

Some results on algorithmic randomness and computability-theoretic strength

By

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Abstract

Algorithmic randomness uses tools from computability theory to give precise formulations for what it means for mathematical objects to be random. When the objects in question are reals (infinite sequences of zeros and ones), it reveals complex interactions between how random they are and how useful they are as computational oracles. The results in this thesis are primarily on interactions of this nature.

Chapter 1 provides a brief introduction to notation and basic notions from computability theory.

Chapter 2 is on *shift-complex sequences*, also known as *everywhere complex sequences*. These are sequences all of whose substrings have uniformly high prefix-free Kolmogorov complexity. Romyantsev showed that the measure of oracles that compute shift-complex sequences is 0. We refine this result to show that the Martin-Löf random sequences that compute shift-complex sequences compute the halting problem. In the other direction, we answer the question of whether every Martin-Löf random sequence computes a shift-complex sequence in the negative by translating it into a question about *diagonally noncomputable* (or DNC) functions.

The key in this result is analyzing how growth rates of DNC functions affect what they can compute. This is the subject of Chapter 3. Using bushy-tree forcing, we show (with J. Miller) that there are arbitrarily slow-growing (but unbounded) DNC functions that fail to compute a Kurtz random sequence. We also extend Kumabe's result that there is a DNC function of minimal Turing degree by showing that for every oracle X , there is a function f that is DNC relative to X and of minimal Turing degree.

Chapter 4 is on how “effective” Lebesgue density interacts with computability-theoretic strength and randomness. Bienvenu, Hölzl, Miller, and Nies showed that if we restrict our attention to the Martin-Löf random sequences, then the positive density sequences are exactly the ones that do not compute the halting problem. We prove several facts around this theorem. For example, one direction of the theorem fails without the assumption of Martin-Löf randomness: Given any sequence X , there is a density-one sequence Y that computes it. Another question we answer is whether a positive density point can have minimal degree. It turns out that every such point is either Martin-Löf random, or computes a 1-generic. In either case, it is nonminimal.

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Chapter 1

Introduction

The results in this thesis align with the program in computability theory that is concerned with the interactions between different types of noncomputability. Broadly speaking, the emphasis here is on two main “flavors” of noncomputability. The first arises in the study of *algorithmic randomness*. Noncomputability notions of this flavor typically involve measure or information-theoretic tools such as Kolmogorov complexity in their formulation. Examples are *shift-complexity* and *positive density*, which are the subjects of chapters 2 and 4, respectively. Notions of the second flavor have their roots in “classical” computability theory. Examples are the ability to compute the *halting problem*, and *diagonal noncomputability* (the subject of chapter 3).

As an illustration of the kind of interaction we are interested in, let us consider Sacks’s theorem [30], which says that given any noncomputable infinite binary sequence X , the set of sequences that compute X has Lebesgue measure 0. One particular consequence of this theorem is that a random infinite binary sequence is almost surely *incomplete*, meaning that it does not possess the ability to compute the halting problem.

On the other hand, there are fairly strong notions of randomness that do not preclude this ability. For example, the Kučera-Gács theorem implies that Martin-Löf randomness is compatible with a form of “coding”: Given any sequence, there is a Martin-Löf random sequence that computes it. In particular, there are Martin-Löf random sequences

that compute the halting problem. A properly stronger form of randomness, *difference randomness*, exactly characterizes the incomplete Martin-Löf randoms. An even stronger notion, *weak 2-randomness*, widens the separation: a weak 2-random sequence and the halting problem have no noncomputable information in common, in the sense that they cannot both compute the same noncomputable sequence. We say that they form a *minimal pair*. Interestingly, the property of being Martin-Löf random and forming a minimal pair with the halting problem exactly characterizes the weak 2-random sequences. Further, the randomness notions just mentioned are formulated in a manner that seems, on first glance, to have little to do with classical forms of noncomputability related to the halting problem. While algorithmic randomness began as an application of computability theory to the philosophical question of what it means for various types of mathematical objects to be random, interactions such as the above demonstrate a deeper, more bilateral, relationship, the study of which has become one of the most active areas in mathematical logic.

Our notation and terminology are standard. Nevertheless, we provide an overview in this section for easy reference. Soare [32] and the first chapters of Downey and Hirschfeldt [11] and Nies [27] provide in-depth accounts of the requisite computability theory.

1.1 Shared notation

Cantor space, Baire space: Let ω denote the set of natural numbers. Baire space, denoted by ω^ω , is the set of infinite sequences of elements of ω (or *functions* from ω to itself). Cantor space, denoted by 2^ω , is the set of all infinite binary sequences (or *reals*).

Strings, prefix-freeness: A *string* is a finite sequence of natural numbers. The set of strings is denoted by $\omega^{<\omega}$. The set of strings that only have binary digits is denoted by $2^{<\omega}$. The empty string is denoted by $\langle \rangle$. If σ is a string, $|\sigma|$ denotes its length. If ρ is another string, we write $\sigma \preceq \rho$ to indicate that σ is a prefix of ρ . If X is a function, and σ a string, we write $\sigma \prec X$ to indicate that σ is an initial segment of X , while $X \upharpoonright n$ denotes the initial segment of X of length n .

A set of strings S is *prefix-free* if for any distinct σ and τ in S , neither is a prefix of the other.

Join: If X and Y are functions, then the *join* $X \oplus Y$ is the function obtained by interleaving the entries of X and Y . The join of two strings of equal length is defined in the same way.

Open sets: Cantor space and Baire space are endowed with topologies generated by basic clopen sets of the form $[\sigma] = \{X : \sigma \prec X\}$, where σ is a string. If S is a set of strings, then $[S]^\prec$ denotes the open set $\bigcup_{\sigma \in S} [\sigma]$.

Measure: The *uniform measure* μ on Cantor space is obtained by setting $\mu([\sigma]) = 2^{-|\sigma|}$. This is essentially the only measure we work with.

1.2 Basic computability notions

Partial computable functions: A partial function from ω to ω is *partial computable* if there is an algorithm that implements the function. There are countably many algorithms, and the ones that implement partial computable functions can be listed effectively. Let $\langle \varphi_e \rangle_{e \in \omega}$ be such a listing. If the partial computable function $\theta = \varphi_e$, then we say e is an *index* for θ .

The diagonal function, the halting problem, completeness: The *diagonal partial computable function* J is of particular importance to us:

$$J(n) = \varphi_n(n).$$

The domain of J is a well-known set, the *halting problem*. We denote it by $0'$. A function that computes $0'$ is said to be *complete*.

Computable sets and functions, c.e. sets: A partial computable function that is total is said to be *computable*. A real is computable if it is computable when viewed as the characteristic function of a set of natural numbers. More generally, we can speak of computable sets of objects that can be coded by natural numbers (e.g., sets of finite strings or rationals). A set is *computably enumerable*, or *c.e.*, if there is an algorithm that lists its elements in some order. The c.e. sets are exactly the domains of the partial computable functions, and so there is an effective listing $\langle W_e \rangle_{e \in \omega}$ of c.e. sets.

Diagonally noncomputable functions, order functions: A total function f is *diagonally noncomputable* (or DNC) if for all $n \in \text{dom}(J)$, $f(n) \neq J(n)$.

Let DNC be the class of diagonally noncomputable functions. For $a \geq 2$, DNC_a denotes $\text{DNC} \cap a^\omega$, i.e., the space of all DNC functions that take values less than a .

An *order function* is a function $h : \omega \rightarrow \omega \setminus \{0, 1\}$ that is computable, non-decreasing and unbounded. For an order function h , let DNC_h denote $\{f \in \text{DNC} : (\forall n) f(n) < h(n)\}$.

Effectively open and closed classes: If S is a c.e. set of strings, then we say that the open set $[S]^\prec$ is *effectively open*, or a Σ_1^0 class. The complement of a Σ_1^0 class is an *effectively closed* or Π_1^0 class. The e -th Σ_1^0 class is then simply $[W_e]^\prec$, where we are

thinking of W_e as listing a set of strings.

Trees: A *tree* is a set of strings that is closed under initial segments. If T is a tree, then $[T]$ denotes the set of *paths through T* , i.e., the functions X such that for every $n \in \omega$, $X \upharpoonright n \in T$.

Turing reducibility, functionals: Suppose that an algorithm Γ , when given black-box (or *oracle*) access to the function X , computes the function Y . We write $\Gamma^X = Y$ and say that Γ is a *Turing reduction* of Y to X . We often omit mention of the specific reduction, saying Y is *Turing reducible* to X , written $Y \leq_T X$, to mean that there is *some* reduction of Y to X . At other times, the emphasis is less on the particular functions X and Y , than on the reduction Γ as a (possibly partial) map from Baire space to itself. In such situations, we refer to Γ as a *Turing functional*. If Γ is total on 2^ω , we say it is a *truth-table functional*.

Turing degrees: The preorder \leq_T on functions induces the equivalence relation \equiv_T . The equivalence classes under this relation are called the *Turing degrees*. If \mathbf{a} and \mathbf{b} are degrees, we write $\mathbf{a} \leq \mathbf{b}$ to mean that functions in \mathbf{b} compute functions in \mathbf{a} . The degree of the computable functions is denoted by $\mathbf{0}$.

PA degrees: A function is of *PA degree* if it computes a member of every nonempty Π_1^0 class. PA degrees are so named because they compute consistent, complete extensions of Peano arithmetic.

Minimal degrees: A Turing degree \mathbf{a} is *minimal* if for every degree $\mathbf{b} \leq \mathbf{a}$, either $\mathbf{b} = \mathbf{a}$ or $\mathbf{b} = \mathbf{0}$.

Hyperimmunity, highness: A degree is *hyperimmune* if it computes a function f such that for every computable function g , there are infinitely many n such that $f(n) > g(n)$. Otherwise, it is *hyperimmune-free*. It is *high* if it computes a function h such that for

every computable function g , for all but finitely many $n \in \omega$, $h(n) > g(n)$.

1-genericity: A real X is 1-generic if for every Σ_1^0 class W in Cantor space, either X is in W , or there is an $n \in \omega$ such that $[X \upharpoonright n]$ is disjoint from W .

Martin-Löf randomness: We say a sequence $\langle A_n \rangle_{n \in \omega}$ of Σ_1^0 classes is *uniform* if there is a computable function f such that $A_n = [W_{f(n)}]^\prec$.

A *Martin-Löf test* is a uniform nested sequence $\langle A_n \rangle_{n \in \omega}$ of Σ_1^0 classes in 2^ω such that $\mu(A_n) \leq 2^{-n}$. A real $X \in 2^\omega$ *passes this test* if it is not contained in $\bigcap_n A_n$. Finally, X is *Martin-Löf random* if it passes every Martin-Löf test.

Kurtz randomness: A real X is *Kurtz random* if it is not contained in any Π_1^0 class that has measure 0. The class of Kurtz random reals contains both the Martin-Löf random reals and the 1-generics.

Chapter 2

Shift-complex sequences

The contents of this chapter have appeared in [20].

2.1 Introduction

We use $K : 2^{<\omega} \rightarrow \omega$ to denote *prefix-free Kolmogorov complexity*. Informally, prefix-free Kolmogorov complexity is a measure of how complicated a string is to describe, or equivalently, how *incompressible* it is. A string of a million zeros ought to have a short description relative to its length and hence Kolmogorov complexity much smaller than a million. On the other hand, a string of random bits will, with high probability, have no description shorter than itself, hence Kolmogorov complexity roughly equal to its length.

One of the major threads in the study of algorithmic randomness has been to determine how information-theoretic formulations of string complexity (such as prefix-free Kolmogorov complexity) interact with measure-theoretic formulations of randomness of infinite sequences. For example, the class of Martin-Löf random sequences was originally defined in terms of *tests* that generalize the idea of statistical tests of randomness. But it also has a precise characterization in terms of prefix-free Kolmogorov complexity. If f and g are functions from some set S to ω , then we write $f \leq^+ g$ to indicate that there exists a constant $c \in \omega$ such that for all $x \in S$, $f(x) \leq g(x) + c$. Schnorr showed¹ that

¹See, for example, [27] Section 3.2.

a sequence $X \in 2^\omega$ is Martin-Löf random if and only if $K(X \upharpoonright n) \geq^+ n$, where $X \upharpoonright n$ is the initial segment of X of length n .

Every Martin-Löf random sequence is *normal*, meaning that finite strings of equal length occur as substrings with equal asymptotic frequency. Therefore, every Martin-Löf random sequence must have substrings that are compressible by more than any given factor². Because of Schnorr’s theorem, we cannot hope to construct sequences with the property that for all substrings σ , $K(\sigma) \geq^+ |\sigma|$. However, it is possible to construct sequences that are uniformly *somewhat* complex everywhere. The following definitions make this idea precise:

Definition 2.1. Fix $\delta \in (0, 1)$ and $b \in \omega$. A sequence $X \in 2^\omega$ is (δ, b) -*shift-complex* if for every substring σ of X , $K(\sigma) \geq \delta|\sigma| - b$.

A sequence is δ -*shift-complex* if it is (δ, b) -shift-complex for some $b \in \omega$.

A sequence is *shift-complex* if it is δ -shift-complex for some $\delta \in (0, 1)$.

In this chapter, we contribute two new results that shed light on how shift-complex sequences relate to Martin-Löf random sequences in the setting of the Turing degrees. Access to a sequence of the latter type implies access to arbitrarily long finite strings of high complexity, so can we exploit such resources to *effectively* produce shift-complex sequences? A result by Romyantsev [29] states that this is not the case: if a sequence is “sufficiently random”, then it cannot compute any shift-complex sequence. We calibrate precisely the level of randomness at which this theorem holds. In a similar vein, we ask if access to a shift-complex sequence enables us to effectively obtain a Martin-Löf random sequence. Again, the answer is no.

²As an extreme example, if 0^n denotes a string of zeros of length n , then $K(0^n) \leq^+ 2 \log(n)$.

2.2 Definitions and notation

In the previous section, we introduced the preordering \geq^+ . The equivalence relation induced by this preordering is denoted by $=^+$.

A *prefix-free machine* is a partial computable function from strings to strings that has prefix-free domain. Let \mathbb{U} be a *universal* prefix-free machine: it can simulate any other prefix-free machine. In particular, this means that every string has a \mathbb{U} -*description*, i.e., for each string σ , there is an input τ to \mathbb{U} that produces σ as the output. Then, for a string σ , $K(\sigma)$ is the length of a shortest \mathbb{U} -description of σ . The string σ^* is a distinguished shortest description of σ : it is the lexicographically least of the strings of length $K(\sigma)$ on which the universal prefix-free machine halts and outputs σ in the least number of steps. If τ is a string, the *conditional prefix-free Kolmogorov complexity* $K(\sigma \mid \tau)$ is defined much as before, except now we imagine that \mathbb{U} has access to τ . For an in-depth account of the theory of Kolmogorov complexity, we refer the reader to Downey and Hirschfeldt [11] or Nies [27].

Finally, if X is a sequence in 2^ω , the *effective Hausdorff dimension* of X , denoted by $\dim(X)$, is

$$\liminf_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n},$$

while the *effective packing dimension* of X , denoted by $\text{Dim}(X)$, is obtained by replacing the \liminf with a \limsup .

We begin by surveying the known constructions of shift-complex sequences in Section 2.3, and show that they lead easily to the existence of bi-infinite shift-complex sequences (i.e., sequences indexed by ξ , the order type of the integers) in Section 2.4. In Section 2.5, we provide a proof of a result by Rumyantsev that shows that the measure of oracles

that compute shift-complex sequences is 0. We also answer the following question that arises from Romyantsev’s theorem:

Question 2.2. How random does an oracle have to be to ensure that it does *not* compute a shift-complex sequence?

In Theorem 2.21, we show that every Martin-Löf random that computes a shift-complex sequence is Turing complete, using the characterization by Franklin and Ng [15] of this class via a notion of randomness they term *difference randomness*. The difference randoms are those sequences that escape being captured by *difference tests*, which are a generalization of Martin-Löf tests where each test element is a *difference* of Σ_1^0 sets. They comprise a class properly between the weak 2-randoms and the Martin-Löf randoms. We remark that Theorem 2.21 could be viewed as a generalization of Stephan’s result in [33] that every Martin-Löf random of PA degree is Turing complete.

In Section 2.6, we turn to the question of whether shift-complex sequences can compute random sequences. In Theorem 2.27, we show that there are shift-complex sequences that compute no Kurtz random real (and hence no Martin-Löf random real). In order to do so, we adapt the technique employed by Greenberg and Miller [16] of using slow-growing diagonally noncomputable functions to avoid computing random reals.

2.3 Constructions of shift-complex sequences

At the time of writing, there are three existence proofs of shift-complex sequences in the literature. The first is due to Durand, Levin and Shen [12]:

Theorem 2.3 (Durand, Levin and Shen [12]). *For any $0 < \delta < 1$, there exists a δ -shift-complex sequence.*

Proof. The construction proceeds by building a sequence using extensions of an appropriately chosen fixed length m (which depends only on δ), such that each extension is sufficiently complex relative to the initial segment constructed so far.

First, note that

$$K(xy) + K(|y|) \geq^+ K(x, y)$$

since there is a prefix-free machine that, from a description of xy concatenated with a description of $|y|$, can recover both x and y , and \mathbb{U} can simulate this machine.

By the prefix-free symmetry of information theorem³, $K(x, y) =^+ K(x) + K(y | x^*)$. Since $K(|y|) \leq^+ 2 \log(|y|)$, we have $K(xy) - K(x) \geq^+ K(y | x^*) - 2 \log(|y|)$. Let c_0 be such that $K(xy) - K(x) \geq K(y | x^*) - 2 \log(|y|) - c_0$.

Now for any length n , and for any x , we can choose a string y of length n so that $K(y | x^*) \geq n$. Let m be large enough so that $m - 2 \log m - c_0 \geq \delta m$. Then for any string x , there exists a string y of length m such that $K(xy) - K(x) \geq \delta m$.

We construct A in blocks of size m , each time choosing a block such that the complexity of the string built so far increases by at least δm . It follows immediately that the desired property holds for initial segments of A of lengths that are multiples of m . We claim that $K(\sigma) \geq \delta |\sigma|$ also holds for substrings σ of A that start and end at indices that are multiples of m . Note that there is a constant c_1 such that

$$K(xy) \leq K(x) + K(y) + c_1.$$

If $\alpha\sigma$ is an initial segment of A where σ starts and ends at indices that are multiples of m , $K(\alpha\sigma) - K(\alpha) \geq \delta |\sigma|$. By the inequality above, $K(\sigma) \geq \delta |\sigma| - c_1$.

³For more details, see Downey and Hirschfeldt [11], chapter 3.

Next, if σ is an arbitrary subsequence of A , we can pad σ at either end to obtain a string $\alpha\sigma\beta$ that starts and ends at indices that are multiples of m . Since both α and β have length less than m , there is a constant c_2 such that

$$K(\sigma) + c_2 \geq K(\alpha\sigma\beta) \geq \delta|\alpha\sigma\beta| - c_1 \geq \delta|\sigma| - c_1,$$

and so $K(\sigma) \geq \delta|\sigma| - (c_1 + c_2)$. Thus A is $(\delta, c_1 + c_2)$ -shift-complex. \square

Rumyantsev and Ushakov in [28] provide an alternative existence proof. They use the Lovász Local Lemma to show that for each δ , there is a $b \in \omega$ such that there exist arbitrarily long *finite* (δ, b) -shift-complex strings. The existence of an infinite (δ, b) -shift-complex sequence then follows from the compactness of 2^ω .

The final construction is due to Miller [25]. We say that $X \in n^\omega$ *avoids* a set $S \subseteq n^{<\omega}$ if no $\sigma \in S$ is a substring of X . The set of all sequences in n^ω that avoid a given set of finite strings is called a *subshift*. Miller provides a condition on the lengths of strings in S that guarantees that the subshift of S is nonempty. We outline the proof of this result since an effective version of this construction is required in the proof of Theorem 2.25. Since we are interested only in binary sequences, we present it for the case $n = 2$:

Theorem 2.4 (Miller [25]). *Let $S \subseteq 2^{<\omega}$. If there is a $c \in (1/2, 1)$ such that*

$$\sum_{\tau \in S} c^{|\tau|} \leq 2c - 1,$$

then there is an $X \in 2^\omega$ that avoids S .

Proof. For $\sigma, \tau \in 2^{<\omega}$, we define

$$T_{\sigma, \tau} = \{\rho \in 2^{<\omega} : |\rho| < |\tau| \text{ and } \sigma\rho \text{ ends in } \tau\}$$

and let

$$w(\sigma) = \sum_{\tau \in S} \sum_{\rho \in T_{\sigma, \tau}} c^{|\rho|}.$$

It is helpful to think of $w(\sigma)$ as a measure of the danger of an extension of σ ending in a forbidden string. Note that if σ ends in a string $\tau \in S$, then $\langle \rangle \in T_{\sigma, \tau}$, and so $w(\sigma) \geq 1$. We build the sequence X a bit at a time, ensuring that for each initial segment σ of X , $w(\sigma) < 1$. Then X avoids S .

Suppose that we have built a string σ that avoids S and that $w(\sigma) < 1$. Note that $w(\sigma 0) + w(\sigma 1) = w(\sigma)/c + (\sum_{\tau \in S} c^{|\tau|})/c$. The second term on the right corresponds to the new threats that emerge as a result of extending σ by either a 0 or a 1, while the first term corresponds to the existing threats to σ , magnified by a factor of $1/c$. So we have

$$w(\sigma 0) + w(\sigma 1) = \frac{w(\sigma)}{c} + \frac{1}{c} \sum_{\tau \in S} c^{|\tau|} < \frac{1}{c} + \frac{2c-1}{c} = 2,$$

from which it follows that either $w(\sigma 0)$ or $w(\sigma 1)$ is strictly less than 1. \square

Theorem 2.3 follows as a corollary.

