η_σ works to define f^σ on progressively higher and higher levels of S^σ . Limit levels are straightforward: η_σ simply defines the function as necessary to preserve continuity, requesting any strings not already present in the ranges of the previous scaffolds. At successor levels, suppose η_σ has defined $f^\sigma(\tau)$ and needs to define $f^\sigma(\tau \smallfrown i)$. For each $\rho \prec \sigma$, let $\theta(\rho)$ be the element in the domain of f^ρ so that $f^\rho(\theta(\rho)) = f^\sigma(\tau)$. Let $f^\sigma(\tau \smallfrown i) = \lim f^\rho(\theta(\rho) \smallfrown i)$, requesting the string if necessary. This limit exists and has the property that every f^ρ has a sequence cofinal along it; so the request is a valid one.

Strategy for Diagonalization Requirements: Let η_σ be a successor worker pursuing requirement D_e . When initialized at stage s , η_σ looks for θ_0 and θ_1 in the range of f^{σ^-} with length at least s differing at some position $\alpha < s$. η_σ sets $S^\sigma = A$, puts $f^\sigma(\langle \rangle) = \theta_0$, and enters outcome 0.

If at some stage $t > s$, $\Phi_{e,t}(\alpha) \downarrow = \theta_0(\alpha)$, then η_σ requires attention. If it receives attention, it deletes its existing scaffold, sets $S^\sigma = A$ and $f^\sigma(\langle \rangle) = \theta_1$, and enters outcome 1.

At any successor stage, η_σ considers each leaf node τ of $\text{dom}(f^\sigma)$. For each such τ , by induction there exists $\theta \in \text{dom}(f^{\sigma^-})$ with $f^{\sigma^-}(\theta) = f^\sigma(\tau)$; if $\theta \smallfrown 0, \theta \smallfrown 1 \in \text{dom}(f^{\sigma^-})$, then η_σ takes $f^\sigma(\tau \smallfrown i) = f^{\sigma^-}(\theta \smallfrown i)$ for each i . Otherwise, it simply waits until the necessary assignments have been made.

At a limit stage, η_σ considers each limit node $\tau \in S^\sigma$ so that f^σ has been defined on every initial segment of τ . f^σ induces an obvious choice of $f^\sigma(\tau)$; if this string is not already in the range of f^{σ^-} , then η_σ requests it. Regardless, it takes this as the assignment of $f^\sigma(\tau)$.

Strategy for \mathcal{K} Workers: Let η_σ be a successor worker with $|\sigma|$ the e th member of \mathcal{K} . Let τ be the length- e subsequence of σ given by $k(\sigma)$. For each $i \in \{0, 1\}$ so that $\tau \frown i \in K$, η_σ activates outcome i . It constructs scaffolds S_0 and S_1 (the left and right subscaffolds of its parent scaffold, respectively) and assigns S_i to outcome i .

Strategy for Minimality Requirements: Let η_σ be a successor worker pursuing requirement M_e . When initialized at stage s , η_σ sets $S^\sigma = A$ and puts $f^\sigma(\langle \rangle)$ some element of $\text{ran}(f^{\sigma^-})$ with length at least s (if none exists, η_σ waits until it does). This scaffold is designated the *splitting scaffold*; it will be the active scaffold whenever η_σ is in outcome S . η_σ begins in outcome S .

η_σ will maintain the condition that if (S, f) is its splitting scaffold and $\tau_0, \tau_1 \in S$ are incomparable, then $\Phi_e^{f(\tau_0)}$ and $\Phi_e^{f(\tau_1)}$ are incomparable. Call this condition (*).

While still in outcome S , η_σ attempts to extend f^σ while preserving (*). At a limit stage, it simply fills in any limit nodes in S^σ , requesting whatever is necessary. At a successor stage, η_σ will attempt to extend each leaf node of $\text{dom}(f^\sigma)$ in both directions. For each leaf $\tau \in \text{dom}(f^\sigma)$, if there exist $\rho_0, \rho_1 \in \text{ran}(f^{\sigma^-})$ with length at least s and $x < s$ so that $\rho_i \succ f^\sigma(\tau) \frown i$ and $\Phi_e^{\rho_0}(x) \not\downarrow \neq \Phi_e^{\rho_1}(x) \downarrow$, then η_σ takes $f^\sigma(\tau \frown i) = \rho_i$. If none exists, then let $\epsilon = f^\sigma(\tau)$. η_σ switches to a new ordinal outcome and associates with it a new scaffold as follows: $f^\sigma(\langle \rangle) = \theta$, where θ is the first extension of $\epsilon \frown 0$ in the range of f^{σ^-} (as usual, if none exists, wait until it does); S^σ is the translation of the part of S^{σ^-} extending that point.

While in an ordinal outcome, η_σ copies the scaffold of η_{σ^-} , just like the diagonalization workers do.

