THE TOPOLOGY OF ULTRAFILTERS AS SUBSPACES OF THE CANTOR SET AND OTHER TOPICS

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Abstract

In the first part of this thesis (Chapter 1), we will identify ultrafilters on ω with subspaces of 2^ω through characteristic functions, and study their topological properties. More precisely, let $\mathcal{P}$ be one of the following topological properties.

- $\mathcal{P}$ = being completely Baire.
- $\mathcal{P}$ = countable dense homogeneity.
- $\mathcal{P}$ = every closed subset has the perfect set property.

We will show that, under Martin's Axiom for countable posets, there exist non-principal ultrafilters $\mathcal{U}, \mathcal{V} \subseteq 2^\omega$ such that $\mathcal{U}$ has property $\mathcal{P}$ and $\mathcal{V}$ does not have property $\mathcal{P}$.

The case '$\mathcal{P}$ = being completely Baire' actually follows from a result obtained independently by Marciszewski, of which we were not aware (see Theorem 1.37 and the remarks following it). Using the same methods, still under Martin's Axiom for countable posets, we will construct a non-principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is countable dense homogeneous. This consistently answers a question of Hrušák and Zamora Avilés. All of Chapter 1 is joint work with David Milovich.

In the second part of the thesis (Chapter 2 and Chapter 3), we will study CLP-compactness and h-homogeneity, with an emphasis on products (especially infinite powers). Along the way, we will investigate the behaviour of clopen sets in products (see Section 2.1 and Section 3.2).

In Chapter 2, we will construct a Hausdorff space X such that X^κ is CLP-compact if and only if κ is finite. This answers a question of Steprāns and Šostak.

In Chapter 3, we will resolve an issue left open by Terada, by showing that h-homogeneity is productive in the class of zero-dimensional spaces (see Corollary 3.27). Further positive results are Theorem 3.17 (based on a result of Kunen) and Corollary 3.15 (based on a result of Steprāns). Corollary 3.29 and Theorem 3.31 generalize results of Motorov and Terada. Finally, we will show that a question of Terada (whether X^ω is h-homogeneous for every zero-dimensional first-countable X) is equivalent to a question of Motorov (whether such an infinite power is always divisible by 2) and give some partial answers (see Proposition 3.38, Proposition 3.42, and Corollary 3.43). A positive answer would give a strengthening of a remarkable result by Dow and Pearl (see Theorem 3.34).

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Contents

Chapter 1

The topology of ultrafilters as subspaces of the Cantor set

This entire chapter is joint work with David Milovich, and the results that it contains originally appeared in [41]. In Section 1.8, we will also point out that the proof of a 'theorem' from the same article is wrong.

Our main reference for descriptive set theory is [30]. For other set-theoretic notions, see [5] or [28]. For notions that are related to large cardinals, see [29]. For all undefined topological notions, see [19].

By identifying a subset of ω with an element of the Cantor set 2^ω through characteristic functions (which we will freely do throughout this chapter), it is possible to study the topological properties of any $\mathcal{X} \subseteq \mathcal{P}(\omega)$. We will focus on the case $\mathcal{X} = \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter on ω. From now on, all filters and ideals are implicitly assumed to be on ω.

It is easy to realize that every principal ultrafilter $\mathcal{U} \subseteq 2^\omega$ is homeomorphic to 2^ω. Therefore, we will assume that every filter contains all cofinite subsets of ω and that every ideal contains all finite subsets of ω. In particular, all ultrafilters and maximal ideals are assumed to be non-principal.

First, we will observe that there are many (actually, as many as possible) non-homeomorphic ultrafilters. However, the proof is based on a cardinality argument, hence it is not 'honest' in the sense of Van Douwen: it would be desirable to find 'quotable' topological properties that distinguish ultrafilters up to homeomorphism. This is consistently achieved in Section 1.3 using the property of being completely Baire (see Corollary 1.9 and Theorem 1.11), in Section 1.4 using countable dense homogeneity (see Theorem 1.15 and Theorem 1.21) and in Section 1.6 using the perfect set property (see Theorem 1.29 and Corollary 1.32).

In Section 1.5, we will adapt the proof of Theorem 1.21 to obtain the countable dense homogeneity of the ω-power, consistently answering a question of Hrušák and Zamora Avilés from [26] (see Corollary 1.27).

In Section 1.7, using a modest large cardinal assumption, we will obtain a strong generalization of the main result of Section 1.6 (see Theorem 1.36).

Finally, in Section 1.8, we will investigate the relationship between the property of being a P-point and the above topological properties.

Proposition 1.1. *Let $\mathcal{U}, \mathcal{V} \subseteq 2^\omega$ be ultrafilters. Define $\mathcal{U} \approx \mathcal{V}$ if the topological spaces $\mathcal{U}$ and $\mathcal{V}$ are homeomorphic. Then the equivalence classes of $\approx$ have size $\mathfrak{c}$.*

Proof. To show that each equivalence class has size at least $\mathfrak{c}$, simply use homeomorphisms of 2^ω induced by permutations of ω and an almost disjoint family of subsets of ω of size $\mathfrak{c}$ (see, for example, Lemma 9.21 in [28]).

By Lavrentiev's lemma (see Theorem 3.9 in [30]), if $g : \mathcal{U} \longrightarrow \mathcal{V}$ is a homeomorphism, then there exists a homeomorphism $f : G \longrightarrow H$ that extends g, where G and H are G_δ subsets of 2^ω. Since there are only $\mathfrak{c}$ such homeomorphisms, it follows that an equivalence class of $\approx$ has size at most $\mathfrak{c}$. $\square$

Corollary 1.2. *There are* $2^{\mathfrak{c}}$ *pairwise non-homeomorphic ultrafilters.*

1.1 Notation and terminology

In this chapter, by *space* we mean separable metrizable topological space, with the only exceptions being Proposition 1.3 and the strong Choquet spaces of Section 1.6.

For every $s \in {}^{<\omega}2$, we will denote by $[s]$ the basic clopen set $\{x \in 2^\omega : s \subseteq x\}$. Given a tree $T \subseteq {}^{<\omega}2$, we will denote by $[T]$ the set of branches of T, that is $[T] = \{x \in 2^\omega : x \upharpoonright n \in T \text{ for all } n \in \omega\}$.

Define the homeomorphism $c : 2^\omega \longrightarrow 2^\omega$ by setting $c(x)(n) = 1 - x(n)$ for every $x \in 2^\omega$ and $n \in \omega$. Using c, one sees that every ultrafilter $\mathcal{U} \subseteq 2^\omega$ is homeomorphic to its dual maximal ideal $\mathcal{J} = 2^\omega \setminus \mathcal{U} = c[\mathcal{U}]$.

We will say that a subset P of a space X is a *perfect set* if P is a homeomorphic copy of 2^ω. Recall that P is a perfect set in 2^ω if and only if P is non-empty, closed and without isolated points. A *Bernstein set* in a space X is a subset B of X such that B and $X \setminus B$ both intersect every perfect set in X. Given such a set B, since 2^ω is homeomorphic to $2^\omega \times 2^\omega$, one actually has $|P \cap B| = \mathfrak{c}$ and $|P \cap (X \setminus B)| = \mathfrak{c}$ for every perfect set P in X.

For every $x \subseteq \omega$, define $x^0 = \omega \setminus x$ and $x^1 = x$. Given a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$, a *word* in $\mathcal{A}$ is an intersection of the form

$$\bigcap_{x \in \tau} x^{w(x)}$$

for some $\tau \in [\mathcal{A}]^{<\omega}$ and $w : \tau \longrightarrow 2$. Recall that $\mathcal{A}$ is an *independent family* if every word in $\mathcal{A}$ is infinite.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ has the *finite intersection property* if $\bigcap \sigma$ is infinite for all $\sigma \in [\mathcal{F}]^{<\omega}$. Given such a family, we will denote by $\langle \mathcal{F} \rangle$ the filter generated by $\mathcal{F}$. Let Cof be the collection of all cofinite subsets of ω. For any fixed $x \in 2^\omega$, define $x\!\uparrow = \{y \in 2^\omega : x \subseteq y\}$.

Whenever $x, y \in \mathcal{P}(\omega)$, define $x \subseteq^* y$ if $x \setminus y$ is finite. Given $\mathcal{C} \subseteq \mathcal{P}(\omega)$, a *pseudointersection* of $\mathcal{C}$ is a subset x of ω such that $x \subseteq^* y$ for all $y \in \mathcal{C}$. Given a cardinal κ, an ultrafilter $\mathcal{U}$ is a P_κ*-point* if every $\mathcal{C} \in [\mathcal{U}]^{<\kappa}$ has a pseudointersection in $\mathcal{U}$. A P*-point* is simply a P_{ω_1}-point. A filter $\mathcal{F}$ is a P*-filter* if every $\mathcal{C} \in [\mathcal{F}]^{<\omega_1}$ has a pseudointersection in $\mathcal{F}$.

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ has the *finite union property* if $\bigcup \sigma$ is coinfinite for all $\sigma \in [\mathcal{I}]^{<\omega}$. Given such a family, we will denote by $\langle \mathcal{I} \rangle$ the ideal generated by $\mathcal{I}$. Let Fin be the collection of all finite subsets of ω. For any fixed $x \in 2^\omega$, define $x\!\downarrow = \{y \in 2^\omega : y \subseteq x\}$.

Given $\mathcal{C} \subseteq \mathcal{P}(\omega)$, a *pseudounion* of $\mathcal{C}$ is a subset x of ω such that $y \subseteq^* x$ for all $y \in \mathcal{C}$. A maximal ideal $\mathcal{J}$ is a P*-ideal* if $c[\mathcal{J}]$ is a P-point.

1.2 Basic properties

In this section, we will notice that some topological properties are shared by all ultrafilters. Recall that a space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$.

Since any maximal ideal $\mathcal{J}$ (actually, any ideal) is a topological subgroup of 2^ω under the operation of symmetric difference (or equivalently, sum modulo 2), every ultrafilter $\mathcal{U} = c[\mathcal{J}]$ is also a topological group. In particular, every ultrafilter $\mathcal{U}$ is a homogeneous topological space.

The following proposition is Lemma 3.1 in [22].

Proposition 1.3 (Fitzpatrick, Zhou)**.** *Let X be a homogeneous topological space. Then X is a Baire space if and only if X is not meager in itself.*

Proof. One implication is trivial. Now assume that X is not a Baire space. Since X is homogeneous, it follows easily that

$$\mathcal{B} = \{U : U \text{ is a non-empty meager open set in } X\}$$

is a base for X. So $X = \bigcup \mathcal{B}$ is the union of a collection of meager open sets. Hence X is meager by Banach's category theorem (see Theorem 16.1 in [57]).

For the convenience of the reader, we sketch the proof in our particular case. Fix a maximal $\mathcal{C} \subseteq \mathcal{B}$ consisting of pairwise disjoint sets. Observe that $X \setminus \bigcup \mathcal{C}$ is closed nowhere dense. For every $U \in \mathcal{C}$, fix nowhere dense sets $N_n(U)$ such that $U = \bigcup_{n\in\omega} N_n(U)$. It is easy to check that $\bigcup_{U\in\mathcal{C}} N_n(U)$ is nowhere dense in X for every $n \in \omega$. □

Given any ultrafilter $\mathcal{U} \subseteq 2^\omega$, notice that c is a homeomorphism of 2^ω such that 2^ω is the disjoint union of $\mathcal{U}$ and $c[\mathcal{U}]$. In particular, $\mathcal{U}$ must be non-meager and non-comeager in 2^ω by Baire's category theorem. Actually, it follows easily from the 0-1 Law that no ultrafilter $\mathcal{U}$ can have the property of Baire (see Theorem 8.47 in [30]). In particular, no ultrafilter $\mathcal{U}$ can be analytic (see Theorem 21.6 in [30]) or co-analytic.

Corollary 1.4. *Let $\mathcal{U} \subseteq 2^\omega$ be an ultrafilter. Then $\mathcal{U}$ is a Baire space.*

Proof. If $\mathcal{U}$ were meager in itself, then it would be meager in 2^ω, which is a contradiction. □

On the other hand, by Theorem 8.17 in [30], no ultrafilter can be a Choquet space (see Section 8.C in [30]).

1.3 Completely Baire ultrafilters

As we will discuss in Section 1.8, the main results of this section can be easily deduced from Theorem 1.37, which was obtained independently by Marciszewski. However, since our methods are different from those of Marciszewski, we hope that this section might still be of some interest. Furthermore, we will need Theorem 1.8 in Section 1.6.

Definition 1.5. *A space X is* completely Baire *if every closed subspace of X is a Baire space.*

For example, every Polish space is completely Baire. For co-analytic spaces, the converse is also true (see Corollary 21.21 in [30]).

In the proof of Theorem 1.11, we will need the following characterization (see Corollary 1.9.13 in [48]). Observe that one implication is trivial.

Lemma 1.6 (Hurewicz)**.** *A space is completely Baire if and only if it does not contain any closed homeomorphic copy of $\mathbb{Q}$.*

The following (well-known) lemma is the first step in constructing an ultrafilter that is not completely Baire.

Lemma 1.7. *There exists a perfect subset P of 2^ω such that P is an independent family.*

Proof. We will give three proofs. The first proof simply shows that the classical construction of an independent family of size $\mathfrak{c}$ (see for example Lemma 7.7 in [28]) actually gives a perfect independent family. Define

$$I = \{(\ell, F) : \ell \in \omega, F \subseteq {}^{\ell}2\}.$$

Since I is a countably infinite set, we can identify 2^I and 2^ω. The desired independent family will be a collection of subsets of I. Consider the function $f : 2^\omega \longrightarrow 2^I$ defined by

$$f(x) = \{(\ell, F) : x \restriction \ell \in F\}.$$

It is easy to check that f is a continuous injection, hence a homeomorphic embedding by compactness. It follows that $P = \mathrm{ran}(f)$ is a perfect set. To check that P is an independent family, fix $\tau \in [P]^{<\omega}$ and $w : \tau \longrightarrow 2$. Suppose that $\tau = f[\sigma]$, where $\sigma = \{x_1, \ldots x_k\}$ and $x_1, \ldots, x_k$ are distinct. Choose ℓ large enough so that $x_1 \restriction \ell, \ldots, x_k \restriction \ell$ are distinct. It follows that

$$(\ell', \{x \restriction \ell' : x \in \sigma \text{ and } w(f(x)) = 1\}) \in \bigcap_{y \in \tau} y^{w(y)}$$

for every $\ell' \geq \ell$, which concludes the proof.

The second proof is also combinatorial. We will inductively construct $k_n \in \omega$ and a finite tree $T_n \subseteq {}^{<\omega}2$ for every $n \in \omega$ so that the following conditions are satisfied.

1. $k_m < k_n$ whenever $m < n < \omega$.

2. $T_m \subseteq T_n$ whenever $m \leq n < \omega$.