Corollary 2.5 (Miller [25]). *Fix $\delta \in (0, 1)$. Let $b = -\log(1 - \delta) + 1$ and $S = \{\tau \in 2^{<\omega} \mid K(\tau) < \delta|\tau| - b\}$. Then there is an $X \in 2^\omega$ that avoids S , and is therefore δ -shift-complex.*

Proof. Let $c = 2^{-\delta}$. Then

$$\sum_{\tau \in S} c^{|\tau|} = \sum_{\tau \in S} 2^{-\delta|\tau|} < \sum_{\tau \in S} 2^{-(K(\tau)+b)} = 2^{-b} \sum_{\tau \in S} 2^{-K(\tau)} \leq 2^{-b}$$

where the last inequality follows from Kraft's inequality⁴. Now $2^{-b} = (1 - \delta)/2$, which for $\delta \in (0, 1)$ is less than $2^{1-\delta} - 1 = 2c - 1$, so we can apply Theorem 2.4. \square

An advantage of the construction in Theorem 2.4 is that it can be effectivized to yield the following:

Proposition 2.6. *Suppose $S \subseteq 2^{<\omega}$ satisfies the condition of Theorem 2.4 and is computable. Then there is a computable $X \in 2^\omega$ that avoids S .*

Before proceeding with the proof, we establish some terminology. We say $c \in \mathbb{R}$ is a *computable real number* if there is an algorithm which, when given a rational ε , outputs a rational d such that $|d - c| \leq \varepsilon$. The computable real numbers form a field. We say $c \in \mathbb{R}$ is *left-c.e.* if there is an algorithm that enumerates an increasing sequence $(q_i)_{i \in \omega}$ of rationals such that $\lim_{i \rightarrow \infty} q_i = c$. It is not difficult to show that if the sum of two left-c.e. real numbers is computable, then both are computable.

Proof of Proposition 2.6. Let $a_n = |S \cap 2^n|$, and let $f(x) = \sum_{n \in \omega} a_n x^n$. We first argue that if there is a $c \in \mathbb{R}$ such that c is computable, $f(c)$ is computable, and $f(c) \leq 2c - 1$, then $w(\sigma)$ is computable, uniformly⁵ in $\sigma \in 2^{<\omega}$. We proceed by induction on the length of σ . First, note that $w(\langle \rangle) = 0$ (hence computable). Now assume that $w(\sigma)$ is

⁴If A is a prefix-free set of strings, then $\sum_{\tau \in A} 2^{-|\tau|} \leq 1$.

⁵We mean that there is a single procedure that, given σ and a rational error ε as input, computes $w(\sigma)$ to within ε .

computable. As observed in the proof of Theorem 2.4,

$$w(\sigma 0) + w(\sigma 1) = \frac{w(\sigma)}{c} + \frac{f(c)}{c},$$

where the right hand side is computable by hypothesis. Both $w(\sigma 0)$ and $w(\sigma 1)$ are left-c.e. reals, and since their sum is uniformly computable, both are computable (uniformly). It follows that we can now *effectively* build X bit by bit as in Theorem 2.4.

Next, we show that if the coefficients a_n are computable, then there is a computable c such that $f(c)$ is also computable. The function $f(x) - 2x + 1$ is concave up for $x > 0$ and so one of the following cases must hold:

Case 1: There is an interval $[p, q] \subseteq (1/2, 1)$ such that for all $x \in [p, q]$, $f(x) \leq 2x - 1$.

In this case, we can choose any rational $c \in [p, q]$. Since $f(q)$ converges,

$$a_i c^i = a_i q^i \frac{c^i}{q^i} \leq f(q) \left(\frac{c}{q}\right)^i \leq L d^i,$$

where L is any rational greater than $f(q)$ and d is any rational in $[c/q, 1)$. In other words, $f(c)$ is dominated term-by-term by the convergent geometric series $\sum_{i \in \omega} L d^i$. It follows that $f(c)$ is computable: to compute it to within ε , choose $k \geq N$ such that $(L d^k)/(1 - d) < \varepsilon$ and compute $\sum_{i=0}^{k-1} a_i c^i$.

Case 2: There is a unique $c \in (1/2, 1)$ such that $f(c) = 2c - 1$. The function $f(x) - 2x + 1$ is computably approximable from below by $f_s(x) - 2x + 1$, where $f_s(x) = \sum_{n \leq s} a_n x^n$. For s large enough, $f_s(x) - 2x + 1$ has two roots, a_s and b_s , both of which are computable. Note that c is between a_s and b_s , so to compute c to within ε , we simply search for s such that $|a_s - b_s| < \varepsilon$. Now $f(c)$, being equal to $2c - 1$, is also

computable. □

The constructions above produce sequences that are *at least* δ -shift-complex. An *exactly δ -shift-complex sequence* is one that is δ -shift-complex but not δ' -shift-complex for any $\delta' > \delta$, while an *almost δ -shift-complex sequence* is one that is δ' -shift-complex for every $\delta' < \delta$, but not δ -shift-complex. Hirschfeldt and Kach have shown that it is possible to adapt the construction in Theorem 2.3 to produce such sequences. Recall that for a sequence X , $\text{Dim}(X)$ denotes the effective packing dimension of X .

Theorem 2.7 (Hirschfeldt and Kach [17]). *For any $\delta \in (0, 1)$, there is a sequence X that is almost δ -shift-complex and a sequence Y that is exactly δ -shift-complex. Moreover, X and Y can be chosen so that $\text{Dim}(X) = \text{Dim}(Y) = \delta$.*

It is easy to see that the effective Hausdorff dimension of a δ -shift-complex sequence is at least δ . Since the effective packing dimension of a sequence bounds its effective Hausdorff dimension, for each of the sequences in Theorem 2.7, the effective Hausdorff dimension is δ .

Hirschfeldt and Kach have also observed that the effective packing dimension of a shift-complex sequence is always less than 1. In fact, this is true of all sequences that have the property that some string never occurs in the sequence.

Proposition 2.8 (Folklore). *Suppose a finite string σ never occurs as a substring of $X \in 2^\omega$. Then $\text{Dim}(X) < 1$.*

Proof. Let $m = |\sigma|$. For $n \in \omega$, let $\pi(n)$ denote the number of strings of length n that occur as substrings of X . Since $\pi(m) \leq 2^m - 1$, for all $j \geq 1$, $\pi(mj) \leq (2^m - 1)^j$. If j is large enough, $\pi(mj) \leq 2^{mj-1}$. So we can represent substrings of X of length mj

using an alphabet consisting of strings of length $m_j - 1$, which allows us to uniformly compress substrings of X by a factor of $m_j/(m_j - 1)$. In particular,

$$K(X \upharpoonright n) \leq^+ K(n) + \frac{m_j - 1}{m_j} n \leq^+ 2 \log(n) + \frac{m_j - 1}{m_j} n,$$

and so $\text{Dim}(X) \leq (m_j - 1)/m_j < 1$. □

A similar argument can be used to show:

Theorem 2.9 (Hirschfeldt and Kach [17]). *Fix a δ -shift-complex sequence A . Then for some $\varepsilon > 0$, there is a $(\delta + \varepsilon)$ -shift-complex sequence $B \leq_T A$.*

We close the section with a couple of observations:

Proposition 2.10. *For every $\delta \in (0, 1)$, there is a nonempty Π_1^0 class of δ -shift-complex sequences.*

Proof. We assume that δ is a rational. By any of the constructions described above, for some b , the class of (δ, b) -shift-complex sequences is nonempty. For each string σ , the set of descriptions of σ is uniformly c.e., by which we mean there is a single program that, given an arbitrary string σ , enumerates its descriptions. It follows that the set W of finite strings that are not (δ, b) -shift-complex is c.e., since we can construct a program that enumerates a finite string γ if any substring of γ has a description that is too short. A real X is (δ, b) -shift-complex if and only if it has no initial segment in W . Hence, the set of (δ, b) -shift-complex sequences is a Π_1^0 class. □

Corollary 2.11. *Every oracle of PA degree computes a δ -shift-complex sequence for every $\delta \in (0, 1)$.*

2.4 Bi-infinite shift-complex sequences

Given a shift-complex sequence $A \in 2^\omega$, it is easy to produce a bi-infinite sequence (in 2^ξ , where ξ denotes the order type of the integers) of lower complexity by treating the even and odd bits of A as two distinct sequences and reversing one of them. Recall that if B and C are sequences in 2^ω , we denote by $B \oplus C$ the *join* of B and C , i.e., the sequence in 2^ω formed by interleaving the bits of B and C .

Proposition 2.12. *Fix $\varepsilon > 0$. Every $(1 - \varepsilon)$ -shift-complex sequence uniformly computes a bi-infinite $(1 - 2\varepsilon)$ -shift-complex sequence.*

Proof. Suppose an infinite sequence A is $(1 - \varepsilon)$ -shift-complex. Letting $A = B \oplus C$, we claim that $Z = \overleftarrow{B}C$ (where \overleftarrow{B} is just B reversed) is $(1 - 2\varepsilon)$ -shift-complex.

First, we consider a substring σ of Z that lies completely in the left or the right half of Z (i.e., it does not overlap the index 0). If σ lies in the right half of Z (the argument is identical in the other case), let τ be the corresponding substring of the left half of Z of the same length such that $\tau \oplus \sigma$ is a substring of A . By assumption, $K(\tau \oplus \sigma) \geq^+ (1 - \varepsilon)2n$, where $n = |\sigma|$. It is easy to see that there is a prefix-free machine that witnesses the inequality $K(\sigma) + K(\tau | \sigma) \geq^+ K(\tau \oplus \sigma)$. A similar argument shows that $K(\tau | \sigma) \leq^+ K(n | \sigma) + n$. So we have:

$$K(\sigma) \geq^+ (1 - \varepsilon)2n - K(n | \sigma) - n.$$

But there is a constant c such that $K(|\sigma| | \sigma) \leq c$, so we obtain:

$$K(\sigma) \geq^+ (1 - \varepsilon)2n - n = (1 - 2\varepsilon)n.$$

Next, we consider the case where σ overlaps 0. Let $\sigma = \tau\gamma$ where τ is the part of σ to the left of 0 and γ the part to the right. If $|\tau| = |\gamma|$, then $K(\sigma) \geq^+ (1-\varepsilon)|\sigma| > (1-2\varepsilon)|\sigma|$, so suppose (without loss of generality) that $n = |\tau| > |\gamma| = m$. We can pad σ on the right by ρ , so that $|\sigma\rho| = 2n$ and $\sigma\rho$ is a substring of Z . Then, $K(\sigma\rho) \geq^+ (1-\varepsilon)2n$, since it corresponds to an initial segment of A of length $2n$. We also have the inequality $K(\sigma\rho) \leq^+ K(\sigma) + K(\rho | \sigma)$. So

$$\begin{aligned} K(\sigma) &\geq^+ (1-\varepsilon)2n - K(\rho | \sigma) \\ &= (1-2\varepsilon)(n+m) + (n-m) + 2\varepsilon m - K(\rho | \sigma). \end{aligned}$$

We know that $K(\rho | \sigma) \leq^+ K(|\rho| | \sigma) + |\rho| = K(n-m | \sigma^*) + (n-m)$. Now, since $|\sigma| = n+m$, $K(n-m | \sigma) \leq^+ K(m) \leq^+ 2\log(m)$. Combining all of the above, we have:

$$K(\sigma) \geq^+ (1-2\varepsilon)(n+m) + 2(\varepsilon m - \log(m)) \geq^+ (1-2\varepsilon)(n+m).$$

□

It follows that for any $\delta \in (0, 1)$, by starting out with a sequence of sufficiently high shift-complexity, we can apply the proposition above to obtain a bi-infinite δ -shift-complex sequence:

Corollary 2.13. *For every $\delta \in (0, 1)$, there exists a bi-infinite δ -shift-complex sequence.*

However, the answer to the following question is as yet unknown:

Question 2.14. Does every δ -shift-complex sequence compute a bi-infinite δ -shift-complex sequence?

2.5 Extracting shift-complexity from randomness

Initial segments of Martin-Löf random reals have high prefix-free Kolmogorov complexity. Might it not be possible to effectively obtain a shift-complex sequence from any Martin-Löf random real? A theorem by Romyantsev shows that this is not the case. In this section we give a proof of Romyantsev's theorem and show something stronger: The Martin-Löf random reals which compute shift-complex sequences do so not because they are random, but because they are of PA degree.

Theorem 2.15 (Romyantsev [29]). *The set of reals that compute shift-complex sequences has measure 0.*

We need a preliminary definition and a lemma.

Definition 2.16. We say that a shift-complex sequence Y is *abundant* if there is an $n \geq 2$ such that Y is δ -shift-complex for some $\delta > 1/n$ and further, for every m , Y contains at least $2^{m(n-1)/n}$ different substrings of length m .

Lemma 2.17. *Every shift-complex sequence computes an abundant shift-complex sequence.*

Proof. We begin by observing that if a sequence Y is δ -shift-complex for some $\delta > 1/n$ but is not abundant with witness m (i.e., Y has fewer than $2^{m(n-1)/n}$ different substrings of length m), then it has fewer than $(2^{m(n-1)/n})^n = 2^{m(n-1)}$ strings of length mn . Thus, substrings of Y of length mn can be represented using an alphabet consisting of strings of length $m(n-1)$, giving us a uniform way to compress substrings of Y whose lengths are a multiple of mn by a factor of $n/(n-1)$.

We show by induction on n that if a sequence X is δ -shift-complex for $\delta > 1/n$, then X computes an abundant shift-complex sequence. First note that if X is δ -shift-complex for some $\delta > 1/2$, then X itself is abundant. For if it is not, fix m such that X has fewer than $2^{m/2}$ strings of length m . By the observation above, we can compress substrings of X whose lengths are a multiple of $2m$ by a factor of 2, which contradicts the fact that X is δ -shift-complex for δ strictly greater than $1/2$.

Next, suppose X is δ -shift-complex for $\delta > 1/n$ and is not abundant with witness m . Again by the observation at the beginning of the proof, $X \equiv_T Y$, where Y is obtained from X by coding segments of X of length mn by strings of length $m(n-1)$. It is not hard to see that Y is δ' -shift-complex for $\delta' > 1/(n-1)$. By the induction hypothesis, Y computes an abundant shift-complex sequence. \square

In the proof of Theorem 2.15 we appeal to the Kraft–Chaitin Theorem, which allows us to compress a set of strings subject to some constraints.

Definition 2.18. A c.e. set $W \subseteq \omega \times 2^{<\omega}$ is a *Kraft–Chaitin set* if

$$\sum_{(n,\tau) \in W} 2^{-n} \leq 1.$$

We refer to elements of W as *requests*.

Theorem 2.19 (Kraft–Chaitin). *For each Kraft–Chaitin set W , there is a constant d such that for all $(n, \tau) \in W$, $K(\tau) \leq n + d$. The constant d is called the coding constant of W . Further, one can effectively obtain d from a procedure for enumerating W .*

In conjunction with the Kraft–Chaitin Theorem, we use the Recursion Theorem from computability theory, which allows us to know the coding constant of a Kraft–Chaitin

set while we are enumerating it. For a detailed exposition of this technique, we refer the reader to Chapter 2 of Nies [27].

Proof of Theorem 2.15. The strategy is to assume, for a contradiction, that a positive measure set of reals computes shift-complex sequences, hence abundant shift-complex sequences by Lemma 2.17. We will argue that since every oracle in this positive measure set computes *lots* of strings of fairly high complexity, some of these strings must be computed by a large measure of oracles. Using the Kraft–Chaitin Theorem, we will compress these strings, thereby invalidating those oracles that compute them.

More precisely, each shift-complex sequence computes an abundant shift-complex sequence for some n , and there are countably many choices for n . So we can assume that for some $n \geq 2$ there is a positive measure set A of oracles every member of which computes an abundant shift-complex sequence for n . By a similar argument, we can further assume that there is a single Turing functional Γ , a rational $\delta > 1/n$ and an integer b such that if $X \in A$, then Γ^X is abundant for n and (δ, b) -shift-complex.

Suppose $\mu(A) > \varepsilon$ where ε is rational. Let α be a rational in the interval $(1/n, \delta)$. We build a Kraft–Chaitin set W , and by the Recursion Theorem, we assume that we know the coding constant d of W in advance. Let m be chosen so that:

1. αm is an integer
2. $\alpha m < \delta m - (b + d)$
3. $2^{(\alpha - \frac{1}{n})m} \varepsilon > 1 - \varepsilon$.

We remark here that the choice of m is effective in ε , δ , n and b . This fact will be relevant to the proof of Theorem 2.21.

If σ is a string of length m , let

$$S_\sigma = \{X \in 2^\omega \mid \Gamma^X \text{ contains } \sigma \text{ as a substring}\}.$$

We say that we *compress* a string τ of length m when we enumerate the pair $(\alpha m, \tau)$ into W . Note that by compressing a string, we ensure that it has a description no longer than $\alpha m + d$. We can compress $2^{\alpha m}$ many strings. If we ensure that $\mu(\bigcup_{(n,\sigma) \in W} S_\sigma) > 1 - \varepsilon$ then we will have obtained a contradiction, since if an oracle X computes a string that has a description strictly smaller than $\delta m - b$, then $X \in \bar{A}$.

Suppose by stage s , we have compressed a set W_s of strings. Let C_s be the set of oracles that we have seen by stage s that compute a string in W_s via Γ . The measure of C_s should be thought of as the measure of oracles that we have eliminated by stage s by determining that they belong to \bar{A} . The key is that we compress a string when we observe that the additional measure eliminated by doing so is large. Note that since $\mu(A) > \varepsilon$ and every element of A computes at least $2^{m(n-1)/n}$ strings of length m , there is a string σ of length m such that

$$\mu(S_\sigma \cap A) > \varepsilon \left(\frac{2^{m(n-1)/n}}{2^m} \right) = \varepsilon \left(2^{-\frac{m}{n}} \right).$$

Since A is disjoint from C_s , eventually we will see a string σ such that $\mu(S_\sigma \setminus C_s) > \varepsilon \cdot 2^{-\frac{m}{n}}$.

When we encounter such a string, we compress it.

Each time we compress a string, we ensure that C_s grows by a measure greater than $\varepsilon \cdot 2^{-\frac{m}{n}}$, so by compressing $2^{\alpha m}$ strings, the measure of oracles we will have eliminated

will be greater than

$$2^{\alpha m} \cdot \varepsilon \cdot 2^{-\frac{m}{n}} = 2^{(\alpha - \frac{1}{n})m} \varepsilon,$$

which exceeds $1 - \varepsilon$ by our choice of m above. \square

Because any property that holds of almost all oracles must hold of sufficiently random oracles, Rumyantsev's theorem could be rephrased as follows: If a sequence X is sufficiently random, then it does not compute a shift-complex sequence. In the remainder of the section, we establish precisely how random an oracle must be for this property to hold. Franklin and Ng in [15] introduce a notion called difference randomness and show it to be strictly stronger than Martin-Löf randomness and strictly weaker than weak 2-randomness.

Definition 2.20. A *difference test* is a uniform sequence of pairs (U_i, V_i) of Σ_1^0 classes such that for all $i \in \omega$, $\mu(U_i \setminus V_i) \leq 2^{-i}$. A real *passes* a difference test $\langle (U_i, V_i) \rangle_{i \in \omega}$ if it is not contained in $\bigcap_{i \in \omega} (U_i \setminus V_i)$. A real is *difference random* if it passes all difference tests.

Theorem 2.21. *No difference random real computes a shift-complex sequence.*

Proof. Suppose a real Y computes a shift-complex sequence X via Γ . Without loss of generality we can assume X is abundant for some n and that it is (δ, b) -shift-complex for some $b \in \omega$ and $\delta > 1/n$. For an arbitrary real A , Γ^A may be partial. We say σ is a substring of Γ^A if Γ^A converges on a contiguous set of positions to σ . Let G_m be the set of reals A such that

1. Γ^A contains no substring σ such that $K(\sigma) < \delta|\sigma| - b$
2. Γ^A contains at least $2^{m(n-1)/n}$ different substrings of length m .

Then G_m is $U_m \setminus V_m$, where U_m is the set of reals A such that Γ^A converges enough to produce $2^{m(n-1)/n}$ substrings of length m and V_m is the set of reals B such that Γ^B converges on a substring σ such that $K(\sigma) < \delta|\sigma| - b$. Clearly, both U_m and V_m are Σ_1^0 .

As remarked in the proof of Theorem 2.15, for fixed δ , b and n , given a rational $\varepsilon > 0$ we can effectively find an m such that $\mu(G_m) \leq \varepsilon$. In other words, there is a computable f such that $\mu(G_{f(i)}) \leq 2^{-i}$. Then $\langle (U_{f(i)}, V_{f(i)}) \rangle_{i \in \omega}$ is a difference test that captures Y . \square

Franklin and Ng also provide the following characterization of the difference random reals:

Theorem 2.22 (Franklin and Ng [15]). *The difference random reals are precisely the incomplete Martin-Löf random reals.*

Since the halting problem has PA degree, any Martin-Löf random real that is not difference random computes a shift-complex sequence, by Corollary 2.11. Together, Theorem 2.21 and Theorem 2.22 imply the following:

Corollary 2.23. *A Martin-Löf random real computes a shift-complex sequence if and only if it is complete.*

We will see shortly that there are shift-complex sequences that are not of PA degree, so Corollary 2.23 can be viewed as a generalization of the following well-known result:

Theorem 2.24 (Stephan [33]). *A Martin-Löf random real is of PA degree if and only if it is complete.*

2.6 Extracting randomness from shift-complexity

We turn our attention now to the strength of shift-complex sequences as oracles. In this section we show that not all shift-complex sequences compute Martin-Löf random reals. The plan is similar to Greenberg and Miller's construction in [16] of a real of effective Hausdorff dimension 1 that computes no Martin-Löf random real. The separation is achieved through an analysis of the computational power of slow-growing diagonally noncomputable (DNC) functions.