If a splitting pair is found within this new scaffold (that is, $\rho_0, \rho_1 \in S^\sigma$ with $\Phi_e^{\rho_0}$ and $\Phi_e^{\rho_1}$ incomparable) then η_σ again switches to a new ordinal outcome and replaces the current scaffold with a new scaffold, defined the same way as before but rooting at an extension of $\epsilon \frown 1$.

If a spitting pair is found within this scaffold as well, then between the two sets of splitting pairs is a pair appropriate for extending the splitting scaffold at τ ; η_σ returns to the S outcome, re-activating the splitting scaffold, and adds those elements.

To ensure that the search for strings of sufficient length is successful (that is, that higher-priority scaffolds are not unexpectedly short) at every stage we allow the active workers to act *in order*, beginning from the root. Responses to requests are immediate, but propagate in the same manner.

This completes the construction.

Claim 2.6.23. *For any α , only countably many requests are made for strings of length $< \alpha$, and none are requested past stage α .*

Proof. Any worker initialized past stage α will, by construction, place its root at a string of length at least α . Again by construction, no worker ever makes requests that do not extend its root; therefore, only the workers initialized before stage α will ever request strings of length less than α .

Let η be such a worker, and suppose that it requests a limit string θ at stage s . In every case, this request is made as soon as the initial segments are available; that is, it must be that there was a sequence $s_0 < s_1 < \dots$ cofinal in s at which η assigned strings in the domain of its scaffold to initial segments of θ . But by construction a worker can only make stage- t assignments of length at least t ; so these initial segments of θ have lengths at least s_0, s_1, \dots . Thus θ has length at least s .

Clearly no worker requests uncountably many strings in a single stage; therefore no individual worker requests uncountably many strings of length $< \alpha$. Since only countably many workers can request these strings at all, only countably many such requests are made. □

Claim 2.6.24. *The tree T is computable and of countable width.*

Proof. This is an immediate consequence of Claim 2.6.23. \square

Claim 2.6.25. *All requests are honored.*

Proof. Again, this is an immediate consequence of Claim 2.6.23; any request made at stage s for a string with length at least s is honored, and by the claim no requests not satisfying this condition are made. \square

Claim 2.6.26. *The set of strings in \mathcal{T} that lie along the execution path unboundedly often forms a path.*

Proof. Observe that limit workers (including the root worker) never change outcomes; diagonalization workers change outcome at most once; and while minimalization workers may change outcome unboundedly often, the only outcome they may visit unboundedly often is S . So every worker visits exactly one outcome unboundedly often; the inductive argument is straightforward. \square

We call this the *true path*, and the workers along this path *true workers*; their outcomes that lie along the true path are *true outcomes*.

Note that the set of strings $f^\sigma(\langle \rangle)$ evaluated for σ with $k(\sigma) \prec Y$, during stages at which η_σ has its true outcome exactly defines a path through the tree T ; we call this path $f[Y]$.

Claim 2.6.27. *The $f[Y]$ are the only paths through T .*

Proof. Suppose for contradiction that $Y \neq X$ is a path through T . Y cannot be “native” - that is, it must be obtained through uncountably many requests, because otherwise it would be part of an Aronszajn tree.

For the following, consider only σ with $k(\sigma) \prec Y$; note that this constrains \mathcal{K} workers to one fixed outcome, rendering them irrelevant.

No worker makes requests incompatible with its root; it therefore cannot be that the requests for initial segments of Y are unbounded along the true path. There is therefore a minimal worker η so that only countably many requests for initial segments of Y are made above the true outcome of η . Note that η is not a limit worker or the root worker, because neither class of worker changes outcomes. Note also that η is not a diagonalization worker; at some stage, a worker pursuing a diagonalization requirement will enter its true outcome and never leave, so only countably many requests for anything can be made above the other outcome. So η is a minimalization worker. Furthermore, η 's true outcome must be S - otherwise, the same argument holds as for the diagonalization case.

Since η 's true outcome is not one of the ordinal outcomes, it cannot be that uncountably many requests for initial segments of Y are made above only one of them; so it must be that an uncountable sequence of ordinal outcomes have at least one request each. The scaffolds for these outcomes must be rooted at points along Y ; but all of them come from points at which the attempt to split in the splitting outcome temporarily failed. So all of these initial segments of Y are in the splitting scaffold - which means uncountably many requests for initial segments of Y must have been made above the S outcome, contradicting our assumption on η . \square

Claim 2.6.28. *Each $f[Y]$ is not computable.*

Proof. Suppose $\Phi_e = f[Y]$. Let η be the (unique) worker along the true path for Y pursuing requirement D_e . When initialized, η selected some pair of strings and an α at which they differed. At some stage t , $\Phi_{e,t}(\alpha) \downarrow$; at that stage, η switched outcomes if necessary and directed construction along the string of the pair that did not agree with Φ_e at α . This is a contradiction. \square

Claim 2.6.29. *Each $f[Y]$ has minimal Turing degree.*

Proof. Let $X = f[Y]$.