3. All maximal elements of T_n have length k_n. We will use the notation $M_n = \{t \in T_n : \mathrm{dom}(t) = k_n\}$.

4. For every $t \in T_n$ there exist two distinct elements of T_{n+1} whose restriction to k_n is t.

5. Given any $v : M_n \longrightarrow 2$, there exists $i \in k_{n+1} \setminus k_n$ such that $t(i) = v(t \upharpoonright k_n)$ for every $t \in M_{n+1}$.

In the end, set $T = \bigcup_{n<\omega} T_n$ and $P = [T]$. Condition (4) guarantees that P is perfect. Next, we will verify that condition (5) guarantees that P is an independent family. Fix $\tau \in [P]^{<\omega}$ and $w : \tau \longrightarrow 2$. For all sufficiently large $n \in \omega$, some $v \in {}^{M_n}2$ satisfies $v(x \upharpoonright k_n) = w(x)$ for all $x \in \tau$. By condition (5), there exists $i \in k_{n+1} \setminus k_n$ such that

$$x(i) = (x \upharpoonright k_{n+1})(i) = v(x \upharpoonright k_n) = w(x)$$

for all $x \in \tau$.

Start with $k_0 = 0$ and $T_0 = \{\varnothing\}$. Given k_n and T_n, define $k_{n+1} = k_n + 2^{|M_n|} + 1$. Fix an enumeration $\{v_j : j \in 2^{|M_n|}\}$ of all functions $v : M_n \longrightarrow 2$. Let T_{n+1} consist of all initial segments of functions $t : k_{n+1} \longrightarrow 2$ such that $t \upharpoonright k_n \in M_n$ and $t(k_n + j) = v_j(t \upharpoonright k_n)$ for all $j < 2^{|M_n|}$. Then, condition (5) is clearly satisfied. Since there is no restriction on $t(k_n + 2^{|M_n|})$, condition (4) is also satisfied.

The third proof is topological. Fix an enumeration $\{(n_i, w_i) : i \in \omega\}$ of all pairs (n, w) such that $n \in \omega$ and $w : n \longrightarrow 2$. Define

$$R_i = \left\{ x \in (2^\omega)^{n_i} : \bigcap_{j \in n_i} x_j^{w_i(j)} \text{ is infinite} \right\}$$

for every $i \in \omega$ and observe that each R_i is comeager. By Exercise 8.8 and Theorem 19.1 in [30], there exists a comeager subset of the Vietoris hyperspace $K(2^\omega)$ consisting of perfect sets $P \subseteq 2^\omega$ such that $\{x \in P^{n_i} : x_j \neq x_k \text{ whenever } j \neq k\} \subseteq R_i$ for every $i \in \omega$. It is trivial to check that any such P is an independent family. $\square$

We remark that, in some sense, the last two proofs that we have given of the above lemma are the same. The Vietoris hyperspace $K(2^\omega)$ is naturally homeomorphic to the space X of pruned subtrees of ${}^{<\omega}2$ with basic open sets of the form $\{T \in X : T \cap {}^{<i}2 = \tau\}$ for a fixed pruned subtree τ of ${}^{<i}2$. Moreover, the set $\{T \in X : [T] \text{ is an independent family}\}$ is comeager in X because the combinatorial proof's rule for constructing T_{n+1} from T_n only needs to be followed infinitely often.

We propose to call the following *Kunen's closed embedding trick.*

Theorem 1.8 (Kunen)**.** *Fix a zero-dimensional space C. There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that contains a homeomorphic copy of C as a closed subset.*

Proof. Fix P as in Lemma 1.7. Since P is homeomorphic to 2^ω, we can assume that C is a subspace of P. Observe that the family

$$\mathcal{G} = C \cup \{\omega \setminus x : x \in P \setminus C\}$$

has the finite intersection property because P is an independent family. Any ultrafilter $\mathcal{U} \supseteq \mathcal{G}$ will contain C as a closed subset. □

Corollary 1.9. *There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is not completely Baire.*

Proof. Simply choose $C = \mathbb{Q}$. □

Since 2^ω is homeomorphic to $2^\omega \times 2^\omega$, one can easily obtain the following strengthening of Theorem 1.8. Observe that, since any space has at most $\mathfrak{c}$ closed subsets, the result cannot be improved.

Theorem 1.10. *Fix a collection $\mathcal{C}$ of zero-dimensional spaces such that $|\mathcal{C}| \leq \mathfrak{c}$. There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that contains a homeomorphic copy of C as a closed subset for every $C \in \mathcal{C}$.*

The next theorem, together with Corollary 1.9, shows that under MA(countable) the property of being completely Baire is enough to distinguish ultrafilters up to homeomorphism.

Theorem 1.11. *Assume that* MA(countable) *holds. Then there exists an ultrafilter* $\mathcal{U} \subseteq 2^\omega$ *that is completely Baire.*

Proof. Enumerate as $\{Q_\eta : \eta \in \mathfrak{c}\}$ all subsets of 2^ω that are homeomorphic to $\mathbb{Q}$. By Lemma 1.6, it will be sufficient to construct an ultrafilter $\mathcal{U}$ such that no Q_η is a closed subset of $\mathcal{U}$.

We will construct $\mathcal{F}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let $\mathcal{U}$ be any ultrafilter extending $\bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
2. $\mathcal{F}_\xi$ has the finite intersection property for every $\xi \in \mathfrak{c}$.
3. $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
4. The potential closed copy of the rationals Q_η is dealt with at stage $\xi = \eta + 1$: that is, either $\omega \setminus x \in \mathcal{F}_\xi$ for some $x \in Q_\eta$ or there exists $x \in \mathcal{F}_\xi$ such that $x \in \mathrm{cl}(Q_\eta) \setminus Q_\eta$.

Start by letting $\mathcal{F}_0 = \mathrm{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta+1$, assume that $\mathcal{F}_\eta$ is given. First assume that there exists $x \in Q_\eta$ such that $\mathcal{F}_\eta \cup \{\omega \setminus x\}$ has the finite intersection property. In this case, simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{F}_\eta \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in Q_\eta$. It is easy to check that this implies $Q_\eta \subseteq \langle \mathcal{F}_\eta \rangle$. Apply Lemma 1.12 with

$\mathcal{F} = \mathcal{F}_\eta$ and $Q = Q_\eta$ to get $x \in \mathrm{cl}(Q_\eta) \setminus Q_\eta$ such that $\mathcal{F}_\eta \cup \{x\}$ has the finite intersection property. Finally, set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x\}$. $\square$

Lemma 1.12. *Assume that* MA(countable) *holds. Let $\mathcal{F}$ be a collection of subsets of ω with the finite intersection property such that $|\mathcal{F}| < \mathfrak{c}$. Let Q be a non-empty subset of 2^ω with no isolated points such that $Q \subseteq \langle \mathcal{F} \rangle$ and $|Q| < \mathfrak{c}$. Then there exists $x \in \mathrm{cl}(Q) \setminus Q$ such that $\mathcal{F} \cup \{x\}$ has the finite intersection property.*

Proof. Consider the countable poset

$$\mathbb{P} = \{s \in {}^{<\omega}2 : \text{there exist } q \in Q \text{ and } n \in \omega \text{ such that } s = q \restriction n\},$$

with the natural order given by reverse inclusion.

For every $\sigma = \{x_1, \ldots, x_k\} \in [\mathcal{F}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma,\ell} = \{s \in \mathbb{P} : \text{there exists } i \in \mathrm{dom}(s) \setminus \ell \text{ such that } s(i) = x_1(i) = \cdots = x_k(i) = 1\}.$$

Using the fact that $Q \subseteq \langle \mathcal{F} \rangle$, it is easy to see that each $D_{\sigma,\ell}$ is dense in $\mathbb{P}$.

For every $q \in Q$, define

$$D_q = \{s \in \mathbb{P} : \text{there exists } i \in \mathrm{dom}(s) \text{ such that } s(i) \neq q(i)\}.$$

Since Q has no isolated points, each D_q is dense in $\mathbb{P}$.

Since $|\mathcal{F}| < \mathfrak{c}$ and $|Q| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma,\ell} : \sigma \in [\mathcal{F}]^{<\omega}, \ell \in \omega\} \cup \{D_q : q \in Q\}$$

has also size less than $\mathfrak{c}$. Therefore, by MA(countable), there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Let $x = \bigcup G \in 2^\omega$. The dense sets of the form $D_{\sigma,\ell}$ ensure that $\mathcal{F} \cup \{x\}$ has the finite intersection property. The definition of $\mathbb{P}$ guarantees that $x \in \mathrm{cl}(Q)$. Finally, the dense sets of the form D_q guarantee that $x \notin Q$. $\square$

1.4 Countable dense homogeneity

Definition 1.13. *A space X is* countable dense homogeneous *if for every pair (D, E) of countable dense subsets of X there exists a homeomorphism $f : X \longrightarrow X$ such that $f[D] = E$.*

We will start this section by consistently constructing an ultrafilter that is not countable dense homogeneous. We will use Sierpiński's technique for killing homeomorphisms (see [47] or Appendix 2 of [15] for a nice introduction). The key lemma is the following.

Lemma 1.14. *Assume that* MA(countable) *holds. Let $\mathcal{D}$ be a countable independent family that is dense in 2^ω. Fix D_1 and D_2 disjoint countable dense subsets of $\mathcal{D}$. Then there exists $\mathcal{A} \subseteq 2^\omega$ satisfying the following requirements.*

- *$\mathcal{A}$ is an independent family.*
- *$\mathcal{D} \subseteq \mathcal{A}$.*
- *If $G \supseteq \mathcal{D}$ is a G_δ subset of 2^ω and $f : G \longrightarrow G$ is a homeomorphism such that $f[D_1] = D_2$, then there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A}$.*

Proof. Enumerate as $\{f_\eta : \eta \in \mathfrak{c}\}$ all homeomorphisms

$$f_\eta : G_\eta \longrightarrow G_\eta$$

such that $f_\eta[D_1] = D_2$, where $G_\eta \supseteq \mathcal{D}$ is a G_δ subset of 2^ω.

We will construct $\mathcal{A}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, set $\mathcal{A} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{A}_\xi$. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{A}_\mu \subseteq \mathcal{A}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.

2. $\mathcal{A}_\xi$ is an independent family for every $\xi \in \mathfrak{c}$.

3. $|\mathcal{A}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.

4. The homeomorphism f_η is dealt with at stage $\xi = \eta + 1$: that is, there exists $x \in G_\eta$ such that $\{x, \omega \setminus f_\eta(x)\} \subseteq \mathcal{A}_\xi$.

Start by letting $\mathcal{A}_0 = \mathcal{D}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that $\mathcal{A}_\eta$ is given.

List as $\{w_\alpha : \alpha \in \kappa\}$ all the words in $\mathcal{A}_\eta$, where $\kappa = |\mathcal{A}_\eta| < \mathfrak{c}$ by (3). It is easy to check that, for any fixed $n \in \omega$, $\alpha \in \kappa$ and $\varepsilon_1, \varepsilon_2 \in 2$, the set

$$W_{\alpha,n,\varepsilon_1,\varepsilon_2} = \{x \in G_\eta : |w_\alpha \cap x^{\varepsilon_1} \cap f_\eta(x)^{\varepsilon_2}| \geq n\}$$

is open in G_η. It is also dense, because $D_1 \setminus (F \cup f_\eta^{-1}[F]) \subseteq W_{\alpha,n,\varepsilon_1,\varepsilon_2}$, where F consists of the finitely many elements of $\mathcal{A}_\eta$ that appear in w_α. Therefore, each $W_{\alpha,n,\varepsilon_1,\varepsilon_2}$ is comeager in 2^ω. Recall that MA(countable) is equivalent to $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ (see Theorem 7.13 in [6] or Theorem 2.4.5 in [5]). It follows that the intersection

$$W = \bigcap \{W_{\alpha,n,\varepsilon_1,\varepsilon_2} : n \in \omega,\ \alpha \in \kappa \text{ and } \varepsilon_1, \varepsilon_2 \in 2\}$$

is non-empty. Now simply pick $x \in W$ and set $\mathcal{A}_\xi = \mathcal{A}_\eta \cup \{x, \omega \setminus f_\eta(x)\}$. □

Theorem 1.15. *Assume that* MA(countable) *holds. Then there exists an ultrafilter* $\mathcal{U} \subseteq 2^\omega$ *that is not countable dense homogeneous.*

Proof. Fix D_1, D_2 and $\mathcal{A}$ as in Lemma 1.14. Let $\mathcal{U} \supseteq \mathcal{A}$ be any ultrafilter. Assume, in order to get a contradiction, that $\mathcal{U}$ is countable dense homogeneous. Let $g : \mathcal{U} \longrightarrow \mathcal{U}$ be a homeomorphism such that $g[D_1] = D_2$. By Lavrentiev's lemma, it is possible to extend g to a homeomorphism $f : G \longrightarrow G$, where G is a G_δ subset of 2^ω (see Exercise 3.10 in [30]). By Lemma 1.14, there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A} \subseteq \mathcal{U}$, contradicting the fact that $f(x) = g(x) \in \mathcal{U}$. $\square$

When first trying to prove Theorem 1.15, we attempted to construct an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that no homeomorphism $g : \mathcal{U} \longrightarrow \mathcal{U}$ would be such that $g[\mathrm{Cof}] \cap \mathrm{Cof} = \varnothing$. This is easily seen to be impossible by choosing g to be the multiplication by any coinfinite $x \in \mathcal{U}$. Actually, something much stronger holds by the following result of Van Mill (see Proposition 3.4 in [49]).

Definition 1.16 (Van Mill)**.** *A space X has the* separation property *if for every countable subset A of X and every meager subset B of X there exists a homeomorphism $f : X \longrightarrow X$ such that $f[A] \cap B = \varnothing$.*

Proposition 1.17 (Van Mill)**.** *Let G be a Baire topological group acting on space X that is not meager in itself. Then, for all subsets A and B of X with A countable and B meager, the set of elements $g \in G$ such that $gA \cap B = \varnothing$ is dense in G.*

Corollary 1.18. *Every Baire topological group has the separation property.*

Corollary 1.19. *Every ultrafilter $\mathcal{U} \subseteq 2^\omega$ has the separation property.*

It is easy to see that, for Baire spaces, being countable dense homogeneous is stronger than having the separation property. On the other hand, the product of 2^ω and the one-dimensional sphere S^1 is a compact topological group that has the separation property but is not countable dense homogeneous (see Corollary 3.6 and Remark 3.7 in [49]). Theorem 1.15 consistently gives a zero-dimensional topological group with the same feature. Notice that such an example cannot be compact (or even Polish) by the following paragraph.