Our main goal in this section is to show that for any $\delta \in (0, 1)$, all sufficiently slow-growing DNC functions compute δ -shift-complex sequences:

Theorem 2.25. *Fix $\delta \in (0, 1)$. There is an order function h such that every DNC_h function computes a δ -shift-complex sequence.*

Then, to see that for every $\delta \in (0, 1)$, there are δ -shift-complex sequences that compute no Martin-Löf random real, we could appeal to a result by Greenberg and Miller that there are arbitrarily slow-growing DNC functions that compute no Martin-Löf random real:

Theorem 2.26 (Greenberg and Miller [16]). *For every order function h , there is an $f \in \text{DNC}_h$ that does not compute a Martin-Löf random real.*

However, we can prove something stronger. In Theorem 3.18, we show (with J. Miller) that Theorem 2.26 can be strengthened to hold for Kurtz randomness.

Theorem 2.25 and Theorem 3.18 then imply:

Theorem 2.27. *For every $\delta \in (0, 1)$ there exists a δ -shift-complex sequence that does not compute a Kurtz random real.*

Before proving Theorem 2.25 we need a preliminary result.

Proposition 2.28. *Let $S \subseteq 2^{<\omega}$ be computable and suppose that for some $\alpha \in (0, 1)$, $|S \cap 2^n| \leq 2^{\alpha n}$ for all n . Then there is a computable $X \in 2^\omega$ that avoids S except for finitely many strings.*

Proof. Pick a rational $c \in (1/2, 1)$ such that $\log(c) < -\alpha$. Then

$$\sum_{\tau \in S} c^{|\tau|} \leq \sum_{n \in \omega} 2^{\alpha n} c^n = \sum_{n \in \omega} 2^{(\alpha + \log(c))n},$$

where the sum on the right converges geometrically. Let $N \in \omega$ be such that

$$\sum_{n \geq N} 2^{(\alpha + \log(c))n} \leq 2c - 1.$$

Now $S' = S \setminus 2^{<N}$ and c satisfy the hypotheses of Theorem 2.4. By Proposition 2.6, there is a computable $X \in 2^\omega$ that avoids S' . \square

If $S \subseteq 2^{<\omega}$ covers $\{\sigma \in 2^{<\omega} : K(\sigma) < \delta|\sigma|\}$, then any sequence which avoids S except for finitely many strings is δ -shift-complex. Relativizing Proposition 2.28 to such an S we obtain the following:

Corollary 2.29. *Fix $\delta \in (0, 1)$. Suppose $S \subseteq 2^{<\omega}$ all but finitely contains $\{\sigma \in 2^{<\omega} : K(\sigma) < \delta|\sigma|\}$ and for some $\alpha \in (0, 1)$, $|S \cap 2^n| \leq 2^{\alpha n}$ for all n . Then S computes a δ -shift-complex sequence.*

The next step is to show that for every $\delta \in (0, 1)$, every sufficiently slow-growing DNC function computes an $S \subseteq 2^{<\omega}$ satisfying the hypotheses of the corollary above.

Lemma 2.30. *Suppose $\delta \in (0, 1)$ and $\alpha \in (\delta, 1)$ are rational. Let $\pi(n) := \max(2, \lfloor 2^{(\alpha-\delta)n} \rfloor)$. Then, for all n such that $2^{(\alpha-\delta)n} \geq 2$, uniformly in n , given access to a function $g \in \text{DNC}_{\pi(n)}$, we can compute a set $S_n \subset 2^n$ such that*

1. $|S_n| \leq 2^{\alpha n}$
2. $\{\sigma \in 2^n : K(\sigma) < \delta n\} \subseteq S_n$.

Moreover, there is a computable function $\theta : \omega \rightarrow \omega$ (independent of the oracle g) such that the use of the computation is bounded by $\theta(n)$.

Instead of working directly with DNC functions, we work with a related class. Let $J(n, m)$ denote $\varphi_n(m)$.

Definition 2.31. For $a \geq 2$ and $c > 0$, let \mathbb{P}_a^c be the class of functions $f \in a^\omega$ such that for all n and all $x < c$, if $(n, x) \in \text{dom}(J)$, then $f(n) \neq J(n, x)$.

We use the following fact, originally due to Cenzer and Hinman [6], in the form presented in Greenberg and Miller [16]:

Theorem 2.32 (Cenzer and Hinman [6]). *For each $a \geq 2$ and $c > 0$, there is a functional Γ such that if $g \in \text{DNC}_a$, then $\Gamma^g \in \mathbb{P}_{ac}^c$. Further, the functional Γ can be obtained effectively from a and c .*

Note that since DNC_a is a Π_1^0 class in a^ω , the functional Γ can be assumed to be total on a^ω , and hence a truth-table functional.

Proof of Lemma 2.30. Let $c = \lfloor 2^{\delta n} \rfloor$. By the preceding theorem, we can obtain $f = \Gamma^g \in \mathbb{P}_{c\pi(n)}^c$. Let B_n denote $\{\sigma \in 2^n : K(\sigma) < \delta n\}$. We construct $S_n \subset 2^n$ by eliminating strings that are not in B_n one at a time. A simple counting argument shows that

$|B_n| \leq 2^{\lfloor \delta n \rfloor} \leq c$. The key is that for an appropriately chosen m , B_n can be covered by the values of $J(m, x)$ for $x \leq c$. Since $f(m)$ avoids these values, it picks out a string that is not in B_n .

More formally, for $T \subseteq 2^n$, we can computably find an $m_T \in \omega$ such that $J(m_T, x) = y$ if and only if the x^{th} element seen in an enumeration of B_n is the y^{th} lexicographically least element of T . If $|T| > 2^{\alpha n}$, then

$$|T| > 2^{(\alpha-\delta)n} \cdot 2^{\delta n} \geq \lfloor 2^{(\alpha-\delta)n} \rfloor \lfloor 2^{\delta n} \rfloor = c\pi(n).$$

Therefore, $f(m_T)$ corresponds to an string in $T \setminus B_n$. So let $T_0 = 2^n$ and, for $0 \leq i < 2^n - \lfloor 2^{\alpha n} \rfloor$, let $T_{i+1} = T_i \setminus \{\sigma_i\}$, where σ_i is the $f(m_{T_i})^{\text{th}}$ lexicographically least element of T_i . Finally, let $S_n = T_{2^n - \lfloor 2^{\alpha n} \rfloor}$.

Since Γ is a truth-table functional, its use γ is computable. We can define $\theta(n)$ to be

$$\max\{\gamma(m_T) : T \subseteq 2^n, |T| > 2^{\alpha n}\}.$$

□

The uniformity in Lemma 2.30 allows us to string together the constructions of S_n for varying n . At the same time, the fact that as n increases the DNC strength needed to compute S_n decreases (π is exponential in n) implies that instead of using the computational power of a function in DNC_a for a fixed $a \in \omega$, we can use a function whose range is allowed to grow unboundedly.

Lemma 2.33. *For every $\delta \in (0, 1)$, there is an order function h such that every $f \in \text{DNC}_h$ computes a set $S \subseteq 2^{<\omega}$ that all but finitely contains $\{\sigma \in 2^{<\omega} : K(\sigma) < \delta|\sigma|\}$*

and satisfies the condition that for some $\alpha \in (\delta, 1)$, $|S \cap 2^n| \leq 2^{\alpha n}$ for all n .

Proof. Without loss of generality, assume δ is rational and choose a rational $\alpha \in (\delta, 1)$. Let θ be the computable bound on the use of the computation in Lemma 2.30. Now, for $i \in \omega$, let $h(i) = \lfloor 2^{(\alpha-\delta)m} \rfloor$, where $m = \min\{n : \theta(n) \geq i \text{ and } 2^{(\alpha-\delta)n} \geq 2\}$. Then h is clearly total, non-decreasing, unbounded, and computable. Moreover, by Lemma 2.30, if $f \in \text{DNC}_h$, then for each n such that $2^{(\alpha-\delta)n} \geq 2$, we can use the values of f on the interval $[0, \theta(n)]$ to compute S_n , since $h(i) \leq \lfloor 2^{(\alpha-\delta)n} \rfloor$ for $i \in [0, \theta(n)]$. \square

To complete the proof of Theorem 2.25, let h be the order function of Lemma 2.33. By Corollary 2.29, every $f \in \text{DNC}_h$ computes a δ -shift-complex sequence.

2.7 Questions

A shift-complex sequence X has positive effective packing dimension, so by a result of Fortnow, Hitchcock, Pavan, Vinogradov, and Wang [14], we know that for every $\varepsilon > 0$, X computes a real Y such that $\text{Dim}(Y) > 1 - \varepsilon$. On the other hand, Conidis [7] has shown that there is a real of positive effective packing dimension that computes no real of effective packing dimension 1. However, the added assumption of shift-complexity might allow us to circumvent the limitation posed by Conidis's result.

Question 2.34 (Kach). Does every shift-complex sequence compute a real of effective packing dimension 1?

We can also ask the analogous question for effective Hausdorff dimension:

Question 2.35. Does every shift-complex sequence compute a real of effective Hausdorff dimension 1? If not, then for every $\varepsilon > 0$, does every shift-complex sequence compute a real of effective Hausdorff dimension $1 - \varepsilon$?

A related question is whether we can extract arbitrarily higher shift-complexity. Theorem 2.9 shows that a shift-complex sequence always computes one of higher shift-complexity. To what extent is this possible? If a shift-complex sequence can have a ‘complexity ceiling’, how is this determined?

Question 2.36. Fix $\delta \in (0, 1)$. Does every δ -shift-complex sequence compute a δ' -shift-complex sequence for every $\delta' \in (\delta, 1)$?

Chapter 3

Bushy tree forcing

The application of bushy tree forcing in computability theory was pioneered by Kumabe [22] who used it to show that there is a DNC function of minimal Turing degree, answering a question of Sacks [31]. We have two main goals in this chapter. First, in Theorem 3.18, we show that given any order function h , there is a function $f \in \text{DNC}_h$ (i.e., bounded pointwise by h) such that f computes no Kurtz random real. Second, we extend Kumabe's theorem to show that given any oracle X there is a function that is DNC relative to X of minimal Turing degree (Theorem 3.24). Along the way, we provide concise proofs of some existing results involving this type of forcing.

Some of the contents of this chapter will appear in a joint publication with J. Miller.

3.1 Definitions and combinatorial lemmas

Definition 3.1. Given $\sigma \in \omega^{<\omega}$, we say that a tree $T \subseteq \omega^{<\omega}$ is *n-bushy above σ* if every element of T is comparable with σ , and for every $\tau \in T$ that extends σ and is not a leaf of T , τ has at least n immediate extensions in T . We will refer to σ as the *stem* of T .

Note that the set of initial segments of σ is actually *n-bushy above σ* according to the definition above.

Definition 3.2. Given $\sigma \in \omega^{<\omega}$, we say that a set $B \subseteq \omega^{<\omega}$ is *n-big above σ* if there is

a finite n -bushy tree T above σ such that all its leaves are in B . If B is not n -big above σ then we say that B is n -small above σ .

We begin by establishing some of the basic combinatorial properties of bushy trees. The first observation is that we can extend the leaves of an n -bushy tree with n -bushy trees to obtain another n -bushy tree:

Lemma 3.3 (Concatenation property). *Suppose that $A \subseteq \omega^{<\omega}$ is n -big above σ . If $A_\tau \subseteq \omega^{<\omega}$ is n -big above τ for every $\tau \in A$, then $\bigcup_{\tau \in T} A_\tau$ is n -big above σ .*

The second property that we use frequently is known as the *smallness preservation property*. This is the *second sparse subset property* of Kumabe and Lewis [23], and Lemma 5.4 of Greenberg and Miller [16].

Lemma 3.4 (Smallness preservation property). *Suppose that B and C are subsets of $\omega^{<\omega}$, that $m, n \in \omega$ and that $\sigma \in \omega^{<\omega}$. If B and C are respectively m -small and n -small above σ then $B \cup C$ is $(n + m - 1)$ -small above σ .*

Proof. Let T be an $(m + n - 1)$ -bushy tree above σ with leaves in $B \cup C$. We show that either B is m -big above or C is n -big above σ . Label a leaf τ of T “B” if it is in B , “C” otherwise. Now if ρ is the immediate predecessor of τ , then ρ has at least $(m + n - 1)$ immediate extensions on T , each of which are labelled either “B” or “C”. Then either m of these are labelled “B”, in which case we label ρ “B”, or n are labelled “C”, in which case we label ρ “C”. Continuing this process leads to σ eventually getting a label. It is clear that if σ is labelled “B” then B is m -big above σ . Otherwise C is n -big above σ . \square

The third property is known as the *small set closure property*:

Lemma 3.5 (Small set closure property). *Suppose that $B \subset \omega^{<\omega}$ is k -small above σ . Let $C = \{\tau \in \omega^{<\omega} : B \text{ is } k\text{-big above } \tau\}$. Then C is k -small above σ . Moreover C is k -closed, meaning that if C is k -big above a string ρ , then $\rho \in C$.*

Proof. Suppose that C is k -big above a string ρ . Then, since B is k -big above every $\tau \in C$, by the concatenation property, B is k -big above ρ , so $\rho \in C$. The lemma follows immediately. \square

The small set closure property is quite useful in the context of a forcing construction. Typically, σ is an approximation to a function that we are building and B is a set of strings that must be avoided in order to ensure that requirements remain met. We refer informally to the set B as the “bad set”. Throughout the construction, we may wish to maintain the property that the bad set B is k -small above σ for some $k \in \omega$. Clearly, if B is k -big above some string ρ , then ρ is off-limits as well. Lemma 3.5 allows us to assume that all such strings are already in the bad set, while preserving its smallness. From now on, whenever we deal with a bad set that is k -small, we also assume that it is k -closed. Note that the k -closure of a c.e. set of strings is also c.e.

3.2 Basic bushy forcing

As a first illustration of the convenience afforded us by these lemmas, we present a proof of a well-known result. Any bounded DNC function (i.e., a function in DNC_k for some $k \geq 2$) computes a function in DNC_2 . However, Jockusch showed in [19] that this is not uniform.

Theorem 3.6 (Jockusch [19]). *For each $n \geq 2$, there is no single functional Γ such that for all $f \in \text{DNC}_{n+1}$, $\Gamma^f \in \text{DNC}_n$.*

Proof. Let us assume that such a Γ exists, i.e., for all $f \in \text{DNC}_{n+1}$, $\Gamma^f \in \text{DNC}_n$. The set of sequences in DNC_{n+1} is a Π_1^0 class in $(n+1)^\omega$, so we may obtain a functional Ξ that is total on $(n+1)^\omega$ and that agrees with Γ on every member of DNC_{n+1} . We may also assume that $\Xi^f \in n^\omega$ for all $f \in (n+1)^\omega$.

For each $m \in \omega$ and for each $i < n$, let $\Lambda_{i,m} = \{\sigma \in (n+1)^{<\omega} : \Xi^\sigma(m) = i\}$. By the compactness of $(n+1)^\omega$, there exists a finite level k such that for every string $\tau \in (n+1)^k$, $\Xi^\tau(m)$ converges. Therefore, $\bigcup_{i < n} \Lambda_{i,m}$ is $(n+1)$ -big above the empty string $\langle \rangle$. It is now easy to see, by repeatedly applying the smallness preservation property, that for some $i < n$, $\Lambda_{i,m}$ must be 2-bushy above $\langle \rangle$.

We specify a partial computable function φ . On input m , φ searches for a 2-bushy tree T above $\langle \rangle$ such that for every leaf τ of T , $\Xi^\tau(m)$ converges to the same value, say i , and when it finds such a tree, itself outputs i . By the argument above, such a tree must exist, and so $\varphi(m)$ is defined for each m . Let e be the index for φ , and let T_e be the 2-bushy tree that φ finds on input e . Since T_e is 2-bushy, there is a leaf τ of T_e that is DNC, and so there is an $f \in (n+1)^\omega$ extending σ that is DNC_{n+1} . But then $\Xi^f(e) = \Xi^\tau(e) = \varphi_e(e)$, which is a contradiction. \square

Finitely iterating this strategy yields the following stronger result:

Theorem 3.7. *For each $n \geq 2$, there is no finite set of functionals $\Gamma_0, \Gamma_1, \dots, \Gamma_k$ such that for all $f \in \text{DNC}_{n+1}$, there exists a $j \leq k$ such that $\Gamma_j^f \in \text{DNC}_n$.*

Proof. Let us assume that such a set of functionals exists. We define a new functional Ξ as follows: on input e , Ξ simulates Γ_0 through Γ_k on input e and outputs the result

of whichever one converges first. Clearly, Ξ is total on the class of DNC_{n+1} sequences, so we can assume that Ξ is total on $(n+1)^\omega$. We then proceed exactly as in the proof of Theorem 3.6, obtaining a string σ_0 that is DNC_{n+1} and an $e \in \omega$ such that $\Xi^{\sigma_0}(e) = \varphi_e(e)$. Then $\Xi^{\sigma_0}(e) = \Gamma_j^{\sigma_0}(e)$ for some $j \leq k$. It follows that Γ_j fails to compute a DNC_n function on any $f \in \text{DNC}_{n+1}$ extending σ_0 . We now repeat the same process above σ_0 with the reduced list of functionals $\{\Gamma_1, \dots, \Gamma_k\} \setminus \{\Gamma_j\}$, obtaining a DNC_{n+1} string σ_1 extending σ_0 that diagonalizes against one of the remaining functionals. After $k+1$ iterations, we will have obtained a contradiction. \square

The previous proof points the way towards more sophisticated constructions involving bushy trees where we satisfy countably many requirements. The next result is our first example of such a construction. It features a simpler variant of bushy tree forcing, which we term *basic bushy forcing*, and is characterized by the fact that the forcing conditions essentially have two components: a finite string that represents an approximation to a function that we are building, and a set of bad strings that we are trying to avoid.

Theorem 3.8 (Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1]). *There is a DNC function that computes no computably bounded DNC function.*

Proof. The forcing conditions are pairs (σ, B) , where $\sigma \in \omega^{<\omega}$, $B \subset \omega^{<\omega}$ and:

- for some $k \in \omega$, B is k -small above σ (and without loss of generality, k -closed)
- B is *upward closed* (i.e., if γ is in B , then all extensions of γ are in B).

The string σ is an approximation to f and the set B is a “bad set”, i.e., a set of strings that must be avoided in order to ensure that requirements remain satisfied.

A condition (σ, B) *extends* another condition (τ, C) if $\tau \preceq \sigma$ and $C \subseteq B$. Let \mathbb{P} denote this partial order. Now if \mathcal{G} is a filter on \mathbb{P} , then for any two elements (σ, B) and (τ, C) of \mathcal{G} , σ and τ are comparable. Hence, $f_{\mathcal{G}} = \bigcup\{\sigma : (\sigma, B) \in \mathcal{G}\} \in \omega^{\leq \omega}$. In fact, we can ensure that $f_{\mathcal{G}}$ is total:

Claim 3.9. *If \mathcal{G} is sufficiently generic with respect to \mathbb{P} , then $f_{\mathcal{G}}$ is total.*

Proof. We show that the collection $\mathcal{T}_m = \{(\sigma, B) \in \mathbb{P} : |\sigma| \geq m\}$ is dense in \mathbb{P} . Suppose $(\sigma, B) \in \mathbb{P}$, where $|\sigma| < m$. Then B is k -small above σ for some $k \in \omega$. Clearly, the set $C = \{\tau \in \omega^{< \omega} : |\tau| \geq m\}$ is k -big above σ . Let τ be any string in $C \setminus B$. Then $(\tau, B) \in \mathbb{P}$. □

Claim 3.10. *If \mathcal{G} is any filter on \mathbb{P} , then for all $(\sigma, B) \in \mathcal{G}$, $f_{\mathcal{G}}$ has no initial segment in B .*

Proof. Suppose that $f_{\mathcal{G}}$ has an initial segment τ in B . Then there is a $(\rho', C') \in \mathcal{G}$ such that ρ' extends τ . Let (ρ, C) be a common extension of (ρ', C') and (σ, B) . Since B is upward closed, $\rho \in B$. But $B \subseteq C$, so $\rho \in C$. This is a contradiction, since it follows that C is k -big above ρ for all $k \in \omega$. □

If Γ is a functional and h a computable function such that Γ is h -valued (in other words, whenever Γ converges with any oracle on input e , its output is less than $h(e)$), let $\mathcal{D}_{\Gamma, h}$ denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $g \in [\sigma] \setminus [B]^{\prec}$, Γ^g is not a DNC_h function.

Claim 3.11. *For each computable function h , and h -valued functional Γ , $\mathcal{D}_{\Gamma, h}$ is dense in \mathbb{P} .*

Proof. Suppose $(\sigma, B) \in \mathbb{P}$ and that B is k -small above σ . As in the proof of Theorem 3.6, we specify a partial computable function φ . On input m , φ searches for a k -bushy tree T above σ such that for every leaf τ of T , $\Gamma^\tau(m)$ converges to the same value $i < h(m)$. Upon finding such a tree, φ outputs i . Let e be the index of φ .

