Descriptive group theory

By

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Abstract

In Chapter 1, we introduce a notion of universality for subgroups of Polish groups that has both algebraic and topological aspects. More precisely, given a class $\mathcal{C}$ of subgroups of a topological group Γ, we say that a subgroup $H \in \mathcal{C}$ is a *universal* $\mathcal{C}$ *subgroup* of Γ if every subgroup $K \in \mathcal{C}$ is a continuous homomorphic preimage of H. Such subgroups may be regarded as complete members of $\mathcal{C}$ with respect to a natural pre-order on the set of subgroups of Γ. In Chapter 2, we show that for any Polish group Γ, the countable power Γ^ω has a universal analytic subgroup. Moreover, if Γ is locally compact, then Γ^ω also contains universal K_σ and compactly generated subgroups. We prove a weaker version of this in the non-locally compact case and provide an example showing that this result cannot readily be improved. Additionally, we show that many standard Banach spaces (viewed as additive topological groups) have universal analytic, K_σ and compactly generated subgroups. As an aside, we explore the relationship between the classes of K_σ and compactly generated subgroups and give conditions under which the two coincide.

In Chapter 3, we study universal dense and co-null sets for the classes of G_δ and F_σ sets, respectively. Specifically, one says that $A \subset X \times Y$ is a *universal dense* G_δ *for* Y (resp. *co-null* F_σ) if the vertical cross-sections A_x are exactly the dense G_δ (resp. co-null F_σ) subsets of Y. We discuss the relatioship between selection theorems for the product space $X \times Y$ and the existence of such universal sets. In the process, we refine a selection theorem of Debs and Saint-Raymond [2]. These results relate to a question of R. D. Mauldin [11].

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Chapter 1

Introduction

1.1 Introducing universal subgroups

The study of definable equivalence relations on Polish spaces has been one of the major threads of descriptive set theory for the past thirty years. In many cases, important equivalence relations arise from algebraic or combinatorial properties of the underlying Polish spaces. A common situation is that of a coset equivalence relation on a Polish group Γ. If $H \subseteq \Gamma$ is a subgroup, one defines the equivalence relation E_H by

$$xE_Hy \iff y^{-1}x \in H.$$

Viewed as a subset of $\Gamma \times \Gamma$, E_H has the same topological complexity (Borel, analytic, etc...) as H and its equivalence classes are the left cosets of H. To give a concrete example, consider the equivalence relation E_0 on 2^ω, defined by

$$xE_0y \iff (\forall^\infty n)(x(n) = y(n)).$$

Identifying 2^ω with the Polish group $\mathbb{Z}_2^\omega$, one recognizes E_0 as the coset equivalence relation of the subgroup

$$\mathrm{Fin} = \{x \in \mathbb{Z}_2^\omega : (\forall^\infty n)(x(n) = 0)\}.$$

Given equivalence relations E, F on a space X, one often asks whether or not there exists a definable map $f : X \to X$ reducing E to F, i.e., such that

$$(\forall x, y)(xEy \iff f(x)Ff(y)).$$

In this situation, "definable" is usually (though not always) interpreted to mean Baire- or Borel-measurable. (In the case that a Borel reduction exists, one writes $E \leq_{\mathrm{B}} F$.)

Returning to the setting of groups, suppose that $H, K \subseteq \Gamma$ are subgroups of a Polish group Γ and $\varphi : \Gamma \to \Gamma$ is a group homomorphism such that

$$(\forall x)(x \in H \iff \varphi(x) \in K).$$

This in turn gives a reduction of E_H to E_K since, by the properties of group homomorphisms,

$$(\forall x, y)(y^{-1}x \in H \iff \varphi(y)^{-1}\varphi(x) \in K).$$

As mentioned above, one is generally interested in reducing maps which are at least Baire-measurable. Recall, however, that Baire-measurable homomorphisms of Polish groups are automatically continuous (Theorem 9.10 in [9].) Taken together, these observations motivate the following definition.

Definition 1.1. Let Γ, Δ be Polish groups. Suppose that $H \subseteq \Gamma$ and $K \subseteq \Delta$ are subgroups. We say that H is *group-homomorphism reducible* to K if, and only if, there exists a continuous homomorphism $\varphi : \Gamma \to \Delta$ such that $\varphi^{-1}(K) = H$. We write $H \leq_{\mathrm{g}} K$.

As mentioned above,

$$H \leq_{\mathrm{g}} K \implies E_H \leq_{\mathrm{B}} E_K. \tag{1.1}$$

In fact, many Borel reductions among coset equivalence relations derive from corresponding group-homomorphism reductions. Each of the Borel reductions $E_0 \leq_{\mathrm{B}} E_1, E_2, E_3$ arises in this way. We give details of these reductions in the following example.

Example 1.2. Recall from above that E_0 is the coset equivalence relation of the subgroup $\mathrm{Fin} \subseteq \mathbb{Z}_2^\omega$. Consider the equivalence relations E_1, E_2 and E_3, where

- $x E_1 y \iff (\forall^\infty n)((x)_n = (y)_n)$,
- $x E_2 y \iff \sum_{x(n) \neq y(n)} \frac{1}{n+1} < \infty$, and
- $x E_3 y \iff (\forall^\infty n)((x)_n E_0 (y)_n)$.

We use the notation $(x)_n$ to indicate the nth sequence coded by x, i.e., $(x)_n(k) = x(\langle n, k \rangle)$, where $\langle \cdot, \cdot \rangle$ is a fixed pairing function. Notice that E_1, E_2 and E_3 are, respectively, the coset equivalence relations of the subgroups

- $H_1 = \{x \in \mathbb{Z}_2^\omega : (\forall^\infty n)((x)_n = 0)\}$,
- $H_2 = \{x \in \mathbb{Z}_2^\omega : \sum_{x(n) \neq 0} \frac{1}{n+1} < \infty\}$, and
- $H_3 = \{x \in \mathbb{Z}_2^\omega : (\forall^\infty n)((x)_n \in H_0)\}$.

A map witnessing the reduction $E_0 \leq_{\mathrm{B}} E_2$ is

$$\varphi(x) = x(0)^\frown x(1)^{2\frown} x(2)^{4\frown} x(3)^{8\frown} \ldots .$$

In other words, φ copies the nth bit of x to a block of 2^n bits of $\varphi(x)$. Observe that φ is actually a continuous group homomorphism of $\mathbb{Z}_2^\omega$ and $\mathrm{Fin} = \varphi^{-1}(H)$, i.e., $\mathrm{Fin} \leq_{\mathrm{g}} H$. (This follows since each nonzero bit of x increases $\sum\{\frac{1}{n+1} : \varphi(x)(n) \neq 0\}$ by more than $\frac{1}{2}$.)

The reductions $E_0 \leq_{\mathrm{B}} E_1, E_3$ may be witnessed by the map $\psi : \mathbb{Z}_2^\omega \to \mathbb{Z}_2^\omega$, where

$$(\psi(x))_n = x(n)^\frown x(n)^\frown \dots .$$

Observe that ψ is a continuous endomorphism of $\mathbb{Z}_2^\omega$ and $H_0 = \psi^{-1}(H_1) = \psi^{-1}(H_3)$. Thus ψ also witnesses $H_0 \leq_{\mathrm{g}} H_1, H_3$.

In general, however, the converse of (1.1) is false. Consider the following situation: suppose that H, K are normal subgroups of a group Γ and $H \leq_{\mathrm{g}} K$, via φ. The map φ induces an injective homomorphism $\tilde{\varphi} : \Gamma/H \to \Gamma/K$, defined by $\tilde{\varphi}(\pi_H(x)) = \pi_K(\varphi(x))$, where π_H and π_K are the quotient maps onto Γ/H and Γ/K, respectively. This observation justifies the following two examples.

Example 1.3. Let

$$H_2 = \{x \in \mathbb{Z}^\omega : (\forall n)(x(n) \text{ is divisible by } 2)\}$$

and

$$H_3 = \{x \in \mathbb{Z}^\omega : (\forall n)(x(n) \text{ is divisible by } 3)\}.$$

Note that $\mathbb{Z}^\omega/H_2 \cong \mathbb{Z}_2^\omega$ and $\mathbb{Z}^\omega/H_3 \cong \mathbb{Z}_3^\omega$. Thus $H_2 \not\leq_{\mathrm{g}} H_3$ and $H_3 \not\leq_{\mathrm{g}} H_2$, since there are no injective homomorphisms $\mathbb{Z}_2^\omega \to \mathbb{Z}_3^\omega$, or *vice versa*.

On the other hand, $E_{H_2} \leq_{\mathrm{B}} E_{H_3}$ via the map $f : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ given by

$$f(x)(n) = \begin{cases} 0 & \text{if } x(n) \text{ is even,} \\ 1 & \text{if } x(n) \text{ is odd,} \end{cases}$$

for each $n \in \omega$. Similarly, $E_{H_3} \leq_{\mathrm{B}} E_{H_2}$.

Example 1.4. In [13], Christian Rosendal showed that the coset equivalence relation of the subgroup

$$\mathcal{B} = \{x \in \mathbb{Z}^\omega : (\exists M)(\forall n)(|x(n)| \leq M)\}$$

is a Borel-complete K_σ equivalence relation. In particular, $E_H \leq_{\mathrm{B}} E_{\mathcal{B}}$, for each K_σ subgroup of $\mathbb{Z}^\omega$. There are, however, K_σ subgroups which are not group-homomorphism reducible to $\mathcal{B}$. For example,

$$2\mathcal{B} = \{x \in \mathcal{B} : (\forall n)(x(n) \text{ is even})\} \ \not\leq_{\mathrm{g}}\ \mathcal{B},$$

since $\mathbb{Z}^\omega/2\mathcal{B}$ has elements of order 2 and $\mathbb{Z}^\omega/\mathcal{B}$ has no elements of finite order.

Our work on $\leq_{\mathrm{g}}$ was motivated in part by the last example. In particular we wondered if there would be an analog of Rosendal's theorem for group-homomorphism reductions. In other words, are there $\leq_{\mathrm{g}}$-complete K_σ subgroups?

Naturally, one can ask this question for classes besides K_σ. This suggests the following definition.

Definition 1.5. Let Γ, Δ be Polish groups and $\mathcal{C}$ a class of subgroups of Γ. We say that a subgroup K of Δ is *universal for subgroups of* Γ *in* $\mathcal{C}$ if, and only if, for each subgroup $H \subseteq \Gamma$, with $H \in \mathcal{C}$, we have $H \leq_{\mathrm{g}} K$.

In the case that $\Gamma = \Delta$ and $K \in \mathcal{C}$, we simply say that K is a *universal* $\mathcal{C}$ *subgroup of* Γ.

In this context, the simplest classes to study are those of compactly generated, K_σ and analytic subgroups. A key property shared by each of these three classes is that membership of a subgroup H in each class is determined by the nature of a generating set for H.

1.2 Results on universal subgroups

Our main results concern the existence of universal compactly generated, K_σ and analytic subgroups in the countable powers and products of various Polish groups.

The following is our principal result for K_σ subgroups:

Theorem (2.9)**.** *Let $(\Gamma_n)_{n\in\omega}$ be a sequence of locally compact Polish groups, each term of which occurs infinitely often (up to isomorphism.) Then $\prod_n \Gamma_n$ has both universal compactly generated and K_σ subgroups.*

Although stated for products, Theorem 2.9 implies that the countable power of any locally compact group has universal K_σ and compactly generated subgroups, e.g. $\mathbb{Z}_2^\omega$, $\mathbb{Z}^\omega$, $\mathbb{R}^\omega$, $\mathbb{Q}^\omega$ (with the discrete topology on $\mathbb{Q}$) and $\mathbb{T}^\omega$ (where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.)

For the case of groups which are not locally compact, we have the following "approximation" of the last theorem:

Theorem (2.16)**.** *If Γ is a Polish group, then there is an F_σ subgroup of Γ^ω which is universal for K_σ subgroups of Γ^ω.*

In particular, S_∞^ω (and S_∞ itself) have "universal F_σ subgroups for K_σ." In Section 2.4, we show that S_∞^ω has no universal compactly generated or K_σ subgroups. This suggests that Theorems 2.9 and 2.16 may be "best possible" results for the class of K_σ subgroups.

Turning to analytic subgroups, there is no similar demarcation between the locally compact and non-locally compact cases. We have the following theorem for arbitrary Polish groups:

Theorem (2.32)**.** *Let Γ be a Polish group. There exists a universal analytic subgroup of Γ^ω.*

Applying this result to a universal Polish group, e.g. $H([0,1]^\omega)$, we obtain

Corollary (2.33)**.** *If $\mathbb{G}$ is universal Polish group, there is an analytic subgroup $H_0 \subseteq \mathbb{G}$, such that $H \leq_{\mathrm{g}} H_0$, for each analytic subgroup H of any Polish group Γ. Moreover, the reduction "$H \leq_{\mathrm{g}} H_0$" is witnessed by an injective map.*

In Section 2.6 we apply the theorems above to some standard Banach and Hilbert spaces, viewed as complete topological groups. In particular, we are able to obtain universal subgroups as in Theorem 2.9 and 2.32 in certain Banach spaces and powers of Banach spaces.

Section 2.2.1 is a brief detour exploring the relationship between K_σ and compactly generated subgroups. We obtain the following result:

Theorem (2.4)**.** *Suppose that Γ is countable discrete group. Every K_σ subgroup of Γ^ω is compactly generated if, and only if, every subgroup of Γ is finitely generated.*

In particular, every K_σ subgroup of the countable power of a finite group is compactly generated. Likewise, in $\mathbb{Z}^\omega$.

In Section 2.7, we apply the methods of Theorems 2.9 and 2.32 to demonstrate the existence of complete F_σ and analytic ideals with respect to a weak form of Rudin-Keisler reduction.

1.3 Further questions

Given the results outlined in the last section, certain questions naturally present themselves. In the first place:

Question 1.6. *Are there classes of subgroups besides those of K_σ, compactly generated and analytic subgroups which admit universal subgroups?*

For instace, $\mathbf{\Sigma}^1_2$ seems a natural candidate as it is closed under existential quantification over the reals (as the class of analytic sets is). Other classes of interest are those of co-analytic, closed and F_σ subgroups.

Based on Corollary 2.33, one can ask if there is an analog for K_σ or compactly generated subgroups:

Question 1.7. *Is there a K_σ (or compactly generated) subgroup of a universal Polish group $\mathbb{G}$ of which every K_σ (resp. compactly generated) subgroup of every Polish group is a continuous homomorphic preimage?*

Given the nature of the proof that S_∞ has neither a universal K_σ subgroup nor a universal compactly generated subgroup, we conjecture that any Polish group Γ into which S_∞ embeds will have no universal K_σ or compactly generated subgroup. If this is the case, then the answer to the last question is negative.

Turning to the relationship between the classes of K_σ and compactly generated subgroups, we would like to develop a characterization of those Polish groups Γ in which the two classes conincide. We also are interested in the following question:

Question 1.8. *Suppose that every K_σ subgroup of Γ is compactly generated. Is this also true in the countable power Γ^ω?*

Regarding this question, Arnold Miller has shown that every K_σ subgroup of $\mathbb{R}^\omega$ is compactly generated. This result is of interest in part because it does not fit in with the scheme of Theorem 2.4, since $\mathbb{R}$ is not discrete.

Along similar lines to the last question, we can ask if the property of having all K_σ subgroups compactly generated is productive.

Question 1.9. *Suppose that Γ_1 and Γ_2 are Polish groups in which every K_σ subgroup is compactly generated. Is the same true in $\Gamma_1 \times \Gamma_2$?*

All of these questions can also be formulated for arbitrary topological groups.

1.4 Selection theorems

Chapter 3 takes up the study of a somewhat different type of universal object. Specifically, suppose that X, Y are Polish spaces, that $\mathcal{C}$ is a class of subsets of Y and that $A \subseteq X \times Y$ is such that

$$\mathcal{C} = \{A_x : x \in X\},$$

where $A_x = \{y \in Y : (x, y) \in A\}$. In this case, A is said to be a *universal set* for $\mathcal{C}$. One referes to X as as the *parameter space* and regards X as "coding" subsets of Y in $\mathcal{C}$. Although we shall provide details in Chapter 3, it known that, for any Polish space Y, there is a G_δ set $A \subseteq \omega^\omega \times Y$ which is universal for dense G_δ subsets of Y, i.e., the vertical sections A_x are exactly the dense G_δ subsets of Y. It follows from results in G. Debs' and J. Saint-Raymond's paper [2] that there is a negative correlation between the existence of universal dense G_δ sets with parameter space X and certain types of selection theorems for product spaces involving X. In particular, Debs and

Saint-Raymond show that if $A \subseteq \omega^\omega \times Y$ is a universal dense G_δ for Y, then A does not contain a *Borel selector*, i.e., a Borel-measurable injection $f : \omega^\omega \to Y$ such that $f \subseteq A$ (viewing the graph of f as a subset of $\omega^\omega \times Y$).

The starting point for this investigation was the following question, posed by Mauldin in [11].

Question 1.10. *Suppose that $B \subseteq [0,1] \times [0,1]$ is Borel with all vertical sections B_x comeager or all B_x are co-null. Does B contain the graph of a Borel injection $f : [0,1] \to [0,1]$?*

By embedding a universal dense G_δ set in $[0,1] \times [0,1]$ (as a $\mathbf{\Delta}^0_3$ set), Debs and Saint-Raymond demonstrated that there are Borel subsets of $[0,1] \times [0,1]$, with comeager sections, which do not contain Borel injections. Mauldin and S. Graf answered this question in case of co-null sections in their paper [6]. In the context of transition kernels, Mauldin and Graf describe a Borel subset of $[0,1] \times [0,1]$ which has a vertical sections co-null, but which does not contain the graph of a Borel injection. In Chapter 3, we present an example of such a subset of the unit square, based on a universal co-null F_σ set.

Although the answer to Mauldin's question was ultimately negative, Debs and Saint-Raymond proved the following positive result for products of Polish spaces where the "horizontal" factor is compact:

Theorem 1.11 (Debs, Saint-Raymond)**.** *Suppose that X is a compact space, Y is a perfect Polish space and $H \subseteq X \times Y$ is a G_δ set such that all vertical sections H_x are comeager. Then H contains the graph of of a Borel injection. If X is zero dimensional, then this Borel map will actually be continuous. Furthermore, the range of this map can* *be made meager.*

Note that by the counterexample described above, the assumption that H be G_δ cannot be weakened to the assumption that H is $\mathbf{\Delta}^0_3$. In fact, the Debs' and Saint-Raymond's counterexample is the union of a G_δ set and an F_σ set, both with comeager sections. Thus Theorem 1.11 is not even true for F_σ sets, as otherwise this would give a selection function for their counterexample. (In fact, Debs and Saint-Raymond prove their result for Borel sets with dense G_δ sections, but they reduce this to the above theorem by a reflection argument.)

Given Debs' and Saint-Raymond's result that a universal dense G_δ set cannot contain a Borel selector, it follows that, for perfect Y and compact X, there is no $A \subseteq X \times Y$ which is a universal dense G_δ set for Y.

In Chapter 3, we begin by proving a refinement of Theorem 1.11 in the case of Polish product spaces $X \times Y$, where X is compact and of finite covering dimension. Specifically, we wished to preserve the continuity of the selector from the zero-dimensional case. Unfortunately, as the following example demonstrates, this is not possible.

Example 1.12. Let H be $[0,1]^2$ minus the union of all lines with slope ± 1 and rational x-intercepts. The G_δ set H has all cross-sections comeager (actually co-countable). By the Intermediate Value Theorem, H does not contain the graph of a continuous function, injective or otherwise.

We noticed, however, that for G_δ subsets of $[0,1]^2$, with comeager sections, it was possible to find selectors, which, though not actually functions, were closed sets, with disjoint vertical sections of cardinality not more than 2. Given that functions with closed graphs are continuous, we felt that "closed" was an appropriate surrogate for "continuous." This led us to Theorem 3.1, the main result of Chapter 3. Before giving the statement of this theorem, we mention the following definitions from general topology which may be found in §50 of [12].

1. An open cover $\mathcal{U}$ of a topological space X has *order* k iff there is a point of X which appears in k members of $\mathcal{U}$ and no element of X appears in more than k members of $\mathcal{U}$

2. An open cover $\mathcal{V}$ of a topological space X *refines* another cover $\mathcal{U}$ iff for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$.

3. A topological space X has *covering dimension* d if every open cover of X is refined by an open cover of order $d+1$ and furthermore, d is the smallest number for which this is true.

The following is our main selection result:

Theorem (3.1)**.** *Suppose that X and Y are Polish spaces where Y is pefect and X is compact with finite covering dimension d. Let $G \subseteq X \times Y$ be a G_δ set such that each vertical section G_x is dense. Then G contains a closed set F such that each F_x is nonempty, with cardinality at most $d+1$ and, for distinct $x, x' \in X$, F_x and $F_{x'}$ are disjoint. Moreover,* $\mathrm{range}(F) = \bigcup_{x \in X} F_x$ *is perfect and nowhere dense.*

Note that the property of having disjoint vertical sections is equivalent to injectivity for sets which are the graphs of functions. Thus, in the zero-dimensional case, our result and that of Debs and Saint-Raymond are the same.

Finally, we will discuss results relating to universal sets. We show, in particular, that for Polish spaces X and Y, with Y perfect, the following hold:

1. If X is compact, then $X \times Y$ contains no universal set for non-empty open subsets of Y.

2. If X is σ-compact, then $X \times Y$ contains no universal dense open or dense G_δ set for Y. (As mentioned above, the latter is a consequence of either our or Debs' and Saint-Raymond's selection results.)

3. If $\alpha > 2$ and X is uncountable, then $X \times Y$ contains universal dense and comeager $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$ sets.

1.5 Preliminaries and notation

The definitions and notation we use are standard and essentially identical to those in the references [9], [8] and [1]. We recall some key points below.

General notation. If x is any sequence, we let $x(n)$ denote the nth term (or *bit*) of x. We denote the length n initial segment of x by $x \restriction n$. If $I \subset \omega$ is the interval $\{k, k+1, \ldots, k+m\}$, then $x \restriction I$ denotes the finite sequence

$$(x(k), x(k+1), \ldots, x(k+m)).$$

For a set A of sequences, we let $A \restriction n$ denote the set $\{x \restriction n : x \in A\}$.

For finite sequences s, t, $s^\frown t$ denotes the concatenation of s and t. If t is the length 1 sequence (a), for some $a \in X$, we simply write $s^\frown a$, for $s^\frown t$.

If X is any set and $a \in X$, a^n denotes the finite sequence $(a, \ldots, a) \in X^n$ and $\bar{a}$ the infinite sequence $(a, a, \ldots) \in X^\omega$.

If $T \subseteq X^{<\omega}$ is a tree, then $[T]$ denotes the set $\{x \in X^\omega : (\forall n)(x \upharpoonright n \in T)\}$ and, for each $s \in X^{<\omega}$, T_s denotes $\{t \in T : t \subseteq s \vee s \subseteq t\}$.

For $\alpha, \beta \in \omega^\omega$, we write $\alpha \leq \beta$ to mean that $(\forall i)(\alpha(i) \leq \beta(i))$. Similarly, if $s, t \in \omega^k$, $s \leq t$ means that $s(i) \leq t(i)$, for each $i < k$.

Finally, if A is a subset of a topological space X, $\overline{A}$ denotes the (topological) closure of A.

Algebra and topology. A *Polish space* is a separable space whose topology is compatible with a complete metric. A *topological group* is a topological space Γ equipped with a group operation and an inverse map, such that the group operation is continuous as a function $\Gamma^2 \to \Gamma$ and the inverse map is continuous as a function $\Gamma \to \Gamma$. Hence a *Polish group* is a topological group, the topology of which is Polish.

Except when working with specific groups, we will always use multiplicative notation for group operations.

It is useful to have the notion of a *group word*. An n-ary group word $\mathcal{W}$ is a function taking n symbols as input and combining these symbols using multiplication and inverses. For example, $\mathcal{W}(a, b, c) = b^{-1}ac^{-1}$ is a ternary group word. For an n-ary group word $\mathcal{W}$ and a topological group Γ, note that $\mathcal{W}$ induces a continuous function $\Gamma^n \to \Gamma$. When there is no ambiguity, we will sometimes write $\mathcal{W}$ for $\mathcal{W}(a_1, \ldots, a_n)$.

For $A \subseteq \Gamma$, we let $\mathcal{W}[A]$ denote the set

$$\{\mathcal{W}(x_1, \ldots, x_m) : x_1, \ldots, x_m \in A\}$$

We let $\langle A \rangle$ denote the *subgroup generated by* A, i.e., the smallest (with respect to containment) subgroup of Γ which contains A. Equivalently,

$$\langle A \rangle = \bigcup \{\mathcal{W}[A] : \mathcal{W} \text{ is a group word}\}.$$

For subsets A, B of a group Γ and $g \in \Gamma$, we let AB denote the set $\{ab : a \in A \;\&\; b \in B\}$, gA denote $\{ga : a \in A\}$ and A^{-1} denote $\{a^{-1} : a \in A\}$. Likewise, define $A + B$ and $-A$, in the case of additive groups.

In Section 2.6, we will discuss examples involving *topological vector spaces*, i.e., topological spaces equipped with continuous addition, additive inverse and scalar multiplication operations. A *Banach space* is a topological vector space, the topology of which is induced by a complete norm. A *Hilbert space* is a Banach space, the norm of which is induced by an inner product. Note that a separable Banach space is a Polish group under its addition operation.

Other relevant notions and examples of Banach spaces will be introduced as appropriate in Section 2.6.

Chapter 2

Universal subgroups

2.1 A universal closed subgroup of $\mathbb{Z}^\omega$

The following is our simplest result. Although it does not fit into the scheme outlined in Section 1.2, it provides an example of the type of "coding" we will use to produce universal subgroups.

Theorem 2.1. *There is a universal closed subgroup of $\mathbb{Z}^\omega$.*

Proof. $\mathbb{Z}^k$ is a free, finitely-generated, Abelian group. Hence all of its subgroup are also finitely generated (see Theorem 7.3 in [10]). In particular, there are only countably many subgroups of $\mathbb{Z}^k$. Enumerate them as $G^k_0, G^k_1, \ldots$. For each n, k, let $I^k_n \subseteq \omega$ be an interval of length k, such that that $\{I^k_n : n, k \in \omega\}$ partition ω.

Define a closed subgroup G of $\mathbb{Z}^\omega$ by

$$x \in G \iff (\forall k, n)(x \restriction I^k_n \in G^k_n).$$

We will show that G is a universal closed subgroup. Let H be an arbitrary closed subgroup of $\mathbb{Z}^\omega$. We show that $H \leq_{\mathrm{g}} G$.

Let T be a pruned tree on $\mathbb{Z}$ such that $H = [T]$. Note that, because T is pruned, $T \cap \mathbb{Z}^k$ is a subgroup of $\mathbb{Z}^k$, for each k. Given k, let n_k be such that $T \cap \mathbb{Z}^k = G^k_{n_k}$.

Define a continuous group homomorphism $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ by

$$\varphi(x) \upharpoonright I_n^k = \begin{cases} x \upharpoonright k & \text{if} \quad n = n_k, \\ 0^k & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{Z}^\omega$ and $y = \varphi(x)$, we have

$$\begin{aligned} x \in H &\iff (\forall k)(x \upharpoonright k \in T \cap \mathbb{Z}^k) \\ &\iff (\forall k)(y \upharpoonright I_{n_k}^k \in G_{n_k}^k) \\ &\iff (\forall k, n)(y \upharpoonright I_n^k \in G_n^k) \\ &\iff \varphi(x) \in G. \end{aligned}$$

The third ' $\iff$ ' follows from the fact that, if $n \neq n_k$, then $y \upharpoonright I_n^k = 0^k \in G_n^k$. This shows that $H \leq_{\mathrm{g}} G$. $\square$

If Γ is a finite group, then there are only finitely many subgroups of Γ^k, for each k. Thus we have the following corollary to the proof of Theorem 2.1.

Corollary 2.2. *If Γ s a finite group, then Γ^ω has a universal closed subgroup.*

2.2 K_σ subgroups

In this section we study the relationship between K_σ and compactly generated subgroups (Section 2.2.1) and produce universal K_σ and compactly generated subgroups in the direct product of any sequence of locally compact Polish groups, with infinitely often repeated factors (Section 2.2.2).

2.2.1 K_σ vs. compactly generated subgroups

A compactly generated subgroup will always be K_σ. Examples of such subgroups in $\mathbb{Z}^\omega$ are

$$\mathcal{B} = \{x : x \text{ is bounded}\}$$

(generated by the set of all 0-1 sequences) and

$$\mathrm{Fin} = \{x : (\forall^\infty n)(x(n) = 0)\}$$

(generated by the set of 0-1 sequences with at most one nonzero bit).

In some cases, the classes of K_σ and compactly generated subgroups coincide. The following two theorems give a sufficient condition for this to be the case. In particular, they imply that every K_σ subgroup of $\mathbb{Z}^\omega$ is compactly generated.

Theorem 2.3. *For a Polish group Γ, every K_σ subgroup of Γ is compactly generated if, and only if, every countable subgroup of Γ is compactly generated.*[1]

Proof. The "only if" part follows from the fact that every countable subgroup is K_σ.