Recall that a space X is *strongly locally homogeneous* if it admits an open base $\mathcal{B}$ such that whenever $U \in \mathcal{B}$ and $x, y \in U$ there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$ and $f \upharpoonright X \setminus U$ is the identity. For example, any homogeneous zero-dimensional space is strongly locally homogeneous. For Polish spaces, strong local homogeneity implies countable dense homogeneity (see Theorem 5.2 in [1]). In [46], Van Mill constructed a homogeneous Baire space that is strongly locally homogeneous but not countable dense homogeneous. Actually, his example does not even have the separation property (see Theorem 3.5 in [46]), so it cannot be a topological group by Corollary 1.18. In this sense, our example from Theorem 1.15 is better than his. On the other hand, his example is constructed in ZFC, while ours needs MA(countable). Furthermore, his example can be easily modified to have any given dimension (see Remark 4.1 in [46]).

Next, we will construct (still under MA(countable)) an ultrafilter that is countable dense homogeneous. In [4], Baldwin and Beaudoin used MA(σ-centered) to construct a homogeneous Bernstein subset of 2^ω that is countable dense homogeneous. Both examples give a consistent answer to Question 389 in [21], which asks whether there exists a countable dense homogeneous space that is not completely metrizable. In [20], using metamathematical methods, Farah, Hrušák and Martínez Ranero showed that the answer to such question is 'yes' in ZFC.

The following lemma will be one of the key ingredients. The other key ingredient is the poset used in the proof of Lemma 1.22, which was inspired by the poset used in the proof of Lemma 3.1 in [4].

Lemma 1.20. *Let $f : 2^\omega \longrightarrow 2^\omega$ be a homeomorphism. Fix a maximal ideal $\mathcal{J} \subseteq 2^\omega$*

and a countable dense subset D of $\mathcal{J}$. Then f restricts to a homeomorphism of $\mathcal{J}$ if and only if $\mathrm{cl}(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}$.

Proof. Assume that f restricts to a homeomorphism of $\mathcal{J}$. It is easy to check that the function $g : 2^\omega \longrightarrow 2^\omega$ defined by $g(x) = x + f(x)$ has range contained in $\mathcal{J}$. Since g is continuous, its range must be compact, hence closed in 2^ω.

Now assume that $\mathrm{cl}(\{d+f(d) : d \in D\}) \subseteq \mathcal{J}$. Let $x \in 2^\omega$. Fix a sequence $(d_n : n \in \omega)$ such that $d_n \in D$ for every $n \in \omega$ and $\lim_{n\to\infty} d_n = x$. By continuity,

$$x + f(x) = \lim_{n\to\infty} (d_n + f(d_n)) \in \mathcal{J}.$$

The proof is concluded by observing that if $a, b \in 2^\omega$ are such that $a+b \in \mathcal{J}$, then either $\{a, b\} \subseteq \mathcal{J}$ or $\{a, b\} \subseteq 2^\omega \setminus \mathcal{J}$. $\square$

Theorem 1.21. *Assume that* MA(countable) *holds. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous.*

Proof. For notational convenience, we will construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ that is countable dense homogeneous. Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω.

We will construct $\mathcal{I}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let $\mathcal{J}$ be any maximal ideal extending $\bigcup_{\xi\in\mathfrak{c}} \mathcal{I}_\xi$. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{I}_\mu \subseteq \mathcal{I}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.

2. $\mathcal{I}_\xi$ has the finite union property for every $\xi \in \mathfrak{c}$.

3. $|\mathcal{I}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.

4. The pair (D_η, E_η) is dealt with at stage $\xi = \eta + 1$: that is, either $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in D_\eta \cup E_\eta$ or there exists $x \in \mathcal{I}_\xi$ and a homeomorphism $f_\eta : 2^\omega \longrightarrow 2^\omega$ such that $f_\eta[D_\eta] = E_\eta$ and $\{d + f_\eta(d) : d \in D_\eta\} \subseteq x\!\downarrow$.

Observe that, by Lemma 1.20, the second part of condition (4) guarantees that any maximal ideal $\mathcal{J}$ extending $\mathcal{I}_\xi$ will be such that $f_\eta : 2^\omega \longrightarrow 2^\omega$ restricts to a homeomorphism of $\mathcal{J}$.

Start by letting $\mathcal{I}_0 = \mathrm{Fin}$. Take unions at limit stages. At a successor stage $\xi = \eta+1$, assume that $\mathcal{I}_\eta$ is given. First assume that there exists $x \in D_\eta \cup E_\eta$ such that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ does not have the finite union property for any $x \in D_\eta \cup E_\eta$. It is easy to check that this implies $D_\eta \cup E_\eta \subseteq \langle \mathcal{I}_\eta \rangle$. Let x and f be given by applying Lemma 1.22 with $\mathcal{I} = \mathcal{I}_\eta$, $D = D_\eta$ and $E = E_\eta$. Finally, set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{x\}$ and $f_\eta = f$. □

Lemma 1.22. *Assume that* MA(countable) *holds. Let* $\mathcal{I} \subseteq 2^\omega$ *be a collection of subsets of* ω *with the finite union property and assume that* $|\mathcal{I}| < \mathfrak{c}$. *Fix two countable dense subsets* D *and* E *of* 2^ω *such that* $D \cup E \subseteq \langle \mathcal{I} \rangle$. *Then there exists a homeomorphism* $f : 2^\omega \longrightarrow 2^\omega$ *and* $x \in 2^\omega$ *such that* $f[D] = E$, $\mathcal{I} \cup \{x\}$ *still has the finite union property and* $\{d + f(d) : d \in D\} \subseteq x\!\downarrow$.

Proof. Consider the countable poset $\mathbb{P}$ consisting of all triples of the form $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following requirements are satisfied.

- $s : n \longrightarrow 2$.

- g is a bijection between a finite subset of D and a finite subset of E.

- π is a permutation of ${}^{n}2$.

Furthermore, we require the following compatibility conditions to be satisfied. Condition (1) will actually ensure that $\{d + f(d) : d \in 2^{\omega}\} \subseteq x\downarrow$. Notice that this is equivalent to $(d + f(d))(i) \leq x(i)$ for all $d \in 2^{\omega}$ and $i \in \omega$.

1. $(t + \pi(t))(i) = 1$ implies $s(i) = 1$ for every $t \in {}^{n}2$ and $i \in n$.

2. $\pi(d \restriction n) = g(d) \restriction n$ for every $d \in \mathrm{dom}(g)$.

Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions are satisfied.

- $s_q \supseteq s_p$.

- $g_q \supseteq g_p$.

- $\pi_q(t) \restriction n_p = \pi_p(t \restriction n_p)$ for all $t \in {}^{n_q}2$.

For each $d \in D$, define

$$D_d^{\mathrm{dom}} = \{p \in \mathbb{P} : d \in \mathrm{dom}(g_p)\}.$$

Given $p \in \mathbb{P}$ and $d \in D \setminus \mathrm{dom}(g_p)$, one can simply choose $e \in E \setminus \mathrm{ran}(g_p)$ such that $e \restriction n_p = \pi_p(d \restriction n_p)$. This choice will make sure that $q = (s_p, g_p \cup \{(d, e)\}, \pi_p) \in \mathbb{P}$. Furthermore it is clear that $q \leq p$. So each D_d^{dom} is dense in $\mathbb{P}$.

For each $e \in E$, define

$$D_e^{\mathrm{ran}} = \{p \in \mathbb{P} : e \in \mathrm{ran}(g_p)\}.$$

As above, one can easily show that each D_e^{ran} is dense in $\mathbb{P}$.

For every $\sigma = \{x_1, \ldots, x_k\} \in [\mathcal{I}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma,\ell} = \{p \in \mathbb{P} : \text{there exists } i \in n_p \setminus \ell \text{ such that } s_p(i) = x_1(i) = \cdots = x_k(i) = 0\}.$$

Next, we will prove that each $D_{\sigma,\ell}$ is dense in $\mathbb{P}$. So fix σ and ℓ as above. Let $p = (s, g, \pi) \in \mathbb{P}$ with $n_p = n$. Find $n' \geq \ell, n$ such that the following conditions hold.

- All $d \upharpoonright n'$ for $d \in \mathrm{dom}(g)$ are distinct.
- All $e \upharpoonright n'$ for $e \in \mathrm{ran}(g)$ are distinct.
- $x_1(n') = \cdots = x_k(n') = d(n') = e(n') = 0$ for all $d \in \mathrm{dom}(g)$, $e \in \mathrm{ran}(g)$.

This is possible because $\mathcal{I}$ has the finite union property and

$$\sigma \cup \mathrm{dom}(g) \cup \mathrm{ran}(g) \subseteq \langle \mathcal{I} \rangle.$$

We can choose a permutation π' of ${}^{n'}2$ such that $\pi'(d \upharpoonright n') = g(d) \upharpoonright n'$ for every $d \in \mathrm{dom}(g)$ and $\pi'(t) \upharpoonright n = \pi(t \upharpoonright n)$ for all $t \in {}^{n'}2$. Extend s to $s' : n' \longrightarrow 2$ by setting $s'(i) = 1$ for every $i \in [n, n')$. It is clear that $p' = (s', g, \pi') \in \mathbb{P}$ and $p' \leq p$.

Now let π'' be the permutation of ${}^{n'+1}2$ obtained by setting

$$\pi''(t) = \pi'(t \upharpoonright n')^\frown t(n')$$

for all $t \in {}^{n'+1}2$. Extend s' to $s'' : n' + 1 \longrightarrow 2$ by setting $s''(n') = 0$. It is easy to check that $p'' = (s'', g, \pi'') \in D_{\sigma,\ell}$ and $p'' \leq p'$.

Since $|\mathcal{I}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma,\ell} : \sigma \in [\mathcal{I}]^{<\omega}, \ell \in \omega\} \cup \{D_d^{\mathrm{dom}} : d \in D\} \cup \{D_e^{\mathrm{ran}} : e \in E\}$$

has also size less than $\mathfrak{c}$. Therefore, by MA(countable), there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Define $x = \bigcup\{s_p : p \in G\}$. To define $f(y)(i)$, for a given $y \in 2^\omega$ and $i \in \omega$, choose any $p \in G$ such that $i \in n_p$ and set $f(y)(i) = \pi_p(y \upharpoonright n_p)(i)$. $\square$

1.5 A question of Hrušák and Zamora Avilés

The second half of Question 387 in [21] asks to characterize the zero-dimensional spaces X such that X^ω is countable dense homogeneous. The first partial answer to such question is Theorem 3.2 in [26], which states that, given a *Borel* subset X of 2^ω, the following conditions are equivalent.

- X^ω is countable dense homogeneous.
- X is a G_δ.

Therefore, it is natural to wonder whether being a G_δ is in fact the characterization that we are looking for. The following is Question 3.2 in [26].

Question 1.23 (Hrušák and Zamora Avilés)**.** *Is there a non-G_δ subset X of 2^ω such that X^ω is countable dense homogeneous?*

By a rather straightforward modification of the proof of Theorem 1.21, we will give the first consistent answer to the above question (see Corollary 1.27).

Observe that, given any ideal $\mathcal{I} \subseteq 2^\omega$, the infinite product $\mathcal{I}^\omega$ inherits the structure of topological group using coordinate-wise addition. The following lemma is proved exactly like the corresponding half of Lemma 1.20.

Lemma 1.24. *Let $f : (2^\omega)^\omega \longrightarrow (2^\omega)^\omega$ be a homeomorphism. Fix a maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense subset D of $\mathcal{J}^\omega$. If $\mathrm{cl}(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}^\omega$ then f restricts to a homeomorphism of $\mathcal{J}^\omega$.*

Theorem 1.25. *Assume that* MA(countable) *holds. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is countable dense homogeneous.*

Proof. For notational convenience, we will construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ such that $\mathcal{J}^\omega$ is countable dense homogeneous. Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of $(2^\omega)^\omega$.

We will construct $\mathcal{I}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. In the end, let $\mathcal{J}$ be any maximal ideal extending $\bigcup_{\xi\in\mathfrak{c}} \mathcal{I}_\xi$. By induction, we will make sure that the following requirements are satisfied. Let $P_\eta = \bigcup_{i\in\omega} \pi_i[D_\eta \cup E_\eta]$, where $\pi_i : (2^\omega)^\omega \longrightarrow 2^\omega$ is the natural projection.

1. $\mathcal{I}_\mu \subseteq \mathcal{I}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
2. $\mathcal{I}_\xi$ has the finite union property for every $\xi \in \mathfrak{c}$.
3. $|\mathcal{I}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.
4. The pair (D_η, E_η) is dealt with at stage $\xi = \eta + 1$: that is, either $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in P_\eta$ or there exists $x_i \in \mathcal{I}_\xi$ for every $i \in \omega$ and a homeomorphism $f_\eta : (2^\omega)^\omega \longrightarrow (2^\omega)^\omega$ such that $f_\eta[D_\eta] = E_\eta$ and $\{d + f_\eta(d) : d \in D_\eta\} \subseteq \prod_{i\in\omega}(x_i\downarrow)$.

Observe that, by Lemma 1.24, the second part of condition (4) guarantees that any maximal ideal $\mathcal{J}$ extending $\mathcal{I}_\xi$ will be such that $f_\eta : (2^\omega)^\omega \longrightarrow (2^\omega)^\omega$ restricts to a homeomorphism of $\mathcal{J}^\omega$.

Start by letting $\mathcal{I}_0 = \mathrm{Fin}$. Take unions at limit stages. At a successor stage $\xi = \eta+1$, assume that $\mathcal{I}_\eta$ is given. First assume that there exists $x \in P_\eta$ such that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{\omega \setminus x\}$.

Now assume that $\mathcal{I}_\eta \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in P_\eta$. It is easy to check that this implies $P_\eta \subseteq \langle \mathcal{I}_\eta \rangle$, hence $D_\eta \cup E_\eta \subseteq \langle \mathcal{I}_\eta \rangle^\omega$. Let x_i

for $i \in \omega$ and f be given by applying Lemma 1.26 with $\mathcal{I} = \mathcal{I}_\eta$, $D = D_\eta$ and $E = E_\eta$. Finally, set $\mathcal{I}_\xi = \mathcal{I}_\eta \cup \{x_i : i \in \omega\}$ and $f_\eta = f$. $\square$

Lemma 1.26. *Assume that* MA(countable) *holds. Let $\mathcal{I} \subseteq 2^\omega$ be a collection of subsets of ω with the finite union property and assume that $|\mathcal{I}| < \mathfrak{c}$. Fix two countable dense subsets D and E of $(2^\omega)^\omega$ such that $D \cup E \subseteq \langle \mathcal{I} \rangle^\omega$. Then there exists a homeomorphism $f : (2^\omega)^\omega \longrightarrow (2^\omega)^\omega$ and $x_i \in 2^\omega$ for $i \in \omega$ such that $f[D] = E$, $\mathcal{I} \cup \{x_i : i \in \omega\}$ still has the finite union property and $\{d + f(d) : d \in D\} \subseteq \prod_{i \in \omega}(x_i \downarrow)$.*

Proof. We will make a natural identification of $(2^\omega)^\omega$ with $2^{\omega \times \omega}$. Namely, we will identify a sequence $(x_i : i \in \omega)$ with the function $x : \omega \times \omega \longrightarrow 2$ given by $x(i,j) = x_i(j)$.