There are now two cases. If the set $A = \{\tau : \Gamma^\tau(e) \downarrow\}$ is $(h(e) \cdot k)$ -small above σ , then $A \cup B$ is $(h(e) \cdot k + k - 1)$ -small above σ . Then $(\sigma, A \cup B) \in \mathbb{P}$ and extends (σ, B) . Note that we have forced Γ to be partial on any $g \in [\sigma] \setminus [A \cup B]^\prec$. Hence, $(\sigma, A \cup B) \in \mathcal{D}_{\Gamma, h}$.

On the other hand, if A is $(h(e) \cdot k)$ -big above σ , then for some $i < h(e)$, $\{\tau : \Gamma^\tau(e) \downarrow = i\}$ is k -big above σ . So $\varphi(e)$ is defined. In this case, we extend σ to any τ not in B such that $\Gamma^\tau(e) \downarrow = \varphi(e)$. This forces Γ^g to fail to be DNC on any g extending τ . Hence, $(\tau, B) \in \mathcal{D}_{\Gamma, h}$. \square

Finally, we observe that the set of finite strings that are not DNC, which we denote by B_{DNC} , cannot be 2-big above the empty string $\langle \rangle$, since any 2-bushy tree contains a string that is DNC. So $(\langle \rangle, B_{\text{DNC}}) \in \mathbb{P}$. Let \mathcal{G} be a filter on \mathbb{P} containing $(\langle \rangle, B_{\text{DNC}})$ that meets \mathcal{T}_m for every $m \in \omega$ and $\mathcal{D}_{\Gamma, h}$ for every computable function h and h -valued functional Γ (note that this is a countable collection of dense sets).

By Claim 3.9, $f_{\mathcal{G}}$ is total. By Claim 3.10 and the fact that $(\langle \rangle, B_{\text{DNC}}) \in \mathcal{G}$, $f_{\mathcal{G}}$ is a DNC function. If $f_{\mathcal{G}}$ computes a function in DNC_h for some computable function h , then it does so via an h -valued functional Γ . Claim 3.11 shows that this is not the case. This concludes the proof of Theorem 3.8. \square

We note that while the bad sets in the previous proof are c.e., we do not make use of this fact. Given an oracle X , let B_{DNC}^X denote the set of finite strings that are not DNC relative to X . Note that B_{DNC}^X is not necessarily c.e., but is nevertheless 2-small above

$\langle \rangle$. This suggests that we could use the same sort of techniques to construct a function that is DNC relative to X . As an example, we prove a theorem that implies the main result in [1], and is slightly stronger.

Theorem 3.12. *Fix an order function h . Suppose X computes no DNC_h function. Then there is an f that is DNC relative to X such that $f \oplus X$ computes no DNC_h function.*

Proof. The forcing partial order is the same as before. If Γ is an h -valued functional, let \mathcal{D}_Γ denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $f \in [\sigma] \setminus [B]^\prec$, $\Gamma^{f \oplus X}$ is not a DNC_h function. We show that \mathcal{D}_Γ is dense in the partial order. Suppose (σ, B) is a condition where B is k -small above σ .

First, if there are $x, l \in \omega$ such that

$$C_x = \{\tau \in \omega^{<\omega} : \Gamma^{\tau \oplus X}(x) \downarrow\}$$

is l -small above σ , then the condition $(\sigma, B \cup C_x)$ extends (σ, B) and forces the divergence of $\Gamma^{f \oplus X}(x)$. Therefore, let us assume that for each $x, l \in \omega$, C_x is l -big above σ .

Next, if there exists an $x \in \omega$ such that $\varphi_x(x)$ converges and

$$N_x = \{\tau \in \omega^{<\omega} : \Gamma^{\tau \oplus X}(x) \downarrow = \varphi_x(x)\}$$

is k -big above σ , then there is a τ extending σ not in B such that $\Gamma^{\tau \oplus X}(x) \downarrow = \varphi_x(x)$, and so the condition (τ, B) extends (σ, B) and forces that $f_{\mathcal{G}}$ is not DNC. Therefore, let us assume that for each $x \in \omega$, either $\varphi_x(x)$ diverges or N_x is k -small above σ .

We now describe how to compute a DNC_h function from X , which yields a contradiction. On input x , search for a k -bushy tree T above σ such that for every leaf τ of T , $\Gamma^{\tau \oplus X}(x)$ converges to the same value $j < h(x)$, then output j . Since for each x , C_x is $(h(x) \cdot k)$ -big above σ , such a tree T exists. So the X -computable function just described is total. Moreover, it disagrees with $\varphi_x(x)$ whenever it is defined, since N_x is k -small above σ .

Therefore, \mathcal{D}_Γ is dense. Let \mathcal{G} be a generic filter including the condition $(\langle \rangle, B_{\text{DNC}}^X)$. Then $f_{\mathcal{G}}$ has the required properties. \square

With a stronger assumption, the technique in the proof of Theorem 3.12 yields a stronger conclusion: If X computes no computably bounded DNC function, then there is an f that is DNC relative to X such that $f \oplus X$ computes no computably bounded DNC function. We omit the proof.

An analysis of the amount of bushiness we require above σ in the diagonalization argument of Claim 3.11 yields the following:

Theorem 3.13 (Ambos-Spies, et al. [1]). *For each order function h there is an order function j and a function $f \in \text{DNC}_j$ that computes no function in DNC_h .*

Proof. If j is an order function, let j^n denote the space

$$\prod_{m < n} \{0, 1, \dots, j(m) - 1\},$$

and let $j^{<\omega}$ and j^ω be defined in the obvious way.

We now fix a computable function h and let $(\Gamma_i)_{i \in \omega}$ be an effective enumeration of all h -valued Turing functionals. We define an order function j by recursion. In order to

define j , we will also define an auxiliary computable function $q : \omega^{<\omega} \times \omega^2$, the definition of which will refer to the index of the function j . This is possible because we can assume, by the recursion theorem, that we have access to the index of j in advance.

On input x , $\varphi_{q(\sigma,i)}$ searches for a $|\sigma|$ -bushy tree T above σ contained in $j^{<\omega}$ such that for every leaf τ of T , $\Gamma_i^\tau(x) \downarrow$ to the same value $k < h(x)$, and upon finding such a tree, itself outputs k . Now let $\bar{q} = \max_{i < n, \sigma \in j^n} q(\sigma, i)$. We define $j(n)$ to be the larger of $\max_{i < n} j(i)$ and $((h(\bar{q}(n)) + 1) \cdot n) + 2$.

The forcing conditions are now pairs (σ, B) where $B \subseteq j^{<\omega}$ and $\sigma \in j^{<\omega} \setminus B$. We require that B be upward-closed and $|\sigma|$ -small above σ . By the small set closure property, we may assume that B is $|\sigma|$ -closed. For $\sigma \in j^{<\omega}$, let $[\sigma]_j$ denote $\{X \in j^\omega : \sigma \prec X\}$.

Claim 3.14. *Let \mathcal{D}_i denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $g \in [\sigma]_j \setminus [B]^\prec$, Γ_i^g is not a DNC_h function. Then for each $i \in \omega$, \mathcal{D}_i is dense in \mathbb{P} .*

Proof. Suppose that $(\sigma, B) \in \mathbb{P}$. By suitably extending σ , we can assume that $|\sigma| > i$. Let $n = |\sigma|$ and

$$A = \{\tau \in j^{<\omega} : \Gamma_i^\tau(q(\sigma, i)) \downarrow\}.$$

As in the proof of Claim 3.11, there are two cases.

If A is $(h(q(\sigma, i)) \cdot n)$ -small above σ , then letting $c = (h(q(\sigma, i)) \cdot n + n - 1)$, $A \cup B$ is c -small above σ . Let C be the c -closure of $A \cup B$. Since $j(n) \geq (h(q(\sigma, i)) + 1) \cdot n > c$ and j is nondecreasing, j^c is c -big above σ . Let τ be any string extending σ in $j^c \setminus C$. Then (τ, C) is a condition. Further, Γ_i^f is partial on any $f \in [\tau]_j \setminus [C]^\prec$, so $(\tau, C) \in \mathcal{D}_i$.

On the other hand, if A is $(h(q(\sigma, i)) \cdot n)$ -big above σ , then for some $k < h(q(\sigma, i))$, the set $\{\tau \in j^{<\omega} : \Gamma_i^\tau(q(\sigma, i)) \downarrow = k\}$ is n -big above σ . It follows that $\varphi_{q(\sigma,i)}(q(\sigma, i))$ is defined. So there is a $\tau \in j^{<\omega} \setminus B$ extending σ such that $\Gamma_i^\tau(q(\sigma, i)) = \varphi_{q(\sigma,i)}(q(\sigma, i))$.

Then $(\tau, B) \in \mathbb{P} \cap \mathcal{D}_i$. □

This concludes the proof of Theorem 3.13. □

Theorem 3.15. *Given any order function g , there is an order function h and an $f \in \text{DNC}_g$ such that f computes no DNC_h function.*

Proof. We define h inductively. Let $n_0 = 0$ and let $h(0) = 2$. At the i th stage of the construction, suppose we have defined it up to n_i . Let $k \geq n_i + 1$ be the least such that $g(k) \geq (h(n_i) + 1) \cdot g(n_i)$. Let $q(\sigma)$ be the computable function such that if $\sigma \in g^k$, then $q(\sigma) \geq k$, and $\varphi_{q(\sigma)}(n)$ searches for a $g(n_i)$ -bushy tree T above σ contained in $g^{<\omega}$ such that for every leaf τ of T , Φ_{i-1}^τ converges to the same value $l < h(n_i)$. Let $m = \max_{\sigma \in g^k} q(\sigma)$. Let $h(n) = h(n_i)$ for all n such that $n_i < n \leq m$ and let $h(m+1) = h(m) + 1$. Finally, let $n_{i+1} = m + 1$, ensuring that h is unbounded. The fact that $k \geq n_i + 1$ ensures that h is total.

It remains to construct f . Let $B_0 = B_{\text{DNC}}$ and let $\sigma_0 \in g^1 \setminus B_{\text{DNC}}$. Assume inductively that $\sigma_i \in g^{n_i} \setminus B_i$ and that B_i is $g(n_i)$ -small above σ_i . Let k and q be defined as above and extend σ to a string $\rho \in g^k \setminus B_i$. For $j < h(q(\rho))$, let

$$A_j = \{\tau \in g^{<\omega} : \Phi_i^\tau(q(\rho)) \downarrow = j\}.$$

If A_j is $g(n_i)$ -big above ρ for some j , then $\varphi_{q(\rho)}(q(\rho))$ is defined. If $\varphi_{q(\rho)}(q(\rho)) = j'$ then there is a $\tau \in A_{j'} \setminus B_i$ extending ρ such that $\Phi_{i-1}^\tau(q(\rho)) = \varphi_{q(\rho)}(q(\rho))$. Otherwise, $C = (\bigcup_{j < h(q(\rho))} A_j) \cup B_i$ is $(h(q(\rho)) + 1) \cdot g(n_i)$ -small above ρ . Since $g(k) \geq (h(n_i) + 1) \cdot g(n_i) = (h(q(\rho)) + 1) \cdot g(n_i)$, C is $g(k)$ -small above ρ . So let $B_{i+1} = C$ and let σ_{i+1} be any string in $g^{n_{i+1}} \setminus B_{i+1}$ extending ρ . Finally, let $f = \bigcup_{i \in \omega} \sigma_i$. □

By alternating the strategies of Theorems 3.13 and 3.15, one can also show:

Theorem 3.16. *Given any order function g_0 , there is another order function g_1 and functions $f_0 \in \text{DNC}_{g_0}$ and $f_1 \in \text{DNC}_{g_1}$ such that f_0 computes no DNC_{g_1} function and f_1 computes no DNC_{g_0} function.*

3.3 Bushy tree forcing

Bounded DNC functions, being of PA degree, compute Martin-Löf random reals. Kučera [21] showed that there is an order function h such that every Martin-Löf random real computes a DNC_h function. Theorem 3.8 then implies that there are unbounded DNC functions that compute no Martin-Löf random real. Greenberg and Miller established a stronger version of this fact:

Theorem 3.17 (Greenberg and Miller [16]). *For each order function h , there is an $f \in \text{DNC}_h$ that computes no Martin-Löf random real.*

The proof uses basic bushy forcing, and does not require that the bad sets be c.e. In fact, the same technique could be used to show that for each order function h and each oracle X , there is an $f \in \text{DNC}_h^X$ that computes no Martin-Löf random real. Our main result in this section (joint with J. Miller) cannot be partially relativized in this manner (it strongly depends on the fact that the bad sets are c.e.) but improves upon the Greenberg-Miller theorem in a different way.

Theorem 3.18 (Khan, Miller). *For each order function h , there is an $f \in \text{DNC}_h$ that computes no Kurtz random real.*

Theorem 3.18 is our first example of *bushy tree forcing*, where the conditions consist of trees, not just finite strings. The atomic step in the forcing is based on the following result of Downey, Greenberg, Jockusch, Milans [10], which we prove here for convenience.

Theorem 3.19 (Downey, et al. [10]). *There is no single functional Γ such that Γ^f is Kurtz random for all $f \in \text{DNC}_3$.*

Proof. Suppose that such a functional Γ exists. As before, we may assume that Γ is total. It will be convenient to assume that Γ satisfies the following additional property:

- If $\sigma \in 3^{<\omega}$ and $\Gamma^\sigma(n)$ converges, then $\Gamma^\sigma(n)$ converges within $|\sigma|$ steps and for all $n' < n$, $\Gamma^\sigma(n')$ also converges.

It is not difficult to see that this assumption can be made without any loss of generality and that if Γ satisfies this property, then $\Gamma^\sigma = \tau$ is a computable relation for $\sigma \in 3^{<\omega}$ and $\tau \in 2^{<\omega}$.

We build a computable 2-bushy subtree S of 3^ω with no leaves such that the image of Γ on S (denoted by $\Gamma(S)$) has measure 0. The tree S will be obtained as the union of a sequence $\{\langle \rangle\} = S_0 \subset S_1 \subset S_2 \dots$ of finite regular¹ binary subtrees of $3^{<\omega}$. Let $\Gamma(S_i)$ denote the set of reals

$$\bigcup \{[\Gamma^\sigma] : \sigma \text{ is a leaf of } S_i\}.$$

In constructing S_{i+1} , we want to ensure that $\mu(\Gamma(S_{i+1})) \leq (3/4)\mu(\Gamma(S_i))$. Let $L = \{\sigma_0, \sigma_1, \dots, \sigma_{|L|-1}\}$ be the set of leaves of S_i and let $m = \max\{|\Gamma^\sigma| : \sigma \in L\}$. Our assumption on Γ above allows us to find m computably. Let l be large enough so that for all $\tau \in 3^l$, $|\Gamma^\tau| \geq m + (2^{2^l} + 1)$. In other words, l is large enough so that we obtain

¹All the leaves are of the same length.

at least $2^{|L|} + 1$ *additional* bits of convergence by extending a leaf of S_i to any ternary string of length l . Note that such an l exists by the compactness of 3^ω and that we can find it computably. Let $T_j = \{\tau \in 3^l : \tau \succ \sigma_j\}$.

Suppose that k is a position corresponding to one of the additional bits of convergence, i.e., $m \leq k < m + 2^{|L|} + 1$. Since each T_j is 3-big above σ_j , by the smallness preservation property, either $\{\tau \in T_j : \Gamma^\tau(k) = 1\}$ is 2-big above σ_j (in which case, we say that we can *force* the k^{th} bit to be 1 above σ_j) or $\{\tau \in T_j : \Gamma^\tau(k) = 0\}$ is 2-big above σ_j (we say that we can force the k^{th} bit to be 0 above σ_j). This allows us to obtain a binary sequence ρ_k of length $|L|$, where $\rho_k(j) = 1$ if we can force the k^{th} bit to be 1 above σ_j , and 0 otherwise. Moreover, we can computably find 2-big sets above σ_j that force the k^{th} bit one way or another, so we can compute ρ_k , given k .

By the pigeonhole principle, there exist r and s such that $m \leq r, s < m + 2^{|L|} + 1$ and $\rho_r = \rho_s$. Note that for each $j < |L|$, even though we can force the r^{th} and s^{th} bits in the same way above σ_j , we may not be able to do so *simultaneously*. We adopt the following strategy above each σ_j : If we can force the r^{th} bit to be 1 above σ_j , we do so, by extending σ_j to a finite 2-bushy tree B_j with leaves in 3^l such that for every leaf τ of B_j , $\Gamma^\tau(r) = 1$. Otherwise, $\rho_r(j) = \rho_s(j) = 0$, so we force the s^{th} bit to be 0 above σ_j . The regular binary tree of height l that results is S_{i+1} .

For any leaf τ of S_{i+1} , it is *not* the case that the r^{th} bit of Γ^τ is 0 and the s^{th} bit is 1: Say τ extends σ_j . By our choice of strategy, if the r^{th} bit is 0, then it must be the case that we could not have forced it to be 1 above σ_j , and so we would have forced the s^{th} bit to be 0 above σ_j .

Let $P = \{X \in \Gamma(S_i) : X(r) = 0 \text{ and } X(s) = 1\}$. Then $\mu(P) = (1/4)\mu(\Gamma(S_i))$, since $r, s \geq m$. Clearly, $\Gamma(S_{i+1}) \subseteq \Gamma(S_i) \setminus P$, so $\mu(\Gamma(S_{i+1})) \leq (3/4)\mu(\Gamma(S_i))$, as desired.

Let $S = \bigcup_{i \in \omega} S_i$. Then $\mu(\Gamma(S)) = \mu(\bigcap_{i \in \omega} \Gamma(S_i)) = 0$. Let f be any path through S that is DNC₃. Then $\Gamma^f \in \Gamma(S)$. But $\Gamma(S)$ is a null Π_1^0 class, which implies that Γ^f is not Kurtz random, contradicting our initial assumption. \square

Note that the construction in Theorem 3.19 starts with a 3-bushy tree and produces a 2-bushy subtree with no leaves.

Definition 3.20. Let j be an order function. We say that a tree $T \subseteq \omega^{<\omega}$ is *j-bushy* above a string $\sigma \in \omega^{<\omega}$ if every element of T is comparable with σ and for each τ extending σ that is not a leaf of T , there are at least $j(|\tau|)$ many immediate extensions of τ . We say T is *exactly j-bushy above σ* if for each nonleaf τ , there are exactly $j(|\tau|)$ immediate extensions of τ in T .

Proof of Theorem 3.18. The forcing conditions have the form (σ, T, B) , where $\sigma \in \omega^{<\omega}$, T is a computable subtree of $\omega^{<\omega}$, $B \subset T$ and:

- T is exactly j -bushy above σ for some order function j
- B is c.e. and upward-closed in T (i.e., if $\tau \in B$ then ρ extending τ on T is also in B)
- B is $j(|\sigma|)$ -small above σ (and, without loss of generality, $j(|\sigma|)$ -closed).

A condition (σ, T, B) *extends* another condition (τ, S, C) if $\sigma \succeq \tau$, $T \subseteq S$ and $B \cap T \supseteq C \cap T$. Let \mathbb{P} denote this partial order. As before, if \mathcal{G} is a filter on \mathbb{P} , then $f_{\mathcal{G}} = \bigcup \{\sigma : (\sigma, T, B) \in \mathcal{G}\} \in \omega^{<\omega}$. It is not difficult to verify that if \mathcal{G} is sufficiently generic, then $f_{\mathcal{G}}$ is total and if $(\sigma, T, B) \in \mathcal{G}$, then $f_{\mathcal{G}}$ contains no initial segment in B .

If Γ is any functional, let \mathcal{D}_{Γ} denote the set of $(\sigma, T, B) \in \mathbb{P}$ such that either

- $g \in [T] \setminus [B]^\prec$ implies that Γ^g is total, or
- there is an $n \in \omega$ such that $g \in [T] \setminus [B]^\prec$ implies that $\Gamma^g(n) \uparrow$.

Claim 3.21. \mathcal{D}_Γ is dense in \mathbb{P} .

Proof. Suppose $(\sigma, T, B) \in \mathbb{P}$, where T is exactly j -bushy above σ . Let $C_x = \{\tau \in T : \Gamma^\tau(x) \downarrow\}$. Note that C_x is c.e. and upward closed in T . As usual, there are two cases.

Case 1. For every $\tau \in T$ extending σ and every $x \in \omega$, $C_x \cup B$ is $j(|\tau|)$ -big above τ . In this case, we build a computable tree $S \subseteq T$ in stages that is exactly j' -bushy above σ for an order function j' . Let S_0 consist of just σ and its initial segments. Suppose inductively that we have $l_i \in \omega$ and $S_i \subset T$ such that

- For each $x < l_i$, $j'(x)$ has already been defined and $j'(x) \leq j(x)$.
- S_i is a finite, regular j' -bushy tree of height l_i above σ .
- For every leaf τ of S_i , either $\Gamma^\tau(x) \downarrow$ for every $x < i$ or $\tau \in B$.

Let τ be a leaf of S_i . By assumption, $C_i \cup B$ is $j(|\tau|)$ -big above τ , so we extend τ to a finite tree with leaves in $C_i \cup B$ that is $j(|\tau|)$ -bushy above τ . Note that since $C_i \cup B$ is c.e., we can find such a tree computably. The tree S'_{i+1} that results from carrying out this operation above each leaf of S_i may not be regular, but since both C_i and B are upward closed in T and T is j -bushy above the leaves of S'_{i+1} , we can extend them $j(l_i)$ -bushily to some common level l_{i+1} , retaining the property that every leaf is in C_i or in B , and producing the tree S_{i+1} . We now let $j'(x) = j(l_i)$ for $l_i \leq x < l_{i+1}$. Note that j' is nondecreasing because of our assumption that $j'(x) \leq j(x)$ for $x < l_i$.