For the "if" part, suppose that $H = \bigcup_n K_n$ is a K_σ subgroup of Γ. Let $U_0 \supseteq U_1 \supseteq \dots$ be a neighborhood base at the identity element $\mathbf{1} \in \Gamma$, with the additional property that each $\overline{U}_{n+1} \subseteq U_n$. For each n

$$\{xU_{n+1} : x \in K_n\}$$

covers K_n. By compactness, there exists a finite set $S_n \subseteq K_n$ such that

$$\{xU_{n+1} : x \in S_n\}$$

[1] *For countable subgroups, note that compactly generated is not the same as finitely generated, e.g. $\mathbb{Q} \subseteq \mathbb{R}$ is generated by $\{\frac{1}{n} : n \in \omega\} \cup \{0\}$, but is not finitely generated.*

still covers K_n. Now let

$$K_n^* = \bigcup_{x \in S_n} x^{-1}((x\overline{U}_{n+1}) \cap K_n).$$

First note that, as the finite union of translates of compact sets, K_n^* is compact. Also, $K_n^* \subseteq H$ and $\mathbf{1} \in K_n^* \subseteq \bar{U}_{n+1} \subseteq U_n$. Furthermore, $K_n \subseteq \langle K_n^* \cup S_n \rangle$. Let $K^* = \bigcup_n K_n^*$. Then $K^* \subseteq H$ and

$$H = \langle K^* \cup \bigcup_n S_n \rangle.$$

We claim that K^* is compact. Indeed, suppose that $z_0, z_1, \ldots \in K^*$. If there is n such that $z_j \in K_n^*$, for infinitely many j, then $(z_j)_{j\in\omega}$ has a subsequential limit in K_n^*, by compactness. On the other hand, suppose that there are only finitely many z_j in each K_n^*. Let $n_0 < n_1 < \ldots$ and $j_0, j_1, \ldots$ be such that for each k, $z_{j_k} \in K_{n_k}^*$. Then for each k, $z_{j_k} \in U_{n_k}$. Hence $z_{j_k} \to \mathbf{1} \in K^*$, as $k \to \infty$.

Let $S \subseteq H$ be the subgroup generated by $\bigcup_n S_n$ (a countable subgroup). By assumption, S is compactly generated. Therefore, take a compact set $C \subseteq S$ with $S = \langle C \rangle$. Then H will be generated by the compact set $K^* \cup C$. $\square$

Theorem 2.4. *Suppose that Γ is countable discrete group. Every K_σ subgroup of Γ^ω is compactly generated if, and only if, every subgroup of Γ is finitely generated.*

Proof. First suppose that there is a subgroup H of Γ which is not finitely generated. Then $H^* = \{\bar{a} : a \in H\}$ is a K_σ subgroup of Γ^ω with no compact generating set.

Suppose now that every subgroup of Γ is finitely generated. We will show that every K_σ subgroup of Γ^ω is compactly generated. By the previous theorem, it suffices to show that every countable subgroup of Γ^ω is compactly generated. Fix a countable subgroup $C = \{x_1, x_2, \ldots\}$. For each n, let $C_n = \{x \in C : x \restriction n = \mathbf{1}^n\}$.

Claim 1. For each n, there is a finite set $F_n \subseteq C_n$ such that if $x \in C_n$, then there exists a group word $\mathcal{W}$ in the elements of F_n such that $x \cdot \mathcal{W}^{-1} \in C_{n+1}$.

Proof of claim. For each C_n there is a finite set $F_n \subseteq C_n$ such that $\{x(n) : x \in F_n\}$ generates $\{x(n) : x \in C_n\}$, since the latter is a subgroup of Γ.

This implies that, for each $x \in C_n$ there is a group word $\mathcal{W}$ in the elements of F_n such that $x(n) = \mathcal{W}(n)$. Hence $x(n) \cdot \mathcal{W}^{-1}(n) = \mathbf{1}$. On the other hand, $x \restriction n = \mathcal{W} \restriction n = \mathbf{1}^n$, since $x, \mathcal{W} \in C_n$. Thus

$$x \cdot \mathcal{W}^{-1} \restriction (n+1) = \mathbf{1}^{n+1}.$$

In other words, $x \cdot \mathcal{W}^{-1} \in C_{n+1}$. This proves the claim.

Claim 2. For each n there exists $\tilde{x}_n \in C_n$ and a group word $\mathcal{W}_n$ in the elements of $F_0 \cup \ldots \cup F_{n-1}$ such that $x_n = \tilde{x}_n \cdot \mathcal{W}_n$.

Proof of claim. The argument is a finite induction. Let $\mathcal{W}_{n,0}$ be a group word in the elements of F_0, as in Claim 1, such that $x_n \cdot \mathcal{W}_{n,0}^{-1} \in C_1$. Set $x_{n,1} = x_n \cdot \mathcal{W}_{n,0}^{-1}$. Now let $\mathcal{W}_{n,1}$ be a group word in the elements of F_1 such that $x_{n,1} \cdot \mathcal{W}_{n,1}^{-1} \in C_2$ and define $x_{n,2} = x_{n,1} \cdot \mathcal{W}_{n,1}^{-1}$. In general, we obtain $x_{n,i} \in C_i$ and group words $\mathcal{W}_{n,i}$ in the elements of F_i such $x_{n,i+1} = x_{n,i} \cdot \mathcal{W}_{n,i}^{-1} \in C_{i+1}$.

Let $\tilde{x}_n = x_{n,n}$ and $\mathcal{W}_n = \mathcal{W}_{n,n-1} \cdot \ldots \cdot \mathcal{W}_{n,0}$. Observe that $\mathcal{W}_n$ is a group word in the elements of $F_0 \cup \ldots \cup F_{n-1}$, $\tilde{x}_n \in C_n$ and $x_n = \tilde{x}_n \cdot \mathcal{W}_n$, as desired.

Claim 2 implies that each x_n is in the subgroup generated by $\tilde{x}_n$ together with $F_0 \cup \ldots \cup F_{n-1}$. Thus the set

$$\tilde{C} = \bigcup_n (\{\tilde{x}_n\} \cup F_n)$$

generates C.

It remains to check that $\tilde{C}$ is compact. For each n, observe that there are only finitely many elements $x \in \tilde{C}$ such that $x(n) \neq \mathbf{1}$, since all such elements are contained in $\{\tilde{x}_i : i \leq n\} \cup F_0 \cup \ldots \cup F_n$. Thus every infinite sequence of distinct elements of $\tilde{C}$ must converge to $\bar{\mathbf{1}}$. This implies that every infinite sequence in $\tilde{C}$ is either eventually constant or has a subsequence converging to $\bar{\mathbf{1}}$. $\square$

We enumerate a couple of direct consequences.

1. Every K_σ subgroup of $\mathbb{Z}^\omega$ is compactly generated. (Since every subgroup of $\mathbb{Z}$ is singly generated.)

2. If Γ is a finite group, then every K_σ subgroup of Γ^ω is compactly generated.

For a Polish group Γ, even if there are non-compactly generated K_σ subgroups, we can still ask whether or not every K_σ subgroup is group-homomorphism reducible to a compactly generated one. The following two examples illustrate the range of possibilities.

Example 2.5. Let $S = \bigoplus_\omega \mathbb{Z}$ be the direct sum of countably many copies of $\mathbb{Z}$. Unlike $\mathbb{Z}$, the countable group S is not finitely generated. Thus, with the discrete topology, S is K_σ, but not compactly generated. (In a discrete space, compact is the same as finite.)

By extension, not all K_σ subgroups of S^ω will be compactly generated. For example, $\{x \in S^\omega : x \text{ is a constant sequence}\}$. On the other hand, we will see that every K_σ subgroup is group-homomorphism reducible to a compactly generated one. We begin by showing that S^ω homomorphically embeds in $\mathbb{Z}^\omega$. Let $\varphi_n : S \to \mathbb{Z}$ be the projection map onto the nth coordinate. Define $\psi : S^\omega \to \mathbb{Z}^\omega$ by

$$\psi(x)(\langle m, n \rangle) = \varphi_n(x(m)),$$

where $\langle \cdot, \cdot \rangle : \omega^2 \longleftrightarrow \omega$ is a fixed bijection. The map ψ is a continuous injective homomorphism whose range is the $\mathbf{\Pi}^0_3$ subgroup

$$\{y \in \mathbb{Z}^\omega : (\forall m)(\forall^\infty n)((x(\langle m, n \rangle) = 0)\}.$$

Now let $H \subseteq S^\omega$ be any K_σ subgroup. The image $\psi(H) \subseteq \mathbb{Z}^\omega$ is also K_σ (because ψ is continuous) hence compactly generated by Theorem 2.4. Say $\psi(H) = \langle K \rangle$. Let $i : \mathbb{Z}^\omega \to S^\omega$ be the natural "inclusion" map. Then $i(K) \subseteq S^\omega$ is compact and $H = (i \circ \psi)^{-1}(\langle i(K) \rangle)$, because $i \circ \psi$ is injective.

For our next example, we introduce some terminology. Suppose that H is a subgroup of an Abelian group Γ (with additive notation) and $x \in H$. We say that x is *divisible in* H to mean that for each $n \in \omega$, there exists $y \in H$ such that $x = ny$. Note that for subgroups $H_1, H_2 \subseteq \Gamma$, if $\varphi : \Gamma \to \Gamma$ is a group homomorphism such that $\varphi^{-1}(H_2) = H_1$ and $x \in H_1$ is divisible in H_1, then $\varphi(x) \in H_2$ is divisible in H_2.

Example 2.6. Consider the group $\mathbb{Q}$ of rational numbers with the discrete topology. We will see that there are K_σ subgroups of $\mathbb{Q}^\omega$ that are not group-homomorphism reducible to any compactly generated subgroup.

We first claim that there are no nonzero divisible elements in a compactly generated subgroup of $\mathbb{Q}^\omega$. Indeed, suppose that, on the contrary, H is generated by the compact set K and there is a nonzero element $x \in H$, with x divisible in H. Let $m \in \omega$ be such that $x(m) \neq 0$. Let

$$A = \{y(m) : y \in K\}.$$

Note that, since x is divisible in H, $x(m)$ will be divisible in $\langle A \rangle \subset \mathbb{Q}$. As K is compact and we have given $\mathbb{Q}$ the discrete topology, A must be finite. Therefore, let $k \in \mathbb{Z}$ be such that $ka \in \mathbb{Z}$, for each $a \in A$. This implies that, for any $b \in \langle A \rangle$, we also have $kb \in \mathbb{Z}$. Let n be large enough that $\frac{k}{n}x(m) \notin \mathbb{Z}$. Thus $\frac{1}{n}x(m) \notin \langle A \rangle$, contradicting the divisibility of $x(m)$ in $\langle A \rangle$.

We now exhibit a K_σ subgroup which is not group-homomorphism reducible to any compactly generated subgroup. Consider the subgroup

$$\mathrm{Fin} = \{x \in \mathbb{Q}^\omega : (\forall^\infty n)(x(n) = 0)\}.$$

Fin is K_σ and every element of Fin is divisible in Fin. Suppose that $\varphi : \mathbb{Q}^\omega \to \mathbb{Q}^\omega$ is a continuous homomorphism and H is a subgroup such $\varphi^{-1}(H) = \mathrm{Fin}$. In the first place, we have that $\ker\varphi \subseteq \mathrm{Fin}$. Note, however, that $\ker\varphi \neq \mathrm{Fin}$, since then we would have $\varphi \equiv 0$ because Fin is dense in $\mathbb{Q}^\omega$. Hence there exists $x \in \mathrm{Fin}$ with $\varphi(x) \neq 0$. Since x is divisible in Fin, we have that $\varphi(x)$ is divisible in H and nonzero. Thus H cannot be compactly generated, by the comments above.

2.2.2 Universal subgroups

The main result of this section is Theorem 2.9, which states that a product $\prod_{n\in\omega}\Gamma_n$ of locally compact Polish groups, each factor of which occurs infinitely often, has universal compactly generated and K_σ subgroups.

The case of $\mathbb{Z}^\omega$

The following theorem and its corollary prove Theorem 2.9 in the case of $\mathbb{Z}^\omega$ and serve to illustrate the main ideas of Theorem 2.9 in a more straightforward setting.

Theorem 2.7. *There is a universal compactly generated subgroup of $\mathbb{Z}^\omega$.*

Proof. We essentially construct a $\leq_{\mathrm{g}}$-complete compact subset of $\mathbb{Z}^\omega$.

For each $m \in \omega$, let $A_0^k, A_1^k, \ldots$ list all finite subsets of $\mathbb{Z}^k$ which contain 0^k and are such that $-A_j^k = A_j^k$. Let I_j^k $(k, j \in \omega)$ partition ω, with each I_j^k an interval of length k. Define $K_0 \subset \mathbb{Z}^\omega$ by

$$x \in K_0 \iff (\forall k, j)(x \restriction I_j^k \in A_j^k).$$

Note that K_0 is compact and $-K_0 = K_0$. Consider $\langle K_0 \rangle$ (the subgroup generated by K_0). We show that $\langle K_0 \rangle$ is universal for compactly generated subgroups of $\mathbb{Z}^\omega$.

Suppose that $\langle K \rangle$ is any compactly generated subgroup. With no loss of generality, we assume that $-K = K$ and $\bar{0} \in K$. There is a pruned tree T on $\mathbb{Z}$ such that $K = [T]$. Since K is compact, all levels of T must be finite. For each k, choose $\tau(k) \in \omega$ such that $A_{\tau(k)}^k = T \cap \mathbb{Z}^k$. Define a homomorphism $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ by

$$\varphi(x) \restriction I_j^k = \begin{cases} x \restriction k & \text{if } j = \tau(k), \\ 0^k & \text{otherwise.} \end{cases}$$

Observe that $\varphi^{-1}(K_0) = K$. The following claim will complete the proof of this theorem.

Claim. $\varphi^{-1}(\langle K_0 \rangle) = \langle K \rangle$.

Proof of claim. Suppose that $x \in \langle K \rangle$, with $x_1, \ldots, x_m \in K$ such that $x = x_1 + \ldots + x_m$. (Note that, since $-K = K$, all elements of $\langle K \rangle$ are finite sums of elements of K.) Then $\varphi(x_1), \ldots, \varphi(x_m) \in K_0$ and hence $\varphi(x) = \varphi(x_1) + \ldots + \varphi(x_m) \in \langle K_0 \rangle$.

Suppose, on the other hand, that $\varphi(x) \in \langle K_0 \rangle$, with $y_1, \ldots, y_m \in K_0$ such that $\varphi(x) = y_1 + \ldots + y_m$. (Again, because $-K_0 = K_0$, $\langle K_0 \rangle$ is the set of finite sums of members of K_0.) We want $x_1, \ldots, x_m \in K$ with $x = x_1 + \ldots + x_m$.

For each $i \leq m$, let $v_i^k = y_i \restriction I_{\tau(k)}^k$. Since each $y_i \in K_0$, the definition of K_0 implies that each

$$v_i^k \in A^k_{\tau(k)} = T \cap \mathbb{Z}^k.$$

Hence (because T is pruned) there exists $x_i^k \in K$ such that

$$x_i^k \restriction k = v_i^k.$$

By the compactness of K, we may iteratively (for $i \leq m$) take convergent subsequences of $(x_i^k)_{k\in\omega}$ to obtain a common subsequence $k_0 < k_1 < \ldots$ such that, for each $i \leq m$, $(x_i^{k_n})_{n\in\omega}$ is convergent, with limit $x_i \in K$. Finally, fix p and let $k_n \geq p$ be large enough that $x_i^{k_n} \restriction p = x_i \restriction p$, for each $i \leq m$. Thus

$$\begin{aligned} x \restriction p &= \sum_{i\leq p} v_i^{k_n} \restriction p \\ &= \sum_{i\leq p} x_i^{k_n} \restriction p \quad \text{(because } k_n \geq p) \\ &= \sum_{i\leq p} x_i \restriction p. \end{aligned}$$

As p was arbitrary, we have $x = \sum_{i\leq m} x_i \in \langle K\rangle$. This completes the proof. □

Corollary 2.8. *There is a universal K_σ subgroup of $\mathbb{Z}^\omega$.*

Proof. Since every K_σ subgroup of $\mathbb{Z}^\omega$ is compactly generated by Theorem 2.4, Theorem 2.7 actually gives a universal K_σ subgroup of $\mathbb{Z}^\omega$. □

Statement of main result

The following is our main existence theorem for universal K_σ and compactly generated subgroups.

Theorem 2.9. *Let $(\Gamma_n)_{n\in\omega}$ be a sequence of locally compact Polish groups, each term of which occurs infinitely often (up to isomorphism). We have the following:*

1. $\prod_n \Gamma_n$ *has a universal compactly generated subgroup.*

2. $\prod_n \Gamma_n$ *has a universal* K_σ *subgroup.*

Note that if every K_σ subgroup of $\prod_n \Gamma_n$ is reducible to a compactly generated subgroup, then (1) of Theorem 2.9 implies (2). On the other hand, in Section 2.2.1 we saw examples of K_σ subgroups of Polish groups (of the form under consideration) which do not reduce to compactly generated subgroups. In such cases, (1) and (2) remain distinct results.

A corollary of Theorem 2.9:

Corollary 2.10. *If* Γ *is locally compact, then* Γ^ω *has universal compactly generated and* K_σ *subgroups.*

For most of the examples we consider in Section 2.6, we will only use the statement of Corollary 2.10.

Our key lemma in the proof of Theorem 2.9 is a restricted, but refined, version of Theorem 2.9(1). (Recall that for an m-ary group word $\mathcal{W}$, we define $\mathcal{W}[K] = \{\mathcal{W}(x_1, \ldots, x_m) : x_1, \ldots, x_m \in A\}$.)

Lemma 2.11. *Let* Γ *be a locally compact Polish group with identity element* $\mathbf{1}$*. There exists a compact set* $K_0 \subseteq \Gamma^\omega$ *with* $\bar{\mathbf{1}} \in K_0$ *and the property that for each compact* $K \subseteq \Gamma^\omega$*, with* $\bar{\mathbf{1}} \in K$*, there is a continuous group homomorphism* $\varphi : \Gamma^\omega \to \Gamma^\omega$ *such that, for each group word* $\mathcal{W}$*,*

$$\varphi^{-1}(\mathcal{W}[K_0]) = \mathcal{W}[K].$$

In particular, $\langle K_0 \rangle$ *is a universal compactly generated subgroup of* Γ^ω*.*

Basic notions

We begin with some notation and facts we will use in the proof of Lemma 2.11. From now on, fix a locally compact Polish group Γ, with identity element $\mathbf{1}$.

The following lemma gives a neighborhood base at $\mathbf{1}$ with the specific properties we require.

Lemma 2.12. *There is a neighborhood base $\{U_k\}$ at $\mathbf{1}$ such that*

1. *Each U_k has compact closure.*
2. $U_0 \supseteq U_1 \supseteq \ldots$.
3. *For each k, $U_k^{-1} = U_k$.*
4. *For each $k > 0$, $\overline{U_k U_k} \subseteq U_{k-1}$.*

Proof. We construct the U_k inductively. Let $V_0 \supseteq V_1 \supseteq \ldots \ni \mathbf{1}$ be any "nested" neighborhood base at $\mathbf{1}$, such that $\overline{V}_0$ is compact. (Such V_k exist since Γ is locally compact.) Let $U_0 = V_0$. Suppose that $U_0 \supseteq \ldots \supseteq U_k$ are given with the desired properties. By the continuity of the group operation, there is a neighborhood V of $\mathbf{1}$ such that $\overline{VV} \subseteq V_k \cap U_k$. By the continuity of the map $(x, y) \mapsto x^{-1}y$, there is a neighborhood W of $\mathbf{1}$ such that $W^{-1}W \subseteq V$. Let $U_{k+1} = W^{-1}W$. Then $(U_{k+1})^{-1} = U_{k+1}$ and

$$\overline{(U_{k+1}U_{k+1})} \subseteq \overline{VV} \subseteq U_k.$$

□

Fix a neighborhood base $\{U_k\}$, as in the lemma above. For $a, b \in \Gamma$, write $a \approx_k b$ ("a k-approximates b") if, and only if, $a^{-1}b \in \overline{U}_k$. Note that, by the properties of the U_k,

1. $a \approx_k a$

2. $a \approx_k b \iff b \approx_k a$

3. $a \approx_k b \approx_k c \implies a \approx_{k-1} c$

4. $(a \approx_k b \;\&\; k' \leq k) \implies a \approx_{k'} b$

5. $\lim_n a_n = a \iff (\forall k)(\forall^\infty n)(a_n \approx_k a)$.

If $x, y \in \Gamma^\omega$ (or Γ^p), we will write $x \approx_k y$ to indicate that $x(i) \approx_k y(i)$, for each $i \in \omega$ (or $i < p$). Item 5 above implies that for $x, x_n \in \Gamma^\omega$

$$\lim_n x_n = x \iff (\forall p, k)(\forall^\infty n)(x_n \restriction p \approx_k x \restriction p) \tag{2.1}$$

Also note that, for each k and fixed $a_0 \in \Gamma$, the set

$$\{a \in \Gamma : a_0 \approx_k a\}$$

is compact.

Fix a countable dense set $D \subseteq \Gamma$, with $\mathbf{1} \in D$. Let $\mathfrak{n} \leq \omega$ be the cardinality of D, and $\# : D \longleftrightarrow \mathfrak{n}$ be a bijection, with $\#\mathbf{1} = 0$.

For $x \in \Gamma^\omega$ and $k \in \omega$, we define a sequence $\beta^k_x \in D^\omega$ (which we call the *least k-approximation* of x) as follows: for each i, let $a_i \in D$ be the element with $\#a_i$ least such that $a_i \approx_k x(i)$. Define $\beta^k_x \in D^\omega$ by

$$(\forall i)(\beta^k_x(i) = a_i).$$

Given a closed set $K \subseteq \Gamma^\omega$ and $k \in \omega$, let

$$\mathcal{B}_k = \{\beta^k_x \restriction k : x \in K\}.$$

Since K is closed, (2.1) above implies that $x \in K$ if, and only if, $(\forall k)(\beta_x^k \upharpoonright k \in \mathcal{B}_k)$. We have the following fact.

Lemma 2.13. *If $K \subseteq \Gamma^\omega$ is compact, then $\{\beta_x^k(n) : x \in K\}$ is finite, for each $k, n \in \omega$. In particular, each $\mathcal{B}_k$ is finite.*

Proof. Since K is compact, so is the set $A = \{x(n) : x \in K\} \subseteq \Gamma$. There is thus a finite set $F_n \subseteq D$ such that, for each $x \in K$, there is an $a \in F_n$ with $x(n) \approx_k a$. As $\beta_x^k(n)$ is the $\#$-least element of D which k-approximates $x(n)$, we conclude that $\#\beta_x^k(n) \leq \max\{\#a : a \in F_n\}$, for each $x \in K$. Hence $\{\beta_x^k(n) : x \in K\}$ is finite.

This implies that each $\mathcal{B}_k$ is finite, since $\mathcal{B}_k \subseteq \prod_{n<k} F_n$. □

Proof of Lemma 2.11

Fix a locally compact group Γ and let D, $\#$, $\approx_k$ be defined as above for Γ. For each $k \in \omega$, let $A_0^k, A_1^k, \ldots \subseteq D^k$ be such that, for each k, j, we have

- A_j^k is finite.
- $\mathbf{1}^k \in A_j^k$.
- For each finite $A \subseteq D^k$, with $\mathbf{1}^k \in A$, there exists j such that $A = A_j^k$.

Let I_j^k (for $k, j \in \omega$) be intervals partitioning ω such that each I_j^k has length k. Define $K_0 \subseteq \Gamma^\omega$ by

$$x \in K_0 \iff (\forall k, j)(\exists u \in A_j^k)(u \approx_k x \upharpoonright I_j^k).$$

Note that K_0 is compact since "$u \approx_k x \upharpoonright I_j^k$" defines a compact subset of Γ^k and the existential quantifier is over a finite set. We shall show that $\langle K_0 \rangle$ has the property that for any compact $K \subseteq \Gamma^\omega$, containing $\bar{1}$, there is a continuous homomorphism $\varphi : \Gamma^\omega \to \Gamma^\omega$ with

$$\varphi^{-1}(\mathcal{W}[K_0]) = \mathcal{W}[K],$$

for each group word $\mathcal{W}$.

Let K be an arbitrary compact subset of Γ^ω, with $\bar{\mathbf{1}} \in K$. For each k, let

$$\mathcal{B}_k = \{\beta_x^k \restriction k : x \in K\}$$

be as above. As we remarked in Lemma 2.13, the compactness of K implies that each $\mathcal{B}_k$ is finite. Since $\bar{\mathbf{1}}$ is its own least k-approximation, each $\mathcal{B}_k$ contains $\mathbf{1}^k$. For each $k \in \omega$, we may therefore choose $\tau(k) \in \omega$ such that

$$A^k_{\tau(k)} = \mathcal{B}_k.$$

Define a continuous group homomorphism $\varphi : \Gamma^\omega \to \Gamma^\omega$ by

$$\varphi(x) \restriction I^k_j = \begin{cases} x \restriction k & \text{if } j = \tau(k), \\ \mathbf{1}^k & \text{otherwise.} \end{cases}$$

Fix an m-ary group word $\mathcal{W}$. The following two claims will complete the proof.

Claim 1. $x \in \mathcal{W}[K] \implies \varphi(x) \in \mathcal{W}[K_0]$.

Proof of claim. Since φ is a group homomorphism, it will suffice to show that $x \in K \implies \varphi(x) \in K_0$. Suppose that $x \in K$. For each k, let

$$u_k = \beta_x^k \restriction k \in \mathcal{B}_k = A^k_{\tau(k)}.$$

Hence $u_k \approx_k x \restriction k = \varphi(x) \restriction I^k_{\tau(k)}$. On the other hand, if $j \neq \tau(k)$, then $\varphi(x) \restriction I^k_j = \mathbf{1}^k \in A^k_j$. Putting these together, we see that

$$(\forall k, j)(\exists u \in A^k_j)(u \approx_k \varphi(x) \restriction I^k_j).$$

Thus $\varphi(x) \in K_0$. This completes the claim.

Claim 2. $\varphi(x) \in \mathcal{W}[K_0] \implies x \in \mathcal{W}[K]$.

Proof of claim. Let $y_1, \ldots, y_m \in K_0$ be such that $\varphi(x) = \mathcal{W}(y_1, \ldots, y_m)$. We will find $x_1, \ldots, x_m \in K$ such that $x = \mathcal{W}(x_1, \ldots, x_m)$ and conclude that $x \in \mathcal{W}[K]$.

For each k, i, let

$$v_i^k = y_i \restriction I_{\tau(k)}^k$$

and let $u_i^k \in A_{\tau(k)}^k = \mathcal{B}_k$ be such that $u_i^k \approx_k v_i^k$. By the definition of $\mathcal{B}_k$, there exist $x_i^k \in K$ such that $u_i^k \approx_k x_i^k \restriction k$, for each k and $i \leq m$. Since K is compact, we may take $k_0 < k_1 < \ldots$ and $x_1, \ldots, x_m \in K$ with $\lim_n x_i^{k_n} = x_i$, for each $i \leq m$.

Let $z_i^k = v_i^k {}^\frown \bar{\mathbf{1}}$. We claim that

$$\lim_n z_i^{k_n} = x_i.$$

Indeed, fix $p, r \in \omega$ and let M be large enough that whenever $k_n \geq M$, we have

$$x_i^{k_n} \restriction r \approx_{p+2} x_i \restriction r.$$

The existence of M follows from (2.1), since $\lim_n x_i^{k_n} = x_i$. We may assume that $M > r, p+2$ and so if $k_n \geq M$, we have

$$\begin{aligned} z_i^{k_n} \restriction r = v_i^{k_n} \restriction r \\ &\approx_{p+2} u_i^{k_n} \restriction r \\ &\approx_{p+2} x_i^{k_n} \restriction r \\ &\approx_{p+2} x_i \restriction r. \end{aligned}$$

Hence $z_i^{k_n} \restriction r \approx_p x_i \restriction r$, for all $k_n \geq M$. As p, r were arbitrary, we conclude (again, by (2.1)) that $z_i^{k_n} \to x_i$ as $n \to \infty$.

We may now finish the claim. Observe that for fixed r and each $k_n > r$, we have

$$\begin{aligned} x \upharpoonright r &= (\varphi(x) \upharpoonright I^{k_n}_{\tau(k_n)}) \upharpoonright r \\ &= \mathcal{W}(v_1^{k_n}, \ldots, v_m^{k_n}) \upharpoonright r \\ &= \mathcal{W}(z_1^{k_n}, \ldots, z_m^{k_n}) \upharpoonright r. \end{aligned}$$

Taking the limit as $n \to \infty$ and using the fact that $\mathcal{W}$ induces a continuous function $\Gamma^{rm} \to \Gamma^r$ we conclude that

$$x \upharpoonright r = \mathcal{W}(x_1, \ldots, x_m) \upharpoonright r.$$

Since r was arbitrary, $x = \mathcal{W}(x_1, \ldots, x_m) \in \mathcal{W}[K]$. This completes the proof. □

Proof of main result

We first prove (1) of Theorem 2.9 and then prove (2) from (1).

Proof of Theorem 2.9(1). Let $(\Gamma_n)_{n\in\omega}$ be a sequence of locally compact Polish groups, with each term occuring infinitely often up to isomorphism. This implies that $\prod_n \Gamma_n \cong \prod_n(\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega) \cong \prod_n \Gamma_n^\omega$. It will therefore suffice to show that there is a compactly generated subgroup of $\prod_n(\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega)$ which is universal for compactly generated subgroups of $\prod_n \Gamma_n^\omega$.

For each n, note that $\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega \cong (\Gamma_0 \times \ldots \times \Gamma_n)^\omega$. As the direct product of finitely many locally compact groups, $\Gamma_0 \times \ldots \times \Gamma_n$ itself is locally compact. Therefore take compact sets $K_n \subseteq \Gamma_0^\omega \times \ldots \times \Gamma_n^\omega$ with $\bar{\mathbf{1}} \in K_n$, as in Lemma 2.11, such that, for any compact $K \subseteq \Gamma_0^\omega \times \ldots \times \Gamma_n^\omega$ with $\bar{\mathbf{1}} \in K$, there is a continuous endomorphism φ of $\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega$ such that $\varphi^{-1}(\mathcal{W}[K_n]) = \mathcal{W}[K]$, for each group word $\mathcal{W}$.