Suppose that Φ_e^X is total but not computable. Let η be the (unique) worker along the true path for Y pursuing requirement M_e . Observe that if the true outcome of η were an ordinal, Φ_e^X would be either partial or computable, because a split would never be found; $\Phi_e^X(x)$ could be computed by taking the first computation $\Phi_e^\tau(x)$ to converge for τ in the range of the scaffold of η . So the true outcome of η is S , and therefore in the limit η constructs a scaffold (U, f) so that if σ and τ are incomparable members of U then $\Phi_e^{f(\sigma)}$ and $\Phi_e^{f(\tau)}$ disagree somewhere. But then given Φ_e^X we can determine a unique branch Y of U so that $\Phi_e^{f(Y)} = \Phi_e^X$; this branch must be the preimage of Y . Thus $\Phi_e^X \geq_T X$.

So for every e we have that Φ_e^X is either partial, computable, or above X ; since we already have that $X >_T \emptyset$, X is of minimal Turing degree. □

□

2.7 Future Directions

The results of this chapter leave one significant question unanswered.

Question 2.7.1. *Is there a degree-theoretic characterization of $\mathcal{P}_{\text{thin}}$? Failing that, is there a characterization of $\mathcal{P}_{\text{thin}}$ that is simpler than the definition in terms of computable trees of countable width?*

An appealing approach might be a positive answer to the following:

Question 2.7.2. *Is there a natural extension of the notion of strong club-approximation that characterizes exactly the members of $\mathcal{P}_{\text{thin}}$?*

Such an extension would have to be somehow intermediate between strong and weak club-approximation, but it is not evident what that intermediate might be.

The work in Section 6 also suggests a sweeping question:

Question 2.7.3. *For which classes of degrees \mathcal{C} is there a computable tree of countable width whose branches realize exactly the members of \mathcal{C} ?*

Certainly such a \mathcal{C} must include a Δ_1^1 member, but the results of Section 2 demonstrate that even if every member of \mathcal{C} is Δ_2^0 there may be no corresponding tree. At the same time, the results of Section 6 show that \mathcal{C} need not consist entirely or even “mostly” of Δ_1^1 degrees; Theorem 2.6.20 in particular demonstrates that \mathcal{C} may be large with very little in its lower cone.

Of further interest to computable structure theorists will be the relationship to the notion of α -true stages introduced by Montalbán [17] as an elaboration of the priority system introduced previously by Ash [1]: the results of Sections 3 and 4 illustrate that there are strong club-approximations to sets of degree $\mathbf{0}^{(\alpha)}$ for every hyperarithmetic α , which serve the same function as the approximations ∇^ξ in the notation of [17]. As a result, the development of Montalbán’s α -true relations and α -true stages will be considerably simpler in the setting of ω_1 . Furthermore, since these strong club-approximations extend considerably further than the hyperarithmetic hierarchy, it is reasonable to suppose that the techniques of [17] could be adapted to perform priority arguments of very large degree.

Chapter 3

Cantor-Bendixson Rank¹

3.1 Introduction

The study of the Cantor-Bendixson derivative and the corresponding rank was initiated by Cantor in 1872, originally as a topological notion regarding subsets of the real line. In general, the definitions are as follows:

Definition 3.1.1. *Let X be a topological space, $A \subseteq X$ closed. A limit point of A is an $x \in A$ such that for any open U containing x , $A \cap (U \setminus \{x\})$ is nonempty. The Cantor-Bendixson derivative A' of A is the set of limit points of A .*

The iterated Cantor-Bendixson derivatives of A are defined inductively:

- $A^{(0)} = A$
- $A^{(\alpha+1)} = (A^\alpha)'$
- For α a limit ordinal, $A^{(\alpha)} = \bigcap_{\beta < \alpha} A^{(\beta)}$.

The Cantor-Bendixson rank of A is the least ordinal α so that $A^{(\alpha)} = A^{(\alpha+1)}$. The

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Cantor-Bendixson rank of a particular $x \in A$ is the least ordinal α so that $x \notin A^{(\alpha+1)}$, or ∞ if there is no such α .

For a computability theorist, the natural application is to subsets of Cantor space, and in particular to Π_1^0 -classes, leading to the general question:

Question 3.1.2. *Which ordinals are possible as the Cantor-Bendixson rank of a Π_1^0 -class (in the sense of ω -computability) and under what conditions? Which ω -Turing degrees are represented by paths of particular Cantor-Bendixson ranks?*

In [2], Cenzer, Clote, Smith, Soare, and Wainer showed that every ω -computable ordinal can be realized as the Cantor-Bendixson rank of a member of an ω - Π_1^0 class, and put precise bounds on the ω -Turing degree of the member in question. In light of the fact that the Cantor-Bendixson derivative of an ω - Π_1^0 class can be obtained by an arithmetic operation on the representing tree, this effectively settled the question in the countable case.