Consider the countable poset $\mathbb{P}$ consisting of all triples of the form $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $m = m_p \in \omega$ and $n = n_p \in \omega$, the following requirements are satisfied.

- $s : m \times n \longrightarrow 2$.
- g is a bijection between a finite subset of D and a finite subset of E.
- π is a permutation of ${}^{m \times n}2$.

Furthermore, we require the following compatibility conditions to be satisfied. Condition (1) will actually ensure that $\{d + f(d) : d \in (2^\omega)^\omega\} \subseteq \prod_{i \in \omega}(x_i \downarrow)$. Notice that this is equivalent to $(d + f(d))(i,j) \leq x(i,j) = x_i(j)$ for all $d \in 2^{\omega \times \omega}$ and $(i,j) \in \omega \times \omega$.

1. $(t + \pi(t))(i,j) = 1$ implies $s(i,j) = 1$ for every $t \in {}^{m \times n}2$ and $(i,j) \in m \times n$.
2. $\pi(d \restriction (m \times n)) = g(d) \restriction (m \times n)$ for every $d \in \mathrm{dom}(g)$.

Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions are satisfied.

- $s_q \supseteq s_p$.

- $g_q \supseteq g_p$.

- $\pi_q(t) \restriction (m_p \times n_p) = \pi_p(t \restriction (m_p \times n_p))$ for all $t \in {}^{m_q \times n_q}2$.

For each $d \in D$, define

$$D_d^{\mathrm{dom}} = \{p \in \mathbb{P} : d \in \mathrm{dom}(g_p)\}.$$

Given $p \in \mathbb{P}$ and $d \in D \setminus \mathrm{dom}(g_p)$, one can simply choose $e \in E \setminus \mathrm{ran}(g_p)$ such that $e \restriction (m_p \times n_p) = \pi_p(d \restriction (m_p \times n_p))$. This choice will make sure that $q = (s_p, g_p \cup \{(d, e)\}, \pi_p) \in \mathbb{P}$. Furthermore it is clear that $q \leq p$. So each D_d^{dom} is dense in $\mathbb{P}$.

For each $e \in E$, define

$$D_e^{\mathrm{ran}} = \{p \in \mathbb{P} : e \in \mathrm{ran}(g_p)\}.$$

As above, one can easily show that each D_e^{ran} is dense in $\mathbb{P}$.

For each $\sigma = \{x_1, \ldots, x_k\} \in [\mathcal{I}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma,\ell} = \{p \in \mathbb{P} : \text{there exists } j \in n_p \setminus \ell \text{ such that}$$

$$s_p(0, j) = \cdots = s_p(m_p - 1, j) = x_1(j) = \cdots = x_k(j) = 0\}.$$

Next, we will prove that each $D_{\sigma,\ell}$ is dense in $\mathbb{P}$. So fix σ and ℓ as above. Let $p = (s, g, \pi) \in \mathbb{P}$ with $m_p = m$ and $n_p = n$. Find $m' \geq m$ and $n' \geq \ell, n$ such that the following conditions hold.

- All $d \restriction (m' \times n')$ for $d \in \mathrm{dom}(g)$ are distinct.

- All $e \restriction (m' \times n')$ for $e \in \mathrm{ran}(g)$ are distinct.

- $x_1(n') = \cdots = x_k(n') = d_i(n') = e_i(n') = 0$ for all $d \in \mathrm{dom}(g)$, $e \in \mathrm{ran}(g)$ and $i \in m'$.

This is possible because $\mathcal{I}$ has the finite union property and

$$\sigma \cup \{d_i : d \in \mathrm{dom}(g), i \in \omega\} \cup \{e_i : e \in \mathrm{ran}(g), i \in \omega\} \subseteq \langle \mathcal{I} \rangle.$$

We can choose a permutation π' of ${}^{m' \times n'}2$ such that $\pi'(d \restriction (m' \times n')) = g(d) \restriction (m' \times n')$ for every $d \in \mathrm{dom}(g)$ and $\pi'(t) \restriction (m \times n) = \pi(t \restriction (m \times n))$ for all $t \in {}^{m' \times n'}2$. Extend s to $s' : m' \times n' \longrightarrow 2$ by setting $s'(i,j) = 1$ for every $(i,j) \in (m' \times n') \setminus (m \times n)$. It is clear that $p' = (s', g, \pi') \in \mathbb{P}$ and $p' \leq p$.

Now let π'' be the permutation of ${}^{m' \times (n'+1)}2$ obtained by setting

$$\pi''(t)(i,j) = \begin{cases} \pi'(t \restriction (m' \times n'))(i,j) & \text{if } (i,j) \in m' \times n' \\ t(i,j) & \text{if } (i,j) \in m' \times \{n'\} \end{cases}$$

for all $t \in {}^{m' \times (n'+1)}2$. Extend s' to $s'' : m' \times (n'+1) \longrightarrow 2$ by setting $s''(i,j) = 0$ for all $(i,j) \in m' \times \{n'\}$. It is easy to check that $p'' = (s'', g, \pi'') \in D_{\sigma,\ell}$ and $p'' \leq p'$.

We will need one last class of dense sets. For any given $\ell \in \omega$, define

$$D_\ell = \{p \in \mathbb{P} : m_p \geq \ell\}.$$

An easier version of the above argument shows that each D_ℓ is in fact dense.

Since $|\mathcal{I}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_{\sigma,\ell} : \sigma \in [\mathcal{I}]^{<\omega}, \ell \in \omega\} \cup \{D_d^{\mathrm{dom}} : d \in D\} \cup \{D_e^{\mathrm{ran}} : e \in E\} \cup \{D_\ell : \ell \in \omega\}$$

has also size less than $\mathfrak{c}$. Therefore, by MA(countable), there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Define $x_i = \bigcup\{s_p(i,-) : p \in G\}$ for every $i \in \omega$. To define $f(y)(i,j)$, for a given $y \in 2^{\omega \times \omega}$ and $(i,j) \in \omega \times \omega$, choose any $p \in G$ such that $(i,j) \in m_p \times n_p$ and set $f(y)(i,j) = \pi_p(y \restriction (m_p \times n_p))(i,j)$. $\square$

Corollary 1.27. *Assume that* MA(countable) *holds. Then there exists a non-G_δ subset X of 2^ω such that X^ω is countable dense homogeneous.*

1.6 The perfect set property

Recall that by *perfect set* we mean homeomorphic copy of 2^ω.

Definition 1.28. *Let X be a space. We will say that $A \subseteq X$ has the* perfect set property *if A is either countable or it contains a perfect set.*

It is a classical result of descriptive set theory, due to Souslin, that every analytic subset of a Polish space has the perfect set property (see, for example, Theorem 29.1 in [30]).

The following is an easy application of Kunen's closed embedding trick.

Theorem 1.29. *There exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ with a closed subset of cardinality $\mathfrak{c}$ that does not have the perfect set property.*

Proof. Fix a Bernstein set B in 2^ω, then apply Theorem 1.8 with $C = B$. $\square$

Next, we will consistently construct an ultrafilter $\mathcal{U}$ such that every closed subset of $\mathcal{U}$ has the perfect set property. Actually, we will get a much stronger result (see Theorem 1.30).

Recall that a play of the *strong Choquet game* on a topological space $(X, \mathcal{T})$ is of the form

$$\begin{array}{c|cccccc} \mathrm{I} & (q_0, U_0) & & (q_1, U_1) & & \cdots & \\ \hline \mathrm{II} & & V_0 & & V_1 & & \cdots, \end{array}$$

where $U_n, V_n \in \mathcal{T}$ are such that $q_n \in V_n \subseteq U_n$ and $U_{n+1} \subseteq V_n$ for every $n \in \omega$. Player II wins if $\bigcap_{n \in \omega} U_n \neq \varnothing$. The topological space $(X, \mathcal{T})$ is *strong Choquet* if II has a winning strategy in the above game. See Section 8.D in [30].

Define an A-*triple* to be a triple of the form $(\mathcal{T}, A, Q)$ such that the following conditions are satisfied.

- $\mathcal{T}$ is a strong Choquet, second-countable topology on 2^ω that is finer than the standard topology.
- $A \in \mathcal{T}$.
- Q is a non-empty countable subset of A with no isolated points in the subspace topology it inherits from $\mathcal{T}$.

By Theorem 25.18 in [30], for every analytic A there exists a topology $\mathcal{T}$ as above. Also, by Exercise 25.19 in [30], such a topology $\mathcal{T}$ necessarily consists only of analytic sets. In particular, all A-triples can be enumerated in type $\mathfrak{c}$.

Theorem 1.30. *Assume that* MA(countable) *holds. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for every analytic $A \subseteq 2^\omega$.*

Proof. Enumerate as $\{(\mathcal{T}_\eta, A_\eta, Q_\eta) : \eta \in \mathfrak{c}\}$ all A-triples, making sure that each triple appears cofinally often. Also, enumerate as $\{z_\eta : \eta \in \mathfrak{c}\}$ all subsets of ω.

We will construct $\mathcal{F}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.
2. $\mathcal{F}_\xi$ has the finite intersection property for every $\xi \in \mathfrak{c}$.

3. $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.

4. By stage $\xi = \eta + 1$, we must have decided whether $z_\eta \in \mathcal{U}$: that is, $z_\eta^\varepsilon \in \mathcal{F}_\xi$ for some $\varepsilon \in 2$.

5. If $Q_\eta \subseteq \mathcal{F}_\eta$ then, at stage $\xi = \eta + 1$, we will deal with A_η: that is, there exists $x \in \mathcal{F}_\xi$ such that $x\uparrow \cap A_\eta$ contains a perfect subset.

In the end, let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. Notice that $\mathcal{U}$ will be an ultrafilter by (4).

Start by letting $\mathcal{F}_0 = \mathrm{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that $\mathcal{F}_\eta$ is given. First assume that $Q_\eta \nsubseteq \mathcal{F}_\eta$. In this case, simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{z_\eta^\varepsilon\}$ for a choice of $\varepsilon \in 2$ that is compatible with condition (2).

Now assume that $Q_\eta \subseteq \mathcal{F}_\eta$. Apply Lemma 1.31 with $\mathcal{F} = \mathcal{F}_\eta$, $A = A_\eta$, $Q = Q_\eta$ and $\mathcal{T} = \mathcal{T}_\eta$ to get a perfect set $P \subseteq A$ such that $\mathcal{F}_\eta \cup \{\bigcap P\}$ has the finite intersection property. Let $x = \bigcap P$. Set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x, z_\eta^\varepsilon\}$ for a choice of $\varepsilon \in 2$ that is compatible with condition (2).

Finally, we will check that $\mathcal{U}$ has the required property. Assume that A is an analytic subset of 2^ω such that $A \cap \mathcal{U}$ is uncountable. By Theorem 25.18 in [30], there exists a second-countable, strong Choquet topology $\mathcal{T}$ on 2^ω that is finer than the standard topology and contains A. Since every second-countable, uncountable Hausdorff space contains a non-empty countable subspace with no isolated points, we can find such a subspace $Q \subseteq A \cap \mathcal{U}$. Since $\mathrm{cf}(\mathfrak{c}) > \omega$, there exists $\mu \in \mathfrak{c}$ such that $Q \subseteq \mathcal{F}_\mu$. Since we listed each A-triple cofinally often, there exists $\eta \geq \mu$ such that $(\mathcal{T}, A, Q) = (\mathcal{T}_\eta, A_\eta, Q_\eta)$. Condition (5) guarantees that $\mathcal{U} \cap A$ will contain a perfect subset. $\square$

Lemma 1.31. *Assume that* MA(countable) *holds. Let $\mathcal{F}$ be a collection of subsets of ω with the finite intersection property such that $|\mathcal{F}| < \mathfrak{c}$. Suppose that $(\mathcal{T}, A, Q)$ is an*

A-*triple with* $Q \subseteq \mathcal{F}$. *Then there exists a perfect subset* P *of* A *such that* $\mathcal{F} \cup \{\bigcap P\}$ *has the finite intersection property.*

Proof. Fix a winning strategy Σ for player II in the strong Choquet game in $(2^\omega, \mathcal{T})$. Also, fix a countable base $\mathcal{B}$ for $(2^\omega, \mathcal{T})$. Let $\mathbb{P}$ be the countable poset consisting of all functions p such that, for some $n = n_p \in \omega$, the following conditions hold.

1. $p : {}^{\leq n}2 \longrightarrow Q \times \mathcal{B}$. We will use the notation $p(s) = (q^p_s, U^p_s)$.

2. $U^p_\varnothing = A$.

3. For every $s, t \in {}^{\leq n}2$, if s and t are incompatible (that is, $s \nsubseteq t$ and $t \nsubseteq s$) then $U^p_s \cap U^p_t = \varnothing$.

4. For every $s \in {}^n2$,

$$\begin{array}{c|cccccccc} \mathrm{I} & (q^p_{s\restriction 0}, U^p_{s\restriction 0}) & & (q^p_{s\restriction 1}, U^p_{s\restriction 1}) & & \cdots & (q^p_{s\restriction n}, U^p_{s\restriction n}) & \\ \hline \mathrm{II} & & V^p_{s\restriction 0} & & V^p_{s\restriction 1} & \cdots & & V^p_{s\restriction n} \end{array}$$

   is a partial play of the strong Choquet game in $(2^\omega, \mathcal{T})$, where the open sets $V^p_{s\restriction i}$ played by II are the ones dictated by the strategy Σ.

Order $\mathbb{P}$ by setting $p \leq p'$ whenever $p \supseteq p'$.

For every $\ell \in \omega$, define

$$D_\ell = \{p \in \mathbb{P} : n_p \geq \ell\}.$$

Since Q has no isolated points and $\mathcal{T}$ is Hausdorff, it is easy to see that each D_ℓ is dense.

For any fixed $\ell \in \omega$, consider the partition of 2^ω in clopen sets $\mathcal{P}_\ell = \{[s] : s \in {}^\ell 2\}$, then define

$$D^{\mathrm{ref}}_\ell = \{p \in \mathbb{P} : \{U^p_s : s \in {}^{n_p}2\} \text{ refines } \mathcal{P}_\ell\}.$$

Let us check that each D_ℓ^{ref} is dense. Given $p \in \mathbb{P}$ and $\ell \in \omega$, let $n = n_p$ and $q_s^0 = q_s^p$ for every $s \in {}^n2$. Since Q has no isolated points, it is possible, for every $s \in {}^n2$, to choose $q_s^1 \neq q_s^0$ such that $q_s^1 \in V_s^p \cap Q$. Fix $j \geq \ell$ big enough to guarantee $[q_s^1 \upharpoonright j] \cap [q_s^0 \upharpoonright j] = \varnothing$ for every $s \in {}^n2$. Now simply extend p to a condition $p' : {}^{\leq n+1}2 \longrightarrow Q \times \mathcal{B}$ by defining $p'(s^\frown\varepsilon) = (q_s^\varepsilon, U_s^\varepsilon)$ for every $s \in {}^n2$ and $\varepsilon \in 2$, where each $U_s^\varepsilon \in \mathcal{B}$ is such that $q_s^\varepsilon \in U_s^\varepsilon \subseteq V_s^p \cap [q_s^\varepsilon \upharpoonright j]$. It is easy to realize that $p' \in D_\ell^{\mathrm{ref}}$.