Define a compact set $K_\infty \subseteq \prod_n(\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega)$ by

$$\xi \in K_\infty \iff (\forall n)(\xi(n) \in K_n)$$

We will show that $\langle K_\infty \rangle$ is universal for compactly generated subgroups of $\prod_n \Gamma_n^\omega$. Indeed, fix an arbitrary compactly generated subgroup $\langle K \rangle \subseteq \prod_n \Gamma_n^\omega$. We may assume that $\bar{\mathbf{1}} \in K$. For each n, Lemma 2.11 gives an endomorphism φ_n of $\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega$ such that

$$\varphi_n^{-1}(\mathcal{W}[K_n]) = \mathcal{W}[K \restriction (n+1)] = \mathcal{W}[K] \restriction (n+1), \tag{2.2}$$

for each group word $\mathcal{W}$. (Recall here that $K \restriction (n+1) = \{x \restriction (n+1) : x \in K\} \subseteq \Gamma_0^\omega \times \ldots \times \Gamma_n^\omega$.)

Define a continuous homomorphism $\varphi : \prod_n \Gamma_n^\omega \to \prod_n(\Gamma_0^\omega \times \ldots \times \Gamma_n^\omega)$ by

$$\varphi(x)(n) = \varphi_n(x \restriction (n+1)),$$

for each n. The following claim will complete the proof.

Claim. $\varphi^{-1}(\langle K_\infty \rangle) = \langle K \rangle$.

Proof of claim. It suffices to show that, for each group word $\mathcal{W}$,

$$\varphi^{-1}(\mathcal{W}[K_\infty]) = \mathcal{W}[K]. \tag{2.3}$$

Fix a group word $\mathcal{W}$. Armed with (2.2) and the fact that $\mathcal{W}[K]$ is compact, we have

$$\begin{aligned} x \in \mathcal{W}[K] &\iff (\forall n)(x \restriction (n+1) \in \mathcal{W}[K] \restriction (n+1)) \\ &\iff (\forall n)(\varphi_n(x \restriction (n+1)) \in \mathcal{W}[K_n]) \\ &\iff (\forall n)(\varphi(x)(n) \in \mathcal{W}[K_n]) \\ &\iff \varphi(x) \in \mathcal{W}[K_\infty]. \end{aligned}$$

The third " $\iff$ " follows from the definition of $\varphi(x)(n)$ as $\varphi_n(x \restriction (n+1))$. This completes the proof. □

Remark. In the proof above, (2.3) and the definition of K_∞ imply that the statement of Lemma 2.11 holds for $\prod_n \Gamma_n$, i.e., $\bar{\mathbf{1}} \in K_\infty$ and, for each compact $K \subseteq \prod_n \Gamma_n$ containing $\bar{\mathbf{1}}$, there is a continuous homomorphism $\varphi : \prod_n \Gamma_n \to \prod_n \Gamma_n$ with $\varphi^{-1}(\mathcal{W}[K_\infty]) = \mathcal{W}[K]$, for each group word $\mathcal{W}$.

Considering the group word $\mathcal{W}_0(a) = a$ and noting that $\langle K \rangle = \bigcup_{\mathcal{W}} \mathcal{W}[K]$, we obtain the following corollary to the proof of Theorem 2.9(1).

Corollary 2.14. *Suppose $(\Gamma_n)_{n\in\omega}$ are as above. There exists a compact set $K_0 \subseteq \prod_n \Gamma_n$ such that $\bar{\mathbf{1}} \in K_0$ and for each compact $K \subseteq \prod_n \Gamma_n$ with $\bar{\mathbf{1}} \in K$, there is a continuous group homomorphism $\varphi : \prod_n \Gamma_n \to \prod_n \Gamma_n$ such that*

$$\varphi^{-1}(K_0) = K \quad \textit{and} \quad \varphi^{-1}(\langle K_0 \rangle) = \langle K \rangle.$$

We will use this in the next proof.

Proof of Theorem 2.9(2). Fix a sequence $(\Gamma_n)_{n\in\omega}$ of locally compact Polish groups, as above. For each n, let $D_n \subseteq \Gamma_n$ be a countable dense set, containing the identity element $\mathbf{1}_n \in \Gamma_n$. For each n, fix an enumeration $\{x_0^n, x_1^n, \ldots\}$ of D_n, with $x_0^n = \mathbf{1}_n$, and fix a neighborhood $U_n \ni \mathbf{1}_n$, with $\overline{U}_n$ compact.

For each n and $x \in \prod_n \Gamma_n$, define $x^* \in \omega^\omega$ by

$$x^*(n) = \min\{i : (x_i^n)^{-1} x(n) \in \overline{U}_n\},$$

for each $n \in \omega$. Define $u^* \in \omega^n$ analogously, for $u \in \prod_{i<n} \Gamma_i$. Observe that, by the argument of Lemma 2.13, if $K \subseteq \prod_n \Gamma_n$ is compact, then $\{x^* : x \in K\}$ has compact

closure in ω^ω. Conversely, since each $\overline{U}_n$ is compact, it follows that

$$\{x \in \prod_n \Gamma_n : x^* \leq \alpha\}$$

is compact, for each $\alpha \in \omega^\omega$.

For notational reasons, we will consider the group

$$\Delta = \prod_{\substack{n\in\omega \\ s\in\omega^{<\omega}}} (\Gamma_0 \times \ldots \times \Gamma_{|s|-1}).$$

Note that n is a "dummy" index, serving only to produce infinitely many copies of the term inside the product. For the sake of clarity, we remark that $\xi(n,s) \in \Gamma_0 \times \ldots \times \Gamma_{|s|-1}$, for each n, s and $\xi \in \Delta$.

Since each Γ_n is isomorphic to infinitely many other Γ_m, we have $\Delta \cong \prod_n \Gamma_n$. To prove our theorem, it will therefore suffice to produce a K_σ subgroup of Δ which is universal for K_σ subgroups of $\prod_n \Gamma_n$.

Let $K_0 \subseteq \prod_n \Gamma_n$ be as in Corollary 2.14. For each n, define

$$A_n = \{\xi \in \Delta : (\forall n' \geq n)(\forall s \in \omega^{<\omega})(\xi(n', s) \in K_0 \restriction |s|)\}.$$

For each n, the subgroup $\langle A_n \rangle$ is F_σ. This follows from the fact that each A_n is the direct product of a compact set with factors of the form $\Gamma_0 \times \ldots \times \Gamma_k$.

Define the set

$$\tilde{A} = \{\xi \in \Delta : (\forall^\infty n, s)(\xi(n,s)^* \leq s)\}.$$

It follows that $\tilde{A}$ is K_σ and hence $\langle \tilde{A} \rangle$ is as well. Let

$$H_0 = \langle \tilde{A} \rangle \cap \bigcup_n \langle A_n \rangle$$

and note that, since the term $\bigcup_n \langle A_n \rangle$ is an increasing union of subgroups, H_0 itself is a subgroup of Δ. As the intersection of an F_σ set with a K_σ set, H_0 is K_σ. We will show that H_0 is universal for K_σ subgroups of $\prod_n \Gamma_n$.

Let $B = \bigcup_n B_n$ be an arbitrary K_σ subgroup of $\prod_n \Gamma_n$, with each B_n compact and $\bar{\mathbf{1}} \in B_0 \subseteq B_1 \subseteq \ldots$. Take continuous endomorphisms ψ_n of $\prod_n \Gamma_n$ such that

$$\psi_n^{-1}(K_0) = B_n \quad \text{and} \quad \psi_n^{-1}(\langle K_0 \rangle) = \langle B_n \rangle,$$

for each n. Each $\psi_m(B_n)$ is compact. As noted above, this implies that the closure of $\{x^* : x \in \psi_m(B_n)\}$ is compact in ω^ω. Thus we may choose $\alpha_n \in \omega^\omega$ such that each α_n is increasing, $\alpha_0 \leq \alpha_1 \leq \ldots$ and $x^* \leq \alpha_n$, for each $x \in \bigcup_{n' \leq n} \psi_{n'}(B_n)$. Define $\psi : \prod_n \Gamma_n \to \Delta$ by

$$\psi(x)(n, s) = \begin{cases} \psi_n(x) \restriction p & \text{if } s = \alpha_{n+p} \restriction p, \\ \\ (\mathbf{1}_0, \ldots, \mathbf{1}_{p-1}) & \text{otherwise,} \end{cases}$$

for each $n \in \omega$ and $s \in \omega^{<\omega}$ with $p = |s|$. It remains to show that $\psi^{-1}(H_0) = B$.

Claim 1. If $\psi(x) \in H_0$, then $x \in B$.

Proof of claim. Let n be such that $\psi(x) \in \langle A_n \rangle$, with $\mathcal{W}$ a group word such that $\psi(x) \in \mathcal{W}[A_n]$. For each p, if $s = \alpha_{n+p} \restriction p$, we have

$$\begin{aligned} \psi_n(x) \restriction p &= \psi(x)(n, s) \\ &\in \{\xi(n, s) : \xi \in \mathcal{W}[A_n]\} \\ &= \mathcal{W}[K_0 \restriction p] \\ &= \mathcal{W}[K_0] \restriction p \end{aligned}$$

and hence $\psi_n(x) \in \mathcal{W}[K_0]$, since the latter is closed. (As the continuous image of a compact set, $\mathcal{W}[K_0]$ is compact.) This implies that $\psi_n(x) \in \langle K_0 \rangle$ and, since ψ_n reduces $\langle B_n \rangle$ to $\langle K_0 \rangle$, we conclude that $x \in \langle B_n \rangle \subseteq B$.

Claim 2. If $x \in B$, then $\psi(x) \in H_0$.

Proof of claim. Suppose that $x \in B$, say $x \in B_{n_0}$. We first verify that $\psi(x) \in A_{n_0}$. Fix $n \geq n_0$ and $s \in \omega^{<\omega}$, with $p = |s|$. If $s \neq \alpha_{n+p} \restriction p$, then $\psi(x)(n,s) = (\mathbf{1}_0, \ldots, \mathbf{1}_{p-1}) \in K_0 \restriction p$, since $\bar{\mathbf{1}} \in K_0$. On the other hand, if $s = \alpha_{n+p} \restriction p$, then

$$\psi(x)(n,s) = \psi_n(x) \restriction p \in K_0 \restriction p,$$

since $\psi_n(B_{n_0}) \subseteq \psi_n(B_n) \subseteq K_0$, by assumption. As $n \geq n_0$ and s were arbitrary, we see that $\psi(x) \in A_{n_0}$.

It remains to see that $\psi(x) \in \langle \tilde{A} \rangle$. Naturally, it suffices to prove that $\psi(x) \in \tilde{A}$. We must show that, for all but finitely many n, s,

$$(\psi(x)(n,s))^* \leq s \tag{2.4}$$

Fix n, s with $p = |s|$. If $s \neq \alpha_{n+p} \restriction p$, then (2.4) follows, since $\psi(x)(n,s) = (\mathbf{1}_0, \ldots, \mathbf{1}_{p-1})$ and $(\mathbf{1}_0, \ldots, \mathbf{1}_{p-1})^* = 0^p$. If $s = \alpha_{n+p} \restriction p$ and $n + p \geq n_0$, then $(\psi_n(x))^* \leq \alpha_{n+p}$, since $x \in B_{n_0} \subseteq B_n$ and $n \leq n + p$. Hence

$$\begin{aligned}(\psi(x)(n,s))^* &= \psi_n(x)^* \restriction p \\ &\leq \alpha_{n+p} \restriction p \\ &= s\end{aligned}$$

and (2.4) holds for n, s. We see that (2.4) only fails when $n+|s| < n_0$ and $s = \alpha_{n+|s|} \restriction |s|$. There are only finitely many such pairs n, s.

We have shown that $\psi(x) \in \tilde{A}$ and hence $\psi(x) \in \tilde{A} \cap A_{n_0} \subseteq H_0$. This completes the proof. $\square$

2.3 Universal F_σ subgroups for K_σ

Theorem 2.9 gives a universal K_σ subgroup of Γ^ω whenever Γ^ω is locally compact. If Γ is arbitrary, we can still prove that there is an F_σ subgroup of Γ^ω which is universal for K_σ subgroups of Γ^ω (Theorem 2.16 below). We prove this result as a consequence of Theorem 2.17, which gives the result of Theorem 2.16 in the case that Γ is the isometry group of a Polish metric space. Theorem 2.16 will then follow via a theorem of Gao and Kechris [4] which states that every Polish group is isomorphic to the isometry group of a Polish metric space.

In Section 2.4, we will show that S_∞^ω does not have a universal K_σ subgroup, implying that the results in this section cannot, in general, be improved.

We will first prove a special case which will indicate the general methods used in the proof of Theorem 2.17.

2.3.1 The case of S_∞

Recall that S_∞ is the group of permutations of ω, regarded as a topological subspace of the Baire space. Hence there is a basis of clopen neighborhoods of the form

$$\mathcal{U}(u) = \{f \in S_\infty : u \subset f\}$$

where $u : \omega \to \omega$ is a finite partial injection. The group operation of S_∞ is composition. A compatible metric is $d(f, g) = \frac{1}{n+1} + \frac{1}{m+1}$, where n is least such that $f(n) \neq g(n)$ and

m is least such that $f^{-1}(m) \neq g^{-1}(m)$.

Note that S_∞ may be regarded as the isometry group of the discrete space ω, with the metric d such that $d(m,n) = 1 \iff m \neq n$.

Theorem 2.15. *There is an F_σ subgroup of S_∞ which is universal for K_σ subgroups of S_∞.*

Proof. It will be enough to show that S_∞^ω contains an F_σ subgroup which is universal for K_σ subgroups of S_∞, since S_∞^ω is isomorphic to a closed subgroup of S_∞. (For example, if $A_0, A_1, \ldots \subset \omega$ are disjoint infinite sets. Then S_∞^ω is isomorphic to the subgroup

$$\{f \in S_\infty : (\forall n)(f(A_n) = A_n)\}.)$$

We will indicate elements of S_∞^ω with bold letters, e.g. $\boldsymbol{f}, \boldsymbol{g}$. For $\boldsymbol{f}, \boldsymbol{g} \in S_\infty^\omega$, $\boldsymbol{fg}$ denotes the "product" of $\boldsymbol{f}$ and $\boldsymbol{g}$, i.e., $\boldsymbol{fg}(n) = \boldsymbol{f}(n) \circ \boldsymbol{g}(n)$, for each n.

We introduce some terminology/notation. Suppose that $u : \omega \to \omega$ is a finite partial injection. We say that u is *n-long* if $n \subseteq \mathrm{dom}(u)$ and $n \subseteq \mathrm{ran}(u)$. Also, if $u, v : \omega \to \omega$ are partial functions, then $v \circ u$ denotes the composite function defined on the largest domain that makes sense, namely $\{n : u(n) \in \mathrm{dom}(v)\}$.

Let $\omega_+^{<\omega} = \omega^{<\omega} \setminus \{\langle\emptyset\rangle\}$ and take $\langle\cdot,\cdot\rangle : \omega \times \omega_+^{<\omega} \leftrightarrow \omega$ to be a fixed bijection.

We fix a family $\{A_s : s \in \omega_+^{<\omega}\}$ of finite sets of finite partial injections on ω, such that

- If $|s| = 1$, then $A_s \ni \mathrm{id}_n$, for some $n > 0$.
- If $u \in A_s$, then $u^{-1} \in A_s$ also.
- For each $i \in \omega$, $A_{s^\frown i} \supseteq A_s$ and if $u, v \in A_s$, then $v \circ u \in A_{s^\frown i}$.

- If $A \supseteq A_s$ is such that $v \circ u \in A$, for each $u, v \in A_s$, and $u^{-1} \in A$, for each $u \in A$, then there exists i such that $A = A_{s^\frown i}$.

For the third property, we permit the composition $v \circ u$ to be the empty function. Also, the first and third properties together imply that each A_s contains id_n, for some $n > 0$.

Define an $G \subseteq S_\infty^\omega$ to be the set of $\boldsymbol{f} \in S_\infty^\omega$ such that

$$(\exists n > 0)(\forall m, s)(m, |s| \geq n \implies (\exists u \in A_{s\restriction n})(u \subset \boldsymbol{f}(\langle m, s\rangle))).$$

Notice that, because each A_s is finite, the innermost condition in the definition of G defines a clopen set. Hence G is F_σ.

Claim. G is a subgroup of S_∞^ω.

Proof of claim. It follows from the properties of the A_s that G contains $\mathbf{id} = (\mathrm{id}, \mathrm{id}, \ldots)$ and is closed under taking inverses.

Suppose that $\boldsymbol{f}, \boldsymbol{g} \in G$, witnessed by n as in the definition of G. Fix m, s with $m, |s| \geq n + 1$. Let $u, v \in A_{s\restriction n}$ with $u \subset \boldsymbol{f}(\langle m, s\rangle)$ and $v \subset \boldsymbol{g}(\langle m, s\rangle)$. Then $u \circ v \in A_{s\restriction(n+1)}$ and

$$u \circ v \subset \boldsymbol{f}(\langle m, s\rangle \circ \boldsymbol{g}(\langle m, s\rangle) = (\boldsymbol{fg})(\langle m, s\rangle).$$

We see that $\boldsymbol{fg} \in G$.

This shows that G is a subgroup and finishes the claim.

Fix a K_σ subgroup $H \subset S_\infty$. We show how to reduce H to G. Let $K_0 \subseteq K_1 \subseteq \ldots$ be compact sets such that $H = \bigcup_n K_n$, $\mathrm{id} \in K_0$, and, for each n and $f, g \in K_n$, $f \circ g \in K_{n+1}$ and $f^{-1}, g^{-1} \in K_n$.

Since the K_n are compact in S_∞, they are also compact in ω^ω. We therefore take increasing functions $h_n : \omega \to \omega$ such that, for each $f \in K_n$, $f(k) \leq h_n(k)$, for every

$k \in \omega$. Since $K_0 \subseteq K_1 \subseteq \ldots$, we may assume that $h_0(k) \leq h_1(k) \leq \ldots$, for each $k \in \omega$. For $m \geq n$, define $a_n^m = (h_m)^{mn}(m)$, where $(h)^k$ denotes h composed with itself, k times.

Observe that, if $u, v : \omega \to \omega$ are finite functions with u, v, u^{-1}, v^{-1} bounded by $(h_m)^m$ and u, v are a_n^m-long, then $u \circ v$ will be a_{n-1}^m-long.

We define finite sets $\mathcal{K}_n^m$ that "approximate" H. These will be such that $\mathcal{K}_n^m$ will only be defined for $n \leq m$, and each $\mathcal{K}_n^m$ will be a set of a_{m-n}^m-long finite injections on ω.

Let $\mathcal{K}_0^m$ be the (finite) set of a_m^m-long finite initial segments of members of K_0. Observe that $\mathcal{K}_0^m$ has the following properties.

1. $\{\mathcal{U}(u) : u \in \mathcal{K}_0^m\}$ covers K_0.
2. If $u \in \mathcal{K}_0^m$, then $u \subset f$ for some $f \in K_0$.
3. If $u \in \mathcal{K}_0^m$, then $u^{-1} \in \mathcal{K}_0^m$.
4. If $u \in \mathcal{K}_0^m$, then u, u^{-1} are bounded by h_0, in particular, they are bounded by $(h_m)^m$.
5. If $u \in \mathcal{K}_0^m$, then $u \in a_m^m$-long.

The third property follows since K_0 is closed under taking inverses and, if u is k-long, for some k, then so is u^{-1}.

Given $\mathcal{K}_n^m$, with $n < m$, let $\mathcal{K}_{n+1}^m$ the set of a_{m-n-1}^m-long initial segments of members of K_{n+1}, together with all $u \circ v$, for $u, v \in \mathcal{K}_n^m$.

Thus each $\mathcal{K}_n^m$ has the following properties.

1. $\{\mathcal{U}(u) : u \in \mathcal{K}_n^m\}$ covers K_n.

2. If $u \in \mathcal{K}_n^m$, then $u \subset f$ for some $f \in K_n$.

3. If $u \in \mathcal{K}_n^m$, then $u^{-1} \in \mathcal{K}_n^m$.

4. If $u, v \in \mathcal{K}_{n-1}^m$, then $u \circ v \in \mathcal{K}_n^m$.

5. If $u \in \mathcal{K}_n^m$, then u, u^{-1} are bounded by $(h_n \circ \ldots \circ h_0)$, in particular, they are bounded by $(h_m)^m$.

6. If $u \in \mathcal{K}_n^m$, then u is at least a_{m-n}^m-long.

Each of these properties is verified by induction. The first, third and fourth properties follow from the definition of $\mathcal{K}_n^m$.

The second property follows from the fact that, if $u \in \mathcal{K}_n^m$, then either u is an initial segment of some $f \in K_n$ or $u = w \circ v$, for some $w, v \in \mathcal{K}_{n-1}^m$. In the latter case, assuming that the second property above holds for $\mathcal{K}_{n-1}^m$, we have $g_1, g_2 \in K_{n-1}$ such that $v \subset g_1$ and $w \subset g_2$. Hence $w \circ v \subset g_2 \circ g_1 \in K_n$.

The fifth property follows since, if u, v are bounded by some function $h : \omega \to \omega$, then $v \circ u$ is bounded by $h \circ h$.

The sixth property holds automatically for each $u \in \mathcal{K}_n^m$ that is an a_{m-n}^m-long initial segment of some $f \in K_n$. If $u = w \circ v$, for some $w, v \in \mathcal{K}_{n-1}^m$, then by properties 5 and 6 for $\mathcal{K}_{n-1}^m$, we conclude that $w \circ v$ is at least a_{m-n}^m-long. (See the comment following the definition of a_n^m.)

We now make the following claim.

Claim. For each $n \in \omega$ and $f \in S_\infty$, $f \in K_n$ if, and only if, for each $m \geq n$, there exists $u \in \mathcal{K}_n^m$ with $u \subset f$.

Proof of claim. "Only if" follows from the fact that, for each $m \geq n$, $\{\mathcal{U}(u) : u \in \mathcal{K}_n^m\}$ covers K_n.

For the "if" part, suppose that $f \in S_\infty$ and $n \in \omega$ are such that, for each $m \geq n$, there exists $u_m \in \mathcal{K}_n^m$ with $u_m \subset f$. By the properties of the $\mathcal{K}_n^m$, we know that $\mathcal{U}(u_m) \cap K_n \neq \emptyset$. Take $f_m \supset u_m$ with $f_m \in K_n$. Then, because each u_m is a_n^m-long, f, f_m and f^{-1}, f_m^{-1} agree on an initial segment of length at least a_n^m. Note that $a_n^m \geq m \to \infty$ as $m \to \infty$. Hence $f_m \to f$ and so $f \in K_n$, because K_n is closed. This proves the claim.

We now define a reduction of H to G. Choose $\alpha_m \in \omega^\omega$ such that for each n, $A_{\alpha_m \restriction (n+1)} = \mathcal{K}_n^m$. Let $\varphi : S_\infty \to S_\infty^\omega$ be given by

$$\varphi(f)(\langle m, s\rangle) = \begin{cases} f & \text{if } s \subset \alpha_m, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

We want to see that $\varphi^{-1}(G) = H$. Suppose $f \in H$, say $f \in K_n$. Write $\boldsymbol{g} = \varphi(f)$. By the claim above, for each $m \geq n$, there exists $u \in \mathcal{K}_n^m = A_{\alpha_m \restriction (n+1)}$ such that $u \subset f$. Hence, for $s \subset \alpha_m$ with $m, |s| \geq (n+1)$, we have $\boldsymbol{g}(\langle m, s\rangle) \supset u$, for some $u \in A_s$. If $s \not\subset \alpha_m$, then $\boldsymbol{g}(\langle m, s\rangle) = \mathrm{id}$. Again, however, there is $u \in A_s$ such that $u \subset g_{\langle m,s\rangle}$, since A_s always contains id_k, for some $k > 0$. We see that $\boldsymbol{g} \in G$.

If $\boldsymbol{g} = \varphi(f) \in G$, then there exists $n > 0$, such that for each $m \geq n$, there is $u \in A_{\alpha_m \restriction n} = \mathcal{K}_{n-1}^m$ with $u \subset \boldsymbol{g}(\langle m, s\rangle) = f$. Thus, by the second claim $f \in K_{n-1}$. □

2.3.2 Arbitrary countable powers

The following theorem is based on Theorem 2.15 and is our most general result of this type.

Theorem 2.16. *If Γ is a Polish group, then there is an F_σ subgroup of Γ^ω which is universal for K_σ subgroups of Γ^ω.*

As mentioned above, we will show in the next section that this result cannot be improved to give a universal K_σ subgroup of an arbitrary countable power. In particular, we show that S_∞^ω does not have a universal K_σ subgroup. We do not yet know if there is a larger class of Polish groups Γ such that Γ^ω has no universal K_σ subgroup.

We will obtain Theorem 2.16 as a consequence of the following.

Theorem 2.17. *For any Polish space X with compatible metric d, there is an F_σ subgroup of $\mathrm{Iso}(X,d)^\omega$ which is universal for K_σ subgroups of $\mathrm{Iso}(X,d)$.*

Before proving Theorem 2.17, we show that it implies Theorem 2.16.

Proof of Theorem 2.16. Let Γ be any Polish group. By Theorem 3.1(i) in [4], there is a Polish space X, with metric d such that $\Gamma^\omega \cong \mathrm{Iso}(X,d)$. Theorem 2.17 implies that there is an F_σ subgroup of $\mathrm{Iso}(X,d)^\omega$ which is universal for K_σ subgroups of $\mathrm{Iso}(X,d)$. Bearing in mind

$$\mathrm{Iso}(X,d) \cong \Gamma^\omega \cong (\Gamma^\omega)^\omega \cong \mathrm{Iso}(X,d)^\omega,$$

it follows that Γ^ω itself has an F_σ subgroup which is universal for K_σ subgroups of Γ^ω. $\square$

It is worth mentioning the following corollary of Theorem 2.16. Recall that a Polish group $\mathbb{G}$ is *universal* if every Polish group is isomorphic to a closed subgroup of $\mathbb{G}$.

Corollary 2.18. *If $\mathbb{G}$ is a universal Polish group, then there is an F_σ subgroup $H_0 \subseteq \mathbb{G}$ such that, for any K_σ subgroup H of a Polish group Γ, there is a continuous injective group homomorphism $\varphi : \Gamma \to \mathbb{G}$ such that $H = \varphi^{-1}(H_0)$.*

Proof. Let $\mathbb{G}$ be a universal Polish group and $\tilde{H}_0 \subseteq \mathbb{G}^\omega$ an F_σ subgroup which universal for K_σ subgroups of $\mathbb{G}^\omega$. By the universality of $\mathbb{G}$, we may identify $\tilde{H}_0$ with an F_σ subgroup $H_0 \subseteq \mathbb{G}$. Observe that, since $\mathbb{G}$ itself is isomorphic to a closed subgroup of $\mathbb{G}^\omega$, H_0 is universal for K_σ subgroups of $\mathbb{G}$.

Fix any Polish group Γ and $H \subseteq \Gamma$, a K_σ subgroup. Let $\pi : \Gamma \to \mathbb{G}$ be an isomorphic embedding. Note that $\pi(H)$ is a K_σ subgroup of $\mathbb{G}$ and hence there is a continuous homomorpism $\varphi : \mathbb{G} \to \mathbb{G}$ such that $\varphi^{-1}(H_0) = \pi(H)$. Inspecting the proof of Theorem 2.17 below, it will become apparent that φ can be chosen to be injective. Since π is injective, it follows that $(\varphi \circ \pi)^{-1}(H_0) = H$. $\square$

It is a theorem of V. V. Uspenskiĭ (Theorem 9.18 in [9]) that there are universal Polish groups. In particular, the homeomorphism group of the Hilbert cube is a universal Polish group.

2.3.3 Preliminary notions

Before giving the proof of Theorem 2.17, we recall some basic facts about isometry groups and introduce terminology we will use in the proof of Theorem 2.17.

Topology on $\mathrm{Iso}(X, d)$

Fix a Polish space X and suppose that d is a compatible complete metric on X. Throughout, we will assume for simplicity that X is infinite. In the case that $|X| = n$, there is a subgroup H (depending on d) of S_n such that $\mathrm{Iso}(X, d) \cong H$. The statement of Theorem 2.17 for (X, d) then follows from Theorem 2.9, since S_n is finite and hence compact.

Recall that the Polish topology on $\mathrm{Iso}(X, d)$ is that of pointwise convergence. This is the weakest topology making all point evaluation maps continuous. By properties of isometries, the topology on $\mathrm{Iso}(X, d)$ is equivalent to the topology of pointwise convergence with respect to any fixed countable dense set. Thus, for each countable dense set $Q \subseteq X$, there is topological basis for $\mathrm{Iso}(X, d)$ consisting of open sets of the form

$$\mathcal{U}(u, \varepsilon) = \{f \in \mathrm{Iso}(X, d) : (\forall p \in \mathrm{dom}(u))(d(f(p), u(p)) < \varepsilon)\},$$

where $u : Q \to Q$ is a finite partial function.

We let $\overline{\mathcal{U}}(u, \varepsilon)$ denote the closure of $\mathcal{U}(u, \varepsilon)$. Note that

$$\overline{\mathcal{U}}(u, \varepsilon) \subseteq \{f \in \mathrm{Iso}(X, d) : (\forall p \in \mathrm{dom}(u))(d(f(p), u(p)) \leq \varepsilon)\}.$$

For the rest of this section, fix a countable dense set $Q \subseteq X$ and a bijection $\# : Q \longleftrightarrow \omega$. Also, fix a compatible complete metric d on X and and simply write $\mathrm{Iso}(X)$ for $\mathrm{Iso}(X, d)$.

Least ε-approximations

We introduce a notion of ε-approximations for isometries on X. These will enable us to work with isometries much as we would work with permutations of a discrete set. For simplicity, we will assume at this stage that X has no isolated points.