Let $S = \cup_{i \in \omega} S_i$ and note that since $j'(|\sigma|) = j(|\sigma|)$, B is already $j'(|\sigma|)$ -closed. So the condition $(\sigma, S, B \cap S)$ extends (σ, T, B) . Finally, if $g \in [S] \setminus [B]^\prec$, then for every i , $g \upharpoonright l_i \in C_i$, so Γ^g is total.

Case 2. Let τ and x be counterexamples to the assumption in Case 1 and let S be the full subtree of T above τ . Let $B' = (C_x \cup B) \cap S$. Then B' is $j(|\tau|)$ -small above τ , so $(\tau, S, B') \in \mathbb{P}$ and if $g \in [S] \setminus [B']^\prec$, then $\Gamma^g(x)$ diverges. \square

Let \mathcal{H}_Γ be the set of all conditions (σ, T, B) such that if $g \in [T] \setminus [B]^\prec$, then Γ^g is not Kurtz random.

Claim 3.22. \mathcal{H}_Γ is dense in \mathbb{P} .

Proof. Let $(\sigma, T, B) \in \mathbb{P}$ and Γ be a $\{0, 1\}$ -valued functional. Claim 3.21 allows us to assume that Γ is total on $[T] \setminus [B]^\prec$, and since B is c.e., we can assume further that Γ is total on $[T]$. Let j be the order function such that T is exactly j -bushy above σ .

The remainder of the proof is a straightforward modification of Theorem 3.19. We build an order function j' and an exactly j' -bushy tree $S \subseteq T$ above σ in stages. Let S_0 consist of σ and its initial segments. Next, suppose inductively that we have $l_i \in \omega$ and $S_i \subset T$ such that

- For each $x < l_i$, $j'(x)$ has already been defined and $j'(x) \leq j(x)$.
- S_i is a finite, regular j' -bushy tree of height l_i above σ .

Let $\Gamma(S_i)$ denote $\{\Gamma^g : g \in [T] \cap [S_i]^\prec\}$.

We first extend S_i $j(l_i)$ -bushily within T to a height $q > l_i$ such that $j(q) \geq 2j(l_i)$, obtaining the tree S'_{i+1} . This ensures that every level of T above q is $2j(l_i)$ -big above each leaf of S'_{i+1} . Clearly, $\mu(\Gamma(S'_{i+1})) \leq \mu(\Gamma(S_i))$. Let L be the set of leaves of S'_{i+1} and

let $m = \max\{|\Gamma^\rho| : \rho \in L\}$. We choose l_{i+1} large enough so that for every $\tau \in T$ of length l_{i+1} , $|\Gamma^\tau| \geq m + 2^{|L|} + 1$. Note that the fact that T is exactly j -bushy ensures that we can find l_{i+1} computably.

For any leaf ρ of S'_{i+1} , let T_ρ be the set of strings of length l_{i+1} in T extending ρ . If k is a position corresponding to one of the additional bits of convergence (i.e., $m \leq k < m + 2^{|L|} + 1$), we say we can force the k^{th} bit to be $c \in \{0, 1\}$ above ρ if $\{\tau \in T_\rho : \Gamma^\tau(k) = c\}$ is $j(l_i)$ -big above ρ . Since T_ρ is $2j(l_i)$ -big above ρ , if we cannot force the k^{th} bit to be 0 above ρ , we can force it to be 1.

As in the proof of Theorem 3.19, we obtain positions r and s such that above each leaf of S'_{i+1} , the r^{th} and s^{th} bits can be forced in the same way. We adopt the same strategy as before for extending S'_{i+1} to S_{i+1} and ensuring that $\mu(\Gamma(S_{i+1})) \leq (3/4)\mu(\Gamma(S_i))$. Finally, we let $j'(x) = j(l_i)$ for $l_i \leq x < l_{i+1}$.

Let $S = \bigcup_{i \in \omega} S_i$. Since $j'(|\sigma|) = j(|\sigma|)$, $B \cap S$ is $j'(|\sigma|)$ -small above σ . So $(\sigma, S, B \cap S) \in \mathbb{P}$ and since $\mu(\Gamma(S)) = \mu(\bigcap_{i \in \omega} \Gamma(S_i)) = 0$, $(\sigma, S, B \cap S) \in \mathcal{H}_\Gamma$. \square

To conclude the proof of Theorem 3.18, let \mathcal{G} be any filter containing $(\langle \rangle, h^{<\omega}, B_{\text{DNC}})$ that meets \mathcal{H}_Γ for each functional Γ as well as the families of conditions that ensure totality. Then $f_{\mathcal{G}} \in \text{DNC}_h$ and does not compute a Kurtz random. \square

Every hyperimmune degree contains a Kurtz random [24], so if the function we are building is to avoid computing a Kurtz random, it must be hyperimmune-free. This is, in fact, the case:

Claim 3.23. *If \mathcal{G} is sufficiently generic, then $f_{\mathcal{G}}$ has hyperimmune-free degree.*

Proof. Suppose $\Gamma^{f_{\mathcal{G}}}$ is a total function. Then if $(\sigma, T, B) \in \mathcal{G} \cap \mathcal{D}_\Gamma$, it must be the case that Γ is total on $[T] \setminus [B]^\prec$. Let Ξ be the functional that on input x and oracle $\tau \in T$,

computes $\Gamma^\tau(x)$ until the computation converges or τ enters B . If the latter occurs first, then let $\Xi^\tau(x) = 0$. Now Ξ is total on $[T]$ and agrees with Γ on $[T] \setminus [B]^\prec$.

Let j be the order function such that T is exactly j -bushy above σ . We define a computable function m that majorizes Γ^{f_G} . To compute $m(i)$, search for a finite tree $S_i \subset T$ that is j -bushy above σ such that for every leaf τ of S_i , $\Xi^\tau(i) \downarrow$. Note that such a finite tree must exist by the compactness of $[T]$ and we can find it computably since T is computable. Now let $m(i)$ be the maximum of the values $\Xi^\tau(i)$ as τ ranges over the leaves of S_i .

Since T is exactly j -bushy above σ and S_i is a subtree of T that is j -bushy above σ , $[T] \subseteq [S_i]^\prec$. So $f_G \in [S_i]^\prec$ and $\Gamma^{f_G}(i) = \Xi^{f_G}(i) \leq m(i)$. \square

3.4 A DNC^X function of minimal degree

In this section, we strengthen Kumabe's result that there is a DNC function of minimal degree.

Theorem 3.24. *Given any oracle X , there is a function that is DNC relative to X and of minimal degree.*

Kumabe and Lewis [23] provided a simplified version of Kumabe's original arguments [22]. Our proof reuses much of the combinatorial machinery developed in the Kumabe-Lewis proof, but differs in several key aspects. Kumabe and Lewis use partial trees with computable domains, hence the function they produce is hyperimmune-free. We use partial trees with noncomputable domains, out of necessity: any DNC function relative to θ' is hyperimmune [18]. Further, it suffices in the Kumabe-Lewis construction to work with bad sets of constant bushiness. This is not the case here; our bad sets are h -small

for some order function h . In our approach to bad sets of varying bushiness, we use ideas from Cai and Greenberg's result in [5] that there exist degrees \mathbf{a} and \mathbf{b} such that \mathbf{a} is minimal and DNC and \mathbf{b} is DNC relative to \mathbf{a} and a strong minimal cover of \mathbf{a} .

3.4.1 Definitions and notation

Definition 3.25. Let h be an order function. Given $\sigma \in \omega^{<\omega}$, we say that a set $B \subseteq \omega^{<\omega}$ is *h -big above σ* if there is a finite h -bushy tree T above σ such that all its leaves are in B . If B is not h -big above σ then we say that B is *h -small above σ* .

It is easy to see that the smallness preservation property, concatenation property and small set closure property all continue to hold when one replaces the constants governing bushiness with order functions.

For an order function g and $l \in \omega$, let $w(g, l)$ denote $\prod_{i < l} g(i)$ and let $r(g, l)$ denote $2^{3+3w_g(l)}$.

In order to simplify our calculations, throughout this proof we restrict ourselves to order functions that only take values that are powers of two.

Definition 3.26. Suppose $h(n) = 2^{h'(n)}$ and $g(n) = 2^{g'(n)}$ are order functions, where $h', g' : \omega \rightarrow \omega$. The *middle* of h and g is the order function $\mathcal{M}(h, g)$ defined by

$$\mathcal{M}(h, g)(n) = 2^{\lfloor \frac{h'(n) + g'(n)}{2} \rfloor}.$$

Definition 3.27. Suppose h and g are order functions. We say the pair (h, g) *allows splitting above $N \in \omega$* if

1. $h(N) \geq g(N)$,

2. for $n \geq N$, $h(n)/g(n)$ is nondecreasing, and
3. there is an increasing sequence $\langle l_i \rangle_{i \in \omega}$ of natural numbers with $l_0 \geq N$ such that

$$h(l_i)/g(l_i) \geq (r(h, l_i))^i.$$

We say (h, g) allows splitting if it allows splitting above some $N \in \omega$. We call the sequence $\langle l_i \rangle$ the *splitting levels* for (h, g) .

Lemma 3.28. *Let h and g be order functions such that (h, g) allows splitting. Let $m = \mathcal{M}(h, g)$. Then (m, g) and (h, m) allow splitting.*

Proof. We provide the argument for (m, g) . Suppose (h, g) allows splitting above N and $\langle l_i \rangle$ is the sequence of splitting levels for (h, g) . Note that conditions (1) and (2) in the definition above are satisfied by (m, g) above N .

We verify condition (3). Suppose that $h(n) = 2^{h'(n)}$ and $g(n) = 2^{g'(n)}$. Note first that for each $n \geq N$,

$$\frac{m(n)}{g(n)} = 2^{\lfloor \frac{h'(n)+g'(n)}{2} \rfloor - g'(n)} = 2^{\lfloor \frac{h'(n)-g'(n)}{2} \rfloor} \geq \frac{2^{\frac{h'(n)-g'(n)}{2}}}{2}.$$

It follows that for each $i \in \omega$,

$$\frac{m(l_{2i+2})}{g(l_{2i+2})} \geq \frac{2^{\frac{h'(l_{2i+2})-g'(l_{2i+2})}{2}}}{2} \geq \frac{(r(h, l_{2i+2}))^{\frac{2i+2}{2}}}{2} = \frac{(r(h, l_{2i+2}))^{i+1}}{2} \geq (r(m, l_{2i+2}))^i,$$

so $\langle l_{2i+2} \rangle_{i \in \omega}$ is a sequence of splitting levels for (m, g) .

A similar calculation shows that (h, m) allows splitting. □

It is not hard to verify that if (h, g) allows splitting then for any $c \in \omega$, so do $(h, 2^c g)$ and $(\max(h/2^c, 2), g)$.

3.4.2 The partial order

The forcing conditions are of the form (σ, T, B, h_T, h_B) , where

- the tree T is partial recursive (some nodes may be terminal) and exactly h_T -bushy above σ
- B includes the terminal nodes in T , is upward closed and is h_B -small above σ
- (h_T, h_B) allows splitting above $|\sigma|$.

Only σ, T and B contribute to the ordering. Let h_M denote $\mathcal{M}(h_T, h_B)$. By extending σ appropriately, we can assume that $h_M(n)/16 \geq h_B(n)$ for all $n \geq |\sigma|$.

Note that we have no access to the set B (it is not c.e.). Since the terminal nodes of T are contained in the bad set B , the conditions that force f_G to be total are dense in this partial order.

As before, we can assume that the bad set is h_B -closed. In other words, if τ is any string in $T \setminus B$ then B is h_B -small above τ .

3.4.3 Forcing Γ^{f_G} to be partial

Let $C_n = \{\tau \in T : \Gamma^\tau(n) \downarrow\}$. Given a condition (σ, T, B, h_T, h_B) and a functional Γ we say we can *force* Γ^{f_G} to be *partial* if there is a τ on T extending σ and an n such that the set $C_n \cup B$ is h_M -small above τ . If this is the case, then we let T' be the full subtree of T above τ . The condition $(\tau, T', C_n \cup B, h_T, h_M)$ extends (σ, T, B, h_T, h_B) , while forcing $\Gamma^{f_G}(n) \uparrow$. From now on we assume that we cannot force Γ^{f_G} to be partial. It follows that for every n , and every $\tau \in T \setminus B$, $C_n \setminus B$ is $h_M/2$ -big above τ . Applying this fact iteratively we obtain the following claim:

Claim 3.29. *For any $\tau \in T \setminus B$ extending σ and any n , there is an $A \subset T \setminus B$, $h_M/2$ -big above τ , such that for every $\rho \in A$, $\Gamma^\rho \upharpoonright n$ is defined.*

3.4.4 Forcing Γ^{f_g} to be computable

It is worth pointing out here how our argument for this case of the forcing differs from the one in Kumabe-Lewis. As we have mentioned, the bad sets in their argument are c.e., and they make strong use of this fact in an *effective* simultaneous construction of a refined subtree and a real such Y that it is the image of Γ on every path on this subtree (and hence computable). We do not have access to the bad set, since we will ultimately want it to include the set of strings that are non-DNC relative to X . So we construct Y *noneffectively*, arguing that it is the image under Γ of a sufficiently bushy subtree. Under the assumption that we make in this case of the forcing, Y will turn out to be computable.

Definition 3.30. Let g be an order function. A g -big splitting above $\tau \in T$ is a pair of sets $A_0 \subset T$ and $A_1 \subset T$, both g -big above τ , such that for any $\tau_0 \in A_0$ and $\tau_1 \in A_1$, $\Gamma^{\tau_0} \upharpoonright \Gamma^{\tau_1}$. We say that A_0 and A_1 are Γ -splitting.

Suppose that there is a $\tau \in T \setminus B$ extending σ such that we cannot find any $h_M/16$ -big splitting above τ . Under this assumption, we construct a real Y with the property that for each $n \in \omega$, the set of ρ on T such that $\Gamma^\rho \upharpoonright n = Y \upharpoonright n$ is $h_M/4$ -big above τ . It follows immediately that Y is computable: to compute it up to n bits, we search for an $h_M/4$ -bushy tree $A \subset T$ above τ every leaf of which gives the same n bits of convergence via Γ . These bits must agree with Y , otherwise we will have obtained an $h_M/16$ -big splitting above τ . Further, if we let $D = \{\rho \in T : \Gamma^\rho \upharpoonright n = Y \upharpoonright n\}$, then D is $h_M/16$ -small

above τ . It follows that $B \cup D$ is h_M -small above τ , so letting T' be the full tree above τ , the condition $(\tau, T', B \cup D, h_T, h_M)$ extends (σ, T, B, h_T, h_B) while forcing $\Gamma^{f_{\mathcal{G}}}$ to be computable.

We construct Y bit by bit (although not effectively), letting $Y_0 = \Gamma^\tau$. We also assume inductively that there is a set $S_i \subset T \setminus B$ that is $h_M/4$ -big above τ and for every $\rho \in S_i$, $\Gamma^\rho \upharpoonright i + |Y_0| = Y_i$. Let S_0 consist of just τ .

Given Y_i and S_i , we proceed as follows. Above each leaf ρ of S_i , there is an $h_M/2$ -big set of strings A_ρ such that for each $\nu \in A_\rho$, $\Gamma^\nu(|Y_i|)$ is defined. A_ρ can then be thinned out to a set A'_ρ that is $h_M/4$ -big above ρ and such that for each $\nu \in A'_\rho$, $\Gamma^\nu(|Y_i|)$ converges to the same value c_ρ . Next, since S_i is $h_M/4$ -big above τ , there is a $V \subset S_i$, $h_M/8$ -big above τ , such that for each $\rho \in V$, c_ρ is the same value, say j . Let $Y_{i+1} = Y_i j$. Note that $V' = \cup\{A'_\rho : \rho \in V\}$ is $h_M/8$ -big above τ and for each $\nu \in V'$, $\Gamma^\nu \succeq Y_{i+1}$. Let $S_{i+1} = \{\nu \in C : \Gamma^\nu \succeq Y_{i+1}\}$. The set $C \setminus S_{i+1}$ must be $h_M/16$ -small above τ , otherwise $C \setminus S_{i+1}$ and V' form an $h_M/16$ -big splitting above τ . It follows that S_{i+1} is $h_M/4$ -big above τ .

3.4.5 Forcing $\Gamma^{f_{\mathcal{G}}} \geq_T f_{\mathcal{G}}$

We work now under the additional assumption that for each $\tau \in T \setminus B$ extending σ there is a $h_M/16$ -big splitting above τ .

We refine T to a subtree S that has the *delayed splitting property*: above each $\tau \in S \setminus B$, there are levels $l' > l > |\tau|$ such that if ρ_0 and ρ_1 are any two extensions of τ on S of length l , and $\rho'_0 \succ \rho_0$ and $\rho'_1 \succ \rho_1$ are extensions on S of length l' , then $\Gamma^{\rho'_0} \mid \Gamma^{\rho'_1}$.

The statement of the following lemma has been slightly modified from the original

in order to apply to trees of varying bushiness:

Lemma 3.31 (Kumabe, Lewis [23]). *Let Γ be a functional. Let A be $4g$ -big above α and B be $4h$ -big above β , where g and h are order functions. Suppose that above every leaf τ of A , there exist $\Delta_{\tau,0}$ and $\Delta_{\tau,1}$, such that they are both $4g$ -big above τ and are Γ -splitting. Let $A' = \cup_{\tau,i} \Delta_{\tau,i}$ and let $v = \max\{|\Gamma^\rho| : \rho \in A'\}$. If for every leaf σ of B , $|\Gamma^\sigma| > v$, then there is an $A'' \subseteq A'$ and a $B' \subseteq B$, g -big above α and h -big above β respectively, that are Γ -splitting.*

Proof. Let $\sigma_0 = \langle \rangle$ and $B_0 = B$.

Assume inductively that we have σ_s of length s and B_s , h -big above β , such that for all $\rho \in B_s$, $\Gamma^\rho \succeq \sigma_s$.

If $\{\tau \in A' : \Gamma^\tau \mid \sigma_s\}$ is g -big above α then we are done. If not, then either

1. $A_1 = \{\tau \in A' : \Gamma^\tau \preceq \sigma_s\}$ is g -big above α or
2. $A_2 = \{\tau \in A' : \Gamma^\tau \text{ properly extends } \sigma_s\}$ is g -big above α .

If (1) holds then let V be the set of leaves of A that have an extension in A_1 . For each $\tau \in V$, the set of strings in A_1 extending τ must lie entirely in one of the $\Delta_{\tau,i}$. Let Δ'_τ denote the other member of the splitting above τ . Then $\cup\{\Delta'_\tau : \tau \in V\}$ is g -big above α and splits with B_s .

Next, assume (2) holds, which implies that $|\sigma_s| < v$. If $\{\tau \in B : \Gamma^\tau \mid \sigma_s\}$ is h -big above β , then we are done. If not, then it must be the case that $D = \{\tau \in B : \Gamma^\tau \succeq \sigma_s\}$ is $h/2$ -big above β . D can be partitioned into the sets $D_i = \{\tau \in D : \Gamma^\tau(|\sigma_s|) = i\}$, one of which must be $h/4$ -big above β , say D_j . Let $B_{s+1} = D_j$ and let $\sigma_{s+1} = \sigma_j$ and continue the construction. Since this process cannot continue indefinitely, we will obtain the required splitting via one of the other alternatives. \square

Claim 3.32. *Suppose τ_0, \dots, τ_k are nodes of length l in $T \setminus B$, $k < w_{h_M}(l)$ and that $h_M(l)/h_B(l) \geq r(h_M, l)$. Then there is a sequence of sets A_0, \dots, A_k , where A_j is $(h_M/2^{3+3k})$ -big above τ_j and which are pairwise Γ -splitting.*

Proof. The proof is by induction on k . Suppose we already have A_0, \dots, A_k , where each A_j is $(h_M/2^{3+3k})$ -big above τ_j and the collection is pairwise Γ -splitting. Let τ_{k+1} be an additional node of length l that is not in B and let $q = h_M/2^3$.

Note that since $w_{h_M}(l) > k + 1$, $h_B(l) < q(l)/2^{3k+1}$. So we first refine each A_j to a Π_j where Π_j is $(q/2^{3k+1})$ -big above τ_j and $\Pi_j \cap B = \emptyset$. If ρ is a leaf of Π_j , then it is not in B and since $q/2^{3k+1} \leq h_M/16$, we can find a $q/2^{3k+1}$ -bushy splitting, say $D_{\rho,0}$ and $D_{\rho,1}$, above ρ . We let $\Pi'_j = \cup_{i,\rho} D_{\rho,i}$.