For any fixed $\sigma = \{x_1, \ldots, x_k\} \in [\mathcal{F}]^{<\omega}$ and $\ell \in \omega$, define

$$D_{\sigma,\ell} = \{p \in \mathbb{P} : \text{there exists } i \in \omega \setminus \ell \text{ such that}$$

$$x(i) = x_1(i) = \cdots = x_k(i) = 1 \text{ for all } x \in U_s^p \text{ for all } s \in {}^{n_p}2\}.$$

Let us check that each $D_{\sigma,\ell}$ is dense. Given $p \in \mathbb{P}$, σ and ℓ as above, let $n = n_p$ and $q_s^0 = q_s^p$ for every $s \in {}^n2$. Notice that

$$\bigcap_{s \in {}^n2} q_s^p \cap \bigcap \sigma$$

is an infinite subset of ω, because $Q \subseteq \mathcal{F}$ by assumption. So there exists $i \in \omega$ with $i \geq \ell$ such that

$$q_s^p(i) = x_1(i) = \cdots = x_k(i) = 1$$

for every $s \in {}^n2$. Since Q has no isolated points, it is possible, for every $s \in {}^n2$, to choose $q_s^1 \neq q_s^0$ such that $q_s^1 \in V_s^p \cap [q_s^p \upharpoonright (i+1)] \cap Q$. Fix $j \geq i+1$ big enough to guarantee $[q_s^1 \upharpoonright j] \cap [q_s^0 \upharpoonright j] = \varnothing$ for every $s \in {}^n2$. Now simply extend p to a condition $p' : {}^{\leq n+1}2 \longrightarrow Q \times \mathcal{B}$ by defining $p'(s^\frown\varepsilon) = (q_s^\varepsilon, U_s^\varepsilon)$ for every $s \in {}^n2$ and $\varepsilon \in 2$, where each $U_s^\varepsilon \in \mathcal{B}$ is such that $q_s^\varepsilon \in U_s^\varepsilon \subseteq V_s^p \cap [q_s^\varepsilon \upharpoonright j]$. It is easy to realize that $p' \in D_{\sigma,\ell}$.

Since $|\mathcal{F}| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_\ell : \ell \in \omega\} \cup \{D_\ell^{\mathrm{ref}} : \ell \in \omega\} \cup \{D_{\sigma,\ell} : \sigma \in [\mathcal{F}]^{<\omega}, \ell \in \omega\}$$

has also size less than $\mathfrak{c}$. Therefore, by MA(countable), there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Let $g = \bigcup G : {}^{<\omega}2 \longrightarrow Q \times \mathcal{B}$. Given $s \in {}^{<\omega}2$, pick any $p \in G$ such that $s \in \mathrm{dom}(p)$ and set $U_s = U_s^p$. For any $x \in 2^\omega$, since Σ is a winning strategy for II, we must have $\bigcap_{n\in\omega} U_{x\restriction n} \neq \varnothing$. Using the dense sets D_ℓ^{ref}, one can easily show that such intersection is actually a singleton. Therefore, letting $f(x)$ be the unique element of $\bigcap_{n\in\omega} U_{x\restriction n}$ yields a well-defined function $f : 2^\omega \longrightarrow A$. Using condition (3) in the definition of $\mathbb{P}$, one sees that f is injective.

Next, we will show that f is continuous in the standard topology, hence a homemorphic embedding by compactness. Fix $x \in 2^\omega$ and let $y = f(x)$. Fix $\ell \in \omega$. Since G is a $\mathcal{D}$-generic filter, there must be $p \in D_\ell^{\mathrm{ref}} \cap G$. Let $n = n_p$. Notice that this implies $U_{x\restriction n} = U_{x\restriction n}^p \subseteq [y \restriction \ell]$, hence $f(x') \in [y \restriction \ell]$ whenever $x' \in [x \restriction n]$.

Therefore $P = \mathrm{ran}(f)$ is a perfect subset of A. Finally, using the dense sets $D_{\sigma,\ell}$ one can show that $\mathcal{F} \cup \{\bigcap P\}$ has the finite intersection property. $\square$

Corollary 1.32. *Assume that* MA(countable) *holds. Then there exists an ultrafilter* $\mathcal{U} \subseteq 2^\omega$ *such that every closed subset of* $\mathcal{U}$ *has the perfect set property.*

1.7 Extending the perfect set property

Assuming $V = L$, there exists an uncountable co-analytic set A that does not contain any perfect set (see Theorem 25.37 in [28]). It follows that MA(countable) is not enough to extend Theorem 1.30 to all co-analytic sets. This section is devoted to attaining a positive result for the co-analytic case. Actually, we will obtain a much stronger result (see Theorem 1.36). We will need a modest large cardinal assumption, a larger fragment of MA, and the negation of CH.

Lemma 1.33. *Assume that $\mathcal{U} \subseteq 2^\omega$ is a* P_{ω_2}*-point. If $A \subseteq 2^\omega$ is such that every closed subspace of A has the perfect set property, then $A \cap \mathcal{U}$ has the perfect set property.*

Proof. Let A be as above, and assume that $A \cap \mathcal{U}$ is uncountable. Choose $B \subseteq A \cap \mathcal{U}$ such that $|B| = \omega_1$. Since $\mathcal{U}$ is a P_{ω_2}-point, there is a pseudointersection x of B in $\mathcal{U}$. For some $n \in \omega$, uncountably many elements of B are in the closed set $C = (x \setminus n)\uparrow$. By hypothesis, $A \cap C$ contains a perfect set P. We now have $A \cap \mathcal{U} \supseteq P$ as desired. Thus, $A \cap \mathcal{U}$ has the perfect set property. □

It is not hard to verify that the hypothesis on A in the above lemma is optimal. Let x_0 and x_1 be complementary infinite subsets of ω. Identify each $\mathcal{P}(x_i)$ with the perfect set $\{x \in 2^\omega : x(n) = 0 \text{ for all } n \in x_{1-i}\}$. Fix a Bernstein subset B_i of $\mathcal{P}(x_i)$ and set $A_i = B_i \cup \mathcal{P}(x_{1-i})$ for each $i \in 2$. Each A_i has the perfect set property. However, if $\mathcal{U} \subseteq 2^\omega$ is an ultrafilter, then some $A_i \cap \mathcal{U}$ lacks the perfect set property. Indeed, if $x_i \in \mathcal{U}$, then $y \in \mathcal{U}$ for some subset $y \subseteq x_i$ such that $x_i \setminus y$ is infinite. The perfect set $y\uparrow \cap \mathcal{P}(x_i)$ contains $\mathfrak{c}$ many elements of B_i, so $A_i \cap \mathcal{U}$ has size $\mathfrak{c}$ as well. However, $A_i \cap \mathcal{U} \subseteq B_i$, so $A_i \cap \mathcal{U}$ does not contain a perfect set.

The following lemma is essentially due to Ihoda (Judah) and Shelah (see Theorem 3.1 in [27]). Given a class Γ, we define $\mathrm{PSP}(\Gamma)$ to mean that every $X \in \Gamma \cap \mathcal{P}(2^\omega)$ has the perfect set property.

Lemma 1.34. *The existence of a Mahlo cardinal is equiconsistent with*

$$\mathrm{MA}(\sigma\text{-centered}) + \neg\mathrm{CH} + \mathrm{PSP}(L(\mathbb{R})).$$

Proof. Any generic extension by the Levy collapse $\mathrm{Col}(\omega, \kappa)$ of an inaccessible cardinal κ to ω_1 satisfies $\mathrm{PSP}(L(\mathbb{R}))$ (see the proof of Theorem 11.1 in [29]). By the proof of

Lemma 1.1 in [27], if κ is inaccessible and $\mathbb{P}$ is a forcing poset that satisfies the following conditions, then every generic extension $V[G]$ of V by $\mathbb{P}$ is such that $L(\mathbb{R})^{V[G]} = L(\mathbb{R})^{V[H]}$ for some V-generic filter $H \subseteq \mathrm{Col}(\omega, \kappa)$, obtained in a further generic extension.

1. $\mathbb{P}$ has the κ-cc.
2. $\mathbb{P}$ forces $\kappa = \omega_1$.
3. For every $R \subseteq \mathbb{P}$ of size less than κ, there exists $Q \subseteq \mathbb{P}$ such that $|Q| < \kappa$, $R \subseteq Q$, and Q is completely embedded in $\mathbb{P}$ by the inclusion map.

Assuming that there exists a Mahlo cardinal κ, the proof of Theorem 3.1 in [27] constructs a generic extension $V[G]$ of V by a forcing $\mathbb{P}$ such that $\mathrm{MA}(\sigma\text{-centered}) + \neg\mathrm{CH}$ holds in $V[G]$, where $\mathbb{P}$ satisfies conditions (1), (2) and (3). Therefore, $\mathrm{PSP}(L(\mathbb{R}))$ also holds in $V[G]$.

Conversely, $\mathrm{PSP}(L(\mathbb{R}))$ implies that all injections of ω_1 into 2^ω are outside of $L(\mathbb{R})$, which in turn implies $\omega_1^{L[r]} < \omega_1$ for all reals r. On the other hand, by the proof of Theorem 3.1 in [27], if $\mathrm{MA}(\sigma\text{-centered}) + \neg\mathrm{CH}$ holds and $\omega_1^{L[r]} < \omega_1$ for all reals r, then ω_1 is Mahlo in L. $\square$

For the convenience of the reader, we include the proof of the following standard lemma.

Lemma 1.35. *Assume that* $\mathrm{MA}(\sigma\text{-centered})$ *holds. Then there exists a* $\mathrm{P}_{\mathfrak{c}}$*-point.*

Proof. Enumerate all subsets of ω as $\{z_\eta : \eta \in \mathfrak{c}\}$. We will construct $\mathcal{F}_\xi$ for every $\xi \in \mathfrak{c}$ by transfinite recursion. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{F}_\mu \subseteq \mathcal{F}_\eta$ whenever $\mu \leq \eta < \mathfrak{c}$.

2. $\mathcal{F}_\xi$ has the finite intersection property for every $\xi \in \mathfrak{c}$.

3. $|\mathcal{F}_\xi| < \mathfrak{c}$ for every $\xi \in \mathfrak{c}$.

4. By stage $\xi = \eta + 1$, we must have decided whether $z_\eta \in \mathcal{U}$: that is, $z_\eta^\varepsilon \in \mathcal{F}_\xi$ for some $\varepsilon \in 2$.

5. At stage $\xi = \eta + 1$, we will make sure that $\mathcal{F}_\xi$ contains a pseudointersection of $\mathcal{F}_\eta$.

Start by letting $\mathcal{F}_0 = \mathrm{Cof}$. Take unions at limit stages. At a successor stage $\xi = \eta + 1$, assume that $\mathcal{F}_\eta$ is given.

Since MA(σ-centered) implies $\mathfrak{p} = \mathfrak{c}$ (see Theorem 7.12 in [6]), there exists an infinite pseudointersection x of $\mathcal{F}_\eta$. Now simply set $\mathcal{F}_\xi = \mathcal{F}_\eta \cup \{x, z_\eta^\varepsilon\}$ for a choice of $\varepsilon \in 2$ that is compatible with condition (2).

In the end, let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. Notice that $\mathcal{U}$ will be an ultrafilter by (4). Since $\mathfrak{p} = \mathfrak{c}$ is regular (see Theorem 7.15 in [6]), condition (5) implies that $\mathcal{U}$ is a $\mathrm{P}_\mathfrak{c}$-point. $\square$

It is well-known that MA(countable) is not a sufficient hypothesis for the above lemma. Consider the Cohen model $W = V[(c_\alpha : \alpha < \omega_2)]$, where $V \vDash \mathrm{CH}$ and each c_α is an element of 2^ω that avoids all meager Borel sets with codes in $V[(c_\beta : \beta < \omega_2, \beta \neq \alpha)]$. Observe that every $x \in 2^\omega$ is in $V[(c_\alpha : \alpha \in I)]$ for some $I \in [\omega_2]^{<\omega_1}$. In this model, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$, so MA(countable) + $\neg$CH holds (see Theorem 7.13 in [6]). However, if $\mathcal{U} \in W$ is an ultrafilter, then $\mathcal{U} \cap V[(c_\alpha : \alpha < \omega_1)]$ is a subset of $\mathcal{U}$ of size ω_1 with no infinite pseudointersection, being a non-meager subset of 2^ω (see Section 11.3 of [6]).

Theorem 1.36. *It is consistent, relative to a Mahlo cardinal, that there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. On the other hand, if there exists such an ultrafilter $\mathcal{U}$, then ω_1 is inaccessible in L.*

Proof. Assume that MA(σ-centered) + $\neg$CH + PSP($L(\mathbb{R})$) holds, which is consistent relative to a Mahlo cardinal by Lemma 1.34. By Lemma 1.35, there exists a $\mathrm{P}_{\mathfrak{c}}$-point $\mathcal{U}$. Since $\neg$CH holds, $\mathcal{U}$ is a P_{ω_2}-point. Fix $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. Every closed subspace C of A is also in $L(\mathbb{R})$ because $C = A \cap [T]$ for some tree $T \subseteq {}^{<\omega}2$. By PSP($L(\mathbb{R})$), all such C have the perfect set property. So $A \cap \mathcal{U}$ has the perfect set property by Lemma 1.33.

For the second half of the theorem, assume that $\mathcal{U} \subseteq 2^\omega$ is a non-principal ultrafilter such that $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$. First, observe that given A as above, $c[A]$ is in $L(\mathbb{R})$ too, so $A \cap \mathcal{U}$ and $c[A] \cap \mathcal{U}$ have the perfect set property. Since

$$A = (A \cap \mathcal{U}) \cup (A \cap c[\mathcal{U}]) = (A \cap \mathcal{U}) \cup c[c[A] \cap \mathcal{U}],$$

it follows that A itself has the perfect set property. So PSP($L(\mathbb{R})$) holds, which implies $\omega_1^{L[r]} < \omega_1$ for all reals r. Therefore ω_1 is inaccessible in L. $\square$

1.8 P-points

Given an ultrafilter $\mathcal{U} \subseteq 2^\omega$, it seems natural to investigate whether there is any relation between the topological properties of $\mathcal{U}$ that we have studied so far and the combinatorial properties of $\mathcal{U}$.

In [41], we constructed a non-P-point that is not completely Baire (see Theorem 39) and a P-point that is completely Baire (see Theorem 40). Question 10 in the same article asks whether the two properties are actually equivalent. We were not aware that Marciszewski had already answered such question fourteen years earlier by proving the following result (see Theorem 1.2 in [36]).