The topology of 2^{ω_1} is in a sense more complicated than 2^ω ; for example, while 2^ω is compact, 2^{ω_1} is not even Lindelöf. It is perhaps reasonable to expect, then, that the analogous question to 3.1.2 in the setting of ω_1 might have a different answer. In this chapter, we approach an answer to the analogy of 3.1.2, ending in a result that is similar to the analogous one in [2]. In Section 2, we consider the ranks of entire Π_1^0 -classes; in Section 3, we concentrate on the ranks of individual members of Π_1^0 -classes. In Section 4, we discuss notable questions left open and possible future directions for this line of research.

3.2 Ranks of Trees

Definition 3.2.1. *For $\sigma \in 2^{<\omega_1}$, let $[\sigma]$ denote the subset of 2^{ω_1} consisting of all extensions of σ .*

A subset $X \subseteq 2^{\omega_1}$ is clopen if it is $\bigcup_{\sigma \in Y} [\sigma]$ for some set $Y \subseteq 2^{\alpha}$ and some countable ordinal α .

X is closed if it is the intersection of clopen sets. X is open if it is the union of clopen sets. $y \in 2^{\omega_1}$ is a limit point of X if every open set containing y intersects X . Equivalently, y is a limit point of X if it is the limit of an ω_1 -sequence of members of X in the lexicographic sense.

X is perfect if it is closed and every point of X is a limit point of X .

The Cantor-Bendixson derivative of X is the set $\{x \in X \mid x \text{ is a limit point of } X\}$.

Definition 3.2.2. $X \subseteq 2^{\omega_1}$ is closed if there exists a tree $T \subseteq 2^{<\omega_1}$ so that X is the set of paths through T .

X is perfect if there exists a tree $T \subseteq 2^{<\omega_1}$ so that every $\sigma \in T$ extends to at least two distinct paths of T and X is the set of paths through T .

Given a tree $T \subseteq 2^{<\omega_1}$, the Cantor-Bendixson derivative of T is the set of $\sigma \in T$ so that at least two distinct paths of T pass through σ .

Observe that, unlike the countable case, it is not easy (computable, arithmetic, hyperarithmetic, or even Δ_1^1) to construct the Cantor-Bendixson derivative of a Π_1^0 -class in the sense of ω_1 -recursion; detecting whether a node of the representing tree should be removed requires determining whether it has a path passing through it, which is a Σ_1^1 -complete task. Thus it is not clear that the rank of a Π_1^0 -class need even be Δ_1^1 .

Lemma 3.2.3. *The above definitions do not conflict, and if T is a tree witnessing that X is closed then the Cantor-Bendixson derivative of T is a tree witnessing that the Cantor-Bendixson derivative of X is closed.*

It is worth noting that this is not the only possible definition. Another natural possibility would be to restrict the union in the definition of *clopen* to countable sets only; in this case, we must replace *tree* in the second definition with *tree of countable width*. For the purposes of this chapter, we will not consider this alternative definition.

Theorem 3.2.4. *Suppose that X is presentable as a path of Cantor-Bendixson rank zero in some computable tree; further suppose that there is a tree T computable in X with Cantor-Bendixson rank α and empty perfect core. Then there is a computable tree \hat{T} that likewise has Cantor-Bendixson rank α and empty perfect core. Furthermore, the rank- α paths in \hat{T} are exactly the joins of the rank- α paths in T with X .*

Proof. We begin by defining, for a Turing functional Γ and a set $Y \subseteq \omega_1$, a tree $T(\Gamma, Y)$ with the following properties:

- There is a Y -computable embedding from the tree given by Γ^Y (as a subset of $2^{<\omega_1}$) into $T(\Gamma, Y)$ that induces a Cantor-Bendixson rank-preserving bijection on the branches of each;

- $T(\Gamma, Y) \cap 2^\alpha$ is uniformly computable in an index for Γ and $Y \upharpoonright \alpha$; and

- $T(\Gamma, Y)$ is total regardless of the totality of Γ^Y , and is unbounded if Γ^Y is partial.

We construct $T(\Gamma, Y)$ as follows, together with the embedding $F : \Gamma^Y \rightarrow T(\Gamma, Y)$ required by the first condition. We approximate Γ^Y at stage α by $\Gamma^Y \upharpoonright \alpha$; computations with use exceeding α are considered to diverge.

Begin construction by mapping the root of Γ^Y to the root of $T(\Gamma, Y)$, and designate the root of $T(\Gamma, Y)$ as the root of an Aronszajn tree. At stage α , extend the part of $T(\Gamma, Y)$ to continue growing an Aronszajn tree above each existing Aronszajn root. If $\sigma \smallfrown i$ enters Γ^Y and $F(\sigma)$ was previously designated an Aronszajn root, remove that designation; set $F(\sigma \smallfrown i)$ to be some maximal-length extension of $F(\sigma)$ in $T(\Gamma, Y)$. Fix another maximal-length extension of $F(\sigma)$, and designate both as Aronszajn roots. At limit stages, if σ is of limit length and every initial segment is in $T(\Gamma, Y)$, include σ in $T(\Gamma, Y)$ and designate it an Aronszajn root.