Given $f \in \mathrm{Iso}(X)$, and a bijection $\alpha : Q \longleftrightarrow Q$, we say that α is an *ε-approximation* of f if, and only if, for each $p \in Q$, $d(f(p), \alpha(p)) < \varepsilon$. It follows that if $u \subset \alpha$ is a finite subfunction, then $f \in \mathcal{U}(u, \varepsilon)$.

We describe a "minimal" (with respect to #) way of defining such an α. For $f \in \mathrm{Iso}(X)$ and fixed $\varepsilon > 0$, we will construct an ε-approximation of f as a union of finite injections $u_n : Q \to Q$.

We begin with u_0. Let $p_0 \in Q$ have $\#p_0 = 0$ and let $q_0 \in Q$ be #-least such that $d(f(p_0), q_0) < \varepsilon$. Let q_0' be #-least with $q_0' \neq q_0$ and $p_0' \neq p_0$ #-least such that $d(f^{-1}(q_0'), p_0') < \varepsilon$. (Hence $d(q_0', f(p_0')) < \varepsilon$.) Note that such a $p_0' \neq p_0$ exists since $f^{-1}(q_0)$ is not isolated and hence there are infinitely many $a \in Q$ such that $d(f^{-1}(q_0'), a) \leq \varepsilon$.[2]

Take $u_0 = \{(p_0, q_0), (p_0', q_0')\}$.

Now suppose that the finite injection $u_n : Q \to Q$ is given. We show how to define u_{n+1}. Let p be the #-least element of $Q \setminus \mathrm{dom}(u_n)$ and let $q \notin \mathrm{ran}(u_n)$ be #-least with $d(f(p), q) < \varepsilon$. Again such a q will always exist because $f(p)$ is not isolated in X. Let q' be the #-least element of $Q \setminus \mathrm{ran}(u_n) \cup \{q\}$ and let $p' \notin \mathrm{dom}(u_n) \cup \{p\}$ be #-least with $d(f^{-1}(q'), p') < \varepsilon$. Once again, we us the fact that X is perfect. Now take $u_{n+1} = u_n \cup \{(p, q), (p', q')\}$.

We call $\alpha = \bigcup_n u_n$ the *least ε-approximation* of f. It follows from the construction above that α is a permutation of Q and for each $p \in Q$, $d(f(p), \alpha(p)) < \varepsilon$.

We call u_n as above the *nth partial ε-approximation* of f.

In the case that X has isolated points, we modify the construction of the least ε-approximation of $f \in \mathrm{Iso}(X)$ as follows: let $\tilde{Q}$ be the (necessarily countable) set of isolated points of X. As Q is dense, $Q \supseteq \tilde{Q}$. As an isometry, f is also a homeomorphism and hence permutes $\tilde{Q}$. Thus we can carry out the construction above in the closed subspace $X \setminus \tilde{Q}$ and then take the union of $\alpha : Q \setminus \tilde{Q} \longleftrightarrow Q \setminus \tilde{Q}$ (obtained as above) with $f \restriction \tilde{Q}$. This will be the least ε-approximation of f.

We give some properties of ε-approximations.

[2] If $f^{-1}(q_0')$ is isolated, the risk is that $f^{-1}(q_0') = p_0$ and there are no $a \in Q \setminus \{p_0\}$ with $d(f^{-1}(q_0'), a) \leq \varepsilon$.

1. If $\alpha, \beta : Q \longleftrightarrow Q$ are ε-approximations of f, g respectively, then $\beta \circ \alpha$ is a 2ε-approximation of $g \circ f$.

2. If α is an ε-approximation of f, then α^{-1} is an ε-approximation of f^{-1}.

3. The least ε-approximation of id is id $\restriction Q$.

Compact subsets of $\mathrm{Iso}(X)$

Let $K \subseteq \mathrm{Iso}(X)$ be a compact set. For $\varepsilon > 0$, we will consider the set of least ε-approximations of members of K.

Lemma 2.19. *For fixed $\varepsilon > 0$, there exists an increasing function $\gamma : \omega \to \omega$ such that for each $f \in K$, if $\alpha : Q \to Q$ is the least ε-approximation of f, then $\#\alpha(p), \#\alpha^{-1}(p) \leq \gamma(\#p)$, for each $p \in Q$. In this case, we say α is* bounded *by γ.*

Proof. As above, let $\tilde{Q} \subseteq Q$ be the set of isolated points of X. Observe that

$$\{f \restriction \tilde{Q} : f \in K\}$$

is a compact subset of $\mathrm{Sym}(\tilde{Q}) \cong S_\infty$. For $p \in \tilde{Q}$, we let $\gamma(\#p) = \max\{\#f(q)+\#f^{-1}(q) : f \in K \ \& \ \#q \leq \#p\}$. We may therefore ignore any isolated points of X and prove the lemma in the case that X is perfect, since the above remarks indicate how to define $\gamma(\#p)$ for $p \in \tilde{Q}$.

Let us therefore assume that X is perfect. Each $f \in \mathrm{Iso}(X)$ determines its sequence of partial ε-approximations $(u_n)_{n \in \omega}$. To prove the lemma, it will be enough to show that, as f ranges over K, there are only finitely many possibilities for u_n, i.e., for each n the set

$$\mathcal{S}_n = \{u : (\exists f \in K)(u \text{ is the } n\text{th partial } \varepsilon\text{-approximation of } f)\}$$

is finite. We prove this by induction on n.

For $n = 0$, recall the definition of u_0: we let $p_0 \in Q$ have $\#p_0 = 0$. By the compactness of K, there exists a finite set $F \subseteq Q$ such that for each $f \in K$, there is some $b \in F$ such that $d(f(p_0), b) < \varepsilon$. In particular, q_0 as in the definition of u_0 must have $\#q_0 \leq \max\{\#b : b \in F\}$. Thus the set of $q \in Q$ which occur as q_0 in the definition of u_0, for some $f \in K$, is a finite set. Suppose now that q_0 has been specified. We let q_0' be the #-least element of $Q \setminus \{q_0\}$. As with q_0, for f varying in K, only finitely many different values will arise for p_0' as the #-least element of $Q \setminus \{p_0\}$ with $d(f^{-1}(q_0'), p_0') < \varepsilon$.

Suppose we are given that $\mathcal{S}_n$ is finite. Fix one of the finitely many $u_n \in \mathcal{S}_n$. Let $p \notin \mathrm{dom}(u_n)$ be #-least. As above, the compactness of K implies that there are only finitely many $q \in Q \setminus \mathrm{ran}(u_n)$ as in the construction of u_{n+1}, for some $f \in K$. Again, given q, the choice of q' is determined and there are only finitely many possible p' for a given q'. Thus, having fixed u_n, there are only finitely many possible u_{n+1}, as f ranges over K. This implies that $\mathcal{S}_{n+1}$ is finite. □

Combinatorics of finite injections on Q

For a finite injection $u : Q \to Q$, we say that u is *m-long* if, and only if, for each $p \in Q$, if $\#p \leq m$, then $p \in \mathrm{dom}(u), \mathrm{ran}(u)$. (Note that for any $f \in \mathrm{Iso}(X)$ and $\varepsilon > 0$, the nth partial ε-approximation of f is at least n-long.)

Suppose that $u, v : Q \to Q$ are finite injections and $\gamma : \omega \to \omega$ is increasing such that, for each $p \in Q$,

$$\#u(p), \#v(p), \#u^{-1}(p), \#v^{-1}(p) \leq \gamma(\#p),$$

whenever the quantities on the left are defined. Again, we say u, v are *bounded* by γ.

Lemma 2.20. *If u, v, γ are as above with u, v bounded by γ and u, v are $\gamma(m)$-long for some m, then $v \circ u$ will be at least m-long. (The domain of $v \circ u$ is $\{p \in Q : u(p) \in \mathrm{dom}(v)\}$.)*

Proof. Fix $p \in Q$ with $\#p \leq m$. We want to see that $p \in \mathrm{dom}(v \circ u), \mathrm{ran}(v \circ u)$. The first statement follows from the fact that, since u is bounded by γ, $\#u(p) \leq \gamma(\#p)$ and hence $u(p) \in \mathrm{dom}(v)$, since v is $\gamma(m)$-long. The second statement follows by applying the same reasoning to $u^{-1} \circ v^{-1}$. $\square$

K_σ **subgroups of** $\mathrm{Iso}(X)$

Let $H = \bigcup_n K_n$ be a subgroup of $\mathrm{Iso}(X)$, with each K_n compact and $\mathrm{id} \in K_0$. By the continuity of the group operations, we may assume that, for each n, if $f, g \in K_n$, then $f^{-1}, g^{-1} \in K_n$ and $g \circ f \in K_{n+1}$. For each n, k, let $\gamma_{n,k}$ be as in Lemma 2.19 such that if $f \in K_n$ and α is the least $\frac{1}{k}$-approximation of f, then α, α^{-1} are bounded by $\gamma_{n,k}$. With no loss of generality, $\gamma_{0,k} \leq \gamma_{1,k} \leq \ldots$, for each k. Now let

$$\delta_{n,k} = \overbrace{\gamma_{n,k} \circ \ldots \circ \gamma_{n,k}}^{2^n \text{times}}.$$

For each n, k and $i \geq n$, let

$$a_i^{(n,k)} = \overbrace{\delta_{n,k} \circ \ldots \circ \delta_{n,k}(n)}^{i \text{ times}}.$$

Observe that, by Lemma 2.20, if u, v are bounded by $\delta_{n,k}$ and are $a_i^{(n,k)}$-long, then $v \circ u$ will be $a_{i-1}^{(n,k)}$-long. Also note that $a_i^{(n,k)} \to \infty$ as $n, k, i \to \infty$.

For $i \leq n$, we will define finite sets $\mathcal{B}_{n,k}^i$ of finite partial injections on Q that will "approximate" the K_n.

Let $\mathcal{B}^0_{n,k}$ be the set of all u, u^{-1}, such that u is the $a^{(n,k)}_n$th partial $\frac{1}{k}$-approximation of some $f \in K_0$.

Given $i < n$ and $\mathcal{B}^i_{n,k}$, we define $\mathcal{B}^{i+1}_{n,k}$ to be the set of all $v \circ u$, where $u, v \in \mathcal{B}^i_{n,k}$, together with the set of all w, w^{-1} for which there exists $f \in K_{i+1}$ such that w is the $a^{(n,k)}_{n-i-1}$-st partial $\frac{1}{k}$-approximation of f.

The following properties are consequences of the definition of $\mathcal{B}^i_{n,k}$.

1. $\mathrm{id}_m \in \mathcal{B}^i_{n,k}$, where $m = a^{(n,k)}_{n-i}$.

2. $u \in \mathcal{B}^i_{n,k} \implies u^{-1} \in \mathcal{B}^i_{n,k}$.

3. $i < n$ & $u, v \in \mathcal{B}^i_{n,k} \implies v \circ u \in \mathcal{B}^{i+1}_{n,k}$.

In the first item, id_m deonotes id restricted to the set of $p \in Q$ with $\#p \leq m$. Note that the first property follows from the fact that each K_n contains id and, for every ε, the least ε-approximation of id is id $\restriction$ Q.

Lemma 2.21. *If $u \in \mathcal{B}^i_{n,k}$, then u is bounded by $\delta_{n,k}$.*

Proof. Each $u \in \mathcal{B}^i_{n,k}$ is obtained as a composite of at most i partial $\frac{1}{k}$-approximations of members of K_j, with $j \leq i$. In particular, since the $\gamma_{j,k}$ are increasing and each $\gamma_{j,k} \leq \gamma_{j+1,k}$, we know that each u is bounded by $\gamma^i_{i,k}$ (the composite of $\gamma_{i,k}$ with itself i times). The lemma follows since $\gamma^i_{i,k} \leq \delta_{n,k}$, if $i \leq n$. $\square$

Lemma 2.22. *For each n, k, i, if $u \in \mathcal{B}^i_{n,k}$, then u is at least $a^{(n,k)}_{n-i}$-long.*

Proof. This follows by induction. The $i = 0$ case follows from the definition of $\mathcal{B}^0_{n,k}$. If $u \in \mathcal{B}^{i+1}_{n,k}$, then either u is the $a^{(n,k)}_{n-i-1}$-st partial ε-approximation of an $f \in K_{n+1}$ (in which case we are done) or $u = w \circ v$, for some $w, v \in \mathcal{B}^i_{n,k}$. In this case, the claim still holds:

since w, v are bounded by $\delta_{n,k}$ (Lemma 2.21) and, by assumption, are $a_i^{(n,k)}$-long, we may conclude that $u = w \circ v$ is $a_{n-i-1}^{(n,k)}$-long, by the observation following the definition of $a_i^{(n,k)}$. $\square$

Lemma 2.23. *For each k, n and $i \leq n$, if $u \in \mathcal{B}_{n,k}^i$, then $\mathcal{U}(u, 2^i/k) \cap K_i \neq \emptyset$.*

Proof. Again, the proof is by induction. The $i = 0$ case is a consequence of the choice of $\mathcal{B}_{n,k}^0$ as a set of partial $\frac{1}{k}$-approximations of elements of K_0. Suppose that the lemma holds for $i < n$. Let $u \in \mathcal{B}_{n,k}^{i+1}$. If u is a partial $\frac{1}{k}$-approximation of an element of K_{i+1}, then there is nothing to prove. On the other hand, if $u = w \circ v$, for some $w, v \in \mathcal{B}_{n,k}^i$, let $f \in \mathcal{U}(v, 2^i/k) \cap K_i$ and $g \in \mathcal{U}(w, 2^i/k) \cap K_i$ be as given by the induction hypothesis. Note that $g \circ f \in K_{i+1}$ and thus it will suffice to show that $g \circ f \in \mathcal{U}(u, 2^{i+1}/k)$. Indeed, fix $p \in \mathrm{dom}(u)$ and observe the following:

$$\begin{aligned}
d((g \circ f)(p), (w \circ v)(p)) &\leq d(g(f(p)), g(v(p))) + d(g(v(p)), w(v(p))) \\
&= d(f(p), v(p)) + d(g(v(p)), w(v(p))) \\
&< \frac{2^i}{k} + \frac{2^i}{k} \\
&= \frac{2^{i+1}}{k}.
\end{aligned}$$

Equality in the second line follows from the fact that g is an isometry. Since $p \in \mathrm{dom}(u)$ was arbitrary, this shows that $g \circ f \in \mathcal{U}(u, 2^{i+1}/k)$. $\square$

Lemma 2.24. *For each i and $f \in \mathrm{Iso}(X)$, $f \in K_i$ if, and only if, for each n, k with $n \geq i$, there exists $u \in \mathcal{B}_{n,k}^i$ such that $f \in \overline{\mathcal{U}}(u, 2^i/k)$.*

Proof. The 'only if' half of the statement follows from the fact that each $\mathcal{B}_{n,k}^i$ contains a partial $\frac{1}{k}$-approximation of f.

For the 'if' part, let $u_{n,k} \in \mathcal{B}^i_{n,k}$ be such that $f \in \overline{\mathcal{U}}(u_{n,k}, 2^i/k)$. Let $v_k = u_{k,k}$. By Lemma 2.23, there exists $f_k \in K_i$ such that $f_k \in \mathcal{U}(v_k, 2^i/k)$. We show that $f_k \to f$ pointwise on Q. Fix $p \in Q$. Since each v_k is $a^{(k,k)}_{k-i}$-long (Lemma 2.22) there exists K_0 such that for each $k \geq K_0$, $p \in \mathrm{dom}(v_k)$. For each $k \geq K_0$, we have

$$\begin{aligned} d(f_k(p), f(p)) &\leq d(f_k(p), v_k(p)) + d(v_k(p), f(p)) \\ &\leq \frac{2^i}{k} + \frac{2^i}{k} \\ &\to 0, \end{aligned}$$

as $k \to \infty$. Thus $f_k \to f$ and we conclude that $f \in K_i$, since K_i is closed. □

2.3.4 The proof

We are now equipped to prove Theorem 2.17.

Proof of Theorem 2.17. We will define an F_σ subgroup of $\mathrm{Iso}(X)^\omega$ which is universal for K_σ subgroups of $\mathrm{Iso}(X)$. Let $\omega_+^{<\omega} = \omega^{<\omega} \setminus \{\langle\emptyset\rangle\}$. Fix a family $\{A_s : s \in \omega_+^{<\omega}\}$, where each A_s is a finite set of finite injections on Q and, for each s, we have

- If $|s| = 1$, then $A_s \ni \mathrm{id}_n$, for some $n > 0$.
- If $u \in A_s$, then $u^{-1} \in A_s$.
- For each $i \in \omega$, $A_{s^\frown i} \supseteq A_s$ and if $u, v \in A_s$, then $v \circ u \in A_{s^\frown i}$.
- If $A \supseteq A_s$ is finite, satisfies the second property above and is such that, for each $u, v \in A_s$, $v \circ u \in A$, then there exists $i \in \omega$ such that $A = A_{s^\frown i}$.

The first and third items together imply that each A_s contains id_n, for some $n > 0$.

Let $\langle \cdot, \cdot, \cdot \rangle : (\omega \times \omega \times \omega_+^{<\omega}) \longleftrightarrow \omega$ be a fixed bijection. Define $H_0 \subseteq \mathrm{Iso}(X)^\omega$ as follows: for $\mathcal{F} \in \mathrm{Iso}(X)^\omega$, let $\mathcal{F} \in H_0$ if, and only if, there exists $n > 0$ such that, for each $m, k \in \omega$ and $s \in \omega_+^{<\omega}$ with $m, |s| \geq n$

$$(\exists u \in A_{s \restriction n})(\mathcal{F}(\langle m, k, s \rangle) \in \overline{\mathcal{U}}(u, 2^{n-1}/k)). \tag{2.5}$$

Observe that H_0 is F_σ, since the formula above defines a finite union of closed sets, i.e., a closed set. Our first step is to check that H_0 is a subgroup.

Claim. H_0 is a subgroup.

Proof of claim. It follows from the properties of the A_s that H_0 contains $\overline{\mathrm{id}}$ and is closed under taking inverses. Suppose that $\mathcal{F}, \mathcal{G} \in H_0$. Let n be as in the definition of H_0, witnessing the membership of $\mathcal{F}$ and $\mathcal{G}$. (Note that we can assume that the same n witnesses the membership of both $\mathcal{F}$ and $\mathcal{G}$, by taking the maximum of their respective n's.) Fix m, k, s with $m, |s| \geq n+1$. Write $f = \mathcal{F}(\langle m, k, s \rangle)$ and $g = \mathcal{G}(\langle m, k, s \rangle)$. Let $u, v \in A_{s \restriction n}$ be such that $f \in \overline{\mathcal{U}}(u, 2^{n-1}/k)$ and $g \in \overline{\mathcal{U}}(v, 2^{n-1}/k)$. Fix $p \in \mathrm{dom}(v \circ u)$ and observe the following:

$$\begin{aligned} d(g(f(p)), v(u(p))) &\leq d(g(f(p)), g(u(p))) + d(g(u(p)), v(u(p))) \\ &\leq 2^{n-1}/k + 2^{n-1}/k \\ &= 2^n/k. \end{aligned}$$

Since p was arbitrary, $g \circ f \in \overline{\mathcal{U}}(v \circ u, 2^n/k)$. Since m, k, s were arbitrary, we conclude that $\mathcal{G} \cdot \mathcal{F} \in H_0$, witnessed by $n+1$. This proves the claim.

Claim. H_0 is universal for K_σ subgroups of $\mathrm{Iso}(X)$.

Proof of claim. Let $H = \bigcup_n K_n$ be an arbitrary K_σ subgroup of $\mathrm{Iso}(X)$. We may assume that $\mathrm{id} \in K_0 \subseteq K_1 \subseteq \ldots$, that each K_n contains the inverses of its members and if $f, g \in K_n$, then $g \circ f \in K_{n+1}$. For each n, m, k, let $\mathcal{B}^n_{m,k}$ be defined for the compact sets K_n, as in the paragraph preceeding Lemma 2.21 above. Comparing the three properties of the $\mathcal{B}^n_{m,k}$ enumerated there with the properties of the A_s, we recognize that, for each pair m, k, there exists $\xi_{m,k} \in \omega^\omega$ such that

$$\mathcal{B}^n_{m,k} = A_{\xi_{m,k} \restriction (n+1)},$$

for each n. We can now define a continuous homomorphism reducing H to H_0. Define $\varphi : \mathrm{Iso}(X) \to \mathrm{Iso}(X)^\omega$ by letting

$$\varphi(f)(\langle m, k, s\rangle) = \begin{cases} f & \text{if } s \subset \xi_{m,k}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

for each triple m, k, s.

First of all, suppose $f \in H$. Say $f \in K_n$. To check that $\varphi(f) \in H_0$, fix m, k, s with $m, |s| \geq n+1$. Let $g = \varphi(f)(\langle m, k, s\rangle)$. If $s \not\subset \xi_{m,k}$, then $g = \mathrm{id}$ and statement (2.5) in the definition of H_0 holds for m, k, s. Suppose now that $s \subset \xi_{m,k}$. In this case, $g = f$. Since $A_{s\restriction(n+1)} = \mathcal{B}^n_{m,k}$, Lemma 2.24 above implies that there exists $u \in A_{s\restriction(n+1)}$ such that $g = f \in \overline{\mathcal{U}}(u, 2^n/k)$. Again, we see that (2.5) holds. Thus $\varphi(f) \in H_0$, witnessed by $n+1$.

Now assume that $\varphi(f) \in H_0$, witnessed by $n > 0$. We will see that $f \in K_{n-1}$. For each m, k and $s \subset \xi_{m,k}$, $\varphi(f)(\langle m, k, s\rangle) = f$ and if $m, |s| \geq n$, there exists $u \in A_{s\restriction n} = \mathcal{B}^{n-1}_{m,k}$ such that $f \in \overline{\mathcal{U}}(u, 2^{n-1}/k)$. Thus Lemma 2.24 implies that $f \in K_{n-1}$.

This completes the proof. $\square$

2.4 The example of S_∞

In this section we prove that there is a countable power with no universal K_σ subgroup. In particular, we show that S_∞^ω has no universal K_σ subgroup. This suggests that Theorem 2.9 cannot readily be expanded to a larger class of Polish groups. In some sense, the example of S_∞ also serves as a complement to Theorem 2.16, again suggesting that this may be a "best possible" result.

We state the main result of this section:

Theorem 2.25. *Their is no K_σ subgroup of S_∞^ω which is universal for compactly generated subgroups of S_∞^ω.*

This theorem shows (in a strong way) that S_∞^ω has neither universal compactly generated nor universal K_σ subgroups. Since S_∞^ω embeds in S_∞ and *vice versa*, it will be enough to prove Theorem 2.25 for S_∞.

Recall that the topology on S_∞ is generated by the basic clopen sets

$$\mathcal{U}(s) = \{s \in S_\infty : s \subset f\},$$

where $s : \omega \to \omega$ is a finite injection. Hence the sets $\mathcal{U}(\mathrm{id} \restriction n)$ form a neighborhood basis at the identity. Becuase we will refer to these open sets several times in what follows, we write $\mathcal{U}_n$ for $\mathcal{U}(\mathrm{id} \restriction n)$.

The fundamental elements of S_∞ are cycles. We use the notation $[a_1, \dots, a_n]$ for n-cycles and $[\dots, a_{-1}, a_0, a_1, \dots]$ for ∞-cycles. For $\pi \in S_\infty$, we let

$$\mathrm{supp}(\pi) = \{n : \pi(n) \neq n\} = \{n : \pi^{-1}(n) \neq n\}.$$

For any $f \in \omega^\omega$ (viewed as a function $\omega \to \omega$) we write f^p for the p-fold composite of f with itself, e.g. $f^2 = f \circ f$. We will use this notation both for permutations of ω as well as arbitrary functions on ω.

For each $\alpha \in \omega^\omega$, define

$$K_\alpha = \{f \in S_\infty : f, f^{-1} \leq \alpha\},$$

where "$f \leq \alpha$" signifies that, for each $n \in \omega$, $f(n) \leq \alpha(n)$. Note that each K_α is compact in S_∞ and that every compact subset of S_∞ is contained in some K_α. Suppose that $H = \bigcup_n K_n$ is a K_σ subset of S_∞, with each K_n compact. To show that H is not universal for compactly generated subgroups of S_∞, it will suffice to find a compact set $K \subseteq S_\infty$ such that no homomorphic image of K is contained in H. For this, it is enough to assume that each K_n has the form K_{β_n}, for some $\beta_n \in \omega^\omega$. We therefore show:

Theorem 2.26. *Given $\beta_0, \beta_1, \ldots \in \omega^\omega$, there exists $\alpha \in \omega^\omega$ such that, for each continuous injective group homomorphism $\Phi : S_\infty \to S_\infty$, we have $\Phi(K_\alpha) \nsubseteq \bigcup_n K_{\beta_n}$.*

We require a few lemmas.

Lemma 2.27. *If Φ is a continuous endomorphism of S_∞ and* $\ker(\Phi) \neq S_\infty$*, then Φ is injective.*

Proof. Since Φ is continuous, $\ker(\Phi)$ is a closed normal subgroup of S_∞. On the other hand, it is well-known that the only normal subgroups of S_∞ are $\{\mathrm{id}\}$, the infinite alternating group, the group of finite support permutations and S_∞ itself. Of these, only $\{\mathrm{id}\}$ and S_∞ are closed. $\square$

Noting that every K_σ subgroup of S_∞ is a proper subgroup, Lemma 2.27 implies that a group-homomorphism reduction between K_σ subgroups of S_∞ must be injective. (This follows from the fact that, if $A = \varphi^{-1}(B)$, then $\ker \varphi \subseteq A$.)

Lemma 2.28. *Suppose that $\alpha \in \omega^\omega$ is such that $(\forall n)(\alpha(n) \geq n)$. If $f \in K_\alpha$ and $s \subset f$ is a finite injection, then there is a finite support permutation $\pi \in K_\alpha$ such that $s \subset \pi$.*

Proof. Let $\mathcal{S}$ be the set of cycles $\sigma \subset f$ such that $\operatorname{supp}(\sigma)$ intersects the domain or range of s. Since s is a finite function, $\mathcal{S}$ is a finite set of disjoint cycles. For each ∞-cycle $\tau \in \mathcal{S}$, we will define a finite cycle $\tau^* \in K_\alpha$ such that τ^* agrees with τ on $\operatorname{dom}(s) \cup \operatorname{ran}(s)$. Write τ as

$$[\ldots a_{-1}, a_0, a_1, \ldots].$$

Let $n_0, n_1 \in \mathbb{Z}$ be such that $n_0 \leq n_1$ and if $a_i \in \operatorname{dom}(s) \cup \operatorname{ran}(s)$, for some i, then $n_0 \leq i \leq n_1$. By taking n_1 large enough, we may assume that $a_{n_1} > a_{n_0}$. Let $m \leq n_0$ be large enough that $a_m < a_{n_1}$ and $a_{m-1} > a_{n_1}$. (Note that we have strict inequalities since τ is an ∞-cycle and hence all a_i are distinct.) Define

$$\tau^* = [a_m, \ldots, a_{n_0}, \ldots, a_{n_1}].$$

We will verify that $\tau^* \in K_\alpha$, i.e., $\tau^*, (\tau^*)^{-1}$ are both bounded by α. Since $\tau \subset f \in K_\alpha$, we know that τ, τ^{-1} are bounded by α. Hence we need only check that $a_{n_1} \leq \alpha(a_m)$ and $a_m \leq \alpha(a_{n_1})$, since τ^* agrees with τ, except at a_{n_1}. That $a_{n_1} \leq \alpha(a_m)$ follows from

$$a_{n_1} < a_{m-1} = \tau^{-1}(a_m) \leq \alpha(a_m).$$

(We are using the fact that $\tau^{-1} \leq \alpha$.) On the other hand, $a_m \leq \alpha(a_{n_1})$ follows from the fact that

$$a_m < a_{n_1} \leq \alpha(a_{n_1}),$$

by our assumption that $(\forall n)(\alpha(n) \geq n)$.

We may now define the desired π as in the statement of the lemma: let π be the product of all finite cycles in $\mathcal{S}$ together with all τ^*, for ∞-cycles $\tau \in \mathcal{S}$. $\square$

Lemma 2.29. *If Φ is a continuous endomorphism of S_∞ and $\alpha, \beta_m \in \omega^\omega$ are such that $(\forall n)(\alpha(n) \geq n)$, $\Phi(K_\alpha) \subseteq \bigcup_n K_{\beta_m}$ and $\{\beta_m : m \in \omega\}$ is closed under composition, then there exist $n, m \in \omega$ such that*

$$\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}.$$

Proof. The following claim is the core of the proof.

Claim. There exists a finite support permutation $\pi \in K_\alpha$ and $m \in \omega$ such that $\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq K_{\beta_m}$.

Proof of claim. Let C be the compact set $\Phi(K_\alpha)$. Applying the Baire Category Theorem to C, it follows that exists $m \in \omega$ such that $K_{\beta_m} \cap C$ is non-meager relative to C. As K_{β_m} is closed, this implies that there exists a nonempty open set $\mathcal{V} \subseteq S_\infty$ such that $\mathcal{V} \cap C \subseteq K_{\beta_m}$. Let $\mathcal{U} = \Phi^{-1}(\mathcal{V})$. Since $\mathcal{U} \cap K_\alpha \neq \emptyset$, there is a finite injection $s : \omega \to \omega$ with $\mathcal{U}(s) \subseteq \mathcal{U}$ and $\mathcal{U}(s) \cap K_\alpha \neq \emptyset$. Lemma 2.28 thus yields a finite support permutation $\pi \in K_\alpha$ such that $\mathcal{U}(\pi) \subseteq \mathcal{U}$. Hence

$$\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq \Phi(\mathcal{U} \cap K_\alpha) \subseteq (\mathcal{V} \cap C) \subseteq K_{\beta_m}.$$

This completes the claim.