Theorem 1.37 (Marciszewski)**.** *A filter $\mathcal{F} \subseteq 2^\omega$ is completely Baire if and only if it is a non-meager* P*-filter.*

In particular, an ultrafilter $\mathcal{U} \subseteq 2^\omega$ is completely Baire if and only if it is a P-point. So Theorem 1.11 follows from the fact that MA(countable) implies that there are P-points (see for example Proposition 5.5, Theorem 7.12 and Theorem 9.25 in [6]), while Theorem 1.9 follows from the existence of a non-P-point (use Lemma 1.38 with $\mathcal{A} = \varnothing$). More importantly, since Shelah showed that it is consistent that there are no P-points (see Theorem 4.4.7 in [5]), it follows that the assumption of MA(countable) cannot be dropped in Theorem 1.11. This answers Question 1 from [41].

In order to construct several kinds of non-P-points, we will use the following lemma, based on an idea of Kunen from [33].

Lemma 1.38. *Let $\mathcal{A}$ be an independent family. Then there exists an ultrafilter $\mathcal{U}$ extending $\mathcal{A}$ that is not a* P*-point.*

Proof. Without loss of generality, assume that $\mathcal{A}$ is infinite. Fix a countably infinite subset $\mathcal{B}$ of $\mathcal{A}$. It is easy to check that

$$\mathcal{F} = \mathcal{A} \cup \{\omega \setminus x : x \subseteq^* y \text{ for every } y \in \mathcal{B}\}$$

has the finite intersection property. Let $\mathcal{U}$ be any ultrafilter extending $\mathcal{F}$. It is clear that $\mathcal{B}$ has no pseudointersection in $\mathcal{U}$. $\square$

Next, we will investigate the relation between P-points and countable dense homogeneity. In [41], we claimed to have constructed a non-P-point $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous (see 'Theorem' 41). Unfortunately, the proof is wrong: Lemma 42 is correct, but it is easy to realize that a stronger lemma is needed. We do not know whether this problem can be fixed. The following two results are the only examples that we were able to produce. Therefore, Question 4.4 and Question 4.5 remain open.

Theorem 1.39. *Assume that* MA(countable) *holds. Then there exists a non-*P*-point* $\mathcal{U} \subseteq 2^\omega$ *that is not countable dense homogeneous.*

Proof. Let $\mathcal{A}$ be as in Lemma 1.14. By the proof of Theorem 1.15, no ultrafilter extending $\mathcal{A}$ is countable dense homogeneous. Now simply apply Lemma 1.38 to $\mathcal{A}$. □

Theorem 1.40. *Assume that* MA(countable) *holds. Then there exists a* P*-point* $\mathcal{U} \subseteq 2^\omega$ *that is countable dense homogeneous.*

Proof. For notational convenience, we will actually construct a maximal ideal $\mathcal{J} \subseteq 2^\omega$ that is countable dense homogeneous and a P-ideal. Enumerate all countable collections of subsets of ω as $\{\mathcal{C}_\eta : \eta \in \mathfrak{c}\}$. The setup of the construction will be as in the proof of Theorem 1.21, but we will do different things at even and odd successor stages.

Start by letting $\mathcal{I}_0 = \mathrm{Fin}$. Take unions at limit stages. At a successor stage $\xi = 2\eta+1$, assume that $\mathcal{I}_{2\eta}$ is given, then take care of (D_η, E_η) as in the proof of Theorem 1.21. At a successor stage $\xi = 2\eta + 2$, assume that $\mathcal{I}_{2\eta+1}$ is given, then take care of $\mathcal{C}_\eta$ as follows.

First assume that there exists $x \in \mathcal{C}_\eta$ such that $\mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$ has the finite union property. In this case, we can just set $\mathcal{I}_\xi = \mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$. Now assume that $\mathcal{I}_{2\eta+1} \cup \{\omega \setminus x\}$ does not have the finite union property for any $x \in \mathcal{C}_\eta$. It is easy to check that this implies

$\mathcal{C}_\eta \subseteq \langle \mathcal{I}_{2\eta+1} \rangle$. As in the proof of Theorem 1.42, it is possible to get a pseudounion x of $\mathcal{C}_\eta$ such that $\mathcal{I}_{2\eta+1} \cup \{x\}$ has the finite union property. Finally, set $\mathcal{I}_\xi = \mathcal{I}_{2\eta+1} \cup \{x\}$. □

Finally, we will investigate the relation between P-points and the perfect set property. The following two results are the only examples that we were able to produce. Therefore, Question 4.6 and Question 4.7 remain open.

Theorem 1.41. *There exists a non-*P*-point* $\mathcal{U} \subseteq 2^\omega$ *with a closed subset of cardinality* $\mathfrak{c}$ *that does not have the perfect set property.*

Proof. We will use the same notation as in the proof of Theorem 1.8. Choose C to be a Bernstein set in 2^ω, so that any ultrafilter extending $\mathcal{G}$ will have a closed subset without the perfect property. Now simply apply Lemma 1.38 to $\mathcal{G}$. □

Theorem 1.42. *Assume that* MA(countable) *holds. Then there exists a* P*-point* $\mathcal{U} \subseteq 2^\omega$ *such that* $A \cap \mathcal{U}$ *has the perfect set property whenever* A *is an analytic subset of* 2^ω*.*

Proof. Enumerate all countable collections of subsets of ω as $\{\mathcal{C}_\eta : \eta \in \mathfrak{c}\}$. The setup of the construction will be as in the proof of Theorem 1.30, but we will do different things at even and odd successor stages.

Start by letting $\mathcal{F}_0 = \mathrm{Cof}$. Take unions at limit stages. At a successor stage $\xi = 2\eta + 1$, assume that $\mathcal{F}_{2\eta}$ is given, then take care of $(\mathcal{T}_\eta, A_\eta, Q_\eta)$ and z_η as in the proof of Theorem 1.30. At a successor stage $\xi = 2\eta + 2$, assume that $\mathcal{F}_{2\eta+1}$ is given, then take care of $\mathcal{C}_\eta$ as follows.

First assume that there exists $x \in \mathcal{C}_\eta$ such that $\mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$ has the finite intersection property. In this case, we can just set $\mathcal{F}_\xi = \mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$. Now assume that $\mathcal{F}_{2\eta+1} \cup \{\omega \setminus x\}$ does not have the finite intersection property for any $x \in \mathcal{C}_\eta$. It is easy to check that this implies $\mathcal{C}_\eta \subseteq \langle \mathcal{F}_{2\eta+1} \rangle$. Recall that MA(countable) implies $\mathfrak{d} = \mathfrak{c}$ (see, for example, Proposition 5.5 and Theorem 7.13 in [6]). So, by Proposition 6.24 in [6], there exists a pseudointersection x of $\mathcal{C}_\eta$ such that $\mathcal{F}_{2\eta+1} \cup \{x\}$ has the finite intersection property. Finally, set $\mathcal{F}_\xi = \mathcal{F}_{2\eta+1} \cup \{x\}$. $\square$

Chapter 2

Products and CLP-compactness

The main results of this chapter originally appeared in [39]. Here, we will give a more comprehensive introduction and say more about CLP-rectangularity.

For all undefined topological notions, see [19]. The following notion was introduced in [64] under the name 'CB-compactness'. Recall that a subset of a topological space is *clopen* if it is closed and open.

Definition 2.1 (Šostak). *A space X is* CLP-compact *if every cover of X consisting of clopen sets has a finite subcover.*

Trivial examples of CLP-compact spaces are given by connected or compact spaces. Also, if X is CLP-compact and Y is connected or compact, then $X \times Y$ is CLP-compact. For zero-dimensional spaces, CLP-compactness is equivalent to compactness.

The Knaster-Kuratowski fan K is an interesting example of CLP-compact space, since it is non-compact, hereditarily disconnected, but not totally disconnected (see Proposition A.2). As noted by Kunen, a Bernstein subset B of $X = 2^\omega \times [0,1]$ will have the same properties, because any clopen subset of B is in the form $(C \times [0,1]) \cap B$, where C is a clopen subset of 2^ω.

Dikranjan constructed the following two classes of examples (see Example 6.13 in [14]), where $0 < n < \omega$.

- An n-dimensional, totally disconnected, pseudocompact, CLP-compact abelian group.

- An n-dimensional, hereditarily disconnected but not totally disconnected, pseudocompact, CLP-compact abelian group.

Since $n > 0$, none of the above spaces can be compact, otherwise it would be zero-dimensional (see Theorem 6.2.9 in [19]).

The article [64] contained the claim that CLP-compactness is productive. Such a claim is false: as Sondore and Šostak notice in [63], Stephenson's example X from [66] is CLP-compact, but X^2 is not. A further counterexample appeared in [68]. The following is the best available positive result (see Corollary 1 in [67]).

Theorem 2.2 (Steprāns)**.** *Let $X = X_1 \times \cdots \times X_n$. If each X_i is CLP-compact and sequential then X is CLP-compact.*

The following is a further positive result, which first appeared as Corollary 3.3 in [14]. We will present a different proof, that uses the material developed in this thesis. Ultimately, though, both proofs rely on Glicksberg's classical Theorem 3.20.

Theorem 2.3 (Dikranjan)**.** *Assume that $X = \prod_{i\in I} X_i$ is pseudocompact. If each X_i is CLP-compact then X is CLP-compact.*

Proof. Apply Proposition 2.10 and Corollary 3.24. □

Corollary 2.4. *If X_i is a CLP-compact pseudocompact group for every $i \in I$ then $X = \prod_{i\in I} X_i$ is CLP-compact.*

Proof. By Theorem 1.4 in [11], a product of pseudocompact groups is pseudocompact.

□

Theorem 2.3 is the only known positive result concerning infinite products of CLP-compact spaces (see Question 4.10). The following is Question 6.3 in [68] (and also Question 7.1 in [14]).

Question 2.5 (Steprāns and Šostak). *Let κ be an infinite cardinal. If $\{X_\xi : \xi \in \kappa\}$ is a collection of spaces such that $\prod_{\xi \in F} X_\xi$ is CLP-compact for every $F \in [\kappa]^{<\omega}$ does it follow that $\prod_{\xi \in \kappa} X_\xi$ is CLP-compact? Does this depend on the size of κ?*

Using the method of Steprāns and Šostak (see Section 4 in [68]), we will construct a family $\{X_i : i \in \omega\}$ of Hausdorff spaces such that $\prod_{i \in n} X_{p(i)}$ is CLP-compact whenever $n \in \omega$ and $p : n \longrightarrow \omega$, while the full product $P = \prod_{i \in \omega} X_i$ is not. In order to make sure that P is non-CLP-compact, we will use an idea from [9]; see also Example 8.28 in [72]. This example answers the first half of the above question.

In the last section, using ideas from [23] and [10], we will convert the example described above into a single Hausdorff space X such that X^n is CLP-compact for every $n \in \omega$, while X^κ is non-CLP-compact for every infinite cardinal κ. This answers the second half of the above question (see Corollary 2.22).

We remark that, given any $n \in \omega$ such that $2 \leq n < \omega$, it is possible to construct a Hausdorff space X such that X^i is CLP-compact for every $i < n$ but X^n is not. This is essentially due to Steprāns and Šostak (see the last part of Section 4 in [68]), but we will sketch a more systematic approach. It is enough to obtain Hausdorff spaces X_i for $i \in n$ such that $\prod_{i \in n} X_{p(i)}$ is CLP-compact whenever $p : n \longrightarrow n$, while the full product $P = \prod_{i \in n} X_i$ is not. In fact, given such spaces, the space

$$X = X_0 \oplus \cdots \oplus X_{n-1}$$

will have the desired properties (this is inspired by the 'Proof of A (using A′)' in [23]).

In Section 2.5, we will use the infinite version of this trick. The reader who understands the methods of this chapter will certainly be able to fill in the details.

2.1 CLP-rectangularity

Definition 2.6. *Given a product space $X = \prod_{i \in I} X_i$, a* rectangle *is a subset of X of the form $R = \prod_{i \in I} R_i$, where $R_i \subseteq X_i$ for each i and $R_i = X_i$ for all but finitely many i.*

By the definition of product topology, every open set in a product can be written as the union of open rectangles. So it seems natural to ask whether it is possible to substitute 'open' with 'clopen'.

Question 2.7 (Šostak, Bauer)**.** *Can every clopen subset of $X \times Y$ be written as the union of clopen rectangles?*

Definition 2.8 (Steprāns and Šostak)**.** *A product $X = \prod_{i \in I} X_i$ is* CLP-rectangular *if for every clopen set $U \subseteq X$ and every $x \in U$ there exists a clopen rectangle $R \subseteq X$ such that $x \in R \subseteq U$.*

In [8], Buzyakova showed that the answer to Question 2.7 is 'no', even for separable metric spaces.

Theorem 2.9 (Buzyakova)**.** *There exist $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}^2$ such that $X \times Y$ is not CLP-rectangular.*

The following proposition shows why CLP-rectangularity is relevant to the study of CLP-compactness. For the proof, see Propositions 2.4 and 2.5 in [68], and Theorem 3.4 in [14].

Proposition 2.10 (Steprāns and Šostak, Dikranjan)**.** *Assume that X_i is CLP-compact for every $i \in I$. Then $\prod_{i \in I} X_i$ is CLP-compact if and only if it is CLP-rectangular.*

In particular, every counterexample to the productivity of CLP-compactness also shows that the answer to Question 2.7 is 'no'. However, such examples cannot be as nice as Buzyakova's (at least in the case of finite products) because of Theorem 2.2.

The rest of this section will not be used in the construction of our counterexample, but we think it might be interesting to mention a positive result about CLP-rectangularity due to Kunen, that first appeared without proof as Theorem 3 in [8].

Theorem 2.11 (Kunen)**.** *Assume that both X and Y are locally compact and σ-compact. Then $X \times Y$ is CLP-rectangular.*

Proof. Fix open covers $\{B_n : n \in \omega\}$ of X and $\{D_n : n \in \omega\}$ of Y such that each $\mathrm{cl}(B_n)$ is compact and each $\mathrm{cl}(D_n)$ is compact. Let C be a clopen subset of $X \times Y$. Fix $(p, q) \in C$. We will find a clopen rectangle $U \times V$ such that $(p, q) \in U \times V \subseteq C$.

Construct open $U_n, S_n \subseteq X$ and $V_n, T_n \subseteq Y$ for every $n \in \omega$ so that the following conditions hold. In the end, set $U = \bigcup_{n \in \omega} U_n$ and $V = \bigcup_{n \in \omega} V_n$. It is not hard to check that U and V will have the required properties.