For $\sigma \in 2^{<\omega_1}$, let $T(\Gamma, \sigma) = T(\Gamma, Y) \cap 2^{<|\sigma|}$ for $\sigma \prec Y$; note that this is independent of the choice of X .

Finally, let X be as named in the statement, and let Γ be a functional so that $T = \Gamma^X$. Let U be a computable tree in which the unique path is X . Let $\hat{T} = \{\sigma \oplus \tau \mid \sigma \in U \wedge \tau \in T(\Gamma, \sigma) \wedge |\sigma| = |\tau|\}$.

Claim 3.2.5. $T(\Gamma, Y)$ has the desired properties.

Proof. The $F : \Gamma^Y \rightarrow T(\Gamma, Y)$ is a Y -computable embedding is immediate. That it induces a Cantor-Bendixson rank-preserving bijection on the branches holds simply because it is an embedding of the trees.

The construction uses only $Y \upharpoonright \alpha$ to construct $T(\Gamma, Y) \cap 2^\alpha$, so the second condition is met.

If Γ^Y is total, $T(\Gamma, Y)$ will clearly be total. If Γ^Y is partial, then for the earliest σ on which Γ^Y is not defined, $F(\sigma)$ or its predecessor will be the root of an Aronszajn subtree; thus $T(\Gamma, Y)$ will be both total and unbounded. \square

Claim 3.2.6. \hat{T} has the desired properties.

Proof. Every path through \hat{T} is a join with X , by construction. Let $Y = X \oplus Z$ be a path through \hat{T} . Z , by construction, is a path through $T(\Gamma, X)$. We therefore have a map $G : T(\Gamma, X) \rightarrow \hat{T}$ which induces a bijection on paths (which is Cantor-Bendixson rank-preserving by virtue of being induced by an embedding). By construction, we have an embedding $F : T \rightarrow T(\Gamma, X)$ which induces a similar bijection. F preserves Turing degree, while G takes Z to $X \oplus Z$. The composition $G \circ F$ then induces a bijection of paths in T with paths in \hat{T} , so that the image of each path Z is Turing-equivalent to $X \oplus Z$. \square

\square

3.3 Ranks of Degrees

Theorem 3.3.1. *There exists a Δ_3^0 degree that can be represented with Cantor-Bendixson rank 1, but not rank 0.*

Proof. We prove the claim by an unbounded injury argument with a tree of strategies. Strategies alternate by level between the following types:

D_e : $X \neq \Phi_e$.

C_e^i : If $\Phi_e^X \in [T_i]$, then T_i has a computable path.

D_e strategies have two outcomes, 0 and 1. C_e^i strategies have outcomes 0 through ω_1 , with the ordering $\omega_1 < 0 < 1 < 2 < \dots$. We assume the reader is familiar with the overall architecture of this style of argument, and with the Aronszajn-root method that has been extensively used previously.

When initialized at stage s , any strategy η chooses (uniformly) an extension σ of the *active* strings of its predecessors that has not yet been excluded from T , and puts σ (and all necessary initial segments) into T . It then identifies $\sigma \frown 1$ as *active*, includes both $\sigma \frown 0$ and $\sigma \frown 1$ in T , designates $\sigma \frown 0$ the root of an Aronszajn tree, and enters outcome 0. If η is a C strategy, then η also designates $\sigma \frown 1$ as its *key node*.

At any stage s , let X_s denote the union of the active strings of the strategies that are active at stage s ; think of this as an approximation to the final path X .

If η is a diagonalization strategy D_e , then η waits for $\Phi_e(|\sigma|) \downarrow = 1$. If this occurs, then η sets $\sigma \frown 0$ as active and enters outcome 1.

If η is instead a C strategy, then while not in outcome ω_1 , η waits for $\Phi_e^{X_s}(x) \downarrow$, where x is least so that this has not already occurred, so that $\Phi_e^{X_s} \upharpoonright (x+1) \in T_i$. If this happens, η requires attention; when permitted to act, η designates X_s as its new key node, switches to outcome ω_1 , and designates $\sigma \frown 1$ as active. While in outcome ω_1 , η waits until there is an $x < s$ so that $\Phi_e^\tau(x) \uparrow$ for every extension τ of its current key node and at least one

complete stage has passed since it entered outcome ω_1 ; when this holds, η designates this τ as active and switches to its first unused countable outcome. If an extension τ exists with $\Phi_e^\tau(x) \downarrow$ for x least so that $\Phi_e^{\prec\tau}(x) \uparrow$, then η takes τ as its new key node but takes no further action.