Let m be the longest length of the image of Γ on any string in any of the Π'_j . Appealing to Claim 3.29, we let Δ_0 be a q -big set above τ_{k+1} such that each leaf of Δ_0 gives at least $m + 1$ bits of convergence via Γ . We now apply Lemma 3.31 on Π'_0 and Δ_0 , obtaining $A'_0 \subset \Pi'_0$ and $\Delta_1 \subset \Delta_0$, which are Γ -splitting and where the former is $q/2^{3(k+1)}$ -big above τ_0 and the latter is $q/4$ -big above τ_{k+1} . Next, we apply Lemma 3.31 to the pair Π'_1 and Δ_1 , obtaining $A'_1 \subset \Pi'_1$ and $\Delta_2 \subset \Delta_1$, which are Γ -splitting and where A'_1 is $q/2^{3(k+1)}$ -big above τ_1 and Δ_2 is $q/4^2$ -big above τ_{k+1} . After $k + 1$ applications of Lemma 3.31, we will have obtained A'_0 through A'_k and Δ_{k+1} , which are pairwise Γ -splitting. Moreover, Δ_{k+1} is $q/2^{2(k+1)}$ -big above τ_{k+1} , so we can let $A'_{k+1} = \Delta_{k+1}$. \square

Our argument here differs once again in a crucial way from Kumabe and Lewis's. Suppose we have defined the delayed splitting tree S up to a certain level and let τ be one of the leaves of this finite tree. In order to continue the construction above τ , we must find a sufficiently bushy splitting above τ . In the Kumabe-Lewis argument, such

a splitting will be found, or τ will be seen to enter the bad set. In either case, the construction of the tree S is in no danger of “stalling”. Here, however, we have no access to the bad set, so we may end up searching in vain for a splitting. In order to get around this, we will only ask for splittings above sufficiently bushy many leaves of the current approximation to S , a situation that we can guarantee, and add the remaining leaves to the bad set. Thus, we will be adding lots of strings to the bad set at each level of the construction. The following lemma is critical to preserving its smallness when we do so:

Lemma 3.33. *Let g be an order function. Suppose $A \subset \omega^{<\omega}$ is g -small above $\sigma \in \omega^{<\omega}$, and suppose $\tau \in \omega^{<\omega}$ extends σ and A contains no extension of τ . If B is a set of strings extending τ that is g -small above τ , then $A \cup B$ is g -small above σ .*

Proof. Suppose otherwise, i.e., there is a g -bushy tree T above σ with leaves in $A \cup B$. Clearly, some leaves of T are in B . Since every string in B extends τ , $\tau \in T$. This means that there is a tree T' that is g -bushy above τ whose leaves are in B , namely the tree consisting of all strings in T that are comparable with τ . This is a contradiction. \square

Let $\langle l_i \rangle$ be the sequence of splitting levels for the pair (h_M, h_B) . We begin by defining h_S . Let $j_i = l_{i+1}$. For $n < j_0$, let $h_S(n) = h_M(n)$. For $j_{i+1} > n \geq j_i$, let $h_S(n) = h_M(j_i)/r(h_S, j_i)$. Then for each i ,

$$\frac{h_S(j_i)}{h_B(j_i)} = \frac{h_M(j_i)}{h_B(j_i)r(h_S, j_i)} \geq \frac{(r(h_M, j_i))^{i+1}}{r(h_S, j_i)} \geq (r(h_S, j_i))^i.$$

Hence the pair (h_S, h_B) allows splitting above $|\sigma|$.

We now describe how we build the partial recursive tree S . We start by letting S_0 be an h_S -bushy subtree of T above σ with leaves of length l_1 or less such that if D_0 is

the set of leaves of S_0 of length strictly smaller than l_1 , then D_0 is h_B -small above σ . Since the terminal nodes of T are contained in B , such a tree must exist. We declare the nodes in D_0 terminal and the leaves of S_0 that are of length l_1 to be the *children of σ* . Throughout the construction we will maintain the property that if $\tau \in S$ has children in S , then they are all of the same length and that length is a splitting level for the pair (h_M, h_B) .

At a stage s of the construction, we will have built a finite approximation S_s of S , and accumulated a set D_s of nodes on S_s that we have declared terminal. D_s will always be h_B -small above σ .

Suppose that $\tau \in S_s$ has a set C_τ of children of length l_i and that they are leaves of S_s . If we have not already done so, we initiate a search for a subset C'_τ of C_τ such that $C_\tau \setminus C'_\tau$ is h_B -small above τ , and for each $\rho \in C'_\tau$, there is a A_ρ , h_S -bushy above ρ such that the collection $\{A_\rho : \rho \in C'_\tau\}$ is pairwise Γ -splitting.

If $\tau \notin B$ then this search must terminate. To see why this is the case note first that B is h_B -small above τ . Let ρ_0, \dots, ρ_k be the strings in $C_\tau \setminus B$. Since l_i is a splitting level for (h_M, h_B) , $h_M(l_i)/h_B(l_i) \geq r(h_M, l_i)$. Moreover, $w_{h_M}(l_i) \geq w_{h_S}(l_i) > k$. By Claim 3.32, there are A_0, \dots, A_k , with A_j $h_M/2^{3+3k}$ -big above ρ_j , that are pairwise Γ -splitting.

Now

$$\frac{h_M(n)}{2^{3+3k}} \geq \frac{h_M(n)}{2^{3+3w_{h_S}(l_i)}} = \frac{h_M(n)}{r(h_S, l_i)} \geq h_S(n)$$

for $n \geq l_i$, so we can refine the A_j to subtrees that are h_S -bushy.

If C'_τ is found, then we extend each $\rho \in C'_\tau$ by A_ρ . Note that by Lemma 3.33, $D_s \cup (C_\tau \setminus C'_\tau)$ is h_B -small above σ , since D_s initially contains no extension of τ and $C_\tau \setminus C'_\tau$ is h_B -small above τ . So we can add $C_\tau \setminus C'_\tau$ to D_s .

Next, for each $\rho \in C'_\tau$ we wish to extend the leaves of A_ρ h_S -bushily to the next splitting level for (h_M, h_B) . Let L_ρ be the set of leaves of A_ρ , and let $m = \max\{|\nu| : \nu \in L_\rho\}$. Let l be least splitting level for (h_M, h_B) greater equal to m . We begin a search for an $L'_\rho \subseteq L_\rho$ such that $L_\rho \setminus L'_\rho$ is h_B -small above ρ and above each $\nu \in L'_\rho$ there is an h_S -bushy tree with leaves of length l . Note that if $\rho \notin B$, this search must terminate. When we find such an L'_ρ , we extend all its elements h_S -bushily to level l , declaring the new leaves to be the children of ρ and add $L_\rho \setminus L'_\rho$ to D_s . The same argument as before shows that D_s remains h_B -small above σ .

The resulting tree S is h_S -bushy and if we let $D = \cup_s D_s$, then the new bad set $D \cup B$ is $2h_B$ -small above σ . It is clear that the construction halts above a node $\tau \in S$ if it is either in B or we have declared it to be terminal by adding it to D , and so $B \cup D$ contains all the terminal nodes of S . By extending σ , we can ensure that $(h_S, 2h_B)$ allows splitting above $|\sigma|$. For such a σ , the condition $(\sigma, S, D \cup B, h_S, 2h_B)$ extends (σ, T, B, h_T, h_B) and forces $\Gamma^{f_{\mathcal{G}}} \geq_T f_{\mathcal{G}}$.

Chapter 4

Lebesgue density and Π_1^0 classes

4.1 Introduction

The Lebesgue density theorem says that if A is any Lebesgue measurable set of reals, for almost every point x of A , the *density of A at x* is 1. Roughly speaking, the more we “zoom in” on x by looking at a smaller and smaller interval containing it, the closer to 1 is the fractional measure of A within that interval.

Suppose that \mathcal{C} is a countable collection of Lebesgue measurable subsets of the unit interval. We say $x \in [0, 1]$ is a *positive density point for \mathcal{C}* if for every $P \in \mathcal{C}$ that contains x , the density of P at x is positive. We say x is a *density-one point for \mathcal{C}* if for every $P \in \mathcal{C}$ that contains x , the density of P at x is 1. It follows from the Lebesgue density theorem (and the countable additivity of Lebesgue measure) that almost every point in the unit interval is a density-one point for \mathcal{C} . Of particular interest are the positive density and density-one points we obtain when \mathcal{C} is the collection of *effectively closed* (or Π_1^0) subsets of the unit interval. These have been at the heart of the solutions of the ML-covering and ML-cupping problems [2, 8, 9]. An interesting fact that emerged from this line of research is a new characterization of the incomplete Martin-Löf random reals:

Theorem 4.1 (Bienvenu, Hölzl, Miller, and Nies [4]). *A Martin-Löf random real is a*

positive density point if and only if it is incomplete.

The positive density points are properly contained within the class of Kurtz random reals, but not within the Martin-Löf random reals. So Theorem 4.1 leads us to ask: Are positive density points computationally weak in general? In the other direction, are the Kurtz random reals that are not positive density computationally powerful?

The 1-generics, which are one of the most widely studied class of reals in computability theory, are closely connected to the density-one points of Π_1^0 classes. In fact, every 1-generic is a density-one point. If the former is a member of a Π_1^0 class P , then P contains an open interval around it. A general density-one point can then be viewed as a *more tolerant 1-generic*. It permits P to have gaps in the interval, as long as the gaps are not too big in fractional measure, and this measure goes down as we shrink the interval. A natural question is, how *unlike* 1-generics can these points be?

Bienvenu, Greenberg, Kučera, Nies, and Turetsky [3] distinguish between *dyadic density* and *full density*. The former is a more natural notion of density in Cantor space, while the latter is more natural on the unit interval. In Section 4.3, we strongly separate the two by constructing a dyadic density-one point that is not a full positive density point (Theorem 4.6). We also show (with J. Miller) that when we restrict our attention to the Martin-Löf random reals, being dyadic density-one is equivalent to being full density-one (Theorem 4.13).

In Section 4.4, we turn to the computational power of dyadic positive density points, showing that one direction of Theorem 4.1 fails dramatically when the assumption of Martin-Löf randomness is removed: There is a dyadic density-one point Turing above any degree (Theorem 4.17). In Section 4.5, we lift Theorem 4.17 to full density on the unit interval (Theorem 4.24).

In Section 4.6, we probe the connection between 1-generics and density-one points further. We find that the “van Lambalgen property” fails for dyadic density-one points. However, no dyadic positive density point can be of minimal Turing degree: Every such point is either Martin-Löf random, or computes a 1-generic (Theorem 4.29).

In Section 4.7, we explore the relationship between randomness and various notions of computability-theoretic strength within the class of reals that are not positive density. We observe (Proposition 4.33) that there is a computably random real that is incomplete and not positive density. On the other hand, the property of being not positive density does imply a weaker form of computational strength on the class of Schnorr random reals. In Proposition 4.35, we show that every such real is high.

4.2 Definitions and notation

We will refer more often to Σ_1^0 classes than the c.e. sets of strings that generate them, so we depart slightly from convention and let $\langle W_e \rangle_{e \in \omega}$ denote a uniform enumeration of Σ_1^0 classes.

If $i \in \{0, 1\}$, \bar{i} denotes the *other* binary digit, namely, $1 - i$.

Any irrational $x \in [0, 1]$ can be identified uniquely with an infinite binary sequence, namely, its binary expansion. Since we are seldom concerned with rationals, we use the term *real* to refer both to infinite binary sequences and elements of $[0, 1]$. The clopen set $[\sigma]$ in Cantor space is similarly identified with the open set $(0.\sigma, 0.\sigma + 2^{-|\sigma|})$ on the unit interval. We can thus speak of Σ_1^0 and Π_1^0 classes on the unit interval.

The symbol μ refers both to the uniform measure on Cantor space and to Lebesgue

measure on the unit interval, which are measure-theoretically isomorphic via the correspondence just described. Given $\sigma \in 2^{<\omega}$ and a measurable set $C \subseteq 2^\omega$, the shorthand $\mu_\sigma(C)$ denotes the relative measure of C in $[\sigma]$, i.e.,

$$\mu_\sigma(C) = \frac{\mu([\sigma] \cap C)}{\mu([\sigma])}.$$

If I and C are measurable subsets of $[0, 1]$, and I is not null, then $\mu_I(C)$ denotes the relative measure of C in I , i.e.,

$$\mu_I(C) = \frac{\mu(I \cap C)}{\mu(I)}.$$

4.3 Dyadic density vs full density

Definition 4.2. Let C be a measurable subset of 2^ω and $X \in 2^\omega$. The (*lower*) *dyadic density* of C at X , written $\varrho_2(C | X)$, is

$$\liminf_n \mu_{X \upharpoonright n}(C).$$

Definition 4.3. A real $X \in 2^\omega$ is a *dyadic positive density point* if for every Π_1^0 class C containing X , $\varrho_2(C | X) > 0$. It is a *dyadic density-one point* if for every Π_1^0 class C containing X , $\varrho_2(C | X) = 1$.

Even though dyadic density seems like the natural notion of density in Cantor space, it is a simplification of the version of density that appears in the classical Lebesgue Density Theorem:

Definition 4.4. Let C be a measurable subset of \mathbb{R} and $x \in \mathbb{R}$. The *(lower) full density* of C at x , written $\varrho(C | x)$, is

$$\liminf_{\gamma, \delta \rightarrow 0^+} \frac{\mu((x - \gamma, x + \delta) \cap C)}{\gamma + \delta}.$$

Definition 4.5. We say $x \in [0, 1]$ is a *full positive density point* if for every Π_1^0 class $C \subseteq [0, 1]$ containing x , $\varrho(C | x) > 0$. It is a *full density-one point* if for every Π_1^0 class $C \subseteq [0, 1]$ containing x , $\varrho(C | x) = 1$.

As pointed out earlier, if x is irrational, we can identify it uniquely with a binary sequence. So it makes sense to ask if x is a dyadic density-one point. Likewise, it makes sense to ask if a sequence $X \in 2^\omega$ is a full density-one point. Clearly, every full density-one point is dyadic density-one. That the converse fails is our main result in this section:

Theorem 4.6. *There is a dyadic density-one point that is not a full positive density point.*

The real described by this theorem is not 1-generic, and as we will see shortly, not Martin-Löf random. Its construction illustrates a method by which we can break out of those classes, and serves as the basic template for the constructions in Sections 4.4 and 4.5. We begin with a lemma that is a restatement of the well-known “Kolmogorov inequality for martingales” (see, for example, [27], 7.1.9):

Lemma 4.7. *Suppose $W \subseteq 2^\omega$ is open. Then for any ε such that $\mu(W) \leq \varepsilon \leq 1$, let U_ε denote the set $\{X \in 2^\omega : \mu_\rho(W) \geq \varepsilon \text{ for some } \rho \prec X\}$. We call U_ε the ε -vicinity of W . Then $\mu(U_\varepsilon) \leq \mu(W)/\varepsilon$.*

Proof. For each $X \in U_\varepsilon$, let ρ_X denote the least initial segment ρ of X such that $\mu_\rho(W) > \varepsilon$. Let $V = \{\rho_X : X \in U_\varepsilon\}$. Note that V is prefix-free and $[V] = U_\varepsilon$. Since W is open, for every $Y \in W$, some initial segment of Y is in V and so $[V]$ covers W . Now, for each $\rho \in V$,

$$\mu_\rho(W) = \frac{\mu(W \cap [\rho])}{2^{-|\rho|}} \geq \varepsilon.$$

So $2^{-|\rho|} \leq \mu(W \cap [\rho])/\varepsilon$ and

$$\mu([V]) = \sum_{\rho \in V} 2^{-|\rho|} \leq \sum_{\rho \in V} \frac{\mu(W \cap [\rho])}{\varepsilon} = \frac{\mu(W)}{\varepsilon}. \quad \square$$

Proof of Theorem 4.6. We build the desired real Y by computable approximation. At each stage s of the construction, we have a sequence of finite strings $\sigma_{0,s} \prec \sigma_{1,s} \prec \dots$ approximating Y . At the same time, we build a Σ_1^0 class B , the complement of which witnesses the fact that Y is not a full positive density point. The main idea for accomplishing this is depicted in Figure 4.1, where σ is the longest initial segment of Y that “sees” the measure that we enumerate into B . This measure is small inside $[\sigma]$, but there is an interval containing Y , namely, the closure of $[\sigma 01^j] \cup [\sigma 10^j]$, in which the measure is quite large.

Recall that W_e denotes the e -th Σ_1^0 class. Each such class represents a requirement that needs to be met by Y . In other words, for each e , if Y is not in W_e , we require that $\lim_{\rho \prec Y} \mu_\rho(W_e) = 0$. Priorities are assigned to Σ_1^0 classes in the usual manner, with W_j being of higher priority than W_i for any $i > j$. We make use of the following shorthand: Let C be a measurable set and τ and τ' two strings such that $\tau \prec \tau'$. If for every ρ such that $\tau \preceq \rho \prec \tau'$, $\mu_\rho(C) < \alpha$, then we say that *between* τ and τ' , $\mu(C) < \alpha$.

At any stage s , for each $k \leq s$, we will be working above $\sigma_{k,s}$ to define $\sigma_{k+1,s}$. We

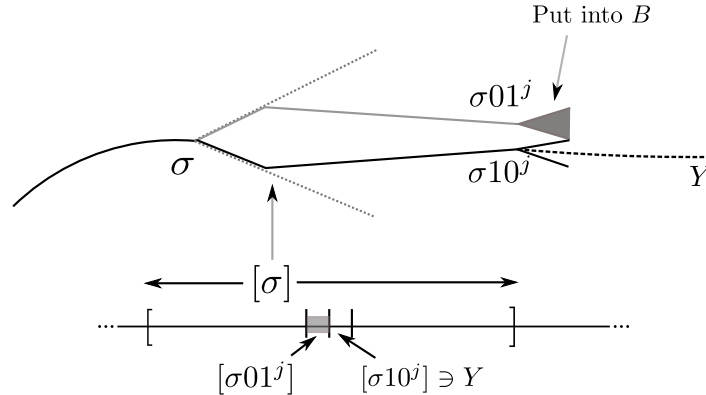


Figure 4.1: Separating dyadic and full density-one

have two goals in mind: First, for any $e < k$ such that $[\sigma_{k,s}]$ is not already contained in W_e , we must keep the measure of W_e between $\sigma_{k,s}$ and $\sigma_{k+1,s}$ below a certain threshold. If the threshold is exceeded, say above a string ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, we will move σ_{k+1} to a string extending ρ so that the cone above it is contained entirely in W_e . Second, we must ensure that there is an interval $I \subseteq [\sigma_{k,s}]$ such that $[\sigma_{k+1,s}] \subseteq I$ and $\mu_I(B)$ is large. Both goals must be satisfied while keeping Y from entering B . Globally, we must maintain the fact that between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, the measure of B remains *strictly below* a threshold $\beta_s(k)$, which is updated each time we act above $\sigma_{k,s}$ by moving $\sigma_{k+1,s}$. We begin the construction by setting $\sigma_{0,0} = \langle \rangle$.

Procedure above $\sigma_{k,s}$

When we first start working above $\sigma_{k,s}$, say at stage s_0 , we set $\beta_{s_0}(k) = \beta^*(k)$ (see below for how $\beta^*(k)$ is defined). If $k > 0$, then we start by choosing a $\nu \succ \sigma_{k,s_0}$ long enough so that between σ_{k-1,s_0} and σ_{k,s_0} , $\mu(B_s \cup [\nu]) < \beta_{s_0}(k-1)$. We let $\sigma_{k+1,s_0} = \nu 10^j$ and enumerate $[\nu 0 1^j]$ into B , where j is chosen large enough so that the measure of B between σ_{k,s_0} and σ_{k+1,s_0} remains below $\beta^*(k)$. If $k = 0$, ν can be chosen to be $\langle \rangle$.

In a subsequent stage s , suppose that C_0, \dots, C_l are those among the first k Σ_1^0 classes in which $[\sigma_{k,s}]$ is not already contained, in order of descending priority. Now if for some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$ and some $j \leq l$, $\mu_\rho(C_j)$ exceeds $\sqrt{\beta_s(k)}$ and no action has yet been taken for a higher priority $C_{j'}$, then we *act* by moving $\sigma_{k+1,s}$ to a string extending ρ . Let $\nu \succeq \rho$ be a string such that $[\nu] \subseteq C_j$ and let it be long enough so that:

1. Between ρ and ν , $\mu(B_s) < \sqrt{\beta_s(k)}$.
2. $B_s \cap [\nu] = \emptyset$.
3. If $k > 0$, then between $\sigma_{k-1,s}$ and $\sigma_{k,s}$, $\mu(B_s \cup [\nu])$ must be strictly less than $\beta_s(k-1)$.

Let j be large enough so that between $\sigma_{k,s}$ and ν , $\mu(B_s \cup [\nu 01^j])$ remains strictly below $\sqrt{\beta_s(k)}$. We set $\sigma_{k+1,s+1} = \nu 10^{j+k}$ and enumerate $[\nu 01^j]$ into B . Finally, we set $\beta_{s+1}(k) = \sqrt{\beta_s(k)}$.