1. $U_0 \subseteq U_1 \subseteq \cdots$, $S_0 \subseteq S_1 \subseteq \cdots$, $V_0 \subseteq V_1 \subseteq \cdots$, $T_0 \subseteq T_1 \subseteq \cdots$.
2. $\mathrm{cl}(U_n) \cap \mathrm{cl}(S_n) = \varnothing$, $\mathrm{cl}(V_n) \cap \mathrm{cl}(T_n) = \varnothing$.
3. $(p, q) \in U_0 \times V_0$.
4. $\mathrm{cl}(U_n) \times \mathrm{cl}(V_n) \subseteq C$.
5. $\{x\} \times \mathrm{cl}(V_n) \nsubseteq C$ whenever $x \in S_n$, and $\mathrm{cl}(U_n) \times \{y\} \nsubseteq C$ whenever $y \in T_n$.

6. $\mathrm{cl}(U_n)$ and $\mathrm{cl}(V_n)$ are compact.

7. $B_i \subseteq U_{2i+1} \cup S_{2i+1}$, $D_i \subseteq V_{2i+2} \cup T_{2i+2}$.

Start the construction by choosing any U_0, V_0 that satisfy (3) and (4). Let S_0, T_0 be empty. Now assume that $n = 2i$ and U_n, S_n, V_n, T_n are given. Let $K = \{x \in X : \{x\} \times \mathrm{cl}(V_n) \subseteq C\}$. Using the fact that $\mathrm{cl}(V_n)$ is compact, one can easily show that K is clopen. Also, $U_n \subseteq K$ by (4), and $S_n \cap K = \varnothing$ by (5). So, let $U_{n+1} = (B_i \cap K) \cup U_n$ and $S_{n+1} = (B_i \setminus K) \cup S_n$. Finally, let $V_{n+1} = V_n$ and $T_{n+1} = T_n$. The case $n = 2i + 1$ is similar. $\square$

Corollary 2.12. *Let $X = X_1 \times \cdots \times X_n$. If each X_i is locally compact and σ-compact then X is CLP-rectangular.*

In Section 3.2, we will obtain a strengthening of Corollary 2.12 (see Corollary 3.16).

2.2 Finite products

For a nice introduction to $\beta\mathbb{N}$, see [50]. Let $\mathcal{F}$ be a family of non-empty closed subsets of $\mathbb{N}^*$. Define $X(\mathcal{F})$ as the topological space with underlying set $\mathbb{N} \cup \mathcal{F}$ and with the coarsest topology satisfying the following requirements.

- The singleton $\{n\}$ is open for every $n \in \mathbb{N}$.
- For every $K \in \mathcal{F}$, the set $\{K\} \cup A$ is open whenever $A \subseteq \mathbb{N}$ is such that $K \subseteq A^*$.

Our example will be obtained by setting $X_i = X(\mathcal{F}_i)$ for every $i \in \omega$, where $\{\mathcal{F}_i : i \in \omega\}$ is the family constructed in Theorem 2.20.

Proposition 2.13 (Steprāns and Šostak)**.** *Assume that $\mathcal{F}$ consists of pairwise disjoint sets. Then $X(\mathcal{F})$ is a Hausdorff space.*

Proof. Use the fact that disjoint closed sets in $\mathbb{N}^*$ can be separated by a clopen set. $\square$

From now on, we will always assume that $\mathcal{F}_i$ is a collection of non-empty pairwise disjoint closed subsets of $\mathbb{N}^*$ for every $i \in \omega$. Given $p \in {}^{<\omega}\omega$, we will denote by $n(p) \in \omega$ the domain of p. We will use the notation

$$X_p = \prod_{i \in n(p)} X(\mathcal{F}_{p(i)})$$

for finite products, where repetitions of factors are allowed. Also, if $i \in n(p)$, we will denote by X_{p-i} the subproduct $\prod_{j \in n(p)\setminus\{i\}} X(\mathcal{F}_{p(j)})$.

The following definitions isolate the multidimensional versions of 'finiteness' and 'cofiniteness' that we need. For every $p \in {}^{<\omega}\omega$ and $N \in \omega$, we will denote the union of the 'initial stripes of height N' as

$$S_p^N = \bigcup_{i \in n(p)} \{x \in X_p : x_i \in N\} \subseteq X_p.$$

Also define

$$T_p^N = X_p \setminus S_p^N = \prod_{i \in n(p)} (X(\mathcal{F}_{p(i)}) \setminus N).$$

Proposition 2.14. *Assume that for every $p \in {}^{<\omega}\omega$ and every clopen set $U \subseteq X_p$, either $U \subseteq S_p^N$ or $T_p^N \subseteq U$ for some $N \in \omega$. Then X_p is CLP-compact for every $p \in {}^{<\omega}\omega$.*

Proof. We will use induction on $n(p)$. The case $n(p) = 1$ is obvious. So assume that X_p is CLP-compact for every $p \in {}^{n}\omega$ and let $p \in {}^{n+1}\omega$. By Proposition 2.10, it is enough to prove that X_p is CLP-rectangular. So let $U \subseteq X_p$ be a clopen set and fix $N \in \omega$

such that $U \subseteq S_p^N$ or $T_p^N \subseteq U$. We will show that for every $x \in U$ there exists a clopen rectangle $R \subseteq X_p$ such that $x \in R \subseteq U$.

First we will assume that $x \in U$ has at least one coordinate in N, say coordinate $i \in n(p)$. It is easy to check that the cross-section

$$V = \{y \in X_{p-i} : y \cup \{(i, x_i)\} \in U\}$$

is clopen in X_{p-i}. Observe that X_{p-i} is homeomorphic to $X_{p'}$ for some $p' \in {}^n\omega$. Therefore, by the inductive hypothesis and Proposition 2.10, there exists a clopen rectangle $Q \subseteq X_{p-i}$ such that $\pi_{p-i}(x) \in Q \subseteq V$, where $\pi_{p-i} : X_p \longrightarrow X_{p-i}$ is the natural projection. It is clear that $R = \{y \in X_p : \pi_{p-i}(y) \in Q \text{ and } y_i = x_i\}$ is the desired clopen rectangle.

On the other hand, if $x \in U$ has no coordinate in N then the case $U \subseteq S_p^N$ is impossible. Therefore $T_p^N \subseteq U$, so that the desired clopen rectangle is $R = T_p^N$ itself. $\square$

We will also need the following definitions. Let

$$\mathbb{S}_p^N = S_p^N \cap (\mathbb{N}^{n(p)}) = \bigcup_{i \in n(p)} \{x \in \mathbb{N}^{n(p)} : x_i \in N\},$$

$$\mathbb{T}_p^N = T_p^N \cap (\mathbb{N}^{n(p)}) = \mathbb{N}^{n(p)} \setminus \mathbb{S}_p^N = (\mathbb{N} \setminus N)^{n(p)}.$$

The next two lemmas show that, in order to achieve what is required by Proposition 2.14, we can just look at the trace of clopen sets on $\mathbb{N}^{n(p)}$.

Lemma 2.15. *Fix $p \in {}^{<\omega}\omega$. Assume that $U \subseteq X_p$ is a clopen set such that $U \cap (\mathbb{N}^{n(p)}) \subseteq \mathbb{S}_p^N$. Then $U \subseteq S_p^N$.*

Proof. Assume, in order to get a contradiction, that $x \in U \setminus S_p^N$. For all $i \in n(p)$ such that $x_i \in \mathbb{N}$, let $N_i = \{x_i\} \subseteq \mathbb{N} \setminus N$. For all $i \in n(p)$ such that $x_i \in \mathcal{F}_{p(i)}$, let

$N_i = \{x_i\} \cup A_i$ be a neighborhood of x_i in $X(\mathcal{F}_{p(i)})$ such that $A_i \subseteq \mathbb{N} \setminus N$. Since U is open, by shrinking each N_i if necessary, we can make sure that $\prod_{i \in n(p)} N_i \subseteq U$. This is a contradiction, because $\varnothing \neq (\prod_{i \in n(p)} N_i) \cap (\mathbb{N}^{n(p)}) \subseteq \mathbb{T}_p^N$. $\square$

Similarly, one can prove the following.

Lemma 2.16. *Fix $p \in {}^{<\omega}\omega$. Assume that $U \subseteq X_p$ is a clopen set such that $\mathbb{T}_p^N \subseteq U \cap (\mathbb{N}^{n(p)})$. Then $T_p^N \subseteq U$.*

Fix $p \in {}^{<\omega}\omega$. We will say that a subset D of $\mathbb{N}^{n(p)}$ is *diagonal* if $D \nsubseteq \mathbb{S}_p^N$ and $\mathbb{T}_p^N \nsubseteq D$ for all $N \in \omega$ and the restriction $\pi_i \restriction D$ of the natural projection $\pi_i : \mathbb{N}^{n(p)} \longrightarrow \mathbb{N}$ is injective for every $i \in n(p)$.

Given $p \in {}^{<\omega}\omega$, we will say that a pair (D, E) is *p-diagonal* if D and E are both diagonal subsets of $\mathbb{N}^{n(p)}$. A pair (D, E) is *diagonal* if it is p-diagonal for some p as above. If (D, E) is such a pair, consider the following statement.

☕(D, E) There exist $K_0, \ldots, K_{n(p)-1}$, with $K_i \in \mathcal{F}_{p(i)}$ for every $i \in n(p)$, such that $D \cap (A_0 \times \cdots \times A_{n(p)-1})$ and $E \cap (A_0 \times \cdots \times A_{n(p)-1})$ are both non-empty whenever $A_0, \ldots, A_{n(p)-1} \subseteq \mathbb{N}$ satisfy $K_i \subseteq A_i^*$ for every $i \in n(p)$.

Proposition 2.17. *Fix $p \in {}^{<\omega}\omega$. Assume that the family $\{\mathcal{F}_i : i \in \omega\}$ is such that condition ☕(D, E) holds for every p-diagonal pair (D, E). If $U \subseteq X_p$ is a clopen set, then there exists $N \in \omega$ such that either $U \cap (\mathbb{N}^{n(p)}) \subseteq \mathbb{S}_p^N$ or $\mathbb{T}_p^N \subseteq U \cap (\mathbb{N}^{n(p)})$.*

Proof. Assume, in order to get a contradiction, that $U \cap (\mathbb{N}^{n(p)}) \nsubseteq \mathbb{S}_p^N$ and $\mathbb{T}_p^N \nsubseteq U \cap (\mathbb{N}^{n(p)})$ for every $N \in \omega$. Then it is possible to construct (in ω steps) diagonal subsets D and E of $\mathbb{N}^{n(p)}$ such that $D \subseteq U$ and $E \subseteq X_p \setminus U$.

Now let $K_0, \ldots, K_{n(p)-1}$ be as given by condition ☕(D, E). Define $x \in X_p$ by setting $x_i = K_i$ for every $i \in n(p)$. It is easy to see that $x \in \mathrm{cl}(U) \cap \mathrm{cl}(X_p \setminus U)$, which contradicts the fact that U is clopen. $\square$

Theorem 2.18. *Assume that the family $\{\mathcal{F}_i : i \in \omega\}$ is such that condition ☕(D, E) holds for every diagonal pair (D, E). Then X_p is CLP-compact for every $p \in {}^{<\omega}\omega$.*

Proof. We will show that the hypothesis of Proposition 2.14 holds. This follows from Proposition 2.17, Lemma 2.15 and Lemma 2.16. $\square$

2.3 The full product

In this section we will show how to ensure that $P = \prod_{i \in \omega} X(\mathcal{F}_i)$ is non-CLP-compact. Consider the following condition. Recall that a family $\mathcal{L}$ is *linked* if $K \cap L \neq \varnothing$ whenever $K, L \in \mathcal{L}$.

🚲 For every $I \in [\omega]^\omega$, the family $\mathcal{L} = \{x_i : i \in I\}$ is not linked for any $x \in \prod_{i \in I} \mathcal{F}_i$.

Theorem 2.19. *Assume that the family $\{\mathcal{F}_i : i \in \omega\}$ is such that condition 🚲 holds. Then $P = \prod_{i \in \omega} X(\mathcal{F}_i)$ can be written as the disjoint union of infinitely many of its non-empty clopen subsets.*

Proof. For each $n \in \omega$, define

$$U_n = \{x \in P : x_i = n \text{ whenever } 0 \leq i \leq n\}.$$

It is easy to check that each U_n is open (actually, clopen), non-empty, and that $U_i \cap U_j = \varnothing$ whenever $i \neq j$. Therefore we just need to show that $V = P \setminus \bigcup_{n \in \omega} U_n$ is open.

So fix $x \in V$ and consider $I = \{i \in \omega : x_i \notin \mathbb{N}\}$. First assume that I is finite. If there exist $i, j \notin I$, say with $i < j$, such that $x_i \neq x_j$ then $\{y \in P : y_i = x_i \text{ and } y_j = x_j\} \setminus \bigcup_{n \in j} U_n$ is an open neighborhood of x which is contained in V. So assume that $x_i = x_j$ whenever $i, j \notin I$. Since $x \in V$, we must have $I \neq \varnothing$. So fix $i \in I$ and $j \notin I$, say with $i < j$ (the other case is similar). Let $N_i = \{x_i\} \cup A_i$ be a neighborhood of x_i such that $x_j \notin A_i$. Then $\{y \in P : y_i \in N_i \text{ and } y_j = x_j\} \setminus \bigcup_{n \in j} U_n$ is an open neighborhood of x which is contained in V.

Finally, assume that I is infinite. An application of condition 🚲 yields $i, j \in I$, say with $i < j$, such that $x_i \cap x_j = \varnothing$. But disjoint closed sets in $\mathbb{N}^*$ can be separated by a clopen set, therefore we can find disjoint clopen neighborhoods N_i and N_j of x_i and x_j respectively. Then $\{y \in P : y_i \in N_i \text{ and } y_j \in N_j\} \setminus \bigcup_{n \in j} U_n$ is an open neighborhood of x which is contained in V. $\square$

2.4 The construction

The next theorem guarantees the existence of our example: finite products will be CLP-compact by Theorem 2.18, while the full product will be non-CLP-compact by Theorem 2.19.

Theorem 2.20. *There exists a family $\{\mathcal{F}_i : i \in \omega\}$ satisfying the following requirements.*

- *Each $\mathcal{F}_i$ consists of pairwise disjoint subsets of $\mathbb{N}^*$ of finite size.*
- *The condition ☕(D, E) holds for every diagonal pair (D, E).*
- *The condition 🚲 holds.*

Proof. Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all diagonal pairs, where D_η and E_η are both diagonal subsets of $\mathbb{N}^{n(p)}$ for some $p = p(\eta) \in {}^{<\omega}\omega$ with domain $n(p) = n(\eta) \in \omega$.

We will construct $\{\mathcal{F}_i : i \in \omega\}$ by transfinite recursion in $\mathfrak{c}$ steps: in the end we will set $\mathcal{F}_i = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_i^\xi$ for every $i \in \omega$. Start with $\mathcal{F}_i^0 = \varnothing$ for each i. By induction, we will make sure that the following requirements are satisfied.

1. $\mathcal{F}_i^\eta \subseteq \mathcal{F}_i^\mu$ whenever $\eta \leq \mu \in \mathfrak{c}$.