When a strategy is inactive, it maintains an Aronszajn tree over its active node and its key node (if it is a C strategy).

At limit stages, the outcome of a strategy is the leftmost outcome held unboundedly often, and limits are taken of all relevant variables stored by the strategy.

This completes the construction.

Claim 3.3.2. *The path X , defined to be the leftmost path that unboundedly often agrees with X_s , has the desired degree.*

Proof. Suppose that T is a computable tree and $Y \equiv_T X$ so that $Y \in [T]$. Then $T = T_i$ for some i and $Y = \Phi_e^X$ for some e .

Suppose further that T has no computable path. Fix a stage s_0 late enough that the C_e^i strategy along the true path has been initialized; denote this strategy η . Note that the sequence of key nodes of η is a computable sequence of pairwise-compatible strings increasing in length, and that their images under Φ_e are likewise computable, pairwise-compatible and increasing in length. Since T does not have a computable path, then, this sequence cannot be unbounded. By construction, the only way this sequence is bounded is if η eventually enters a countable outcome and never leaves; this means that η never finds another convergence that remains within the tree and extends the active node of η . But X extends this node, so Φ_e^X either must fail to be total or must land outside of T . This contradicts the initial supposition, so the claim is proven. \square

That X is Δ_3^0 is readily seen from the definition, so this concludes the proof. \square

Note that in fact the above proof shows something slightly stronger; we label it here as a “corollary” because it is a corollary of the proof, though not of the theorem.

Corollary 3.3.3. *There is a Δ_3^0 degree which can be represented with Cantor-Bendixson rank 1, but does not compute any noncomputable element of rank zero.*

In light of the existence of degrees like the one specified in Theorem 3.3.1, we introduce some additional terminology.

Definition 3.3.4. *Say X has proper Cantor-Bendixson rank α if there is a computable tree in which X has rank α but none in which X has rank $< \alpha$.*

Theorem 3.3.5. *Let α be any computable ordinal. Then there is a Δ_3^0 degree, uniformly in a notation for α , which has proper Cantor-Bendixson rank α .*

We prove the theorem by way of the following lemma:

Lemma 3.3.6. *If α is a computable ordinal, then there is (uniformly in a notation of α) a computable tree T so that the following conditions hold:*

- (i) *T has Cantor-Bendixson rank α .*
- (ii) *Every path through T is computable.*
- (iii) *T has exactly one path of rank α .*

Proof. We prove the lemma by effective transfinite induction on α ; the case $\alpha = 0$ is trivial.

Suppose that $\alpha = \beta + 1$ and the claim holds for β ; fix the appropriate T_β . Let $T = \{0^\gamma \frown \sigma \mid \sigma \in T_\beta \wedge \gamma < \omega_1\}$. Then T is the desired tree.

Suppose now that $\alpha = \lim_{i < \omega} \alpha_i$, and the claim holds for each α_i . Then let T_i be the appropriate tree corresponding to α_i , and let $T = \{0^\gamma \frown \sigma \mid \sigma \in T_{f(\gamma)} \wedge \gamma < \omega_1\}$, where f

is a fixed computable function $f : \omega_1 \rightarrow \omega$ so that $f^{-1}(n)$ is unbounded for every n . Now T is the desired tree.

Finally, suppose that $\alpha = \lim_{\beta < \omega_1} \alpha_\beta$, and the claim holds for each α_β . Let T_β be the appropriate tree corresponding to α_β , and let $T = \{0^\gamma \frown \sigma \mid \sigma \in T_\gamma \wedge \gamma < \omega_1\}$. T is the desired tree. \square

Proof. (Proof of Theorem 3.3.5) We prove the theorem by constructing, given (a notation for) α , a tree T with a distinguished path X , so that X will be of the desired degree and will have Cantor-Bendixson rank α in $[T]$. The approach will be to ensure that whenever X is equivalent to a path through a computable tree \hat{T} , there will be a series of embeddings of trees of increasing Cantor-Bendixson rank into the neighborhoods around that point in \hat{T} ensuring that the point has Cantor-Bendixson rank at least α .

Given α , fix a computable sequence of computable ordinals $\langle \alpha_\beta \mid \beta < \omega_1 \rangle$ so that $\alpha = \limsup_{\beta \rightarrow \omega_1} \alpha_\beta + 1$; note that the constant sequence $\alpha_\beta = \gamma$ will suffice in the event that $\alpha = \gamma + 1$.

Fix computable trees T_{α_β} corresponding to α_β by the Lemma; also fix an effective enumeration $\langle \mathcal{T}_e \rangle_{e < \omega_1}$ of computable trees so that every computable tree appears in this enumeration unboundedly often.

We prove the theorem by an unbounded injury argument on a tree of strategies. Let $\Lambda = \omega_1 + 1$, endowed with the order $\omega_1 < 0 < 1 < \dots$. We operate on the tree of outcomes $\Lambda^{< \omega_1}$; the strategies assigned to each level pursue requirements as follows.