Suppose that π, m are as in the claim, such that $\Phi(\mathcal{U}(\pi) \cap K_\alpha) \subseteq K_{\beta_m}$. If n is an upper bound for the support of π, then $\pi^{n!} = \mathrm{id}$. Note that each permutation in $\mathcal{U}(\pi) \cap K_\alpha$ has the form $\pi \circ f$, for some $f \in K_\alpha$, with $\mathrm{supp}(f)$ disjoint from $\mathrm{supp}(\pi)$. With this in mind, fix an arbitrary $\pi \circ f \in \mathcal{U}(\pi) \cap K_\alpha$ and observe that

$$\pi^{n!-1} \circ \pi \circ f = f$$

and hence $\Phi(f)$ is the composite of $n!$ elements of K_{β_m}, since $\Phi(\pi), \Phi(\pi \circ f) \in K_{\beta_m}$. On the other hand, any composite of $n!$ elements of K_{β_m} is bounded by the $n!$-fold composite of β_m with itself. As we assumed that $\{\beta_m : m \in \omega\}$ is closed under composition, we conclude that $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_r}$, for an appropriate $r \in \omega$. This completes the proof. $\square$

Lemma 2.30. *Given increasing $\beta_m \in \omega^\omega$, there exists $\alpha \in \omega^\omega$ such that, for each $n, m \in \omega$, there is no continuous injective group homomorphism Φ of S_∞ with $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}$.*

Proof. For the sake of the present proof, if $f \in S_\infty$, we define a *chain of roots of length n for f* to be a sequence $f_0, \ldots, f_n \in S_\infty$ such that $f_0 = f$ and $f_j^2 = f_{j-1}$, for each $1 \leq j \leq n$.

Suppose that $f \in K_\alpha$ is a product of disjoint k-cycles, for some $k \geq 2$. If $n \in \operatorname{supp}(f)$ (i.e., $f(n) \neq n$), then f has no chain of roots in K_α, of length greater than $\alpha(n)$. This follows from the fact that, were $f_0, \ldots, f_p$ a chain of roots of length $p > \alpha(n)$, then at least one f_j is not a member of K_α, as $f_0(n), \ldots, f_p(n)$ are all distinct. Recall here that "square-roots" of products of cycles are obtained by interleaving terms of distinct cycles. For example,

$$f_0 = [0, 1]\,[2, 3]\,[4, 5]\,[6, 7]$$

$$f_1 = [0, 2, 1, 3]\,[4, 6, 5, 7]$$

$$f_2 = [0, 4, 2, 6, 1, 5, 3, 7]$$

is a chain of roots for f_0, of length 2.

Let $\alpha \in \omega^\omega$ be such that, for each $k \in \omega$, the permutation

$$f_k = [k, k + 1]\,[k + 2, k + 3] \ldots$$

has a chain of roots in K_α of length at least

$$\max_{i\leq k}(\beta_i^{4k+1}(k))+1.$$

We may further assume that $(\forall k)(\alpha(k)\geq k+2)$.

Suppose, towards a contradiction, that Φ is a continuous endomorphism of S_∞ with $\Phi(\mathcal{U}_n\cap K_\alpha)\subseteq K_{\beta_m}$, for some $\alpha\in\omega^\omega$ and $m,n\in\omega$. For simplicity, write $\beta_m=\beta$. Let $a\in\omega$ be least such that $\Phi([n,n+1,n+2])(a)\neq a$.

For each $k\geq n+2$, we have that

$$\Phi([n,n+1,k])=\Phi([n+2,k])\circ\Phi([n,n+1,n+2])\circ\Phi([n+2,k]).$$

Observe that

$$[n+2,k]=[n+2,n+3]\,[n+3,n+4]\ldots[k-1,k]\,[k-2,k-1]\ldots[n+2,n+3]$$

and hence $[n+2,k]$ is a product of fewer than $2k$ members of K_α, since each $[j,j+1]$ is in K_α. Thus $\Phi([n+2,k])$ is a product of fewer than 2k members of K_β. (Since each $\Phi([j,j+1])\in K_\beta$, for each $j\geq n$.) In particular, $\Phi([n+2,k])$ is bounded by β^{2k}, the $2k$-fold composite of β with itself. (This follows in part from the fact that β was assumed to be increasing.) Hence we have

$$\Phi([n+2,k])^{-1}(a)=\Phi([n+2,k])(a)\leq\beta^{2k}(a)$$

and thus there exists $b_0\in\operatorname{supp}(\Phi([n,n+1,k]))$ with $b_0\leq\beta^{2k}(a)$.

As noted above, the choice of α guarantees that each $[j,j+1]\in K_\alpha$. Hence

$$[k,k+1]\,[k+2,k+3]\ldots\in K_\alpha$$

and thus

$$h=\Phi([k,k+1]\,[k+2,k+3]\ldots)\in K_\beta.$$

Observe now that

$$\begin{aligned}
&\Phi([n, n+1, k, k+1]\,[k+2, k+3]\,[k+4, k+5]\dots) \qquad (*)\\
&= \Phi([n, n+1, k]\,[k, k+1]\,[k+2, k+3]\dots)\\
&= \Phi([n, n+1, k]) \circ \Phi([k, k+1]\,[k+2, k+3]\dots)\\
&= \Phi([n, n+1, k]) \circ h
\end{aligned}$$

As can be seen from the line marked $(*)$, this permutation has order 4, while $\Phi([n, n+1, k])$ has order 3. Thus $\mathrm{supp}(h)$ must intersect each orbit of $\Phi([n, n+1, k])$, as otherwise the permutation above will contain a 3-cycle and not be of order 4. In particular, $\mathrm{supp}(h)$ contains an element of the orbit of b_0 under $\Phi([n, n+1, k])$. This implies that $\mathrm{supp}(h)$ contains an element $b_1 \leq \beta^{4k}(a)$. We now conclude that h has no chain of roots in K_β, of length greater than $\beta(b_1) \leq \beta^{4k+1}(a)$. (Again, we are using the fact that β is increasing to obtain this inequality.)

On the other hand, if $k \geq m, a$, then

$$\beta^{4k+1}(a) \leq \beta^{4k+1}(k) = \beta_m^{4k+1}(k) \leq \max_{i \leq k}(\beta_i^{4k+1}(k))$$

and $[k, k+1]\,[k+2, k+3]\dots$ has a chain of roots in K_α of length at least

$$\max_{i \leq k}(\beta_i^{4k+1}(k)) + 1.$$

This is a contradiction since Φ maps K_α into K_β and, being a homomorphism, must preserve chains of roots. □

We may now complete the proof of Theorem 2.26.

Proof of Theorem 2.26. Suppose that $\beta_0, \beta_1, \ldots \in \omega^\omega$ are given. With no loss of generality, we may assume that $\{\beta_m : m \in \omega\}$ is closed under compositions and that each β_m is strictly increasing. (Making these assumptions only enlarges the K_σ set $\bigcup_m K_{\beta_m}$. Also, these two assumptions do not conflict as the composite of increasing functions remains increasing.)

Let $\alpha \in \omega$ be as in Lemma 2.30, for $\{\beta_m : m \in \omega\}$. Here we may assume that $\alpha(n) \geq n$, for each n. If there is a continuous endomorphism Φ of S_∞ such that $\Phi(K_\alpha) \subseteq \bigcup_m K_{\beta_m}$, then Lemma 2.29 yields m, n such that $\Phi(\mathcal{U}_n \cap K_\alpha) \subseteq K_{\beta_m}$. This contradicts the properties of α. $\square$

2.5 Universal analytic subgroups

2.5.1 The case of $\mathbb{Z}^\omega$

Theorem 2.32 below gives a universal analytic subgroup of the countable power of any Polish group. In this section we consider a special case.

Theorem 2.31. *There is a universal analytic subgroup of $\mathbb{Z}^\omega$.*

Proof. The proof is similar in spirit to that of Theorem 2.7.

For $s \in \omega^{<\omega}$, let $A^s_0, A^s_1, \ldots$ list all finite subsets of $\mathbb{Z}^{|s|}$ that contain the zero sequence $0^{|s|}$. For each s, j, let $I^s_j \subset \omega$ be an interval of length $|s|$ such that, taken together, the I^s_j partition ω. Define an analytic set $A_0 \subset \mathbb{Z}^\omega$ by

$$x \in A_0 \iff (\exists \alpha \in \omega^\omega)(\forall s, j)((s \geq (\alpha \restriction |s|) \implies x \restriction I^s_j \in A^s_j)).$$

Let $H_0 = \langle A_0 \rangle$ be the subgroup generated by A_0. As the class of analytic sets is closed under continuous images and countable unions, we have that H_0 is also analytic. We will show that H_0 is universal for analytic subgroups of $\mathbb{Z}^\omega$.

Fix an analytic subgroup $H \subset \mathbb{Z}^\omega$. Let $S \subset (\mathbb{Z} \times \omega)^{<\omega}$ be a tree such that $H = p[S]$. For $u \in \mathbb{Z}^{<\omega}$, let u^* be the sequence defined by $u^*(i) = |u(i)|$. Likewise, define x^*, for $x \in \mathbb{Z}^\omega$.

We define a new tree $T \subset (\mathbb{Z} \times \omega)^{<\omega}$ by

$$T = \{(u, s + u^*) : (\exists t \leq s)((u,t) \in S)\} \cup \{(0^{|s|}, s) : s \in \omega^{<\omega}\}.$$

Claim 1. $p[T] = p[S]$.

Proof of claim. If $(x, \alpha) \in [S]$, then $(x, \alpha + x^*) \in [T]$. Hence $p[S] \subseteq p[T]$. On the other hand, suppose that $(x, \alpha) \in [T]$. If $x = \bar{0}$, then $x \in p[S]$, since $p[S]$ is a subgroup. If $x \neq \bar{0}$, then, for each k, there exists $t_k \in \omega^k$ such that $t_k + (x \upharpoonright k)^* \leq \alpha \upharpoonright k$ and $(x \upharpoonright k, t_k) \in S$. By compactness (the t_k are all bounded by α) there exist $k_1 < k_2 < \dots$ and $\beta \leq \alpha$ such that $t_{k_i} \to \beta$, as $i \to \infty$. Thus $(x \upharpoonright k, \beta \upharpoonright k) \in S$, for all k. In other words, $(x, \beta) \in [S]$ and hence $x \in p[S]$. This proves the claim.

For $s \in \omega^{<\omega}$, let T_s denote the set $\{u \in \mathbb{Z}^{<\omega} : (u, s) \in T\}$. The tree T has the property that, for $s_0, s_1 \in \omega^k$ with $s_0 \leq s_1$,

$$T_{s_0} \cap \mathbb{Z}^k \subseteq T_{s_1} \cap \mathbb{Z}^k. \tag{2.6}$$

Observe that each $T_s \cap \mathbb{Z}^{|s|}$ contains $0^{|s|}$ and is finite (since $u \in T_s \cap \mathbb{Z}^{|s|}$ implies $u^* \leq s$). Thus, for each $s \in \omega^{<\omega}$, we may take $\tau(s) \in \omega$ such that $A^s_{\tau(s)} = T_s \cap \mathbb{Z}^{|s|}$. Define a continuous homomorphism $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ by

$$\varphi(x) \upharpoonright I^s_j = \begin{cases} x \upharpoonright k & \text{if } j = \tau(s), \\ 0^k & \text{otherwise,} \end{cases}$$

for $s \in \omega^{<\omega}$ with $k = |s|$ and $j \in \omega$. The following two claims will complete the proof.

Claim 2. If $x \in H$, then $\varphi(x) \in A_0$ and hence $\varphi(x) \in H_0$.

Proof of claim. Let $x \in H$ and $\alpha \in \omega^\omega$ be such that $(x, \alpha) \in [T]$. Fix k and $s \in \omega^k$. If $s \geq (\alpha \restriction k)$, then

$$\begin{aligned} \varphi(x) \restriction I^s_{\tau(s)} &= x \restriction k \\ &\in T_{\alpha \restriction k} \cap \mathbb{Z}^k \\ &\subseteq T_s \cap \mathbb{Z}^k \quad \text{(by (2.6))} \\ &= A^s_{\tau(s)}. \end{aligned}$$

If $j \neq \tau(s)$, then $\varphi(x) \restriction I^s_j = 0^k \in A^s_j$. We see that $\varphi(x) \in A_0$, witnessed by α. This finishes the claim.

Claim 3. If $\varphi(x) \in H_0$, then $x \in H$.

Proof of claim. Since $\varphi(x) \in H_0$, there are $y_1, \ldots, y_m \in A_0$ and a group word $\mathcal{W}$ such that $\varphi(x) = \mathcal{W}(y_1, \ldots, y_m)$.

Let $\alpha_1, \ldots, \alpha_m \in \omega^\omega$ be such that $y_i \restriction I^s_j \in A^s_j$, for each $i \leq m$, each $s \geq (\alpha_i \restriction |s|)$ and each j. (This is the definition of membership in A_0.) Let $\alpha = \alpha_1 + \ldots + \alpha_m$. If $s \in \omega^k$ and $s \geq (\alpha \restriction k)$, then also $s \geq (\alpha_i \restriction k)$ and so $y_i \restriction I^s_j \in A^s_j$. Write $I_k = I^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$, $A_k = A^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$ and define

$$u^k_i = (y_i \restriction I_k). \tag{2.7}$$

For each k, i, we have $u^k_i \in A_k = T_\alpha \cap \mathbb{Z}^k$. Since $u^* \leq \alpha \restriction |u|$, for each $u \in T_\alpha$, we have that T_α is finite branching and hence there are $k_0 < k_1 < \ldots$ and $x_i \in [T_\alpha]$ such that $u^{k_n}_i \to x_i$, as $n \to \infty$, for each $i \leq m$.

Finally, we check that $x = \mathcal{W}(x_1, \ldots, x_m)$. Fix p and let $k_n \geq p$ be such that

$u_i^k \upharpoonright p = x_i \upharpoonright p$. Thus

$$\begin{aligned}
x \upharpoonright p &= (\varphi(x) \upharpoonright I_{k_n}) \upharpoonright p \\
&= (\mathcal{W}(y_1, \ldots, y_m) \upharpoonright I_{k_n}) \upharpoonright p \\
&= \mathcal{W}(u_1^{k_n}, \ldots u_m^{k_n}) \upharpoonright p \qquad \text{(by (2.7))} \\
&= \mathcal{W}(x_1, \ldots, x_m) \upharpoonright p.
\end{aligned}$$

Since p was arbitrary, we see that $x = \mathcal{W}(x_1, \ldots, x_m)$ and so $x \in H$, since H is a subgroup. This completes the claim and finishes the proof. □

2.5.2 Arbitrary countable powers

The following is our main result on the existence of universal analytic subgroups.

Theorem 2.32. *Let Γ be a Polish group. There exists a universal analytic subgroup of Γ^ω.*

As with Theorem 2.16, applying this result to a universal Polish group yields the following corollary.

Corollary 2.33. *If $\mathbb{G}$ is universal Polish group, there is an analytic subgroup $H_0 \subseteq \mathbb{G}$, such that for each analytic subgroup H of a Polish group Γ there is a continuous injective group homomorphism $\varphi : \Gamma \to \mathbb{G}$ such that $H = \varphi^{-1}(H_0)$.*

Again, the injectivity of φ follows from an inspection of the proof of Theorem 2.32.

Before proving Theorem 2.32, we will introduce some notation reminiscent of that in the proofs of Theorems 2.9 and 2.17.

Basic notions

Fix a Polish group Γ with identity element $\mathbf{1}$, a compatible complete metric d and a countable dense set $D \subseteq \Gamma$ such that $\mathbf{1} \in D$. Let $\mathfrak{n}$ be the cardinality of D (either a finite number or ω). Let $\# : D \longleftrightarrow \mathfrak{n}$ be a bijection such that $\#\mathbf{1} = 0$. If $\beta \in D^\omega$ (or $D^{<\omega}$), define $\beta^* \in \mathfrak{n}^\omega$ (or $\mathfrak{n}^{<\omega}$) to be the sequence with $\beta^*(i) = \#\beta(i)$. For $u, v \in \Gamma^k$, write $u \sim v$ to indicate that $d(u(i), v(i)) < 2^{-(i+1)}$, for each $i < k$.

For $x \in \Gamma$, we the define a sequence $\beta_x \in D^\omega$ by letting $\beta_x(i)$ be the $\#$-least element $a \in D$ such that $d(x, a) < 2^{-(i+1)}$. We call β_x the *D-approximation* of x. It follows that $\beta_x(i) \to x$, as $i \to \infty$. Notice that $\beta_{\mathbf{1}} = \bar{\mathbf{1}} = (\mathbf{1}, \mathbf{1}, \ldots)$. With the notation above, $\beta_x \restriction k \sim x^k = (x, \ldots, x)$, for each k.

Analytic subgroups

Fix an analytic subgroup H of Γ. Let $F : \omega^\omega \to \Gamma$ be a continuous map with $H = \operatorname{ran}(F)$. With no loss of generality, $F(\bar{0}) = \mathbf{1}$. Otherwise, we could replace F with the function $\alpha \mapsto (F(\bar{0})^{-1}F(\alpha))$.

For $s \in \omega^{<\omega}$, with $k = |s|$, define

$$P_s = (\bigcup\{F([t]) : |t| = k \ \&\ t \leq s\}) \cap \{x : (\beta_x \restriction k)^* \leq s\}.$$

This Suslin scheme is very similar the one in Theorem 25.13 of [9] and the next claim is more or less verbatim from its proof. (Recall that $[t] = \{\alpha \in \omega^\omega : t \subseteq \alpha\}$.)

Claim. $H = \mathcal{A}_s P_s$.

Proof of claim. Observe that $H \subseteq \mathcal{A}_s P_s$, since if $x = F(\alpha)$, then $x \in \bigcap_n P_{\gamma\restriction n}$, with $\gamma = \alpha + (\beta_x)^*$. To see that $H \supseteq \mathcal{A}_s P_s$, fix $x \in \mathcal{A}_s P_s$, with $\alpha \in \omega^\omega$ such that $x \in \bigcap_n P_{\alpha\restriction n}$.

By definition of the P_s, there are $\alpha_n \in \omega^\omega$ such that, for each n, we have $\alpha_n \upharpoonright n \leq \alpha \upharpoonright n$ and $F(\alpha_n) = x$. By compactness, there is a subsequential limit $\gamma \leq \alpha$ of $(\alpha_n)_{n\in\omega}$. The continuity of F implies that $x = F(\alpha_n) = F(\gamma)$. Hence $x \in H$, as desired.

Now let

$$B_s = \{\beta_x \upharpoonright k : x \in P_s\}.$$

Each B_s is a finite subset of D^k such that $u^* \leq s$, for every $u \in B_s$. We state the key properties of the B_s as a lemma.

Lemma 2.34. *For each $s \in \omega^{<\omega}$ with $k = |s|$, we have the following:*

1. $\mathbf{1}^k \in B_s$.
2. *If $t \in \omega^k$ and $s \leq t$, then $B_s \subseteq B_t$.*
3. *If $m \leq k$, then $B_s \upharpoonright m \subseteq B_{s\upharpoonright m}$.*
4. *If $u \in B_s$, then $d(u(i), u(i+1)) < 2^{-i}$, for each $i < k-1$.*

Proof. We prove each statement in turn.

1. Since $F(\bar{0}) = \mathbf{1}$, we have that $\mathbf{1} \in P_s$, for each s. This implies that $\mathbf{1}^k \in B_s$, because $\beta_{\mathbf{1}} = \bar{\mathbf{1}}$.

2. It follows from the definition of P_s that $P_s \subseteq P_t$, whenever $s \leq t$. Thus also $B_s \subseteq B_t$, if $s \leq t$.

3. Suppose that $u \in B_s$. Let $x \in P_s$ be such that $u = \beta_x \upharpoonright k$. From the definition of the P_s, we see that $P_{s\upharpoonright m} \supseteq P_s$ and hence $x \in P_{s\upharpoonright m}$. Thus $u \upharpoonright m = \beta_x \upharpoonright m \in B_{s\upharpoonright m}$ and therefore $B_s \upharpoonright m \subseteq B_{s\upharpoonright m}$.

4. Let $x \in H$ be such that $u = \beta_x \upharpoonright k$. For each $i < k - 1$,

$$
\begin{aligned}
d(u(i), u(i+1)) &\leq d(u(i), x) + d(x, u(i+1)) \\
&< 2^{-(i+1)} + 2^{-(i+2)} \\
&< 2^{-i}
\end{aligned}
$$

□

Lemma 2.35. *For $x \in \Gamma$, we have $x \in H$ if, and only if, there exist $\alpha \in \omega^\omega$, $\gamma \in D^\omega$ such that $\gamma \upharpoonright k \in B_{\alpha \upharpoonright k}$, for each k, and $\lim_k \gamma(k)$ exists and equals x.*

Proof. For the 'only if' part, suppose that $x \in H$, with $x \in \bigcap_n P_{\alpha \upharpoonright n}$. Let $\gamma = \beta_x$. Then for each k, we have $\gamma \upharpoonright k \in B_{\alpha \upharpoonright k}$ and $\lim_k \gamma(k) = x$, since $d(\gamma(k), x) < 2^{-(k+1)}$.

For the 'if' part, suppose $\alpha \in \omega^\omega$ and $\gamma \in D^\omega$ are such that $(\forall k)(\gamma \upharpoonright k \in B_{\alpha \upharpoonright k})$ and $\lim_k \gamma(k) = x$.

For each k, let $x_k \in P_{\alpha \upharpoonright k}$ be such that $\gamma \upharpoonright k = \beta_{x_k} \upharpoonright k$. By the definition of the P_s, there exist $\alpha_k \in \omega^\omega$ such that $\alpha_k \upharpoonright k \leq \alpha \upharpoonright k$ and $x_k = F(\alpha_k)$. By compactness, there is a convergent subsequence $(\alpha_{k_n})_{n \in \omega}$ of $(\alpha_k)_{k \in \omega}$, with limit $\delta \leq \alpha$.

Claim. $F(\delta) = x$ and hence $x \in H$.

Proof of claim. Fix $\varepsilon > 0$. Let i be such that $2^{-i} < \varepsilon/3$ and $d(\gamma(i), x) < \varepsilon/3$. Since $2^{-i} < \varepsilon/3$, it follows from the definition of D-approximations that $d(x_k, \beta_{x_k}(i)) < \varepsilon/3$, for each k. By the continuity of F, we may choose $k_n > i$ such that $d(F(\delta), F(\alpha_{k_n})) < \varepsilon/3$. Since $F(\alpha_k) = x_k$, this is equivalent to $d(F(\delta), x_{k_n}) < \varepsilon/3$. Also observe that, since $k_n > i$, we have $\gamma(i) = \beta_{x_{k_n}}(i)$ and hence $d(x_{k_n}, \gamma(i)) < \varepsilon/3$, by our choice of i. We now conclude that

$$
\begin{aligned}
d(F(\delta), x) &\leq d(F(\delta), x_{k_n}) + d(x_{k_n}, \gamma(i)) + d(\gamma(i), x) \\
&< 3\varepsilon/3 \\
&= \varepsilon.
\end{aligned}
$$

Since ε was arbitrary, we must have $F(\delta) = x$. $\square$

Proof of Theorem 2.32

We will prove that, given a Polish group Γ, there is an analytic subgroup of Γ^ω which is universal for analytic subgroups of Γ. If Γ is itself a countable power, then we have $\Gamma \cong \Gamma^\omega$ and the statement of the theorem follows.

Let D, $\#$ and $\sim$ be as above for Γ, with $\mathfrak{n} = |D|$.

We begin by defining the desired universal subgroup. For each k and $s \in \omega^k$, let $A_0^s, A_1^s, \ldots$ enumerate all finite subsets of D^k that contain $\mathbf{1}^k$. For $s \in \omega^{<\omega}$ and $j \in \omega$, let $I_j^s \subseteq \omega$ be an interval of length $|s|$, such that the I_j^s partition ω. Define $A_0 \subset \Gamma^\omega$ to be the set of all $\xi \in \Gamma^\omega$ such that

$$
\begin{aligned}
&(\exists \alpha \in \omega^\omega)(\exists \beta \in D^\omega)(\forall s, j, k)(k = |s| \;\&\; s \geq \alpha \restriction k \\
&\qquad\qquad \implies \beta \restriction I_j^s \in A_j^s \;\text{ and }\; \beta \restriction I_j^s \sim \xi \restriction I_j^s).
\end{aligned}
$$

Let H_0 be the subgroup generated by A_0.

We now show that H_0 is a universal analytic subgroup. Fix an arbitrary analytic subgroup $H \subseteq \Gamma$ and let F, P_s and B_s, for $s \in \omega^{<\omega}$, be defined as above for H. For each $s \in \omega^{<\omega}$, choose $\tau(s) \in \omega$ such that $B_s = A_{\tau(s)}^s$. Let $\psi : \Gamma \to \Gamma^\omega$ be the continuous group homomorphism such that for each s, with $k = |s|$, and $j \in \omega$,

$$\psi(x) \restriction I^s_j = \begin{cases} x^k & \text{if } j = \tau(s), \\ \mathbf{1}^k & \text{otherwise.} \end{cases}$$

We will show that $\psi^{-1}(H_0) = H$.

Claim 1. If $x \in H$, then $\psi(x) \in A_0$ and hence $\psi(x) \in H_0$.

Proof of claim. Given $x \in H$, there is $\alpha \in \omega^\omega$ such that $x \in \bigcap_k P_{\alpha \restriction k}$. Define $\beta \in D^\omega$ by

$$\beta \restriction I^s_j = \begin{cases} \beta_x \restriction k & \text{if } j = \tau(s), \\ \mathbf{1}^k & \text{otherwise,} \end{cases}$$

for each $s \in \omega^{<\omega}$ of length k and $j \in \omega$.

We will see that α and β witness the membership of $\psi(x)$ in A_0. Indeed, fix $s \in \omega^{<\omega}$ with $k = |s|$. If $j \neq \tau(s)$, then $\beta \restriction I^s_j = \mathbf{1}^k$ and $\psi(x) = \mathbf{1}^k$. Hence $\beta \restriction I^s_j \in A^s_j$ and $\beta \restriction I^s_j \sim \psi(x) \restriction I^s_j$. Now assume $s \geq \alpha \restriction k$ and $j = \tau(s)$. By the definition of β_x,

$$\beta \restriction I^s_j = \beta_x \restriction k \;\sim\; x^k = \psi(x) \restriction I^s_j.$$

Also, since $x \in P_{\alpha \restriction k}$, we have $\beta_x \restriction k \in B_{\alpha \restriction k}$ and so

$$\beta \restriction I^s_j = \beta_x \restriction k \in B_{\alpha \restriction k} \subseteq B_s = A^s_j.$$

The containment "$B_{\alpha \restriction k} \subseteq B_s$" is a consequence of $s \geq \alpha \restriction k$. It now follows that $\psi(x) \in A_0$, by definition.

It remains to show that $x \in H$, whenever $\psi(x) \in H_0$. Suppose $\psi(x) \in H_0$. There must be $\eta_1, \ldots, \eta_m \in A_0$ and an m-ary group word $\mathcal{W}$ such that

$$\psi(x) = \mathcal{W}(\eta_1, \ldots, \eta_m)$$

Let $\alpha_1, \ldots, \alpha_m \in \omega^\omega$ and $\beta_1, \ldots, \beta_m \in D^\omega$ be as in the definition of A_0, witnessing the membership of $\eta_1, \ldots, \eta_m$ in A_0. Note that we may replace all of the α_i's with $\alpha = \alpha_1 + \ldots + \alpha_m$ and α, β_i will still witness $\eta_i \in A_0$, for each $i \leq m$.

For simplicity, we write $I_k = I^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$ and $A_k = A^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$. Recall that $A_k = B_{\alpha \restriction k}$, by our choice of $\tau(\alpha \restriction k)$.

Let $u_i^k = \beta_i \restriction I_k$. The definition of A_0 implies that $u_i^k \in A_k$ and

$$u_i^k \sim \eta_i \restriction I_k,$$

for each k. Since $u_i^k \in A_k = B_{\alpha \restriction k}$, we also have $(u_i^k)^* \leq \alpha \restriction k$, by the definition of $B_{\alpha \restriction k}$. Define $\gamma_i^k = u_i^k {}^\frown \bar{\mathbf{1}}$. Recall that $\#\mathbf{1} = 0$, and so $(\gamma_i^k)^* \leq \alpha$.

By the compactness of $\{\delta \in \mathfrak{n}^\omega : \delta \leq \alpha\}$,[3] we iteratively choose subsequences of the $(\gamma_i^k)_{k \in \omega}$ to obtain $k_0 < k_1 < \ldots$ and $\gamma_i \in D^\omega$ such that, for each $i \leq m$, $(\gamma_i)^* \leq \alpha$ and $(\gamma_i^{k_p})^* \to (\gamma_i)^*$, as $p \to \infty$. By taking a further subsequence of the k_p, we may assume that $k_p \geq p$ and

$$\gamma_i \restriction p = \gamma_i^{k_p} \restriction p, \tag{2.8}$$

for each $i \leq m$ and $p \in \omega$. Note that $\gamma_i^{k_p} \restriction p = u_i^{k_p} \restriction p$, since $k_p \geq p$.

Claim 2. For each i, the sequence γ_i is Cauchy.

Proof of claim. Since $\sum_n 2^{-n} < \infty$, it will suffice to show that $d(\gamma_i(n), \gamma_i(n+1)) \leq 2^{-n}$, for each n. If n is fixed and $p > n+2$, then, by (2.8),

$$\gamma_i \restriction (n+2) = \gamma_i^{k_p} \restriction (n+2) = u_i^{k_p} \restriction (n+2).$$

Since $u_i^{k_p} \in B_{\alpha \restriction k_p}$, Lemma 2.34(4) implies that $d(\gamma_i(n), \gamma_i(n+1)) \leq 2^{-n}$, as desired.

[3] Of course, if $\mathfrak{n} < \omega$, then $\mathfrak{n}^\omega$ itself is compact.