Choosing $\beta^*(k)$

We move $[\sigma_{k+1,s+1}]$ into C_j when the following is seen to occur at some stage s : For some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, $\mu_\rho(C_j)$ exceeds the measure of the $\sqrt{\beta_s(k)}$ -vicinity of B_s above ρ , i.e., if $\mu_\rho(C_j) > \beta_s(k)/\sqrt{\beta_s(k)} > \mu_\rho(B_s)/\sqrt{\beta_s(k)}$. If this does not occur, we wish to limit the measure of C_j to 2^{-k} between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. Each time we act above $\sigma_{k,s}$, the value of $\beta_{s+1}(k)$ is magnified by a power of $1/2$, so we require that $\beta^*(k)$ satisfy

$$(\beta^*(k))^{1/2^{k+1}} \leq 2^{-k}.$$

Verification

Claim 4.8. *Unless we act immediately above $\sigma_{k,s}$, the measure of B remains strictly below $\beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$.*

Proof. Condition (2) above ensures that if $\sigma_{k,s}$ is redefined at stage s due to an action above $\sigma_{l,s}$ for some $l < k$, then $\mu(B_s \cap [\sigma_{k,s}]) = 0$. If we act above $\sigma_{k+1,s}$, then condition (3) ensures that $\mu(B_s)$ remains below $\beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. Note that there is a string ν such that $\sigma_{k+1,s} \prec \nu \prec \sigma_{k+2,s}$ and $\mu(B_s \cup [\nu]) < \beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. So if we act above $\sigma_{l,s}$ for some $l > k + 1$, then we add some measure to B , but this measure is contained entirely in $[\nu]$. \square

Claim 4.9. *We can act above $\sigma_{k,s}$ while satisfying requirements (1) through (3) above.*

Proof. By Claim 4.8, $\mu(B_s) < \beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. So if at stage s , for some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, $\mu(C_j)$ exceeds $\sqrt{\beta_s(k)}$ then by Lemma 4.7 there is an $X \in C_j$ extending ρ such that for every α such that $\rho \preceq \alpha \prec X$, $\mu_\alpha(B) < \sqrt{\beta_s(k)}$. Thus there are arbitrarily long strings extending ρ satisfying condition (1). Conditions (2) and (3) are met by simply choosing a long enough such string. \square

Claim 4.10. *For each $k \in \omega$, $\sigma_k = \lim_s \sigma_{k,s}$ exists, and $Y = \bigcup_k \sigma_k$ is total.*

Proof. Assume that $\sigma_{k,s}$ has stabilized by stage s . Then σ_{k+1} is redefined above $\sigma_{k,s}$ at most k times. \square

Claim 4.11. *Y is a dyadic density-one point.*

Proof. Suppose that $Y \notin W_e$. Let k be large enough so that $k > e$ and for all $e' < e$, if $Y \in W_{e'}$, then $[\sigma_k] \subseteq W_{e'}$. Fixing a $k' > k$, let s be large enough so that $\sigma_{k',s}$ has

stabilized. By our choice of k , we never act above $\sigma_{k',s}$ for the sake of $W_{e'}$ for any $e' < e$, and by the assumption that $Y \notin W_e$, we never act for the sake of W_e . Let $t > s$ be such that $\sigma_{k'+1,t}$ has stabilized. For all $t' > t$, between $\sigma_{k',t'}$ and $\sigma_{k'+1,t'}$, $\mu(W_e)$ does not exceed $\sqrt{\beta_{\nu'}(k')}$, which is always bounded by $2^{-k'}$. \square

Claim 4.12. *Y is not a full positive density point.*

Proof. Let σ_k and σ_{k+1} be the final values of $\sigma_{k,s}$ and $\sigma_{k+1,s}$ respectively. Then by construction there is a string ν such that $\sigma_k \prec \nu \prec \sigma_{k+1} \prec Y$, and $\sigma_{k+1} = \nu 10^{j+k}$ for some j and $[\nu 01^j] \subseteq B$. Let $l = |\nu| + j + 1$ and let I be the interval $(0.\nu 1 - 2^{-l}, 0.\nu 1 + 2^{-(l+k)})$. Since Y is a dyadic density-one point, Y is not a rational and so $Y \in (0.\nu 1, 0.\nu 1 + 2^{-(l+k)}) \subset I$, and $\mu_I(B) \geq 1/(1 + 2^{-k})$. \square

This completes the proof of Theorem 4.6. \square

Bienvenu, et al. have observed (see [4], Remark 3.4) that Theorem 4.1 remains true if full density is replaced by dyadic density. It follows that a Martin-Löf random real is dyadic positive density if and only if it is full positive density. We now show that the notions of dyadic density-one and full density-one also coincide on the class of Martin-Löf random reals.

Theorem 4.13 (Khan, J. Miller). *Suppose X is Martin-Löf random. Then X is a dyadic density-one point if and only if X is a density-one point.*

In order to prove Theorem 4.13, we need to introduce *non-porosity points*.

Definition 4.14. We say that a Π_1^0 class C is *porous* at $X \in 2^\omega$ if there is an $\varepsilon > 0$ such that for every $\alpha > 0$, there is a $0 < \beta < \alpha$ such that $(X - \beta, X + \beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from C .

We say $Y \in 2^\omega$ is a *non-porosity point* if every Π_1^0 class to which Y belongs is non-porous at Y .

Lemma 4.15 (Khan, J. Miller). *If $X \in 2^\omega$ is dyadic density-one but not full density-one, then there is a Π_1^0 class that is porous at X .*

Proof. Suppose that the Σ_1^0 class W witnesses that X is not a full density-one point, i.e., there is an $\varepsilon > 0$ such that for all $\delta > 0$, there is an open interval $I \subseteq (X - \delta, X + \delta)$ containing X such that $\mu_I(W) > \varepsilon$. Since X is a dyadic density-one point, there is an initial segment σ of X such that for all $\tau \succeq \sigma$, $\mu_\tau(W) \leq \varepsilon/6$, and σ is not all zeros or all ones.

If ρ is a string of length k that is not all zeros or all ones, let ρ^- and ρ^+ denote the lexicographically preceding and succeeding strings of length k . The intervals $[\rho^-]$, $[\rho]$ and $[\rho^+]$ are all of the same length and adjacent.

Now let $I \subseteq [\sigma]$ be any open interval containing X , and let ρ be the longest initial segment of X extending σ such that the closure of $[\rho^-] \cup [\rho] \cup [\rho^+]$ covers I and denote this closure by I' . Then $\mu(I) \geq \mu(I')/6$. To see this, assume without loss of generality, that $X \succ \rho 0$. By the maximality of ρ , it cannot be the case that I is contained in the closure of $[\rho 0^-] \cup [\rho 0] \cup [\rho 0^+]$. So I must overlap either half of the interval $[\rho^-]$ or half of the interval $[\rho]$, which means that $\mu(I) > \mu([\rho])/2 = \mu(I')/6$.

Next, assume that $\mu_{\rho^-}(W) \leq \varepsilon/6$ and $\mu_{\rho^+}(W) \leq \varepsilon/6$. Since ρ extends σ , $\mu_\rho(W) \leq \varepsilon/6$. It follows that $\mu_{I'}(W) \leq \varepsilon/6$. Then

$$\mu_I(W) = \frac{\mu(W \cap I)}{\mu(I)} \leq \frac{6\mu(W \cap I)}{\mu(I')} \leq \frac{6\mu(W \cap I')}{\mu(I')} = 6\mu_{I'}(W) \leq \varepsilon.$$

We have shown that if $\mu_I(W) > \varepsilon$, then either $\mu_{\rho^-}(W)$ or $\mu_{\rho^+}(W)$ must exceed $\varepsilon/6$.

We build C as follows: whenever we see a $\rho \succeq \sigma$ such that $\mu_\rho(W) > \varepsilon/6$, enumerate $[\rho]$ into the complement of C . Note that we never enumerate an initial segment of X into the complement of C , so C contains X . Moreover, C is porous at X : Given an $\alpha > 0$, there is an open interval $I \subseteq [\sigma]$ containing X such that $I \subseteq (X - \alpha/24, X + \alpha/24)$ and $\mu_I(W) > \varepsilon$. Let ρ be chosen as above, and I' accordingly. Then $I' \subseteq (X - \alpha/4, X + \alpha/4)$. Finally, let $\beta = 2 \cdot 2^{-|\rho|}$. Then $(X - \beta, X + \beta) \subseteq (X - \alpha/2, X + \alpha/2)$, and there is a subinterval of $(X - \beta, X + \beta)$ of size $\beta/2$, namely, one of $[\rho^-]$ or $[\rho^+]$, that lies in the complement of C . \square

Theorem 4.13 now follows from two facts. By Theorem 4.1 (and the fact that it holds for dyadic density), X in Lemma 4.15 is incomplete, while the Π_1^0 class C is porous at X . But then X cannot be Martin-Löf random:

Theorem 4.16 (Bienvenu, et al. [4]). *Every incomplete Martin-Löf random real is a non-porosity point.*

Nies [26] has extended Lemma 4.15 to show that if X is a non-porosity point, then for each Π_1^0 class C , $\varrho(C | X) = \varrho_2(C | X)$.

4.4 A dyadic density-one point above any degree

We have seen that the Martin-Löf random positive density points are incomplete. Every 1-generic G satisfies $G \oplus 0' \equiv_T G'$ and is therefore also incomplete. However, the proof of Theorem 4.6 suggests a way of constructing dyadic density-one points outside of those classes. In this section, we use this framework to show that general dyadic density-one points can be arbitrarily powerful as oracles. Our ultimate goal is Theorem 4.24 which

shows this to be true of full density-one points, but working on the unit interval presents complications that obscure the idea behind the proof of that theorem. For this reason, we first present the dyadic version.

Theorem 4.17. *For every $X \in 2^\omega$, there is a dyadic density-one point $Y \in 2^\omega$ such that $X \leq_T Y \leq_T X \oplus \emptyset'$.*

Proof. We build a Δ_2^0 perfect tree $F : 2^{<\omega} \rightarrow 2^{<\omega}$ and a functional Γ such that for every $X \in 2^\omega$, $F(X)$ is a density-one point and $\Gamma^{F(X)} = X$. F will be obtained as the limit of partial computable functions $F_s : 2^{<\omega} \rightarrow 2^{<\omega}$. For each s , we will ensure that if $F_s(\sigma)$ is defined, then $\Gamma^{F_s(\sigma)} = \sigma$. If, at any stage s , we set $F_s(\sigma)$ to a new value, it should be assumed that for any σ' properly extending σ , we undefine $F_s(\sigma')$. Each Σ_1^0 class now represents a requirement that needs to be met by each path on the tree. In other words, for each e and for each $X \in 2^\omega$, if $F(X)$ is not in W_e , we require that $\lim_{\rho \prec F(X)} \mu_\rho(W_e) = 0$. Priorities are assigned as before.

Above $F_s(\sigma)$, we work to define $F_s(\sigma i)$ for $i \in \{0, 1\}$. We want to ensure that for each $e < |\sigma|$, if $[F_s(\sigma i)]$ is not already contained in W_e , then between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu(W_e)$ remains below a certain threshold. If the threshold is exceeded above some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, we will move $F(\sigma i)$ to a new string ν extending ρ such that $[\nu]$ is contained in W_e . Complications arise because ν cannot be such that Γ^ν properly extends σi or is incompatible with σi . In the proof of Theorem 4.6, we built a single forbidden Σ_1^0 class B , the measure of which we had to keep small along the approximation. Here, we maintain a Σ_1^0 class B_σ for every nonempty string σ : if $\sigma = \alpha i$, then B_σ consists of the union of the set of current or previous values of $[F(\sigma 0)]$, $[F(\sigma 1)]$ and $[F(\alpha \bar{i})]$. We also maintain thresholds $\beta_s(\sigma)$, and the fact that at every stage s ,

for every nonempty string σ , between $F_s(\sigma^-)$ and $F_s(\sigma)$, the measure of $B_{\sigma,s}$ is strictly below $\beta_s(\sigma)$.

We begin the construction by setting $F_0(\langle \rangle) = \langle \rangle$.

Procedure for $F_s(\sigma i)$

Let t be the stage at which $F_s(\sigma)$ is first set to its current value. Both $\beta_t(\sigma 0)$ and $\beta_t(\sigma 1)$ are set to the same initial value $\beta^*(|\sigma|)$. The strings $F_t(\sigma i)$ for $i \in \{0, 1\}$ are chosen initially so that:

- The measure of $[F_t(\sigma i)]$ between $F_t(\sigma)$ and $F_t(\sigma \bar{i})$ is strictly below $\beta^*(|\sigma|)$.
- If σ is not the empty string, $F_t(\sigma 0)$ and $F_t(\sigma 1)$ must be long enough so that between $F_t(\sigma^-)$ and $F_t(\sigma)$, $\mu(B_{\sigma,t}) < \beta_t(\sigma)$.

Suppose that C_0, \dots, C_l are those among the first $|\sigma|$ many Σ_1^0 classes in which $[F_s(\sigma)]$ is not already contained, in order of descending priority. Now if for some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$ and some $j \leq l$, $\mu_\rho(C_j)$ exceeds $\sqrt{\beta_s(\sigma i)}$ and no action has yet been taken for a higher priority $C_{j'}$, then we *act*: Let ν be a string extending ρ such that $[\nu] \subseteq C_j$ and

1. between ρ and ν , $\mu(B_{\sigma i,s}) < \sqrt{\beta_s(\sigma i)}$.
2. ν is long enough so that $\mu(B_{\sigma \bar{i},s} \cup [\nu]) < \beta_s(\sigma \bar{i})$ between $F_s(\sigma)$ and $F_s(\sigma \bar{i})$, and $\mu(B_{\sigma,s} \cup [\nu]) < \beta_s(\sigma)$ between $F_s(\sigma^-)$ and $F_s(\sigma)$.
3. $B_{\sigma i,s} \cap [\nu] = \emptyset$.

We set $F_{s+1}(\sigma i) = \nu$ and $\beta_{s+1}(\sigma i) = \sqrt{\beta_s(\sigma i)}$.

Choosing $\beta^*(|\sigma|)$

We should move $[F_s(\sigma i)]$ into C_j when the following is seen to occur at stage s : For some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu_\rho(C_j)$ exceeds the measure of the $\sqrt{\beta_s(\sigma i)}$ -vicinity of $B_{\sigma i, s}$, i.e., if $\mu_\rho(C_j) > \beta_s(\sigma i) / \sqrt{\beta_s(\sigma i)} > \mu_\rho(B_{\sigma i, s}) / \sqrt{\beta_s(\sigma i)}$. If this does not occur, we wish to limit the measure of C_j to $2^{-|\sigma|}$ between $F_s(\sigma)$ and $F_s(\sigma i)$. Each time we act by moving $F_s(\sigma i)$ above $F_s(\sigma)$, the value of $\beta(\sigma i)$ is magnified by a power of $1/2$, so we require that $\beta^*(|\sigma|)$, the initial value of $\beta(\sigma i)$, satisfy

$$(\beta^*(|\sigma|))^{1/2^{|\sigma|+1}} \leq 2^{-|\sigma|}.$$

Verification

Claim 4.18. *For every $\sigma \in 2^{<\omega}$, $\lim_s F_s(\sigma)$ exists.*

Proof. Assume that $F_s(\sigma)$ has stabilized by stage s_0 . For each $i \in \{0, 1\}$, $F_s(\sigma i)$ is redefined after stage s_0 at most $|\sigma|$ times. \square

Claim 4.19. *We can act to redefine $F_s(\sigma i)$ while satisfying requirements (1) through (3) above.*

Proof. Suppose we redefine $F_s(\sigma i)$ for the sake of C_j , i.e., for some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu_\rho(C_{j,s}) > \sqrt{\beta_s(\sigma i)}$. By Lemma 4.7, there is a $Y \in 2^\omega$ extending ρ such that for each α such that $\rho \preceq \alpha \prec Y$, $\mu_\alpha(B_{\sigma i, s}) < \sqrt{\beta_s(\sigma i)}$. Thus there are arbitrarily long strings α extending ρ satisfying condition (1). To satisfy (2) and (3), we simply choose an α long enough and designate it $F_{s+1}(\sigma i)$. \square

Claim 4.20. *Suppose at stage $s + 1$, we set $F_{s+1}(\sigma i) = \tau$ and set $\Gamma_{s+1}^\tau = \sigma i$. Then $\Gamma_s^\tau \preceq \sigma i$. In other words, setting $\Gamma^\tau = \sigma i$ keeps Γ consistent.*

Proof. We first show by induction that if t is the stage when $F_s(\sigma)$ is first set to its current value, then for all $\rho \succeq F_t(\sigma)$, $\Gamma_t^\rho = \sigma$. The base case is trivial since $F_s(\langle \rangle) = \langle \rangle$ for all s . Suppose $\sigma = \alpha j$ for some $j \in \{0, 1\}$. When $F_s(\alpha)$ is first set to its current value, say at stage t_0 , then for all $\nu \succeq F_{t_0}(\alpha)$, $\Gamma_{t_0}^\nu = \alpha$. Now suppose at some subsequent stage t_1 , we set $F_{t_1}(\alpha j) = \tau$, then because of requirement (3), $B_{\alpha j, t_1} \cap [\tau]$ is empty, and hence for all $\rho \succeq \tau$, $\Gamma^\rho = \sigma$.

Subsequent to initialization, $[F_s(\sigma i)]$ is always disjoint from $B_{\sigma i, s}$, hence $\Gamma^{F_s(\sigma i)}$ never properly extends σi or becomes incompatible with σi . \square

Claim 4.21. *For each $X \in 2^\omega$, $F(X) = \bigcup_{k \in \omega} \lim_{s \rightarrow \infty} F_s(X \upharpoonright k)$ is a dyadic density-one point.*

Proof. Suppose that $F(X) \notin W_e$. Let $\sigma \prec X$ be long enough so that $|\sigma| > e$ and for all $e' < e$, if $F(X) \in W_{e'}$, then $F(\sigma) \in W_{e'}$. Let ρ be any initial segment of X that properly extends σ and let t be large enough so that $F_t(\rho)$ has stabilized. By our choice of σ , we never act to redefine $F_s(\rho)$ for the sake of $W_{e'}$ for any $e' < e$, and by the assumption that $F(X) \notin W_e$, we never act for the sake of W_e . Hence, for all $t' \geq t$, between $F_{t'}(\rho^-)$ and $F_{t'}(\rho)$, $\mu(W_e)$ never exceeds $\sqrt{\beta_{t'}(\rho)}$, which is always bounded by $2^{-(|\rho|-1)}$. \square

This concludes the proof of Theorem 4.17. \square

4.5 A full density-one point above any degree

We can adapt the previous construction to produce a full density-one point above any degree. We will need a version of Lemma 4.7 for the unit interval:

Lemma 4.22 (Bienvenu, et al. [4]). *Suppose $W \subseteq [0, 1]$ is open. Then for any ε such that $\mu(W) \leq \varepsilon \leq 1$, let $U_\varepsilon(W)$ denote the set*

$$\{X \in [0, 1] : \exists \text{ an interval } I, X \in I, \text{ and } \mu_I(W) \geq \varepsilon\}.$$

We call $U_\varepsilon(W)$ the ε -vicinity of W . Then $\mu(U_\varepsilon(W)) \leq 2\mu(W)/\varepsilon$.

Lemma 4.22 has a subtle shortcoming. When relativizing it to an interval $J \subset [0, 1]$, we obtain a bound on the measure of the ε -vicinity of $W \cap J$, but in our construction we will be concerned about the ε -vicinity of W , not merely its intersection with J . Fortunately, this is easily remedied:

Lemma 4.23. *Let $W \subseteq [0, 1]$ be open, and let K be an open interval such that for all open intervals L containing K , $\mu_L(W) < \delta$. Then for any interval I containing K , and any ε such that $\mu(W) \leq \varepsilon \leq 1$, $\mu_I(U_\varepsilon(W)) < 6\delta/\varepsilon$.*

Proof. By Lemma 4.22, $\mu_I(U_\varepsilon(W \cap I)) \leq 2\delta/\varepsilon$. Let $S = U_\varepsilon(W) \setminus U_\varepsilon(W \cap I)$ and $c = \mu_I(S)$. There must exist an $X \in S$ such that X is at least $\mu(I)c/4$ away from the nearest endpoint of I . Let J be an interval containing X such that $\mu_J(W) \geq \varepsilon$. Since $X \notin U_\varepsilon(W \cap I)$, J cannot be contained in I , so $\mu(J \cap I) > \mu(I)c/4$. We now have $\mu(J)/\mu(J \cup I) \geq \mu(J \cap I)/\mu(I) > c/4$, and so:

$$\mu_{I \cup J}(W \cap J) = \frac{\mu(W \cap J)}{\mu(I \cup J)} = \frac{\mu(W \cap J)}{\mu(J)} \cdot \frac{\mu(J)}{\mu(I \cup J)} > \mu_J(W) \frac{c}{4} \geq \frac{\varepsilon c}{4}.$$

On the other hand, $\mu_{I \cup J}(W \cap J) < \delta$ by assumption, so $c < 4\delta/\varepsilon$. \square

The following shorthand is convenient: Let C be a measurable set and I and I' intervals such that $I' \subseteq I$. If for every interval J such that $I' \subseteq J \subseteq I$, $\mu_J(C) < \alpha$, then we say that *between I and I' , $\mu(C) < \alpha$* .

We briefly outline the obstacles to lifting Theorem 4.17 to the unit interval. The first is that what was an advantage in the proof of Theorem 4.6 now works against us. In building a full density-one point X , we can no longer restrict our attention to relative measures of Σ_1^0 classes within dyadic cones of the form $[X_s \upharpoonright n]$. As an example, consider the intervals we enumerate into B in the proof of Theorem 4.6, which appear small in dyadic cones along the approximation, but big when we consider their fractional measure within arbitrary intervals around X_s .

The second obstacle is subtler. In the proof of Theorem 4.17 we decompose a density requirement with respect to a single Σ_1^0 class into countably many subrequirements. At each level of the construction, we attempt to satisfy stronger and stronger subrequirements with respect to W_ε that, when taken together, ensure that the limiting density requirement is satisfied. The key is that if $\nu_0 \preceq \nu_1 \preceq \nu_2$ are strings, and the measure of the set W is below ε between ν_0 and ν_1 , and also between ν_1 and ν_2 , then the measure of W is below ε between ν_0 and ν_2 . However, if $I_0 \subseteq I_1 \subseteq I_2$ are intervals, it may be the case that the measure of W is below ε between I_0 and I_1 , and also between I_1 and I_2 , but not between I_0 and I_2 .