$R_{e,i}$: If $\Phi_e^X \in [\mathcal{T}_i]$, then T_{α_e} computably embeds into the intersection of \mathcal{T}_i and the set of extensions of $\Phi_e^X \upharpoonright e$.

We assign limit levels to the least $R_{e,i}$ requirement not yet assigned.

Any strategy $\eta \in \Lambda^{< \omega_1}$ has an *anchor point* $\sigma \in T$; this is a string so that every string added to T on behalf of η or any strategy extending η will extend σ . The *true path* is the

leftmost element $P \in \Lambda^{\omega_1}$ so that the current sequence of outcomes is unboundedly often an initial segment of P ; X will be the limit of the anchor points of the strategies along P , and hence will be Δ_3^0 .

When a strategy η *directs construction along* a string σ , it requires that all strategies extending η choose anchor points that extend σ .

When initialized, an $R_{e,i}$ strategy *anchors* itself at a string $\sigma \in T$ and includes both $\sigma \frown 0$ and $\sigma \frown 1$ in the tree. It begins in outcome ω_1 , and directs construction along $\sigma \frown 0$, designating $\sigma \frown 1$ an Aronszajn root. It begins building an embedding $F : T_{\alpha_e} \rightarrow T$ by setting $F(\langle \rangle) = \sigma \frown 1$. At each stage following, while the strategy is in outcome ω_1 , it extends the embedding if possible to include the next string in T_{α_e} of minimal length not yet in the domain of F , so that $\Phi_e \circ F$ is a partial embedding of T_{α_e} into \mathcal{T}_i . In order to accomplish this, it may add a necessary string to T at each stage. Note that if this process continues indefinitely, then $\Phi_e \circ F$ will be a total embedding of T_{α_e} into \mathcal{T}_i .

While in outcome ω_1 , we say an $R_{e,i}$ strategy *requires attention* at stage s if either of the following hold:

(i) For the first τ in the domain of F , there is no way to extend F to $\tau \frown 0, \tau \frown 1$ so that $\Phi_e \circ F$ remains an embedding into \mathcal{T}_i , or

(ii) For the first string of τ of limit length that is in T_{α_e} , not in the domain of F , and every proper initial segment of which is in the domain of F , there is no way to extend F to τ so that $\Phi_e \circ F$ remains an embedding into \mathcal{T}_i .

In either case, the $R_{e,i}$ strategy deactivates all strategies extending its present outcome and changes its outcome to its first unused countable ordinal outcome. In case (i), it directs construction to extend $F(\tau)$. In case (ii), it directs construction to extend $F(\prec \tau) = \sup_{\rho \prec \tau} F(\rho)$.

Regardless, the $R_{e,i}$ strategy may require attention again if the condition that caused it to leave outcome ω_1 ceases to hold, in which case it returns to outcome ω_1 and designates

all maximal-length nodes deactivated this way as Aronszajn roots. The extensions to F which prompted the change are incorporated into F .

Finally, let X be the limit of the sequence of anchor strings of the strategies falling along the leftmost path P in $\Lambda^{<\omega_1}$ that was visited unboundedly often during the construction.

This completes the construction; it remains to verify that the tree T and the path X are as desired.

Claim 3.3.7. *Every requirement is satisfied.*

Proof. Let $R_{e,i}$ be a requirement, η a strategy along the true path pursuing that requirement; by induction, suppose that all higher-priority requirements are satisfied.

Suppose that $\Phi_e^X \in [\mathcal{T}_i]$ (otherwise the satisfaction of $R_{e,i}$ is vacuous). If η never requires attention, then at every stage during which it is active it will be able to add to the embedding F it is building so that $\Phi_e \circ F$ will be an embedding into \mathcal{T}_i ; in the limit, $\Phi_e \circ F$ will be a (computable) embedding of T_{α_e} into \mathcal{T}_i , and $R_{e,i}$ will be satisfied. If, instead, η eventually requires attention, then this was because $\Phi_e(F(\tau))$ failed to be in \mathcal{T}_i for some particular τ ; η then directed construction along $F(\tau)$, guaranteeing (as long as $\Phi_e(F(\tau))$ continued to fail to be in \mathcal{T}_i) that Φ_e^X would extend $\Phi_e(F(\tau)) \notin \mathcal{T}_i$. This contradicts our initial supposition, so $R_{e,i}$ is satisfied. \square

Claim 3.3.8. *The true path is the only path in the tree of strategies that is visited unboundedly often.*

Proof. It suffices to show that for each strategy, there is at most one outcome that is visited unboundedly often.

Let η be a strategy that is itself visited unboundedly often (otherwise, the result is trivial). Let σ be the anchor point of η .

η is pursuing the requirement $R_{e,i}$ for some e and some i . By construction, if η ever leaves a particular countable outcome, then it will not return; therefore, if η visits outcome α unboundedly often, then either $\alpha = \omega_1$ or η enters outcome α and remains there for the remainder of the construction. In either case, η visits only one outcome unboundedly often.