Since the metric d is complete, it follows from Claim 2 that there are $x_1, \dots, x_m \in \Gamma$ such that $\lim_n \gamma_i(n) = x_i$, for each $i \le m$. Combining this with the fact that, for each p, we have

$$\gamma_i \restriction p = u_i^{k_p} \restriction p \in B_{\alpha \restriction k_p} \restriction p \subseteq B_{\alpha \restriction p},$$

we conclude from Lemma 2.35 that each $x_i \in H$. Note that the statement "$B_{\alpha \restriction k_p} \restriction p \subseteq B_{\alpha \restriction p}$" follows from Lemma 2.34(3).

Claim 3. $x = \mathcal{W}(x_1, \dots, x_m)$ and hence $x \in H$.

Proof of claim. Let n_p be the $(p-1)$-st element of the interval I_{k_p}. (Note that I_{k_p} has length $k_p > p$.) Recall that $u_i^k \sim \eta_i \restriction I_k$, for each k, i. Hence $d(u_i^{k_p}(p-1), \eta_i(n_p)) < 2^{-p}$ and, since $u_i^{k_p}(p-1) = \gamma_i(p-1)$, we have

$$\begin{aligned} d(\eta_i(n_p), x_i) &\le d(\eta_i(n_p), u_i^{k_p}(p-1)) + d(u_i^{k_p}(p-1), x_i) \\ &\le 2^{-p} + d(\gamma_i(p-1), x_i), \end{aligned}$$

for each $i \le m$. Since $\lim_p \gamma_i(p-1) = x_i$, we conclude that $\eta_i(n_p) \to x_i$, as $p \to \infty$.

By the continuity of the group operations, the group word $\mathcal{W}$ induces a continuous function $\Gamma^m \to \Gamma$. Thus

$$\mathcal{W}(\eta_1(n_p), \dots, \eta_m(n_p)) \to \mathcal{W}(x_1, \dots, x_m),$$

as $p \to \infty$. On the other hand,

$$x = \mathcal{W}(\eta_1(n_p), \dots, \eta_m(n_p))$$

is constant, for all p. This implies that, in fact,

$$W(\eta_1(n_p), \dots, \eta_m(n_p)) = \mathcal{W}(x_1, \dots, x_m),$$

for each p. Thus $x = \mathcal{W}(x_1, \dots, x_m)$, completing the claim and proof. $\square$

2.6 Examples

The following observation enables us to apply our main results in a somewhat broader setting.

Proposition 2.36. *Suppose that Γ, Δ are topological groups and $\mathcal{C}$ is a class of subgroups that is closed under continuous homomorphic images. Suppose that Δ has a universal subgroup for $\mathcal{C}$ and $\Gamma \hookrightarrow \Delta \hookrightarrow \Gamma$, where "$\hookrightarrow$" denotes continuous homomorphism embedding. Then Γ has a universal subgroup for $\mathcal{C}$.*

Remark. The classes of compactly generated, K_σ and analytic subgroups are all closed under continuous homomorphic images.

Proof of Proposition 2.36. Let $\Gamma \xrightarrow{\varphi} \Delta \xrightarrow{\psi} \Gamma$ be continuous injective group homomorphisms. Let $H \subseteq \Delta$ be a universal $\mathcal{C}$ subgroup of Δ. To see that $\tilde{H} = \psi(H)$ is a universal $\mathcal{C}$ subgroup of Γ, observe that if $K \subseteq \Gamma$ and $K \in \mathcal{C}$, then $\varphi(K) \in \mathcal{C}$ and hence $\varphi(K) = \theta^{-1}(H)$, for some continuous endomorphism θ of Δ. Thus we have $K = (\psi \circ \theta \circ \varphi)^{-1}(\tilde{H})$, becuase φ, ψ are injective. $\square$

2.6.1 Basic examples

The following examples are direct applications of Proposition 2.36.

Example 2.37. By Theorem 2.32, S_∞^ω has a universal analytic subgroup. Note that S_∞ embeds isomorphically in S_∞^ω and, in fact, S_∞^ω embeds in S_∞ as well: if $A_0, A_1, \ldots$ are disjoint infinite subsets of ω, then S_∞^ω is isomorphic to the closed subgroup

$$\{f \in S_\infty : (\forall n)(f(A_n) = A_n)\}.$$

Proposition 2.36 thus gives a universal analytic subgroup of S_∞.

Example 2.38. Let $\mathbf{c}_0 \subset \mathbb{R}^\omega$ be the subgroup

$$\{x \in \mathbb{R}^\omega : \lim_n x(n) = 0\}.$$

Recall that $\mathbf{c}_0$ is a separable Banach space (hence a Polish group) when equipped with the sup-norm (denoted by $\|\cdot\|_{\sup}$). Let $\mathcal{C}$ be either the class of compactly generated or K_σ subgroups. Since $\mathbf{c}_0$ is nowhere locally compact, Theorem 2.9 does not immediately give universal a $\mathcal{C}$ subgroup of $\mathbf{c}_0^\omega$. (Theorem 2.32 still applies, of course.) Nonetheless, we shall see that $\mathbf{c}_0^\omega$ has a universal $\mathcal{C}$ subgroup.

Observe that the Banach space topology on $\mathbf{c}_0$ refines the subspace topology inherited from $\mathbb{R}^\omega$: suppose $U = I_0 \times \ldots \times I_{k-1} \times \mathbb{R}^\omega$ is a basic open set in $\mathbb{R}^\omega$ (where $I_0, \ldots, I_{k-1} \subseteq \mathbb{R}$ are bounded open intervals) and $x_0 \in U \cap \mathbf{c}_0$. Let $\varepsilon > 0$ be small enough that, for each $n < k$, the open interval $(x_0(n) - \varepsilon, x_0(n) + \varepsilon)$ is contained in I_n. If

$$B = \{x \in \mathbf{c}_0 : \|x - x_0\|_{\sup} < \varepsilon\},$$

then B is open in $\mathbf{c}_0$ and $x \in B \subseteq U \cap \mathbf{c}_0$. Hence $U \cap \mathbf{c}_0$ is open with respect to the Banach space topology on $\mathbf{c}_0$. This implies that the inclusion map $\mathbf{c}_0 \hookrightarrow \mathbb{R}^\omega$ is a continuous homomorphic embedding and hence so is the inclusion $\mathbf{c}_0^\omega \hookrightarrow \mathbb{R}^{\omega\times\omega} \cong \mathbb{R}^\omega$.

To apply Proposition 2.36, we also need to check that $\mathbb{R}^\omega \hookrightarrow \mathbf{c}_0^\omega$. This embedding is witnessed by the map $\varphi : \mathbb{R}^\omega \to \mathbf{c}_0^\omega$ where

$$\varphi(x)(n) = (x(n), 0, 0, \ldots).$$

By Proposition 2.36 we conclude that $\mathbf{c}_0^\omega$ has a universal $\mathcal{C}$ subgroup, since $\mathbb{R}^\omega$ does.

By similar arguments using the fact that the Banach space topologies of $\ell^p, \ell^\infty, \mathbf{c} \subset \mathbb{R}^\omega$ refine their subspace topologies, we can also conclude that the groups $(\ell^p)^\omega$, $(\ell^\infty)^\omega$

and $\mathbf{c}^\omega$ contain universal subgroups for the classes compactly generated, K_σ and analytic subgroups. The case of $(\ell^\infty)^\omega$ is interesting because ℓ^∞ (with the sup-norm) is complete, but not separable. Theorem 2.32 does not even apply to $(\ell^\infty)^\omega$, but Proposition 2.36 still enables us to conclude that $(\ell^\infty)^\omega$ contains a universal analytic subgroup.[4]

It is also worth mentioning the case of ℓ^2. Since ℓ^2 is a separable Hilbert space and, by Corollary 5.5 in [1], all separable Hilbert spaces (over $\mathbb{R}$) are isomorphic, we have that all separable Hilbert spaces are isomorphic to ℓ^2. The comments above thus imply the following.

Proposition 2.39. *The countable power of every separable Hilbert space (over $\mathbb{R}$) contains universal K_σ, compactly generated and analytic subgroups.*

Remark. The arguments above apply equally to $\mathbb{C}$ in place of $\mathbb{R}$. (I.e., $\mathbb{C}^\omega$ also has universal subgroups in our three classes.) Thus the proposition above applies to complex Hilbert spaces as well.

The following example shows the existence of universal subgroups in another non-separable topological group.

Example 2.40. Let S be a separable space and $C(S)$ be the additive group of continuous real-valued functions on S, with the topology of uniform convergence. The group $C(S)$ is metrizable, but not separable if S is not compact. A compatible metric is

$$\rho(f,g) = \sup\{\min\{|f(x) - g(x)|, 1\} : x \in S\}.$$

The distance function ρ is the so-called "uniform metric" on $C(S)$.[5]

[4] Definitions of the Banach spaces ℓ^p, ℓ^∞ and $\mathbf{c}$ may be found in Conway's book [1].

[5] See p. 266 in Munkres' book [12].

Let $A \subseteq S$ be a countable dense set. Consider the Polish group $\mathbb{R}^A$, equipped with the product topology, i.e., $\mathbb{R}^A \cong \mathbb{R}^\omega$. The map $\psi : C(S) \to \mathbb{R}^A$ defined by

$$f \mapsto f \restriction A$$

is a group homomorphism. To see that ψ is continuous it suffices to check that $\psi^{-1}(U)$ is open when U is a basic neighborhood of $\bar{0}$. Given a basic neighborhood $U \ni \bar{0}$, we may assume that, for some finite set $F \subseteq A$ and $\varepsilon > 0$,

$$U = \{x \in \mathbb{R}^A : (\forall a \in F)(|x(a)| < \varepsilon)\}.$$

Let $\mathcal{F} = \{f \in C(S) : (\forall a \in F)(f(a) = 0)\}$ and take

$$\mathcal{V} = \bigcup_{f \in \mathcal{F}} \{g \in C(S) : \rho(f, g) < \varepsilon\}.$$

As the union of open sets, $\mathcal{V}$ is open in $C(S)$ and $\psi^{-1}(U) = \mathcal{V}$. Also, φ is injective because A is dense and thus $f \restriction A = g \restriction A$ implies $f = g$. It follows that $C(S)^\omega$ embeds in $\mathbb{R}^{A\times\omega} \cong \mathbb{R}^\omega$ as well, via a continuous group homomorphism.

Finally, note that $\mathbb{R}^\omega$ embeds in $C(S)^\omega$ via the map $\varphi : \mathbb{R}^\omega \to C(S)^\omega$, where $\varphi(x)(n)$ is the constant function $f \equiv x(n)$. Proposition 2.36 now lets us conclude that $C(S)^\omega$ contains universal compactly generated, K_σ and analytic subgroups. If S is such that $\omega \times S \approx S$ (for example, if $S = \omega^\omega$) then $C(S)^\omega \cong C(S)$ and thus $C(S)$ itself contains universal subgroups for each of these classes.

As noted on page 79 in [9], every separable Banach space is isomorphic to a closed subspace of $C(2^\omega)$. Since $C(2^\omega)^\omega \cong C(\omega \times 2^\omega)$, this implies that the countable power of any Banach space is isomorphic to a closed subgroup of $C(\omega \times 2^\omega)$. By the previous example, we therefore have

Proposition 2.41. *Let $\mathcal{C}$ be one of the classes of compactly generated, K_σ or analytic subgroups. There is a subgroup $H_0 \subseteq C(\omega \times 2^\omega)$, with $H_0 \in \mathcal{C}$, such that for any separable Banach space $\mathfrak{B}$ and any subgroup $H \subseteq \mathfrak{B}^\omega$ in $\mathcal{C}$, there is a continuous group homomorphism $\varphi : \mathfrak{B}^\omega \to C(\omega \times 2^\omega)$ such that $H = \varphi^{-1}(H_0)$.*

The next example relates directly to Theorem 2.9.

Example 2.42. Let $(\Gamma_n)_{n\in\omega}$ be a sequence of locally compact Polish groups. Consider $\bigoplus_n \Gamma_n$ with the subspace topology from $\prod_n \Gamma_n$. Although separable, the direct sum $\bigoplus_n \Gamma_n$ is, in general, not Polishable.[6]

The product $\prod_n \Gamma_n^\omega$ is isomorphic to a closed subgroup of $(\bigoplus_n \Gamma_n)^\omega$. Furthermore, $(\bigoplus_n \Gamma_n)^\omega$ is isomorphic the the $\mathbf{\Pi}^0_3$ subgroup

$$\{\xi : (\forall k)(\forall^\infty n)(\xi(n)(k) = \mathbf{1}_n)\}$$

of $\prod_n \Gamma_n^\omega$. Theorem 2.9 and 2.32 together with Proposition 19 therefore imply that $(\bigoplus_n \Gamma_n)^\omega$ has universal compactly generated, K_σ and analytic subgroups.

2.6.2 Separable Banach spaces

In this section we show that every separable infinite-dimensional Banach space with an unconditional basis (we give the definition below) has universal compactly generated, K_σ and analytic subgroups. The key facts will be Proposition 2.36 along with the following.

Theorem 2.43. *The Banach space $\mathbf{c}_0$ has*

[6] To see this with $\Gamma_n = \mathbb{R}^n$, suppose that $\mathcal{T}$ is a Polishing topology on $\bigoplus_n \mathbb{R}^n$. By the Baire Category Theorem, there is an n such that $\mathbb{R}^n$ is $\mathcal{T}$-non-meager in $\bigoplus_n \mathbb{R}^n$. Being a subgroup, $\mathbb{R}^n$ is thus open in $\bigoplus_n \mathbb{R}^n$, by Pettis' theorem. This gives a contradiction to separability, since $\mathbb{R}^n$ has uncountable index in $\bigoplus_n \mathbb{R}^n$.

A. a universal compactly generated subgroup,

B. a universal K_σ subgroup and

C. a universal analytic subgroup.

In each case, we obtain the desired universal subgroup of $\mathbf{c}_0$ by "shrinking" an appropriate universal subgroup of $\mathbb{R}^\omega$. Note that we could also prove these facts directly by modifying the proofs of Theorems 2.9 and 2.32. We begin with a lemma.

Lemma 2.44. *Suppose $\alpha : \omega \to \mathbb{R}^+$ is such that $\lim_n \alpha(n) = 0$. If $F \subseteq \mathbf{c}_0$ is closed and $|x(n)| \leq \alpha(n)$, for each $x \in F$ and $n \in \omega$, then F is compact in $\mathbf{c}_0$.*

Proof. Suppose that $(x_i)_{i\in\omega}$ is a sequence of elements of F. Let $i_0, i_1, \ldots$ be a subsequence such that $(x_{i_n}(k))_{n\in\omega}$ is convergent, for each $k \in \omega$. Such a subsequence may be obtained by succesively choosing subsequences to guarantee that $(x_{i_n}(j))_{n\in\omega}$ is Cauchy for all $j \leq k$ and taking $(i_n)_{n\in\omega}$ to be a pseudo-intersection of these subsequences. Let $x \in \mathbf{c}_0$ be given by $x(k) = \lim_n x_{i_n}(k)$, for each k. Note that $|x(k)| \leq \alpha(k)$, for each $k \in \omega$.

To see that $\|x_{i_n} - x\|_{\sup} \to 0$, as $n \to \infty$, fix $\varepsilon > 0$ and let k_0 be large enough that $|\alpha(k)| < \frac{\varepsilon}{2}$, for each $k \geq k_0$. Let n_0 be large enough that $|x_{i_n}(k) - x(k)| < \varepsilon$, for each $n \geq n_0$ and $k < k_0$. It follows that $\|x_{i_n} - x\|_{\sup} < \varepsilon$, for each $n \geq n_0$. $\square$

Proof of A. Let $\langle K\rangle \subseteq \mathbb{R}^\omega$ be a universal compactly generated subgroup of $\mathbb{R}^\omega$. (Such a subgroup exists by Theorem 2.9(1).) With no loss of generality, we assume that the compact set K contains $\bar{0}$. Let $\{I_{n,p} : n, p \in \omega\}$ be intervals partitioning ω such that each $I_{n,p}$ has length n. Define $K' \subseteq \mathbb{R}^\omega$ by

$$x \in K' \iff (\forall n, p)(x \restriction I_{n,p} \in (1/np)K \restriction n).$$

Where $(1/np)K \restriction n$ denotes the set of scalar multiples by $(1/np)$ of elements of $K \restriction n$. It follows from Lemma 2.44 that K' is compact in $\mathbf{c}_0$.

We will show that $\langle K' \cap \mathbf{c}_0 \rangle$ is a universal compactly generated subgroup of $\mathbf{c}_0$. Indeed, fix an arbitrary compact $A \subseteq \mathbf{c}_0$. Since A is also compact in $\mathbb{R}^\omega$, there is a continuous group homomorphism $\varphi : \mathbb{R}^\omega \to \mathbb{R}^\omega$ such that $\langle A \rangle = \varphi^{-1}(\langle K \rangle)$.[7]

For each $n \in \omega$, let $\tau(n) \in \omega \setminus \{0\}$ be such that, for every $x \in [-1,1]^\omega$ and $i < n$, we have $|\varphi(x)(i)| \leq \tau(n)$. (Such $\tau(n)$ exist by the compactness of $[-1,1]^\omega$ and the continuity of φ.) Define $\psi : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$\psi(x) \restriction I_{n,p} = \begin{cases} (1/np)\varphi(x) \restriction n & \text{if } p = \tau(n)^2 \\ 0^n & \text{otherwise} \end{cases}$$

Claim 1. $\psi(\mathbf{c}_0) \subseteq \mathbf{c}_0$.

Proof of claim. Note that all continuous group homomorphisms of $\mathbb{R}^\omega$ are automatically linear, hence ψ is linear. Thus, to prove the claim, it will suffice to show that $\psi(x) \in \mathbf{c}_0$, for all $x \in \mathbf{c}_0$ with $\|x\|_{\sup} \leq 1$. Fix such an x and an $\varepsilon > 0$. For $i \in \omega$, $\psi(x)(i) \neq 0$ only if $i \in I_{n,\tau(n)^2}$, for some n. For $i \in I_{n,\tau(n)^2}$, we have

$$\begin{aligned} |\psi(x)(i)| &\leq (1/n\tau(n)^2) \max_{j<n} |\varphi(x)(j)| \\ &\leq 1/n\tau(n) \end{aligned}$$

Thus $|\psi(x)(i)| \geq \varepsilon$ only if $i \in I_{n,\tau(n)^2}$ and $1/n\tau(n) \geq \varepsilon$. There are only finitely many such i.

Claim 2. For each $x \in \mathbf{c}_0$, we have $x \in \langle A \rangle \iff \psi(x) \in \langle K' \cap \mathbf{c}_0 \rangle$.

[7] As noted earlier the Banach space topology of $\mathbf{c}_0$ refines the subspace topology inherited from $\mathbb{R}^\omega$ and hence compactness is "preserved upwards."

Proof of claim. To prove the claim, it will suffice to show that $\psi(x) \in \langle K' \cap \mathbf{c}_0 \rangle \iff \varphi(x) \in \langle K \rangle$, since we already have $x \in \langle A \rangle \iff \varphi(x) \in \langle K \rangle$.

Fix a group word $\mathcal{W}$,

$$\begin{aligned}
\varphi(x) \in \mathcal{W}[K] &\iff (\forall n)(\varphi(x) \restriction n \in \mathcal{W}[K] \restriction n) \\
&\iff (\forall n)(\psi(x) \restriction I_{n,\tau(n)^2} \in (1/n\tau(n)^2)(\mathcal{W}[K] \restriction n)) \\
&\iff \psi(x) \in \mathcal{W}[K'].
\end{aligned}$$

The first and last " $\iff$ " use the fact that $\mathcal{W}[K]$ is closed (since K is compact). As $\mathcal{W}$ was arbitrary, this completes the claim and proof. $\square$

Proof of B. Let $H = \bigcup_n K_n$ be a universal K_σ subgroup of $\mathbb{R}^\omega$, as given by Theorem 2.9(1). We may assume that

$$(\bar{0} \in K_0) \text{ and } (\forall n)(-K_n = K_n \text{ and } K_n + K_n \subseteq K_{n+1}). \tag{2.9}$$

Let $\{I_{m,p} : m, p \in \omega\}$ be a family of intervals partitioning ω such that each $I_{m,p}$ has length m. Define $K'_n \subseteq \mathbb{R}^\omega$ by

$$x \in K'_n \iff (\forall m, p)(x \restriction I_{m,p} \in (1/mp)K_n \restriction m)$$

and let $H' = \bigcup K'_n$. Again, Lemma 2.44 implies that each K'_n is compact in $\mathbf{c}_0$. Observe that (2.9) holds for the K'_n as well. In particular, H' is a subgroup of $\mathbb{R}^\omega$. We will show that $H' \cap \mathbf{c}_0$ is in fact a universal K_σ subgroup of $\mathbf{c}_0$.

Let $A = \bigcup_n A_n$ be an arbitrary K_σ subgroup of $\mathbf{c}_0$. Again, A is still K_σ in $\mathbb{R}^\omega$. Hence there is a continuous homomorphism $\varphi : \mathbb{R}^\omega \to \mathbb{R}^\omega$ such that $\varphi^{-1}(H) = A$. Let $\tau(m) \in \omega \setminus \{0\}$ be such that, for each $x \in [-1, 1]^\omega$ and $i < m$, we have $|\varphi(x)(i)| \leq \tau(m)$.

Define $\psi : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$\psi(x) \restriction I_{m,p} = \begin{cases} (1/mp)\varphi(x) \restriction m & \text{if } p = \tau(m)^2, \\ 0^m & \text{otherwise.} \end{cases}$$

As in proof of part A, it follows that $\psi(\mathbf{c}_0) \subseteq \mathbf{c}_0$. Finally, to see that $\psi^{-1}(H') = A$, it will suffice to show that

$$(\forall x \in \mathbf{c}_0)(\forall n)(\psi(x) \in K'_n \iff \varphi(x) \in K_n).$$

To see this, observe that, for each n,

$$\begin{aligned} \psi(x) \in K'_n &\iff (\forall m)(\psi(x) \restriction I_{m,\tau(m)^2} \in (1/m\tau(m)^2)K_n \restriction m \\ &\iff (\forall m)(\varphi(x) \restriction m \in K_n \restriction m) \\ &\iff \varphi(x) \in K_n. \end{aligned}$$

□

Proof of C. Let $H \subseteq \mathbb{R}^\omega$ be a universal analytic subgroup of $\mathbb{R}^\omega$, as given by Theorem 2.32. Let $F : \omega^\omega \to \mathbb{R}^\omega$ be continuous with $H = \operatorname{ran}(F)$ and let $P_s = \bigcup\{F([t]) : |t| = |s| \wedge t \leq s\}$. The proof of Theorem 25.13 in [9] shows that $H = \mathcal{A}_s P_s$ and, for each $\alpha \in \omega^\omega$, the set $P_\alpha = \bigcap_n P_{\alpha\restriction n}$ is compact. Take $\{I_{s,p} : s \in \omega^{<\omega}, p \in \omega\}$ to be a set of intervals partitioning ω such that each $I_{s,p}$ has length $|s|$. Let $\# : \omega^{<\omega} \leftrightarrow \omega$ be a bijection and define $H' \subseteq \mathbb{R}^\omega$ by

$$x \in H' \iff (\exists\alpha)(\forall s \geq \alpha \restriction |s|)(\forall p)(x \restriction I_{s,p} \in (1/p\#s)P_s \restriction |s|).$$

We will show that $\langle H' \cap \mathbf{c}_0\rangle$ is a universal analytic subgroup of $\mathbf{c}_0$. Fix an analytic subgroup $A \subseteq \mathbf{c}_0$. As A is analytic in $\mathbb{R}^\omega$, there is a continuous homomorphism φ :

$\mathbb{R}^\omega \to \mathbb{R}^\omega$ such that $\varphi^{-1}(H) = A$. As before, let $\tau(n) \in \omega \setminus \{0\}$ be such that, for each $x \in [-1,1]^\omega$ and $i < n$, we have $|\varphi(x)(i)| \leq \tau(n)$. Define $\psi : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$\psi(x) \restriction I_{s,p} = \begin{cases} (1/p\#s)\varphi(x) \restriction |s| & \text{if } p = \tau(|s|)^2, \\ 0^{|s|} & \text{otherwise.} \end{cases}$$

Again, it follows that $\psi(\mathbf{c}_0) \subseteq \mathbf{c}_0$.

To check that $A = \psi^{-1}(\langle H' \cap \mathbf{c}_0 \rangle)$ it will be enough to check that

$$(\forall x \in \mathbf{c}_0)(\psi(x) \in \langle H' \rangle \iff \varphi(x) \in H). \tag{2.10}$$

For the " $\Longleftarrow$ " part of (2.10), suppose that $\varphi(x) \in P_\alpha$. Then

$$\begin{aligned} (\forall n)(\varphi(x) \in P_{\alpha\restriction n}) &\implies (\forall n)(\varphi(x) \restriction n \in P_{\alpha\restriction n} \restriction n) \\ &\implies (\forall s \geq \alpha \restriction |s|)(\varphi(x) \restriction |s| \in P_s \restriction |s|) \\ &\implies \psi(x) \in H', \text{ witnessed by } \alpha, \end{aligned}$$

since $\psi(x) \restriction I_{s,p} = (1/p\#s)\varphi(x) \restriction |s|$.

For the " $\implies$ " half of (2.10), suppose that $\psi(x) \in \langle H' \cap \mathbf{c}_0 \rangle$. Say $\mathcal{W}$ is a group word and $y_1, \ldots, y_m \in H' \cap \mathbf{c}_0$ are such that $\psi(x) = \mathcal{W}(y_1, \ldots, y_m)$. We may assume that there is a single α witnessing the membership of $y_1, \ldots, y_m$ in H', i.e., for each $i \leq m$, $p, n \in \omega$ and $s \geq \alpha \restriction n$, we have $y_i \restriction I_{s,p} \in (1/p\#s)P_s \restriction n$. We will see that $\varphi(x) \in \mathcal{W}[P_\alpha]$. For notational simplicity, let I_n denote the interval $I_{\alpha\restriction n, \tau(n)^2}$ and r_n denote $1/n\tau(n)^2$. By definition, $\psi(x) \restriction I_n = (1/r_n)\varphi(x) \restriction n$. For each n and $i \leq m$, let $\alpha_n^i \in \omega^\omega$ be such that $\alpha_n^i \restriction n \leq \alpha \restriction n$ and $F(\alpha_n^i) \restriction n = r_n y_i \restriction I_n$. Hence

$$\mathcal{W}(F(\alpha_n^1), \ldots, F(\alpha_n^m)) \restriction n = \varphi(x) \restriction n.$$

By compactness, take $n_0 < n_1 < \ldots$ and $\alpha_i \leq \alpha$ such that, for each $i \leq m$, we have $\lim_p \alpha^i_{n_p} = \alpha_i$. Finally, we will see that $\varphi(x) = \mathcal{W}(F(\alpha_1), \ldots, F(\alpha_m))$ and conclude that $\varphi(x) \in H$. Fix $\ell \in \omega$ and observe that, for each $n_p \geq \ell$,

$$\begin{aligned}\varphi(x) \restriction \ell &= \mathcal{W}(F(\alpha^1_{n_p}), \ldots, F(\alpha^m_{n_p})) \restriction \ell \\ &= \mathcal{W}(F(\alpha_1), \ldots, F(\alpha_m)) \restriction \ell.\end{aligned}$$

The second equality is obtained by taking the limit as $p \to \infty$. As ℓ was arbitrary, we have the desired result and conclude the proof. □

We now proceed to the main result of this section. The following definition may be found at the beginning of [5].

Definition 2.45. Let $\mathfrak{B}$ be an infinite-dimensional Banach space (over $\mathbb{R}$). An *unconditional basis* for $\mathfrak{B}$ is a set $\{e_n\}_{n\in\omega} \subseteq \mathfrak{B}$ such that

1. each e_n is a unit vector,
2. for each $x \in \mathfrak{B}$, there is a unique sequence $a_0, a_1, \ldots \in \mathbb{R}$ with $x = \sum_{n\in\omega} a_n e_n$ (convergence in norm) and
3. any permutation of $\{e_n\}_{n\in\omega}$ still has the previous property.

The following fact (also mentioned in [5]) gives a useful property of unconditional bases.

Proposition 2.46 ([5], Theorem 1)**.** *If $\{e_n\}_{n\in\omega}$ is an unconditional basis for a Banach space $\mathfrak{B}$, then there is a constant C such that for each $x = \sum_{n\in\omega} a_n e_n \in \mathfrak{B}$ and $(\varepsilon_n)_{n\in\omega} \in [-1,1]^\omega$, we have*

$$\Big\| \sum_{n\in\omega} \varepsilon_n a_n \Big\| \leq C \Big\| \sum_{n\in\omega} a_n e_n \Big\|.$$

The following lemma is consequence of this proposition.

Lemma 2.47. *If $\mathfrak{B}$ is an infinite-dimensional Banach space with an unconditional basis, then $\mathfrak{B}$ and $\mathbf{c}_0$ are mutually embeddable, as topological groups.*

Proof. Let $\{e_n\}_{n\in\omega}$ be an unconditional basis for $\mathfrak{B}$, with C as in the previous proposition.

We first show that $\mathfrak{B}$ embeds in $\mathbf{c}_0$. Define $\varphi : \mathfrak{B} \to \mathbf{c}_0$ by $\varphi(\sum_n a_n e_n) = (a_n)_{n\in\omega}$. Since the sum $\sum_n a_n e_n$ is convergent, it is, in particular Cauchy and hence the norm of the nth term converges to 0. It follows that φ maps $\mathfrak{B}$ into $\mathbf{c}_0$. We must now see that φ is continuous. Since φ is linear, it will suffice to show that φ is continuous at the zero element of $\mathfrak{B}$. Fix $x = \sum_n a_n e_n \in \mathfrak{B}$. For each n, let $\varepsilon_n = 1$ and $\varepsilon_k = 0$, for $k \neq n$, and observe that

$$|a_n| = \|a_n e_n\| = \Big\| \sum_{n\in\omega} \varepsilon_n a_n \Big\| \leq C\|x\|.$$

Thus $\|\varphi(x)\|_{\sup} \leq C\|x\|$, showing that φ is continuous at $0 \in \mathfrak{B}$.