Theorem 4.24. *For every $X \in 2^\omega$, there is a density-one point $Y \in 2^\omega$ such that $X \leq_T Y \leq_T X \oplus \emptyset'$.*

Proof. Let \mathcal{I} denote the collection of closed subintervals of the unit interval with dyadic

rational endpoints. By computable approximation, we build a tree $F: 2^{<\omega} \rightarrow \mathcal{I}$ of intervals and a functional Γ such that for all every σ in $2^{<\omega}$, and every $Y \in F(\sigma)$, $\Gamma^Y \upharpoonright |\sigma| = \sigma$. F is obtained as a limit of partial computable functions F_s such that if $\sigma \preceq \sigma'$ and $F_s(\sigma)$ and $F_s(\sigma')$ are both defined, then $F_s(\sigma') \subseteq F_s(\sigma)$. If at stage s , $F_s(\sigma)$ is redefined, it should be assumed that we set $\Gamma_s^X = \sigma$ for all X in $F_s(\sigma)$ and we undefine $F_s(\sigma')$ for any σ' properly extending σ .

As in the previous construction, we will be working within $F_s(\sigma)$ to define $F_s(\sigma i)$ for $i \in \{0, 1\}$. A key difference is that we now maintain a proper subinterval $J_s(\sigma)$ of $F_s(\sigma)$ within which $F_s(\sigma 0)$ and $F_s(\sigma 1)$ reside. If we act at stage s by setting $F_{s+1}(\sigma i)$ to a new value, we are allowed to move it outside $J_s(\sigma)$, in which case we expand $J_s(\sigma)$ to a larger interval $J_s(\sigma)^+ = J_{s+1}(\sigma)$ that contains $F_{s+1}(\sigma i)$. We postpone explaining how the initial value of $J_s(\sigma)$ is chosen and how $J_s(\sigma)^+$ is defined.

Let $B_{\sigma i, s}$ denote the union over all $t \leq s$ of $F_t(\sigma i 0) \cup F_t(\sigma i 1) \cup F_t(\sigma \bar{i})$. By acting to redefine $F(\sigma i 0)$, say, we contribute measure to $B_{\sigma i}$. We shall have to ensure that we can do this without violating the measure constraint $\beta(\sigma)$ for $B_{\sigma i}$.

We begin the construction by setting $F_0(\langle \rangle) = [0, 1]$.

Procedure for $F_s(\sigma i)$

Let t be the stage at which $F_s(\sigma)$ is first set to its current value. We set $\beta_t(\sigma 0)$ and $\beta_t(\sigma 1)$ to the same initial value $\beta^*(|\sigma|)$. We set $J_t(\sigma) = \text{Int}(F_t(\sigma), |\sigma|)$ (we define Int later) and choose $F_t(\sigma 0)$ and $F_t(\sigma 1)$ to satisfy the following conditions:

- Both are contained in $J_t(\sigma)$.
- Between $J_t(\sigma)^+$ and $J_t(\sigma 0)$, $\mu(F_t(\sigma 1)) < \beta^*(|\sigma|)$.

- Between $J_t(\sigma)^+$ and $J_t(\sigma 1)$, $\mu(F_t(\sigma 0)) < \beta^*(|\sigma|)$.
- If σ is not the empty string, let $\alpha = \sigma^-$. Then between $J_t(\alpha)^+$ and $J_t(\sigma)$, $\mu([B_{\sigma,t}]) < \beta_t(\sigma)$.

It is not hard to see that these conditions can be met by ensuring that the intervals are small enough and far enough apart relative to their width.

In a subsequent stage s , let C_0, \dots, C_l be those among the first $|\sigma|$ many Σ_1^0 classes that $F_s(\sigma)$ has not already entered, in order of descending priority. Suppose that for some interval I such that $J_s(\sigma)^+ \supseteq I \supseteq J_s(\sigma i)$, $\mu_I(C_j)$ exceeds $6\sqrt{\beta_s(\sigma i)}$, and no action has yet been taken within $F_s(\sigma)$ for a higher priority $C_{j'}$. Then there is an interval $L \subseteq C_j$ such that:

1. For every $Z \in L$, and every interval $K \subseteq J_s(\sigma)^+$ such that $Z \in K$, $\mu_K(B_{\sigma i, s}) < \sqrt{\beta_s(\sigma i)}$. To see that such an interval exists, note that we inductively maintain the property that between $J_s(\sigma)^+$ and $J_s(\sigma i)$, $\mu(B_{\sigma i, s}) < \beta_s(\sigma i)$. By Lemma 4.23,

$$\mu_I(U_{\sqrt{\beta_s(\sigma i)}}(B_{\sigma i, s} \cap J_s(\sigma)^+)) < 6\sqrt{\beta_s(\sigma i)}.$$

Now let L be an interval contained in $C_j \cap I$ that is disjoint from $U_{\sqrt{\beta_s(\sigma i)}}(B_{\sigma i, s} \cap J_s(\sigma)^+)$.

2. L is small enough so that between $J^s(\sigma)^+$ and $J_s(\sigma \bar{i})$, $\mu(B_{\sigma \bar{i}, s} \cup L) < \beta_s(\sigma \bar{i})$.
3. If σ is not the empty string, let $\alpha = \sigma^-$. Then between $J_s(\alpha)^+$ and $J_s(\sigma)$, $\mu(B_{\sigma, s} \cup L) < \beta_s(\sigma)$.
4. $L \cap B_{\sigma i, s} = \emptyset$.

In this case, we let $F_{s+1}(\sigma i) = L$, $J_{s+1}(\sigma) = J_s(\sigma)^+$, and $\beta_{s+1}(\sigma i) = \sqrt{\beta_s(\sigma i)}$.

Choosing $\beta^*(|\sigma|)$

If W_j is a Σ_1^0 class that $F_s(\sigma)$ has not already entered, then if $F_s(\sigma i)$ never enters W_j , we wish to limit the measure of W_j to 2^{-k} between $J_s(\sigma)^+$ and $J_s(\sigma i)$. The idea is the same as in the proof of Theorem 4.17, the only difference being the factor of 6 in the statement of Lemma 4.23. It suffices to pick $\beta^*(|\sigma|)$ small enough so that

$$6(\beta_\sigma^*)^{1/2^{k+1}} \leq 2^{-k}.$$

Defining Int and $+$

For an interval I , let I^+ be obtained by padding I on either side with intervals of the same length. For an interval J , let $Int(J, k)$ be a subinterval I of J small enough so that $2k$ applications of the $+$ operation to I result in an interval still contained in J .

Verification

Claim 4.25. *We can act to redefine $F_s(\sigma i)$ without violating any measure constraints.*

Proof. Given a string ν , there are two ways in which the measure constraint for $B_{\nu, s}$ could be affected: by the direct addition of measure to $B_{\nu, s}$, or by the expansion of $J_s(\sigma)$.

It is easy to see that the only such ν are σ and $\sigma\bar{i}$.

Property (2) of L above ensures that $\mu(B_{\sigma\bar{i}, s} \cup L) < \beta_s(\sigma\bar{i})$ between $J_s(\sigma)^+$ and $J_s(\sigma\bar{i})$, and since $F_s(\sigma) \cap (B_{\sigma\bar{i}, s} \cup L)$ is contained entirely in $J_s(\sigma)^+ = J_{s+1}(\sigma)$, also between $J_{s+1}(\sigma)^+$ and $J_{s+1}(\sigma\bar{i})$.

Property (3) of L ensures that $\mu(B_{\sigma,s} \cup L) < \beta_s(\sigma)$ between $J_s(\sigma^-)^+$ and $J_s(\sigma)$, and hence between $J^{s+1}(\sigma^-)^+$ and $J_{s+1}(\sigma) = J_s(\sigma)^+$. \square

The argument for the following claim is virtually the same as for Claim 4.20.

Claim 4.26. *Suppose at stage $s+1$, we set $F_{s+1}(\sigma i) = I$. Then for any $X \in I$, $\Gamma_s^X \preceq \sigma i$. In other words, setting $\Gamma_{s+1}^X = \sigma i$ for all $X \in I$ keeps Γ consistent.*

Claim 4.27. *Let X be any real, and let Y be a real in $\bigcap_{\sigma \prec X} F(\sigma)$. Then Y is a density-one point.*

Proof. For $\sigma \in 2^{<\omega}$, let $F(\sigma)$, $J(\sigma)$, and $\beta(\sigma)$ denote the limiting values of $F_s(\sigma)$, $J_s(\sigma)$, and $\beta_s(\sigma)$, respectively. Suppose that Y is not in W_j . Let σ be an initial segment of X such that $|\sigma| > j$ and if $Y \in W_l$ for any $l < j$, then $F(\sigma) \subseteq W_l$. We claim that for any $I \subseteq J(\sigma)^+$ such that $Y \in I$, $\mu_I(W_j) \leq 2^{-|\sigma|+1}$.

Let $\rho \succeq \sigma$ be the longest initial segment of X such that I is entirely contained in $J(\rho)^+$. Suppose $X \succ \rho i$. Let $I' = I \cup J(\rho i)$. Now,

$$\frac{\mu(I')}{\mu(I)} \leq \frac{\mu(I \cup J(\rho i))}{\mu(I)} \leq 1 + \frac{\mu(J(\rho i))}{\mu(I)}.$$

By the maximality of ρ , $I \not\subseteq J(\rho i)^+$. Since $J(\rho i)^+$ is obtained by pasting a copy of $J(\rho i)$ on either side of $J(\rho i)$, $\mu(I) \geq \mu(J(\rho i))$. So the ratio above is bounded by 2. Since I' is an interval between $J(\rho)^+$ and $J(\rho i)$, $\mu_{I'}(W_j)$ never exceeds $\beta(\rho i) \leq 2^{-|\rho|}$. Therefore, $\mu_I(W_j)$ never exceeds $2^{-|\rho|+1}$. \square

This concludes the proof of Theorem 4.24. \square

4.6 Nonminimality

It is easy to see that if $A \oplus B$ is dyadic positive density, then so are A and B (and if $A \oplus B$ is dyadic density-one, so are A and B). The similarities with 1-generics seem to end here. It can be shown using the techniques of the constructions above that the “van Lambalgen property” fails badly for density-one points:

Proposition 4.28. *There is a dyadic density-one point $A \oplus B$ such that $A \equiv_T B$.*

Proof sketch. The construction is similar to the one in Theorem 4.6. We build a real $X = X_0 \oplus X_1$ and reductions Γ_0 and Γ_1 by computable approximation, satisfying the usual density requirements as well as

$$P_{n,i}: \Gamma_i^{X_i}(n) = X_{\bar{i}}(n),$$

for $n \in \omega$ and $i \in \{0, 1\}$.

Acting on $P_{n,i}$ requirements produces Σ_1^0 classes that we must avoid:

$$B_0 = \{X \oplus Y : \exists n \Gamma_0^X(n) \neq Y(n)\},$$

with B_1 defined analogously. Let $B = B_0 \cup B_1$. As in previous constructions, we will have to keep the measure of B small along the approximation.

At each stage s of the construction, we have a sequence of finite strings $\langle \rangle = \sigma_{0,s} \prec \sigma_{1,s} \prec \dots$ approximating X . If at stage $s + 1$, we set $\sigma_{k+1,s+1} = \tau \oplus \rho$, then we also set $\Gamma_0^\tau(k) = \rho(k)$ and $\Gamma_1^\rho(k) = \tau(k)$. We claim that we can control the resulting addition of measure to B_s . Let us consider the effect of setting $\Gamma_0^\tau(k) = \rho(k)$. The additional

measure is contained in the set

$$\Delta B_0 = \{X \oplus Y : \tau \prec X \text{ and } Y(k) \neq \rho(k)\}.$$

Let ν be the initial segment of $\sigma_{k+1,s+s}$ of length $2k+2$. Then for any α such that $\nu \preceq \alpha \preceq \sigma_{k+1,s}$, $\mu_\alpha(\Delta B_0) = 0$, because if $\alpha = \tau' \oplus \rho'$, then $\rho'(k) = \rho(k)$. Next, suppose that $\alpha \prec \nu$, then

$$\mu_\alpha(\Delta B_0) \leq 2^{-\left(\lfloor \frac{|\sigma_{k+1,s+1}| - |\alpha|}{2} \rfloor\right)} \leq 2^{-\left(\lfloor \frac{|\sigma_{k+1,s+1}| - (2k+2)}{2} \rfloor\right)}.$$

Controlling the measure of ΔB_0 along the approximation, then, is a matter of ensuring that $\sigma_{k+1,s+1}$ is long enough. Defining ΔB_1 analogously, let $\Delta B = \Delta B_0 \cup \Delta B_1$. We must now maintain the fact that for each $l \leq k$ and each α between $\sigma_{l,s+1}$ and $\sigma_{l+1,s+1}$, $\mu_\alpha(B_{s+1}) \leq \mu_\alpha(B_s) + \mu_\alpha(\Delta B) < \beta_{s+1}$.

In other respects, the construction and verification are identical to those in Theorem 4.6. We omit the details. \square

What the previous proposition demonstrates is that settling the question of whether there is a positive density point of minimal degree is not simply a matter of pointing to the even and odd bits (or in fact, any computable sampling) of the sequence. It is nevertheless a consequence of the main result of this section that no positive density point can be of minimal Turing degree.

Theorem 4.29. *Every dyadic positive density point is either Martin-Löf random or computes a 1-generic.*

Proof. Let $\langle U_n \rangle_{n \in \omega}$ be a Martin-Löf test such that $X \in \bigcap_n U_n$. For each n , let S_n be a

prefix-free c.e. set of strings such that $U_n = [S_n]^\prec$. We can assume that $S_{0,s} = \{\langle \rangle\}$ for all s , and if $\tau \in S_{j+1,s}$, then there is some $\sigma \preceq \tau$ such that $\sigma \in S_{j,s}$. Let V_e denote the e -th c.e. set of strings.

We define a functional Γ such that for each $Y \in \bigcap_n U_n$,

1. Γ^Y is total, and
2. if Y is a dyadic positive density point, Γ^Y is 1-generic.

We define Γ inductively on a sequence $\langle R_n \rangle_{n \in \omega}$ of c.e. sets of strings. Let $R_0 = S_0$, and let $\Gamma^\langle \rangle = \langle \rangle$. When a string τ enters R_n , we choose m large enough so that $2^{-m} \leq 2^{-|\tau|-n}$, and so $\mu_\tau(U_m) \leq 2^{-n}$. Then, whenever a string ν extending τ enters S_m at stage s , we extend the definition of Γ as follows: If there exists an e such that $[\Gamma^\tau]$ is not already contained in $[V_{e,s}]^\prec$ and there is an extension of Γ^τ in $V_{e,s}$, then let e' be the least such index and let σ be an extension of Γ^τ in $V_{e',s}$. We set $\Gamma^\nu = \sigma 0$. On the other hand, if no such e exists, we set $\Gamma^\nu = \Gamma^\tau 0$. In either case, we enumerate ν into R_{n+1} . This completes the definition of Γ .

Consider a $Y \in \bigcap_n U_n$. To see that (1) holds, note that for each n , Y has a unique initial segment σ_n in R_n , and $\Gamma^{\sigma_{n+1}}$ properly extends Γ^{σ_n} . It remains to verify (2). If Γ^Y is not 1-generic, then let e be the least index such that V_e is dense along it. Let M be large enough so that for each $e' < e$, if $\Gamma^{Y'} \in [V_{e'}]^\prec$, then $[\Gamma^{\sigma_M}] \subseteq [V_{e'}]^\prec$, otherwise $[\Gamma^{\sigma_M}] \cap [V_{e'}]^\prec = \emptyset$. We exhibit a Σ_1^0 class B such that $Y \in \overline{B}$ and $\varrho_2(\overline{B} | Y) = 0$. For each $n \geq M$ and for each $\tau \in R_n$, we wait for a stage s such that an extension of Γ^τ appears in $V_{e,s}$. If this occurs, we enumerate the open set $[\tau] \setminus [R_{n+1,s}]^\prec$ into B .

If τ is an initial segment of Y , then since V_e is dense along Γ^Y , such a stage s must occur. Let ν be the initial segment of Y in R_{n+1} , and let t be the first stage at which

$\nu \in R_{n+1,t}$. By our choice of M , if an extension σ of Γ_τ occurs in $V_{e,t}$, we would have set $\Gamma^\nu = \sigma 0$. Therefore, $t < s$, which implies that $Y \in \overline{B}$. Moreover, $\mu_\tau([R_{n+1}]^{\prec}) \leq 2^{-n}$, and so $\varrho_2(\overline{B} | Y) = 0$. \square

Corollary 4.30. *No dyadic positive density point is of minimal degree.*

Theorem 4.29 has an interesting consequence. Bienvenu, et al. [3] introduce *Oberwolfach randomness* and show that every Oberwolfach random real is a full density-one point. Based on earlier work by Figueira, Hirschfeldt, Miller, Ng, and Nies [13], they observe that one “half” of every Martin-Löf random real is always Oberwolfach random, hence full density-one¹:

Proposition 4.31 (Bienvenu, et al. [3]; Figueira, et al. [13]). *If $A \oplus B$ is Martin-Löf random, then either A or B is a full density-one point.*

Thus, every Martin-Löf random real computes a full density-one point, which, together with Theorem 4.29, implies:

Corollary 4.32. *Every dyadic positive density point computes a full density-one point.*

4.7 Randomness and computational strength

We have already mentioned that Theorem 4.1 holds regardless of whether we use dyadic density or full density, so one direction of that theorem can be rephrased as follows: Every Martin-Löf random point that is not dyadic positive density computes $0'$. Theorem 4.29 implies that we cannot weaken the hypothesis from Martin-Löf randomness

¹The author thanks A. Kuč for bringing this fact to his attention.

to computable randomness. To see this, note that there is a computably random real of minimal degree. By Corollary 4.30, it cannot be dyadic positive density.

Proposition 4.33. *There is a computably random real that is not dyadic positive density and is incomplete.*

In this section, we show that the property of not being positive density does imply some form of computational strength on the computably random reals, and in fact, on a more general randomness class, the *Schnorr random* reals.

Definition 4.34. A *Schnorr test* is a Martin-Löf test $\langle G_n \rangle_{n \in \omega}$ where $\mu(G_n)$ is uniformly computable in n . A real X is *Schnorr random* if there is no Schnorr test $\langle G_n \rangle_{n \in \omega}$ such that X is contained in G_n for infinitely many n .

Proposition 4.35. *Every Schnorr random real that is not full positive density is high.*

We will need the following lemma:

Lemma 4.36 (Bienvenu, et al. [4]). *Let $W \subseteq [0, 1]$ be open. Fixing an $\varepsilon \in (0, 1)$, let*

$$U_\varepsilon(W) = \{z : \exists \text{ an open interval } I, z \in I, \text{ and } \mu_I(W) > 1 - \varepsilon\}.$$

Then $\mu(U_\varepsilon(W) \setminus W) < 2\varepsilon$.

Proof of Proposition 4.35. Fix $z \in 2^\omega$ and B a Σ_1^0 class such that $z \in \overline{B}$ and $\varrho(\overline{B} \mid z) = 0$. Let $f \leq_T z$ be the function such that $f(n)$ is the least stage s such that there is an open interval I containing z with $\mu_I(B_s) > 1 - 2^{-n}$. Note that f is total. Suppose the computable function g is not dominated by f .

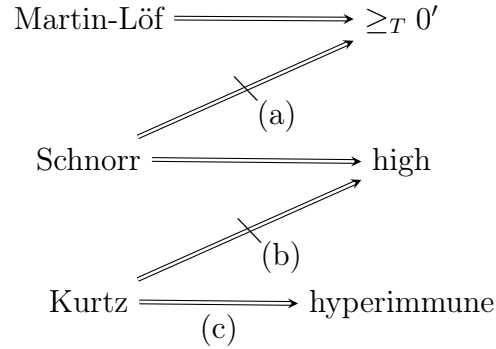


Figure 4.2: Relationships between randomness and notions of computability-theoretic strength within the reals that are not positive density

Then for each $n \in \omega$, let

$$G_n = U_{2^{-n-1}}(B_{g(n)}) \setminus B_{g(n)}.$$

Each G_n is a Σ_1^0 class modulo the rationals. By Lemma 4.36, $\mu(G_n) < 2^{-n}$. It is not hard to see that $\mu(G_n)$ is uniformly computable in n , and in fact, that $U_{2^{-n-1}}(B_{g(n)})$ is the union of a finite collection of open intervals with rational endpoints that can be computed from $B_{g(n)}$. Moreover, there are infinitely many n such that $z \in G_n$. Therefore, z is not Schnorr random. \square

Figure 4.2 shows the relationship between three important randomness classes and three forms of classical computability-theoretic strength within the class of reals that are not positive density². Every computably random real is Schnorr random, and so Proposition 4.33 yields nonimplication (a).

To see nonimplication (b), let X be a minimal degree below $0'$. Every minimal degree is GL_2 , and so X satisfies $(X \oplus 0')' \equiv_T X''$, which implies that it is not high. Because it

²It does not matter whether we use dyadic or full density.

is minimal, X cannot compute a 1-generic, so by Theorem 4.29, it is not dyadic positive density. However, every hyperimmune degree contains a Kurtz random real [24], and so $X \equiv_T Y$, where Y is Kurtz random, not dyadic positive density, and not high.

For implication (c), we appeal to a result by L. Yu (see, for example, [11], Theorem 8.11.12) that every hyperimmune-free Kurtz random is weakly 2-random. For each Π_1^0 class C and each rational $\varepsilon > 0$, the set of points $\{X \in C : \varrho(C|X) < 1 - \varepsilon\}$ is a null Π_2^0 set. The weakly 2-random reals are exactly those which avoid every null Π_2^0 set. Therefore, every hyperimmune-free Kurtz random real is, in fact, full density-one.

We conclude with a question. In Theorems 4.17 and 4.24, we saw that general positive density points (in fact, density-one points) can be arbitrarily powerful as oracles. It is unknown whether this remains true under the assumption of any form of randomness intermediate between Kurtz and Martin-Löf randomness.

Question 4.37. Is there a positive density real which is Schnorr random and complete?

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