Since every strategy pursues one of these two requirements, this completes the construction. \square

Claim 3.3.9. *Every path in T other than X has Cantor-Bendixson rank $< \alpha$ in T .*

Proof. Let $Y \neq X$ be a path in T , and let σ be the maximal string so that $\sigma \prec Y$ and $\sigma \prec X$. By construction, σ is the anchor node of some strategy η along the true path. Let $R_{e,i}$ be the requirement pursued by η .

There are then two possibilities: either η is successful in defining an embedding $F : T_{\alpha_e} \rightarrow T$ above $\sigma \frown 1$, or it is not. If η is successful, then all of the paths extending $\sigma \frown 1$ are computable images of paths in T_{α_e} ; so Y is contained in a clopen subset of $[T]$ with Cantor-Bendixson rank $\alpha_e < \alpha$, and therefore itself has Cantor-Bendixson rank less than α .

If η is unsuccessful, then eventually the construction will be directed along some extension of $\sigma \frown 1$. Then $\sigma \frown 1 \prec X$, contradicting our choice of Y . \square

Claim 3.3.10. *X has Cantor-Bendixson rank α in $[T]$.*

Proof. T is a computable tree, and is therefore \mathcal{T}_i for some index i . Fix $\beta < \alpha$. Let e be an index for the identity functional (i.e., $\Phi_e^A = A$ for any oracle A) so that $\beta < \alpha_e$. Certainly $\Phi_e^X = X$, and $X \in [\mathcal{T}_i]$; so by the satisfaction of the requirement $R_{e,i}$, T_{α_e} computably embeds into $\mathcal{T}_i \cap [X \upharpoonright e]$. Since $[T_{\alpha_e}]$ has Cantor-Bendixson rank $\alpha_e > \beta$, it must be that $[\mathcal{T}_i] \cap [X \upharpoonright e]$ has Cantor-Bendixson rank greater than β . But by construction, there are unboundedly many i so that $\mathcal{T}_i = T$; so unboundedly many neighborhoods about X have

elements of Cantor-Bendixson rank at least β . Therefore X has Cantor-Bendixson rank at least β .

Since X is the only element of $[T]$ that may not have Cantor-Bendixson rank $< \alpha$, X must therefore have Cantor-Bendixson rank exactly α in X . \square

Claim 3.3.11. *X has proper Cantor-Bendixson rank α .*

Proof. Suppose that \hat{T} is a computable tree and $Y \equiv_T X$ so that Y has Cantor-Bendixson rank $\beta < \alpha$ in $[\hat{T}]$. Since \hat{T} is computable, there exists an i so that $\mathcal{T}_i = \hat{T}$. Fix e so that $\Phi_e^X = Y$; by the Padding Lemma, there are unboundedly many of these.

By the satisfaction of the requirement $R_{e,i}$, T_{α_e} computably embeds into $\mathcal{T}_i \cap \{\tau \mid \tau \succcurlyeq Y \upharpoonright e\}$. Because each T_{α_e} has Cantor-Bendixson rank α_e , Y is therefore the sole member of the intersection of a decreasing sequence of neighborhoods with Cantor-Bendixson ranks cofinal in α ; thus Y itself has Cantor-Bendixson rank at least α . \square

This completes the proof of Theorem 3.3.5. \square

Corollary 3.3.12. *Let α be an ordinal computable from the unique path X of a tree T . Then there exists (uniformly in an X -notation for α) a set $Y \leq_T X''$ with proper Cantor-Bendixson rank α .*

Proof. By Theorem 3.3.5 relativized to X , there exists a $Z \leq_T X''$ which has Cantor-Bendixson rank α in some X -computable tree U , and never has rank less than α in any computable tree. It is further evident that we can choose Z so that $X <_T Z$. By Theorem 3.2.4, there is a computable tree \hat{U} so that the paths of \hat{U} are in one-to-one rank-preserving correspondence with those of U , and furthermore that the image of any path P in U under this correspondence is equivalent to $P \oplus X$. Since $Z >_T X$, $Z \oplus X \equiv_T Z$, so the image of Z is of the same degree as Z . Z therefore has proper Cantor-Bendixson rank α . \square

Corollary 3.3.13. *Let α be any Δ_1^1 ordinal. Then there exists a set X with proper Cantor-Bendixson rank α . Furthermore, both a Δ_1^1 -code for X and an index for a computable tree witnessing the rank of X may be found uniformly in a Δ_1^1 -code for a presentation of α .*

Proof. This is an immediate consequence of the previous result, together with Theorem 2.1.16. □

3.4 Future Directions

The immediate question that presents itself regarding the results of Sections 2 and 3 is whether these results are sharp. In particular:

Question 3.4.1. *Is there a set with properly Δ_2^1 Cantor-Bendixson rank?*

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