We now wish to embed $\mathbf{c}_0$ into $\mathfrak{B}$. Define $\psi : \mathbf{c}_0 \to \mathfrak{B}$ by $\psi((a_n)_{n\in\omega}) = \sum_n \frac{a_n}{2^n} e_n$. Since $(a_n)_{n\in\omega}$ is a bounded sequence, this latter sum is always well-defined. To see that ψ is continuous, observe that, if $\|(a_n)_{n\in\omega}\|_{\sup} \leq 1$, then by Proposition 2.46

$$\|\psi((a_n)_{n\in\omega})\| = \Big\| \sum_n \tfrac{a_n}{2^n} e_n \Big\| \leq C \Big\| \sum_n \tfrac{1}{2^n} e_n \Big\|.$$

Thus ψ is a bounded linear map and hence continuous. □

Combining this lemma with Proposition 2.36, we obtain the following theorem.

Theorem 2.48. *Let $\mathfrak{B}$ be an infinite-dimensional Banach space with an unconditional basis. Then $\mathfrak{B}$ has universal compactly generated, K_σ and analytic subgroups.*

Remark. To put this theorem in context, recall that (among many others) all ℓ^p spaces ($1 \leq p < \infty$) have unconditional bases. (On the other hand, Per Enflo [3] and later Gowers-Maurey [5] showed that there exist separable Banach spaces with no unconditional bases.)

The following serves as an addendum to the last theorem.

Theorem 2.49. *The following Banach spaces (viewed as topological groups) have universal compactly generated, K_σ and analytic subgroups:*

1. ℓ^∞,

2. $C(X)$, *if X is infinite, Polish and compact, and*

3. $C_0(X)$, *if X is infinite, Polish and locally compact.*

Remark. In general, the spaces listed in this theorem may not have unconditional bases (ℓ^∞ is not even separable) and so Theorem 2.48 does not necessarily apply.

Proof of Theorem 2.49. In each case, we will apply Proposition 2.36 and Theorem 2.43.

1. The embedding $\mathbf{c}_0 \hookrightarrow \ell^\infty$ is via the inclusion map, while $\ell^\infty \hookrightarrow \mathbf{c}_0$ is by means of the map $(a_n)_{n\in\omega} \mapsto ((1/n)a_n)_{n\in\omega}$.

2. Let $\{x_n\}_{n\in\omega}$ be a discrete sequence of distinct points in X. For each n, let $f_n \in C(X)$ have sup-norm 1 and be such that $f_n(x_n) = 1$ and $f_n(x_k) = 0$, if $k \neq n$. Such functions exist by the Tietze Extension Theorem. Then $\mathbf{c}_0$ embeds in $C(X)$ via the map $x \mapsto \sum_{n\in\omega}(x(n)/2^n)f_n$.

Let $\{y_n\}_{n\in\omega}$ be a countable dense subset of X. Then $C(X)$ embeds in $\mathbf{c}_0$ via the map $f \mapsto ((1/n)f(y_n))_{n\in\omega}$.

3. Use the same functions as in 2. $\square$

2.6.3 A negative example

The following example gives our only instances of perfect Polish groups without universal subgroups in any of the three classes we consider. The key fact is that any nontrivial group homomorphism of $\mathbb{R}^n$ is in fact an automorphism. At this point, we do not know if there are Polish groups which have no universal analytic subgroups and do not have this property.

Example 2.50. By Theorem 2.9 there is a universal K_σ subgroup of $\mathbb{R}^\omega$. On the other hand, we shall see that there is no universal K_σ subgroup of $\mathbb{R}^n$, for $n \in \omega$. First, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous group homomorphism, then φ is automatically a linear transformation. To see this, observe that, since φ is a group homomorphism, one can show that $\varphi(q\mathbf{r}) = q\varphi(\mathbf{r})$, for any $q \in \mathbb{Q}$ and $\mathbf{r} \in \mathbb{R}^n$. One then concludes that $\varphi(a\mathbf{r}) = a\varphi(\mathbf{r})$, for any $a \in \mathbb{R}$, by the density of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of φ.

Towards a contradiction, suppose that $H_0 \subseteq \mathbb{R}^n$ is a universal K_σ subgroup of $\mathbb{R}^n$. Let $\tilde{A}, \tilde{B} \subsetneq \mathbb{R}$ be nontrivial K_σ subgroups such that $\tilde{A}$ is countable and $\tilde{B}$ is uncountable. Let

$$A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \in \tilde{A} \ \&\ x_2 = x_3 = \ldots = x_n = 0\}$$

and

$$B = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \in \tilde{B} \ \&\ x_2 = x_3 = \ldots = x_n = 0\}.$$

A and B are K_σ subgroups of $\mathbb{R}^n$ that contain no linear (over $\mathbb{R}$) subspaces of $\mathbb{R}^n$ other than $\{0^n\}$. Let φ_A, φ_B be continuous endomorphisms of $\mathbb{R}^n$ reducing A, B to H_0. As φ_A and φ_B are actually linear transformations, $\ker \varphi_A$ and $\ker \varphi_B$ are linear subspaces of $\mathbb{R}^n$. Since φ_A and φ_B are reductions between subgroups, we must have that

$\ker \varphi_A \subseteq A$ and $\ker \varphi_B \subseteq B$, in particular, both kernels are trivial. Hence φ_A and φ_B are actually automorphisms. Thus A and B have the same cardinality, a contradiction.

By the same reasoning, there are no universal compactly generated or analytic subgroups of $\mathbb{R}^n$.

2.7 An application to ideals

Recall that an *ideal* on ω is a set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ that is closed under finite unions and closed downwards (i.e., if $x \subseteq y \in \mathcal{I}$, then $x \in \mathcal{I}$). Also recall that $\mathcal{P}(\omega)$ becomes a Polish group when equipped with the addition operation

$$x \triangle y = (x \setminus y) \cup (y \setminus x).$$

In particular, every ideal is a subgroup of $\mathcal{P}(\omega)$, since $x \triangle y \subseteq x \cup y$, for $x, y \subseteq \omega$.

By identifying each $x \subseteq \omega$ with its characteristic function, one can regard $(\mathcal{P}(\omega), \triangle)$ as $(\mathbb{Z}_2^\omega, +)$. With this identification, the relation $x \subseteq y$ agrees with the pointwise $x \leq y$. We use the latter when dealing with $\mathbb{Z}_2^\omega$ to avoid confusion with the "$\subset$" (extension) relation on $\mathbb{Z}_2^{<\omega}$.

In this section, we study the following weak form of Rudin-Keisler reduction.

Definition 2.51. For ideals $\mathcal{I}, \mathcal{J}$ on ω, we write $\mathcal{I} \leq^+_{\mathrm{RK}} \mathcal{J}$ if, and only if, there is a subset $A \subseteq \omega$ and a function $\beta : A \to \omega$ such that $x \in \mathcal{I} \iff \beta^{-1}(x) \in \mathcal{J}$, for each $x \subseteq \omega$.[8]

Theorems 2.54 and 2.55 will use the methods of our earlier results to show that there are $\leq^+_{\mathrm{RK}}$-complete F_σ and analytic ideals. In a personal communication, Michael Hrušák

[8]We use the notation $\leq^+_{\mathrm{RK}}$ as a parallel with $\leq_{\mathrm{RB}}$ versus $\leq^+_{\mathrm{RB}}$. See pp. 41-42 in [8] for definitions.

has informed us that, though unpublished, the former result is already known to him.[9]

The only difference between $\leq^+_{\mathrm{RK}}$ and the usual Rudin-Keisler order is that the reducing map in the case of $\leq^+_{\mathrm{RK}}$ need not be defined on all of ω. As with Rudin-Keisler reduction, if $\mathcal{I} \leq^+_{\mathrm{RK}} \mathcal{J}$ and $\mathcal{J}$ is an ideal, then $\mathcal{I}$ is an ideal as well. We call a map β as in the definition above a *weak RK-reduction.* Observe that the map

$$x \mapsto \beta^{-1}(x)$$

defines a continuous homomorphism of $\mathcal{P}(\omega)$ (equivalently, of $\mathbb{Z}_2^\omega$). This implies that, for ideals $\mathcal{I}, \mathcal{J}$, if $\mathcal{I} \leq^+_{\mathrm{RK}} \mathcal{J}$, then automatically $\mathcal{I} \leq_{\mathrm{g}} \mathcal{J}$.

Before proceeding, we verify that $\leq^+_{\mathrm{RK}}$ is indeed weaker than $\leq_{\mathrm{RK}}$. Consider the following example.

Example 2.52. For $x \subseteq \omega$, let

$$\mathrm{Fin}(x) = \{y \in \mathcal{P}(\omega) : y \text{ is finite and } y \subseteq x\}.$$

With this notation, the ideal Fin is $\mathrm{Fin}(\omega)$. If x is infinite, then any bijection $\beta : x \longleftrightarrow \omega$ witnesses $\mathrm{Fin} \leq^+_{\mathrm{RK}} \mathrm{Fin}(x)$. On the other hand, if $x \neq \omega$, then $\mathrm{Fin} \not\leq_{\mathrm{RK}} \mathrm{Fin}(x)$. To see this, suppose otherwise and let $\beta : \omega \to \omega$ be such for each $y \subseteq \omega$, $y \in \mathrm{Fin} \iff \beta^{-1}(y) \in \mathrm{Fin}(x)$. Let $a \in \omega \setminus x$ and let $b = \beta(a)$. We have $\{b\} \in \mathrm{Fin}$, but $\beta^{-1}(\{b\}) \notin \mathrm{Fin}(x)$, since $a \in \beta^{-1}(\{b\})$ and $a \notin x$.

We also remark on the fact that $\leq_{\mathrm{g}}$ is weaker than $\leq^+_{\mathrm{RK}}$.

Example 2.53. Consider $H = \{\emptyset, \{0,1\}\}$ and the ideal Fin. Both are subgroups of

[9]See Proposition 5.4 of [7] for a similar result.

$(\mathcal{P}(\omega), \triangle)$ and $H \leq_{\mathrm{g}} \mathrm{Fin}$, via the map $\varphi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by

$$\varphi(x) = \begin{cases} \emptyset & \text{if } 0, 1 \in x \text{ or both } 0, 1 \notin x, \\ \omega & \text{otherwise.} \end{cases}$$

It is easier to see that this is a group homomorphism by viewing $\mathcal{P}(\omega)$ as $\mathbb{Z}_2^\omega$. With this identification, φ is given by

$$\varphi(x)(n) = x(0) + x(1),$$

for all $x \in \mathbb{Z}_2^\omega$ and $n \in \omega$.

On the other hand, we cannot have $H \leq_{\mathrm{RK}}^+ \mathrm{Fin}$, since this would imply that H is an ideal.

The next two theorems establish the existence of the $\leq_{\mathrm{RK}}^+$-complete ideals mentioned above.

Theorem 2.54. *There is a $\leq_{\mathrm{RK}}^+$-complete F_σ ideal in $\mathbb{Z}_2^\omega$.*

Remark on F_σ ideals. Since every ideal on ω is a subgroup of the compact group $\mathbb{Z}_2^\omega$, Theorem 2.4 implies that every F_σ (i.e., K_σ) ideal is compactly generated. Since the downward closure of a compact set is also compact, we conclude that every F_σ ideal on ω is the set of finite unions of elements of a downward closed compact subset of $\mathcal{P}(\omega)$.

Proof of Theorem 2.54. For $k \in \omega$ and $s \in \omega^{<\omega}$, let A_s^k be subsets of $\mathbb{Z}_2^k$ such that

- Each A_s^k is closed downward, i.e., $u \leq v \in A_s^k \implies u \in A_s^k$.
- If $A \subseteq \mathbb{Z}_2^k$ is closed downward and $A \supseteq A_s^k$, then there exists i such that $A = A_{s^\frown i}^k$.

For each k, j, let I_j^k be an interval in ω of length k, such that the I_j^k partition ω. Define $A \subseteq \mathbb{Z}_2^\omega$ by

$$x \in A \iff (\exists n)(\forall k, s)(|s| \geq n \implies x \restriction I_s^k \in A_s^k).$$

Observe that A is F_σ and hence so is the ideal $\mathcal{I}_0$, generated by A. Note that A is already closed downward and thus $\mathcal{I}_0$ is the set of finite unions of elements of A. We will show that $\mathcal{I}_0$ is $\leq_{\mathrm{RK}}^+$-complete among F_σ ideals.

Let $\mathcal{I} = \bigcup_n F_n$ be an arbitrary F_σ ideal. We may assume that $F_0 \subseteq F_1 \subseteq \ldots$ and that each F_n is closed downward. (Since the downward closure of a closed set is also closed.) For each k, choose $\alpha_k \in \omega^\omega$ such that for each n,

$$F_n \restriction k = A^k_{\alpha_k \restriction n}.$$

Let $S = \bigcup_{s \subset \alpha_k} I_s^k$. We will define a weak RK-reduction $\beta : S \to \omega$ which will witness $\mathcal{I} \leq_{\mathrm{RK}}^+ \mathcal{I}_0$. For each I_s^k, with $s \subset \alpha_k$, if i is the jth element of I_s^k, we set $\beta(i) = j$. We can re-write the map $x \mapsto \beta^{-1}(x)$ in a way that will be easier to work with. Observe that

$$\beta^{-1}(x) \restriction I_s^k = \begin{cases} x \restriction k & \text{if } s \subset \alpha_k, \\ 0^k & \text{otherwise.} \end{cases}$$

The following two claims will complete the proof.

Claim 1. If $x \in \mathcal{I}$, then $\beta^{-1}(x) \in \mathcal{I}_0$.

Proof of claim. Suppose that $x \in \mathcal{I}$, with $x \in F_{n_0}$. This implies that, for each k and

$s \subset \alpha_k$, with $n = |s| \geq n_0$, we have

$$\begin{aligned} \beta^{-1} \restriction I_s^k &= x \restriction k \\ &\in F_n \restriction k \\ &= A^k_{\alpha_k \restriction n}. \end{aligned}$$

If $s \not\subset \alpha_k$, then $\beta^{-1}(x) \restriction I_s^k = 0^k \in A_s^k$, since A_s^k is closed downwards. Putting these two cases together, we see that

$$(\forall k, s)(|s| \geq n_0 \implies \beta^{-1}(x) \restriction I_s^k \in A_s^k).$$

Hence $\beta^{-1}(x) \in A \subseteq \mathcal{I}_0$. This proves our first claim.

Claim 2. If $\beta^{-1}(x) \in \mathcal{I}_0$, then $x \in \mathcal{I}$.

Proof of claim. Suppose that $\beta^{-1}(x) \in \mathcal{I}_0$ and $y_1, \ldots, y_m \in A$ are such that $\beta^{-1}(x) = y_1 \cup \ldots \cup y_m$. We will find $x_1, \ldots, x_m \in \mathcal{I}$ such that $x = x_1 \cup \ldots \cup x_m$. Let n be such that for each $i \leq m$,

$$(\forall k, s)(|s| \geq n \implies y_i \restriction I_s^k \in A_s^k).$$

Let $v_i^k = y_i \restriction I^k_{\alpha_k \restriction n}$. For each k and all $i \leq m$, $v_i^k \in A^k_{\alpha_k \restriction n} = F_n \restriction k$. Hence there exists $x_i^k \in F_n$ such that $v_i^k = x_i^k \restriction k$. By repeated use of the compactness of $\mathbb{Z}_2^\omega$, we choose a subsequence $k_0 < k_1 < \ldots$ and $x_i \in F_n$ such that, for each $i \leq m$

$$\lim_{p \to \infty} x_i^{k_p} = x_i.$$

To check that $x = x_1 \cup \ldots \cup x_m$, observe that, for each fixed ℓ and p with $k_p \geq \ell$, we have

$$\begin{aligned} x \restriction \ell &= (\beta^{-1}(x) \restriction I^{k_p}_{\alpha_{k_p} \restriction n}) \restriction \ell \\ &= (v_1^{k_p} \cup \ldots \cup v_m^{k_p}) \restriction \ell \\ &= (x_1^{k_p} \cup \ldots \cup x_m^{k_p}) \restriction \ell. \end{aligned}$$

Taking the limit as $p \to \infty$, we see that

$$x \restriction \ell = (x_1 \cup \ldots \cup x_m) \restriction \ell.$$

Since ℓ was arbitrary, we must have $x = x_1 \cup \ldots \cup x_m$. This shows that $x \in \mathcal{I}$ and completes the proof. $\square$

Theorem 2.55. *There exists a $\leq^+_{\mathrm{RK}}$-complete analytic ideal in $\mathbb{Z}_2^\omega$.*

Notation. As in the proof of Theorem 2.31, if T is a tree on $2 \times \omega$ and $s \in \omega^{<\omega}$, we let T_s denote the set $\{u \in \mathbb{Z}^{<\omega} : (u, s) \in T\}$.

Lemma 2.56. *Suppose that $\mathcal{I}$ is an analytic ideal in $\mathbb{Z}_2^\omega$. There exists a tree T on $2 \times \omega$ such that*

1. *$\mathcal{I} = p[T]$.*
2. *If $s, t \in \omega^k$ and $s \leq t$, then $T_s \subseteq T_t$.*
3. *For each $s \in \omega^k$, if $u \leq v \in T_s \cap \mathbb{Z}_2^k$, then $u \in T_s \cap \mathbb{Z}_2^k$.*

Proof. Let S be any tree on $2 \times \omega$ such that $\mathcal{I} = p[S]$. Define T by

$$(u, s) \in T \iff (\exists (v, t) \in S)(v \geq u \;\&\; t \leq s).$$

It follows that T satisfies properties 2 and 3. We must verify $\mathcal{I} = p[T]$. That $\mathcal{I} \subseteq p[T]$ derives from the inclusion $S \subseteq T$. For the other direction of containment, suppose that $(x, \alpha) \in [T]$. By definition, there exist pairs $(u_k, s_k) \in S \cap (2 \times \omega)^k$ such that $u_k \geq x \upharpoonright k$ and $s_k \leq \alpha \upharpoonright k$, for each k.

By compactness, there exist $y \in \mathbb{Z}_2^\omega$, $\beta \leq \alpha$ and $k_0 < k_1 < \ldots$ such that $s_{k_n} \to \beta$ and $u_{k_n} \to y$, as $n \to \infty$. It follows that $(y, \beta) \in [S]$ and $x \leq y$. Thus $x \in \mathcal{I}$, since $\mathcal{I}$ is an ideal. □

Proof of Theorem 2.55. For $s \in \omega^{<\omega}$ with $k = |s|$ and $j \in \omega$ let $A_j^s \subseteq \mathbb{Z}_2^k$ be such that

- If $u \leq v \in A_j^s$, then $u \in A_j^s$. (A_j^s is closed downwards.)
- For any $A \subseteq \mathbb{Z}_2^k$ which is closed downwards, there exists j such that $A = A_j^s$.

Let I_j^s be intervals, partitioning ω, such that each I_j^s has length equal to $|s|$. Define an analytic set $A_0 \subset \mathbb{Z}_2^\omega$ by

$$x \in A_0 \iff (\exists \alpha \in \omega^\omega)(\forall s, j, k)(k = |s| \ \&\ s \geq \alpha \upharpoonright k \implies x \upharpoonright I_j^s \in A_j^s).$$

Since each A_j^s is closed downward, if $x \leq y \in A_0$, then $x \in A_0$. Thus, taking $\mathcal{I}_0$ to be the ideal generated by A_0, we note that $\mathcal{I}_0$ is the set of finite unions of members of A_0. We will show that $\mathcal{I}_0$ is $\leq_{\mathrm{RK}}^+$-complete for analytic ideals. To this end, fix an analytic ideal $\mathcal{I} \subseteq \mathbb{Z}_2^\omega$. Let T be a tree on $2 \times \omega$ as in Lemma 2.56, with $p[T] = \mathcal{I}$. By item 3 of Lemma 2.56, we may choose, for each k and $s \in \omega^k$, a $\tau(s) \in \omega$ such that $A_{\tau(s)}^s = T_s \cap \mathbb{Z}_2^k$.

We will now define a weak Rudin-Keisler reduction of $\mathcal{I}$ to $\mathcal{I}_0$. Let

$$S = \bigcup_{s \in \omega^{<\omega}} I_{\tau(s)}^s.$$

This will be the domain of our reducing map. Define $\beta : S \to \omega$ by $\beta(i) = p$, if i is the pth element of $I^s_{\tau(s)}$, for some s. Note that the map $x \mapsto \beta^{-1}(x)$ is given by

$$\beta^{-1}(x) \restriction I^s_j = \begin{cases} x \restriction k & \text{if } j = \tau(s), \\ 0^k & \text{otherwise,} \end{cases}$$

for each $s \in \omega^k$ and $j \in \omega$. The following two claims will verify that β witnesses $\mathcal{I} \leq^+_{\mathrm{RK}} \mathcal{I}_0$.

Claim 1. If $x \in \mathcal{I}$, then $\beta^{-1}(x) \in \mathcal{I}_0$.

Proof of claim. It will suffice to show that $x \in \mathcal{I} \implies \beta^{-1}(x) \in A_0$, since $A_0 \subseteq \mathcal{I}_0$. Assuming $x \in \mathcal{I}$, let $\alpha \in \omega^\omega$ be such that $(x, \alpha) \in [T]$. We will see that α witnesses $\beta^{-1}(x) \in A_0$. Indeed, fix $s \in \omega^k$, with $s \geq \alpha \restriction k$ and consider $\beta^{-1}(x) \restriction I^s_j$. If $j \neq \tau(s)$, then $\beta^{-1}(x) \restriction I^s_j = 0^k \in A^s_j$, since A^s_j is closed downward. On the other hand, if $j = \tau(s)$, then

$$\begin{aligned} \beta^{-1}(x) \restriction I^s_j &= x \restriction k \\ &\in T_{\alpha \restriction k} \cap \mathbb{Z}^k_2 \\ &\subseteq T_s \cap \mathbb{Z}^k_2 \qquad \text{(since } s \geq \alpha \restriction k) \\ &= A^s_{\tau(s)}. \end{aligned}$$

This proves the claim.

Claim 2. If $\beta^{-1}(x) \in \mathcal{I}_0$, then $x \in \mathcal{I}$.

Proof of claim. Given that $\beta^{-1}(x) \in \mathcal{I}_0$, we take $y_1, \ldots, y_m \in A_0$ be such that $\beta^{-1}(x) = y_1 \cup \ldots \cup y_m$. For each $i \leq m$, let $\alpha_i \in \omega^\omega$ be as in the definition of A_0, witnessing the membership of y_i in A_0. It follows from the definition of A_0 that $\alpha = \alpha_1 + \ldots + \alpha_m$ also witnesses the membership of $y_1, \ldots, y_m$ in A_0. For each k, write $I_k = I^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$, $A_k = A^{\alpha \restriction k}_{\tau(\alpha \restriction k)}$ and let

$$u_i^k = y_i \restriction I_k.$$

Note that $u_i^k \in A_k = T_\alpha \restriction k$. By compactness, there exist $k_0 < k_1 < \ldots$ and $x_1, \ldots, x_m \in [T_\alpha]$ such that, for each $i \leq m$, we have $u_i^{k_n} \to x_i$, as $n \to \infty$.

Finally, we check that $x = x_1 \cup \ldots \cup x_m$. Observe that for each $p \in \omega$ and $k_n \geq p$ large enough that $(\forall i \leq m)(u_i^{k_n} \restriction p = x_i \restriction p)$, we have

$$\begin{aligned} x \restriction p &= (\beta^{-1}(x) \restriction I_{k_n}) \restriction p \\ &= u_1^{k_n} \cup \ldots \cup u_m^{k_n} \restriction p \\ &= x_1 \cup \ldots \cup x_m \restriction p. \end{aligned}$$

Since p was arbitrary, we conclude that $x = x_1 \cup \ldots \cup x_m$ and hence $x \in \mathcal{I}$, since ideals are closed under finite unions. This completes the proof. $\square$

Chapter 3

Selectors and universal sets

3.1 A selection theorem

Recall from Chapter 1 the definition of covering dimension:

1. An open cover $\mathcal{U}$ of a topological space X has *order* k iff there is a point of X which appears in k members of $\mathcal{U}$ and no element of X appears in more than k members of $\mathcal{U}$
2. An open cover $\mathcal{V}$ of a topological space X *refines* another cover $\mathcal{U}$ iff for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$.
3. A topological space X has *covering dimension* d if every open cover of X is refined by an open cover of order $d+1$ and furthermore, d is the smallest number for which this is true.

We state the main result of this chapter. As mentioned in the introduction, this refines work of Debs and Saint-Raymond, in the case of compact spaces of finite covering dimension.

Theorem 3.1. *Suppose that X and Y are Polish spaces where Y is pefect and X is compact with finite covering dimension d. Let $G \subseteq X \times Y$ be a G_δ set such that each vertical section G_x is dense. Then G contains a closed set F such that each F_x is* *nonempty, with cardinality at most $d+1$ and, for distinct $x, x' \in X$, F_x and $F_{x'}$ are disjoint. Moreover,* $\operatorname{range}(F) = \bigcup_{x \in X} F_x$ *is perfect and nowhere dense.*

Lemma 3.2. *Let X and Y be Polish and $\varepsilon > 0$ be fixed. Suppose that $U \subseteq X \times Y$ is open with each U_x dense, that $A_0 \times B_0 \subseteq X \times Y$ is an open rectangle and that $V \subseteq Y$ is a fixed open set. Then there exist families of open sets $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ such that*

1. *$A_0 = \bigcup \mathcal{A}$*
2. *For each $A \in \mathcal{A}$ and $B \in \mathcal{B}$,* $\operatorname{diam}(A), \operatorname{diam}(B) \leq \varepsilon$
3. *For each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\overline{A} \subseteq A_0$ and $\overline{B} \subseteq B_0$*
4. *For each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $\overline{A} \times \overline{B} \subseteq U$*

Proof. Since each U_x is dense, each $U_x \cap B \neq \emptyset$ and so we may choose, for each $x \in A_0$, open sets A_x and B_x such that $x \in A_x$ and $\overline{A}_x \times \overline{B}_x \subseteq U$. By shrinking the B_x as appropriate, we may ensure that each B_x has diameter less than ε. Let $\mathcal{A} = \{A_x : x \in A_0\}$ and $\mathcal{B} = \{B_x : x \in A_0\}$. □

Proof of Theorem 3.1. By assumption, X has finite covering dimension, let $d = \dim(X)$. Let $G \subseteq X \times Y$ be a G_δ set such that each G_x is dense. Let $U_0, U_1, \ldots$ be dense open sets such that $G = \bigcap_{n \in \omega} U_n$. Finally, let $\{V_0, V_1, \ldots\}$ be a countable topological basis for X.

We will construct a finite branching tree $T \subseteq \omega^{<\omega}$ and open sets $A_s \subseteq X$ and $B_s \subseteq Y$ (for $s \in T$) such that the following hold:

1. For each $s \in T \cap \omega^n$, A_s and B_s have diameter $\leq 1/(n+1)$

2. For each $n \in \omega$, $\{A_s : s \in T \cap \omega^n\}$ covers X and has order at most $d+1$

3. For each $n \in \omega$, $\{B_s : s \in T \cap \omega^n\}$ have disjoint closures

4. If $s, t \in T$ with $s \subseteq t$, then $A_s \supset \overline{A}_t$ and $B_s \supset \overline{B}_t$

5. For each $s \in T \cap \omega^n$, $\overline{A}_s \times \overline{B}_s \subseteq U_n$

6. For each $n \in \omega$, $V_n \setminus \bigcup_{s \in T \cap \omega^n} \overline{B}_s \neq \emptyset$

The construction is inductive. First of all, we let $A_{\langle\rangle} = X$ and $B_{\langle\rangle} = Y$. Suppose now that we have constructed $T \cap \omega^n$ and A_s and B_s for all $s \in T \cap \omega^n$, satisfying the conditions above.

Apply Lemma 3.2 separately to each $A_s \times B_s$ ($s \in T \cap \omega^n$) and combine the resulting covers to obtain families of open sets $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ such that

- $\mathcal{A}$ covers X

- For each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\operatorname{diam}(A), \operatorname{diam}(B) \leq 1/(n+1)$

- For each $A \in \mathcal{A}$, there exists $s \in T \cap \omega^n$ and $B \in \mathcal{B}$ such that $\overline{A} \times \overline{B} \subseteq (A_s \times B_s) \cap U_{n+1}$

Because X is compact, there is a finite subcover $A_0, \ldots, A_k \in \mathcal{A}$ of X, which (by passing to a refinement) we may assume to have order at most $d+1$. We now select k appropriate elements of $\mathcal{B}$ and (possibly shrinking them in the process) obtain open sets $B_0, \ldots, B_k \subseteq Y$, having disjoint closures, such that for each $i = 0, \ldots, k$

- There exists a (unique) $s \in T \cap \omega^n$ such that $\overline{A}_i \times \overline{B}_i \subseteq (A_s \times B_s) \cap U_{n+1}$

- $\overline{B}_0, \ldots, \overline{B}_k$ do not cover $V_n \cap B_s$

- $\{A_0, \ldots, A_k\}$ has order at most $d+1$ as a cover of X

Note that the disjointness of distinct B_s's (for $s \in T \cap \omega^n$) together with the second condition above ensure that property 6 will be satisfied.

We can now continue our construction of T. For each i, A_i and B_i will be assigned as $A_{s^\frown m}$ and $B_{s^\frown m}$ for some m iff s is the unique element of $T \cap \omega^n$ such that $\overline{A}_i \times \overline{B}_i \subseteq (A_s \times B_s) \cap U_{n+1}$. Note that there may be some $A_s \times B_s$ which do not get assigned any successors and for a given $s \in T \cap \omega^n$, A_s is not necessarily covered by $\{A_{s^\frown m} : s^\frown m \in T\}$.

Now define

$$F = \bigcap_{n \in \omega} \bigcup \{\overline{A}_s \times \overline{B}_s : s \in T \cap \omega^n\}.$$

Observe that F is closed and, by property 5 above, $F \subseteq G$. Properties 1 and 6 guarantee that range(F) is perfect and nowhere dense. We make the following three additional claims and conclude proof of this theorem.

A. For each $x \in X$, $F_x \neq \emptyset$

B. For distinct $x, x' \in X$, F_x and $F_{x'}$ are disjoint

C. For each $x \in X$, $|F_x| \leq d+1$

Let us say that $\alpha \in [T]$ "leads to (x, y)" iff $\{(x, y)\} = \bigcap_{n \in \omega} \overline{A}_{\alpha \restriction n} \times \overline{B}_{\alpha \restriction n}$. Note that this intersection will allways be a singleton since the A_s and B_s have vanishing diameter and are strongly nested (condition 4). Note that for a given $y \in Y$, by condition 3, there is at most one pair α, x, with $\alpha \in [T]$ and $x \in X$, such that α leads to (x, y).

A. This claim holds because, for each $n \in \omega$, $\{A_s : s \in T \cap \omega^n\}$ covers X and hence, for each each $x \in X$, there exists $\alpha \in [T]$ and $y \in Y$ such that α leads to (x, y).

B. Suppose that $\alpha, \beta \in [T]$, are such that α leads to (x, y) and β leads to (x', y'). If $x \neq x'$, then $\alpha \neq \beta$ and $y \neq y'$, since for some $n \in \omega$ $B_{\alpha\restriction n}$ and $B_{\beta\restriction n}$ have disjoint closures (by condition 3.) Claim B follows from this.

C. Suppose that, on the contrary, there exist $x \in X$ and distinct $y_0, \ldots, y_k \in Y$ with $k > d$ and $(x, y_i) \in F$, for each $i = 0, \ldots, k$. Let $\alpha_0, \ldots, \alpha_k \in [T]$ be such that each α_i leads to (x_i, y_i). The α_i are all distinct and so there exists $n \in \omega$ such that the $\alpha_i \restriction n$ are all distinct. We now have a contradiction, since x appears in each $A_{\alpha_i\restriction n}$, i.e., in more than $d + 1$ of the A_s, for $s \in T \cap \omega^n$. □

This theorem implies Theorem 1.11 in the case that X has finite covering dimension.

Corollary 3.3. *Suppose that X, Y and G are as in the statement of the theorem above, then there exists a Borel injection $f : X \to Y$ such that the graph of f is contained in G.*

Proof. Let $F \subseteq G$ be as obtained from Theorem 3.1. By the Lusin-Novikov Uniformization Theorem (Theorem 18.10 in [9]), F has a Borel uniformizing function f. The function f is one-to-one because all of the vertical sections of F are disjoint. □

Corollary 3.4. *Suppose that X is a σ-compact Polish space with finite covering dimension and Y is any uncountable Polish space. If $G \subseteq X \times Y$ is a G_δ set such that each G_x is dense, then there is a Borel injection $f \subseteq G$ with meager range.*

Proof. Let $K_0, K_1, \ldots \subseteq X$ be compact such that $X = \bigcup_{n\in\omega} K_n$. Let $U_0, U_1, \ldots \subseteq Y$ be disjoint open sets. For each n, apply Corollary 3.3 to $G \cap (K_n \times U_n)$ to obtain Borel injections $f_n : K_n \to U_n$ such that, for each n, $f_n \subseteq G$ and range(f_n) is nowhere dense. Define $f : X \to Y$ by

$$f(x) = \begin{cases} f_0(x) \text{ if } x \in K_0, \\ \\ f_{n+1}(x) \text{ if } x \in K_{n+1} \setminus K_n. \end{cases}$$

Observe that f is a one-to-one Borel map with meager range and $f \subseteq G$. $\square$

3.2 Examples

Theorem 3.1 guarantees that if $H \subseteq X \times Y$ is a dense G_δ such that all H_x are dense, then there is a closed set $F \subseteq H$ such that all F_x are nonempty, but have cardinality $\leq \dim(X) + 1$. The examples below show that this upper bound of $\dim(X) + 1$ cannot be improved.

The following fact is standard.

Lemma 3.5. *Suppose that X and Y are compact Polish spaces such that X is zero-dimensional and Y has covering dimension d. Then there is no continuous surjection $f : X \to Y$ which is better than $(d+1)$-to-one. (i.e., there is some $y \in Y$ such that $|f^{-1}(y)| \geq d + 1$.)*

Proof. Supose otherwise. Say $f : X \to Y$ is a continuous surjection and for each $y \in Y$, $|f^{-1}(y)| \leq d$, where d is the dimension of Y. Let $\mathcal{U}$ be any open cover of Y. We will show that $\mathcal{U}$ has an order d refinement, contradicting the assumption that Y has dimension d. By compactness, we may assume that $\mathcal{U}$ is finite. Let $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$. Since $\mathcal{V}$ is an open cover of X it has a refinement $\mathcal{V}'$, consisting of disjoint clopen sets. Let

$\mathcal{K} = \{f(V) : V \in \mathcal{V}'\}$. Observe that $\mathcal{K}$ is a finite cover of Y, consisting of closed sets, with the property that no point of Y occurs in more that d distinct members of $\mathcal{K}$.

List $\mathcal{K}$ as $\{K_0, \ldots, K_n\}$ and note that K_0 is disjoint from the intersection of any distinct $F_1, \ldots, F_d \in \mathcal{K}$ (with each $F_i \neq K_0$). Hence there exists an open set $U \supset K_0$ such that $\overline{U}$ is disjoint from $F_1 \cap \ldots \cap F_d$. Repeatedly shrink U to obtain an open set $U_0 \supset K_0$ such that $\overline{U}_0$ is disjoint from every the intersection of every sequence of d distinct members of $\mathcal{K}$ which are different from K_0. Replace $\mathcal{K}$ with $\mathcal{K}_0 = \{\overline{U}_0, K_1, \ldots, K_n\}$. Through the same process, obtain $U_1 \supset K_1$ such that $\overline{U}_1$ is disjoint from the intersection of any d distinct members of $\mathcal{K}_0$. Take $\mathcal{K}_1 = \{\overline{U}_0, \overline{U}_1, K_2, \ldots, K_n\}$.

Proceeding in this way, we obtain $\mathcal{K}_n = \{\overline{U}_0, \ldots, \overline{U}_n\}$, where $U_0, \ldots, U_n$ form an order d open cover of Y and for each i, $K_i \subseteq U_i$. Note that since each member of $\mathcal{K}$ was contained in some member of the original cover $\mathcal{U}$, we may assume (by intersecting with an appropriate member of $\mathcal{U}$) that each U_i is contained in some member of $\mathcal{U}$. We have now obtained our desired contradiction. $\square$

Example 3.6. Let $G_n \subseteq [0,1]^n \times [0,1]$ be a G_δ set such that each vertical and horizontal section is dense, but G_n is zero-dimensional in its subspace topology. (This could be achieved by letting G_n be $[0,1]^{n+1}$ minus a countable collection of hyperplanes.) Hence, by Theorem 3.1, there exists a closed $F \subseteq G_n$, such that for all $\mathbf{x} \in [0,1]^n$, $F_x \neq \emptyset$. Suppose F could be chosen such that for each $\mathbf{x} \in [0,1]$, $|F_x| \leq n$. Let $p : [0,1]^n \times [0,1] \to [0,1]^n$ be the projection map. Consider the restriction $p \restriction F : F \to [0,1]^n$. The map $p \restriction F$ is a continuous surjection and is n-to-one or better. This contradicts the lemma above.

The following example shows that no analogue of Theorem 3.1 can be proved in the case that X is not of finite covering dimension.

Example 3.7. We can combine the G_n to produce an analogous example in $[0,1]^\omega \times [0,1]$. For each $n \in \omega$, let $\hat{G}_n = \{(\xi, y) \in [0,1]^\omega \times [0,1] : (\xi \upharpoonright n, y) \in G_n\}$. Again, all vertical and horizontal sections of $\hat{G}_n$ are dense and $\hat{G}_n$ itself is G_δ.

If $F \subseteq \hat{G}_n$ is such for each $\xi \in [0,1]^\omega$, F_ξ is nonempty, then there must be some ξ such that $|F_\xi| > n$. Otherwise, we could fix $\xi_0 \in [0,1]^\omega$ and define $F' = \{(\mathbf{x}, y) \in [0,1]^n \times [0,1] : (\mathbf{x}^\frown \xi_0, y) \in F\}$. Then $F' \subseteq G_n$ and for each $x \in [0,1]^n$, $|F'_x| \leq n$, a contradiction.

Now take $G = \bigcap_{n\in\omega} \hat{G}_n$ and note that G is G_δ with all vertical and horizontal sections dense. If $F \subseteq G$ is a closed set such that, for each $\xi \in [0,1]^\omega$, $F_\xi \neq \emptyset$, then there there is no finite bound on the cardinality of F_ξ.

3.3 An example for measure

As mentioned in the introduction, we prove the following theorem of Graf and Mauldin using the example of a universal co-null F_σ set.

Theorem 3.8 (Graf, Mauldin). *There is a Borel set $B \subseteq [0,1] \times [0,1]$ such that each B_x and B_y is co-null, but B does not contain the graph of a Borel injection.*

3.3.1 A universal co-null F_σ set

We begin with a coding of co-null F_σ sets which is quite similar to a standard coding of dense G_δ sets. Recall that a set $A \subseteq X \times Y$ is *universal* for a class $\mathcal{C}$ of subsets of Y if, for each $B \in \mathcal{C}$, there exists $x \in X$ such that $A_x = B$.

Let Y be any Polish space with an associated σ-finite Borel measure μ. We will describe an F_σ set $F \subseteq 2^\omega \times Y$ which is universal for the co-null F_σ subsets of Y. It turns out to be easier to accomplish this by first coding null G_δ sets and then taking compliments. Let $\{V_n : n \in \omega\}$ be a basis for the topology of Y.

Fix $\varepsilon > 0$, for $x \in 2^\omega$ define

$$U_x^\varepsilon = \bigcup \{V_n : x(n) = 1 \ \& \sum_{\substack{k \le n \\ x(k)=1}} \mu(V_k) \le \varepsilon\}.$$

Observe that, for each $x \in 2^\omega$, $\mu(U_x^\varepsilon) \le \varepsilon$. Every open set $U \subseteq Y$, with $\mu(U) \le \varepsilon$ appears as U_x^ε, for some $x \in 2^\omega$. Also note that $U^\varepsilon = \{(x, y) : y \in U_x^\varepsilon\}$ is itself an open set.

Suppose now that $H = \bigcap_{n \in \omega} V_n$ is a null G_δ set. Then $\mu(V_n) \to 0$ as $n \to \infty$. Hence there is a subsequence $k_0, k_1, \ldots$ such that, for each i, $\mu(V_{k_i}) \le 1/i$. Therefore, by replacing V_n with $V_0 \cap \ldots \cap V_{k_n}$, we may assume that $\mu(V_n) \le 1/n$, for each n. Thus we may code null G_δ sets by taking

$$G_x = \bigcap_{n \in \omega} U^{\frac{1}{n}}_{(x)_{n+1}}.$$

The set $G = \{(x, y) : y \in G_x\}$ is G_δ and universal for null G_δ sets. If $F = 2^\omega \times Y \setminus G$, then F is universal for co-null F_σ sets.

Remark. It follows from the definition of G that each vertical section F_x of F is co-null. That each horizontal section F^y is co-null follows from the next lemma.

Lemma 3.9. $\lim\limits_{\varepsilon \to 0} \mu(\{x : U_x^\varepsilon \neq \emptyset\}) = 0$

Proof. Let $\beta > 0$ be arbitrary. Choose k such that $2^{-k} \le \beta$ and $\varepsilon > 0$ such that $V_0, \ldots, V_k$ all have measure greater than ε. Then for each $x \in 2^\omega$, U_x^ε is nonempty only if $x \restriction k = 0^k$. Thus $\lambda(\{x : U_x^\varepsilon \neq \emptyset\}) < 2^{-k} < \beta$. This proves the lemma. $\square$

3.3.2 The proof

We begin with a lemma:

Lemma 3.10. *Suppose that $F \subseteq X \times Y$ is universal for co-null F_σ sets. If $f : X \to Y$ is such that $f \subseteq F$, then* $\mathrm{range}(f)$ *is not a null set.*

Proof. Suppose otherwise. Say $f \subseteq F$ has null range. Hence $Y \setminus \mathrm{range}\,(f)$ is co-null and thus contains a co-null F_σ set. Let $x \in X$ be such that $F_x = Y \setminus \mathrm{range}\,(f)$. We now have a contradiction, since $f(x) \in F_x$ by the assumption that $f \subseteq F$, but $F_x \cap \mathrm{range}\,(f) = \emptyset$, by choice of x. □

Proof of Theorem 3.8. Let F be the universal co-null F_σ set decribed above.

Note that, in this coding, the F_σ set coded by $x \in 2^\omega$ is unaffected by the value of $(x)_0$. This lets us obtain $\mathfrak{c}$ many disjoint perfect sets P_z (for $z \in 2^\omega$) such that each null G_δ is coded by a real in every P_z by taking $P_z = \{x : (x)_0 = z\}$.

Suppose that $f \subseteq F$ is a Borel injection. For each $z \in 2^\omega$, $f(P_z)$ must be non-null, by Lemma 3.10. The images $f(P_z)$ are then $\mathfrak{c}$ many disjoint non-null Borel (and hence measurable) sets. This contradicts the σ-finiteness of μ. This proves that F does not contain the graph of a Borel injection.

We complete the proof by taking an embedding $\varphi : 2^\omega \to [0, 1]$ of the Cantor space into the unit interval. Let $F \subseteq 2^\omega \times [0, 1]$, be a universal co-null F_σ set as described above and define

$$B = \{(\varphi(x), y) : (x, y) \in F\} \cup \{(a, y) : a \notin \mathrm{range}(\varphi) \wedge y \in [0, 1]\}.$$

Observe that B is $\mathbf{\Sigma}^0_2$ and does not contain the graph of a Borel injection. Indeed, suppose on the contrary that $f \subseteq B$ is a Borel injection. Define $g : 2^\omega \to [0, 1]$ by

$g(x) = f(\varphi(x))$. We have $g \subseteq F$ and g is injective, a contradiction. $\square$

3.4 Universal sets

We derive a few more results using the methods and theorems above. We remind the reader of the definition of a universal set.

Definition 3.11. $U \subseteq X \times Y$ is *universal* for a class $\mathcal{C}$ of subsets of Y if, and only if, for each $A \in \mathcal{C}$, there is $x \in X$ such that $U_x = A$

The following is a related notion.

Definition 3.12. $U \subseteq X \times Y$ is *semi-universal* for a class $\mathcal{C}$ of subsets of Y if, and only if, for each $A \in \mathcal{C}$, there is $x \in X$ such that $U_x \subseteq A$

Note that a universal set for a class $\mathcal{C}$ is also semi-universal for $\mathcal{C}$.

Lemma 3.13. *Suppose that X, Y are Polish spaces with X compact. If $U \subseteq X \times Y$ is an open set such with each $U_x \neq \emptyset$, then there is a finite set $F \subseteq Y$ such that for all $x \in X$, $F \cap U_x \neq \emptyset$*

Proof. For each $x \in X$, let $U(x) \subseteq X$ and $V(x) \subseteq Y$ be open sets such that $x \in U(x) \times V(x) \subseteq U$. By compactness, there is a finite family of the $U(x)$'s, say $U_0, \ldots, U_n$ which cover X. Let $V_0, \ldots V_n$ be the corresponding $V(x)$'s. Then

$$U_0 \times V_0 \cup \ldots \cup U_n \times V_n \subseteq U.$$

Pick any elements $y_0, \ldots, y_n$, with each $y_i \in V_i$. Taking $F = \{y_0, \ldots, y_n\}$ we have a finite set which meets each vertical section U_x of U. $\square$

Theorem 3.14. *If X, Y are Polish with X compact, there is no open set $U \subseteq X \times Y$ which is semi-universal for nonempty open sets and such that each U_x is nonempty.*

Proof. Suppose on the contrary that U is such a semi-universal set. Since each U_x is nonempty, Lemma 3.13 implies that there is a finite set F such that for all $x \in X$, $F \cap U_x \neq \emptyset$. Let $V = Y \setminus F$ and observe that V is a nonempty open set which does not contain any of the U_x, since it is disjoint from F. This contradicts semi-universality. $\square$

The following lemma will yield a similar result for the class of dense open sets.

Lemma 3.15. *Suppose that X, Y are Polish spaces with X σ-compact and Y perfect. If $U \subseteq X \times Y$ is an open set with each U_x* dense, *then there exists a countable, closed nowhere dense set $C \subseteq Y$ such that, for each $x \in X$, $C \cap U_x \neq \emptyset$.*

Proof. Let $A_0, A_1, \ldots$ be compact, with $X = \bigcup_n A_n$. Take $y_n \to y$, distinct elements of Y. Let $B_0, B_1, \ldots \subseteq Y$ be open such that for each n, $y_n \in B_n$. Also, choose the B_n to have disjoint closures and have the property that

$$\operatorname{diam}(\{y\} \cup B_n \cup B_{n+1} \cup \ldots) \to 0 \text{ as } n \to \infty.$$

Since each U_x is dense, $B_n \cap U_x \neq \emptyset$, for each $x \in A_n$. By Lemma 3.13 there are finite sets $F_n \subseteq B_n$ such that for each $x \in A_n$, $F_n \cap U_x \neq \emptyset$. Now take

$$C = (\bigcup_{n \in \omega} F_n) \cup \{y\}.$$

By our choice of the B_n, y is the only limit point of C and since C is a countable closed set, it is nowhere dense. It is clear that $C \cap U_x \neq \emptyset$, for each $x \in X$. $\square$

Theorem 3.16. *If X, Y are Polish with X σ-compact, there is no semi-universal dense open set $U \subseteq X \times Y$.*

Proof. As in the proof of Theorem 3.14, we suppose that there was such a semi-universal set U. By Lemma 3.15 there is a closed nowhere dense set C which intersects each U_x. Then $V = Y \setminus C$ is open dense and for each $x \in X$, $U_x \nsubseteq V$. □

Theorems 3.14 and 3.16 also imply that there are no corresponding universal sets either.

Our next result concerns universal dense G_δ sets. It is known that one can parameterize the dense G_δ subsets of a second countable space with ω^ω. As it turns out, it is not possible to use 2^ω or even $\omega \times 2^\omega$ instead.

Theorem 3.17. *Suppose that X is a σ-compact Polish space and Y is an uncountable Polish space. There is no G_δ set $G \subseteq X \times Y$ such that G is semi-universal for dense G_δ subsets of Y.*

Lemma 3.18. *Suppose that $G \subseteq X \times Y$ is semi-universal for dense G_δ subsets of Y and $f : X \to Y$ is any map with $f \subseteq G$. The range of f must be non-meager.*

Proof. Suppose otherwise. Let $A = Y \setminus \mathrm{range}(f)$ and note that A is comeager. Hence there exists $x \in X$ such that $G_x \subseteq A$. By assumption that $f \subseteq G$ we must have $f(x) \in G_x$. On the other hand, G_x is disjoint from range(f), by our choice of x. This is a contradiction. □

Proof of Theorem 3.17. Suppose otherwise, say G is such a semi-universal set. Let $\varphi : \omega \times 2^\omega \to X$ be a continuous, open surjection. Define $G' \subseteq (\omega \times 2^\omega) \times Y$ by

$$G' = \{(\xi, y) : (\varphi(\xi), y) \in G\}.$$

G' is a G_δ set and is semi-universal for dense G_δ subsets of Y. By Corollary 3.4, there is a Borel injection $f : \omega \times 2^\omega \to Y$ such that $f \subseteq G'$ and range(f) is meager. This contradicts Lemma 3.18. $\square$

Although we used Corollary 3.4, we did not require that X have finite covering dimension in the statement of the Theorem 3.17

Analogous results do not hold for higher ranked Borel classes.

Lemma 3.19. *Suppose that Y is a Polish space. There is an F_σ set $C \subseteq \omega^\omega \times Y$ which is universal for countable dense subsets of Y*

Proof. For each $x \in \omega^\omega$, we can code a countable dense set as follows: Let $n \mapsto s_n$ be an enumeration of $\omega^{<\omega}$ and $\varphi : \omega^\omega \to Y$ a continuous surjection. The set coded by x will be $C_x = \{\varphi(s_n{}^\frown(x)_n) : n \in \omega\}$. Note that for all x, C_x is dense and that every countable dense set may be represented in this way. Also note that $C = \{(x, y) : y \in C_x\}$ is F_σ, since $y \in C_x$ iff there exists $n \in \omega$ such that $y = \varphi(s_n{}^\frown(x)_n)$ $\square$

Theorem 3.20. *For each $\alpha \geq 3$ and Polish space Y, there exists a universal $\mathbf{\Pi}^0_\alpha$ (resp. $\mathbf{\Sigma}^0_\alpha$) set $A \subseteq 2^\omega \times Y$ for dense (resp. comeager) $\mathbf{\Pi}^0_\alpha$ (resp. $\mathbf{\Sigma}^0_\alpha$) subsets of Y.*

Proof. We give the proof for $\mathbf{\Pi}^0_\alpha$. The proof for the $\mathbf{\Sigma}^0_\alpha$ case is exactly the same.

Comeager set case. Let $G \subseteq \omega^\omega \times Y$ be universal for dense G_δ subsets of Y. Note that ω^ω embeds in 2^ω homeomorphically as a G_δ set. We may therefore map G to a G_δ set $G' \subseteq 2^\omega \times Y$ such that every dense G_δ subset of Y appears as G'_x for some $x \in 2^\omega$, but for an F_σ set of x's, G'_x is empty. By making the empty sections of G' all of Y, we replace G' with a $\mathbf{\Delta}^0_3$ set G^* such that every vertical section of G^* is a dense G_δ and every dense G_δ appears as one of the vertical sections of G^*.

Now let $A \subseteq 2^\omega \times Y$ be a universal $\mathbf{\Pi}^0_\alpha$ set (Theorem 22.3 in Kechris) and define B^* by

$$B^*_{\langle x,y\rangle} = G^*_x \cup A_y.$$

Then B^* is $\mathbf{\Pi}^0_\alpha$ and universal for comeager $\mathbf{\Pi}^0_\alpha$ subsets of Y.

Dense set case. Let $A \subseteq 2^\omega \times Y$ be universal for $\mathbf{\Pi}^0_\alpha$ subsets of Y and let C be as obtained in Lemma 3.19.

As above, C embeds in $2^\omega \times Y$ as a $\mathbf{\Delta}^0_3$ set C' such that all vertical section of C' are either countable dense or empty. Replacing each of the empty sections with all of Y, we obtain another $\mathbf{\Delta}^0_3$ set C^*.

Define A^* by

$$A^*_{\langle x,y\rangle} = C^*_x \cup A_y.$$

A^* is $\mathbf{\Pi}^0_\alpha$ and is universal for dense $\mathbf{\Pi}^0_\alpha$ sets. □

Note that we could have used any perfect Polish space X as the coding space in the theorem above, since 2^ω would embed in X as a closed set.

3.5 Appendix

To the best of our knowledge, [2] has not been translated. For the sake of completeness, we therefore give a version of Debs' and Saint-Raymond's proof of Theorem 1.11 and their example, answering the category version of Mauldin's original question about the existence of Borel selectors.

Proof of Theorem 1.11. Suppose that $G \subseteq X \times Y$ is as in the statement of the Theorem. Let $U_0 \supset U_1 \supset \ldots$ be dense open sets such that $G = \bigcap_{n\in\omega} U_n$ and let $\{V_n : n \in \omega\}$ be a topological basis for Y. We will construct inductively a finite branching tree $T \subseteq \omega^{<\omega}$ and for each $s \in T$, sets $A_s \subseteq X$ (a difference of open sets) and $B_s \subseteq Y$ (an open set) such that the following hold

1. $A_s = \bigcup\{A_{s^\frown m} : s^\frown m \in T\}$
2. $\mathrm{diam}(A_s), \mathrm{diam}(B_s) < 1/(|s|+1)$
3. For each $s, t \in T$, if $s \subseteq t$, then $\overline{B}_t \subseteq B_s$
4. For each $s, t \in T$, if $s \perp t$, then $A_s \cap A_t = \emptyset$ and $\overline{B}_s \cap \overline{B}_t = \emptyset$
5. $A_s \times \overline{B}_s \subseteq U_{|s|}$ and $V_{|s|} \setminus \overline{B}_s \neq \emptyset$

To start the induction, we let $A_{\langle\rangle} = X$ and $B_{\langle\rangle} = Y$. Suppose now that we have defined the first n levels of T and corresponding A_s and B_s satisfying items 1-5 above. For each $s \in T \cap \omega^n$, we carry out the following construction. For convenience, write $A = A_s$ and $B = B_s$. For each $x \in \overline{A}$, there exist open sets $A_x \subseteq X$ and $B_x \subseteq Y$ such that

- $\mathrm{diam}(A_x), \mathrm{diam}(B_x) < 1/(n+2)$
- $x \in A_x$
- $\overline{B}_x \subseteq B$
- $A_x \times \overline{B}_x \subseteq U_{n+1}$

The A_x's cover $\overline{A}$. By compactness, there is a finite subcover $A_0, \ldots, A_n$. Let $B_0, \ldots, B_n$ be the corresponding B_x's. For each i, let $A_i' = (A \cap A_i) \setminus (A_0 \cup \ldots \cup A_{i-1})$.

(Let $A'_0 = A \cap A_0$.) The A'_i are differences of open sets. For each $i = 0, \ldots, n$, we will put $s^\frown i \in T$ and let $A_{s^\frown i} = A'_i$.

Now shrink $B_0, \ldots, B_n$ to $B'_0, \ldots, B'_n$ so that $\overline{B}'_0, \ldots, \overline{B}'_n$ are all disjoint and for each i, $V_{n+1} \setminus \overline{B}'_i \neq \emptyset$. Let $B_{s^\frown i} = B'_i$

For each $n \in \omega$, let

$$F_n = \bigcup \{A_s \times \overline{B}_s : s \in T \cap \omega^n\}$$

and define $f = \bigcap_{n \in \omega} F_n$. We have that $f \subseteq G$ and f is the graph of a Borel injection with closed, nowhere dense range.

Note that if X had been zero-dimensional, then we could have taken the A_s to be clopen and, consequently, each F_n would have been closed and f would be continuous.

□

Theorem 3.21. *There is a $\mathbf{\Delta}^0_3$ set $B \subseteq [0,1]^2$, such that all vertical and horizontal sections of B are comeager and B does not contain the graph of a Borel injection.*

First we desribe a coding of dense G_δ sets. Fix an enumeration $\{s_n : n \in \omega\}$ of $\omega^{<\omega}$. Fix a Polish space Y and let $\varphi : \omega^\omega \to Y$ be a continuous, open surjection. Define $U \subseteq \omega^\omega \times Y$ as follows: for each $x \in \omega^\omega$ the vertical section U_x of U is the open dense set

$$\bigcup_{n \in \omega} \varphi([s_n{}^\frown s_{x(n)}]).$$

Note that U itself is open and universal for dense open subsets of Y. We code dense G_δ sets by setting

$$G_x = \bigcap_{n \in \omega} U_{(x)_{n+1}}.$$

Let $G = \{(x, y) : y \in G_x\}$ and observe that G is a universal dense G_δ set for Y.

We make the observation that G_x is unaffected by $(x)_0$. Take $P_z = \{x \in \omega^\omega : (x)_0 = z\}$, for $z \in \omega^\omega$. The P_z are pairwise disjoint closed sets and, for each z, $G \cap (P_z \times Y)$ is still universal for dense G_δ subset of Y – a given dense G_δ subset of Y is coded by a real in each P_z.

Lemma 3.22. *If $G \subseteq \omega^\omega \times Y$ is a universal dense G_δ set as described above, then G does not contain the graph of a Borel injection.*

Proof. Suppose that $f : \omega^\omega \to Y$ is a Borel injection with $f \subseteq G$. For each z, we must have that $f(P_z)$ is non-meager, by Lemma 3.18. Also note that each $f(P_z)$ has the Property of Baire, since the P_z are closed and f is Borel. We now have a contradiction, since the sets $f(P_z)$ are all disjoint (by the injectivity of f) and there cannot be $\mathfrak{c}$ many disjoint non-meager sets with the Property of Baire in a Polish space. □

We now finish the proof of Theorem 3.21

Proof of Theorem 3.21. Let $G \subseteq \omega^\omega \times [0,1]$ be a universal dense as described above. Take $\varphi : \omega^\omega \to [0,1]$ to be a continuous embedding of ω^ω as a G_δ subset of $[0,1]$. Define

$$A = \{(\varphi(x), y) : (x,y) \in G\} \cup \{(a,y) : a \notin \text{range}(\varphi) \wedge y \in [0,1]\}.$$

Then A is $\mathbf{\Delta}^0_3$ and does not contain the graph of a Borel injection $f : [0,1] \to [0,1]$. Otherwise, $\varphi \circ f$ would be a Borel injection contained in G. □

Remark. We can see that each G^y is comeager by noting that, for a given $y \in Y$, $\{x \in \omega^\omega : y \in U_{(x)_n}\}$ is a dense open set, for each $n \in \omega$. Hence

$$\begin{aligned} G^y &= \{x \in \omega^\omega : y \in G_x\} \\ &= \bigcap_{n \in \omega} \{x \in \omega^\omega : y \in U_{(x)_n}\} \end{aligned}$$

must be comeager.

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