2. $|\bigcup_{i \in \omega} \bigcup \mathcal{F}_i^\eta| < 2^{\mathfrak{c}}$ for every $\eta \in \mathfrak{c}$.

3. The condition ☕(D_η, E_η), where (D_η, E_η) is a p-diagonal pair, is satisfied at stage $\xi = \eta + 1$: that is, the witness K_i is already in $\mathcal{F}_{p(i)}^{\eta+1}$ for each $i \in n(p)$.

At a limit stage ξ, just let $\mathcal{F}_i^\xi = \bigcup_{\eta \in \xi} \mathcal{F}_i^\eta$ for every $i \in \omega$.

At a successor stage $\xi = \eta + 1$, assume that $\mathcal{F}_i^\eta$ is given for each i. Let $p = p(\eta)$. First, define $W = \bigcup_{i \in \omega} \bigcup \mathcal{F}_i^\eta$ and observe that $|W| < 2^{\mathfrak{c}}$ by (2). Set $\tau_i = \pi_i \restriction D_\eta$ for every $i \in n(p)$. Since each τ_i is injective, it makes sense to consider the induced function $\tau_i^* : D_\eta^* \longrightarrow \mathbb{N}^*$. Recall that the explicit definition is given by

$$\tau_i^*(\mathcal{U}) = \{S \subseteq \mathbb{N} : \tau_i^{-1}[S] \in \mathcal{U}\}.$$

It is easy to check that each τ_i^* is injective. Therefore, since $|D_\eta^*| = 2^{\mathfrak{c}}$, it is possible to choose

$$\mathcal{U}^\eta \in D_\eta^* \setminus ((\tau_0^*)^{-1}[W] \cup \cdots \cup (\tau_{n(p)-1}^*)^{-1}[W]).$$

Let $\mathcal{U}_i^\eta = \tau_i^*(\mathcal{U}^\eta)$ for every $i \in n(p)$.

Now, define $Z = W \cup \{\mathcal{U}_i^\eta : i \in n(p)\}$. Set $\sigma_i = \pi_i \restriction E_\eta$ for every $i \in n(p)$. As above, it is possible to choose

$$\mathcal{V}^\eta \in E_\eta^* \setminus ((\sigma_0^*)^{-1}[Z] \cup \cdots \cup (\sigma_{n(p)-1}^*)^{-1}[Z]).$$

Let $\mathcal{V}_i^\eta = \sigma_i^*(\mathcal{V}^\eta)$ for every $i \in n(p)$.

We conclude the successor stage by setting

$$\mathcal{F}_k^{\eta+1} = \mathcal{F}_k^\eta \cup \{\{\mathcal{U}_i^\eta : i \in p^{-1}(k)\} \cup \{\mathcal{V}_i^\eta : i \in p^{-1}(k)\}\}$$

for every $k \in \operatorname{ran}(p)$ and $\mathcal{F}_k^{\eta+1} = \mathcal{F}_k^\eta$ for every $k \in \omega \setminus \operatorname{ran}(p)$.

Next, we will verify that condition 🚲 holds. Assume, in order to get a contradiction, that $I \in [\omega]^\omega$ and $x \in \prod_{i\in I} \mathcal{F}_i$ are such that $\mathcal{L} = \{x_i : i \in I\}$ is linked. Observe that the only possible equalities among points of $\mathbb{N}^*$ produced in our construction are those in the form $\mathcal{U}_i^\eta = \mathcal{U}_j^\eta$ or $\mathcal{V}_i^\eta = \mathcal{V}_j^\eta$ for some $i, j \in n(\eta)$. Therefore each element of $\mathcal{L}$ must have been added to some $\mathcal{F}_k$ at the same stage $\xi = \eta + 1$. Since I is infinite, we can pick $k \in I \setminus \operatorname{ran}(p(\eta))$. It is clear from the construction that $x_k \in \mathcal{L}$ cannot have been added to $\mathcal{F}_k$ at stage $\xi = \eta + 1$.

Finally, we will verify that (3) holds. For every $i \in n(p)$, set

$$K_i = \{\mathcal{U}_j^\eta : j \in p^{-1}(p(i))\} \cup \{\mathcal{V}_j^\eta : j \in p^{-1}(p(i))\} \in \mathcal{F}_{p(i)}^{\eta+1}.$$

Suppose $A_0, \ldots, A_{n(p)-1} \subseteq \mathbb{N}$ are such that $K_i \subseteq A_i^*$ for every $i \in n(p)$. In particular $A_i \in \mathcal{U}_i^\eta$ for every $i \in n(p)$. By the definition of the induced functions, we have $\tau_i^{-1}[A_i] \in \mathcal{U}^\eta$ for every $i \in n(p)$. Therefore

$$D_\eta \cap (A_0 \times \cdots \times A_{n(p)-1}) = D_\eta \cap \tau_0^{-1}[A_0] \cap \cdots \cap \tau_{n(p)-1}^{-1}[A_{n(p)-1}]$$

is non-empty. By the same argument, using $\mathcal{V}^\eta$, one can show that $E_\eta \cap (A_0 \times \cdots \times A_{n(p)-1})$ is non-empty. $\square$

2.5 Arbitrarily large products

The main idea behind the next theorem is due to Frolík (see the 'Proof of B (using B′)' in [23]). To show that X^ω can be written as the disjoint union of infinitely many of its non-empty clopen subsets, we will proceed as in the proof of Theorem 9.10 in [10].

Theorem 2.21. *There exists a Hausdorff space X such that X^n is CLP-compact for every $n \in \omega$, while X^ω can be written as the disjoint union of infinitely many of its non-empty clopen subsets.*

Proof. Let $\{\mathcal{F}_i : i \in \omega\}$ be the family given by Theorem 2.20 and set $X_i = X(\mathcal{F}_i)$ for every $i \in \omega$. It follows from Theorem 2.18 and Theorem 2.19 that $\{X_i : i \in \omega\}$ is a collection of Hausdorff spaces such that $X_p = \prod_{i \in n(p)} X_{p(i)}$ is CLP-compact for every $p \in {}^{<\omega}\omega$, while $\prod_{i\in\omega} X_i$ can be written as the disjoint union of infinitely many of its non-empty clopen subsets.

Define X as the topological space with underlying set the disjoint union $\{0\} \oplus X_0 \oplus X_1 \oplus \cdots$ and with the coarsest topology satisfying the following requirements.

- Whenever U is an open subset of X_i for some $i \in \omega$, the set U is also open in X.
- The tail $\{0\} \cup \bigcup_{i \leq j < \omega} X_j$ is open in X for every $i \in \omega$.

It is easy to check that X is Hausdorff.

We will prove that X^n is CLP-compact by induction on n. The case $n = 1$ is obvious. So assume that X^n is CLP-compact and consider a cover $\mathcal{C}$ of X^{n+1} consisting of clopen sets. Since

$$S = \bigcup_{i \in n+1} \{x \in X^{n+1} : x_i = 0\}$$

is a finite union of subspaces of X^{n+1} that are homeomorphic to X^n, there exist $\mathcal{D} \in [\mathcal{C}]^{<\omega}$ such that $S \subseteq \bigcup \mathcal{D}$. It follows that there exists $N \in \omega$ such that $X^{n+1} \setminus (X_0 \oplus \cdots \oplus X_{N-1})^{n+1} \subseteq \bigcup \mathcal{D}$. But

$$T = (X_0 \oplus \cdots \oplus X_{N-1})^{n+1}$$

is homeomorphic to a finite union of spaces of the form X_p for some $p \in {}^{n+1}\omega$, hence it is CLP-compact. Therefore there exists $\mathcal{E} \in [\mathcal{C}]^{<\omega}$ such that $T \subseteq \bigcup \mathcal{E}$. Hence $\mathcal{D} \cup \mathcal{E}$ is the desired finite subcover of $\mathcal{C}$.

Finally, we will show that X^ω can be written as the disjoint union of infinitely many of its non-empty clopen subsets by constructing a continuous surjection $f : X^\omega \longrightarrow \prod_{i \in \omega} X_i$. Since every X_i is clopen in X, we can get a continuous surjection $f_i : X \longrightarrow X_i$ by letting f_i be the identity on X_i and constant on $X \setminus X_i$. Now simply let $f = \prod_{i \in \omega} f_i$ (that is, for every $x \in X^\omega$, define $y = f(x)$ by setting $y_i = f_i(x_i)$ for every $i \in \omega$). $\square$

Corollary 2.22. *For every infinite cardinal κ, there exists a collection $\{X_\xi : \xi \in \kappa\}$ of Hausdorff spaces such that $\prod_{\xi \in F} X_\xi$ is CLP-compact for every $F \in [\kappa]^{<\omega}$, while $\prod_{\xi \in \kappa} X_\xi$ is non-CLP-compact.*

Proof. Let $X_\xi = X$ for every $\xi \in \kappa$, where X is the space given by Theorem 2.21. $\square$

Chapter 3

Products and h-homogeneity

Most of the results in this chapter originally appeared in [40], but we will give a more complete and systematic exposition here. For example, the concept of strong CLP-rectangularity is not explicitly treated in [40].

All spaces in this chapter are implicitly assumed to be Tychonoff. It is easy to see that every zero-dimensional space is Tychonoff. For all undefined topological notions, see [19]. For all undefined boolean algebraic notions, see [32]. Recall that a subset of a topological space is *clopen* if it is closed and open.

Definition 3.1. *A space X is* h-homogeneous *(or* strongly homogeneous*) if every non-empty clopen subset of X is homeomorphic to X.*

The Cantor set 2^ω, the rationals $\mathbb{Q}$ and the irrationals $\mathbb{R} \setminus \mathbb{Q} \approx \omega^\omega$ are examples of h-homogeneous spaces. This follows easily from their respective characterizations (see Theorem 1.5.5, Theorem 1.9.6 and Theorem 1.9.8 in [48]). Obviously, every connected space is h-homogeneous. Also, it is easy to see that any product of an h-homogeneous space with a connected space is h-homogeneous. A finite space is h-homogeneous if and only if it has size at most 1.

An interesting example of h-homogeneous space is given by the Knaster-Kuratowski fan K, which is hereditarily disconnected but not totally disconnected (see Proposition A.2).

Another interesting example of h-homogeneous space is the famous Erdős space. Let ℓ^2 be the Hilbert space consisting of all sequences $x \in \mathbb{R}^\omega$ such that $\sum_{n=0}^{\infty} x_n^2 < \infty$, with the distance induced by the usual norm (see Example 4.1.7 in [19]). The subspace

$$\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$$

of ℓ^2 is known as *Erdős space*. The space $\mathfrak{E}$ is totally disconnected but not zero-dimensional (see Example 1.5.18 in [48]). The fact that $\mathfrak{E}$ is h-homogeneous follows from a deep result due to Dijkstra and Van Mill (see the next paragraph).

In fact, the spaces $\mathbb{Q}$, ω^ω and $\mathfrak{E}$ are more than h-homogeneous: each one of their non-empty open sets is homeomorphic to the whole space (such spaces are called *divine* or *of diversity one*, see [58]). For $\mathbb{Q}$ and ω^ω, this follows from their characterizations, as above. For $\mathfrak{E}$, this is the statement of Corollary 8.15 in [13], which answers Question 6.8 from [58]. Since 2^ω has non-compact open subsets, it is not divine. (Actually, Schoenfeld and Gruenhage showed in [60] that 2^ω is the only compact metric space with exactly two homeomorphism classes of non-empty open sets.) The Knaster-Kuratowski fan K is not divine by Proposition 6.7 in [58].

One might wonder whether a dichotomy theorem with respect to connectedness holds for h-homogeneous spaces: is every h-homogeneous space either connected or hereditarily disconnected? It is not hard to see that the answer is 'no'. Let $\{q_n : n \in \omega\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and consider the subspace

$$Z = ([0,1] \times \{0\}) \cup \{(q_n, 2^{-n}) : n \in \omega\}$$

of $\mathbb{R}^2$. The space Z^ω is h-homogeneous by Theorem 3.31, and it is obviously disconnected but not hereditarily disconnected.

The notion of h-homogeneity was introduced 'instrumentally', independently by Van Mill in [45] and by Ostrovskiĭ in [55]. Subsequently, this concept has been studied by several authors: consider for example the seminal papers [51] and [52] by Motorov, the article [69] by Terada, and the article [37] by Matveev. We conclude this introduction with a list of results in the literature that involve h-homogeneity. More will be mentioned in the rest of this chapter.

Medvedev published several papers on h-homogeneous metric spaces X such that $\dim X = 0$. See for example the recent articles [42] and [43], where he employed h-homogeneity to generalize to the non-separable case results of Keldysh, and Van Douwen and Van Mill respectively. See also [44].

In [71], assuming CH, De la Vega constructed a zero-dimensional compact space that has a base consisting of pairwise homeomorphic clopen sets but is not h-homogeneous (this consistently answers Question 4 from [37]).

Let $H(X)$ denote the set of all homeomorphisms $f : X \longrightarrow X$. Balcar and Dow showed that, given an infinite extremally disconnected compact space X, the dynamical system $(X, H(X))$ is minimal if and only if X is h-homogeneous (see Theorem 13 in [3]).

Di Concilio showed that if X_i is zero-dimensional and h-homogeneous for every $i \in I$, then the upper semi-lattice $\mathcal{L}_H(X)$ of all admissible group topologies on $H(X)$, where $X = \prod_{i \in I} X_i$, is actually a complete lattice (see Theorem 3.8 in [12], whose proof can be simplified using our Corollary 3.27).

Let $\mathrm{Clop}(X)$ denote the Boolean algebra of the clopen subsets of X. Recall that a Boolean algebra $\mathcal{A}$ is *homogeneous* if $\mathcal{A} \upharpoonright a$ is isomorphic to $\mathcal{A}$ for every non-zero $a \in \mathcal{A}$, where $\mathcal{A} \upharpoonright a$ denotes the *relative algebra* $\{x \in \mathcal{A} : x \leq a\}$. If X is h-homogeneous then $\mathrm{Clop}(X)$ is homogeneous; the converse holds if X is compact and zero-dimensional (see the remarks following Definition 9.12 in [32]). Shelah and Geschke studied h-homogeneity from this point of view in [61].

3.1 The productivity of h-homogeneity, part I

In [69], the productivity of h-homogeneity is stated as an open problem (see also [37] and [38]), and it is shown that the product of zero-dimensional h-homogeneous spaces is h-homogeneous provided it is compact or non-pseudocompact (see Theorem 3.3 in [69]). The following theorem, proved by Terada under the additional assumption that X is zero-dimensional (see Theorem 2.4 in [69]), is the key ingredient in the proof. Recall that a collection $\mathcal{B}$ consisting of non-empty open subsets of a space X is a *π-base* if for every non-empty open subset U of X there exists $V \in \mathcal{B}$ such that $V \subseteq U$. (Notice that having a π-base consisting of clopen sets is strictly weaker than being zero-dimensional: consider for example the space Z described in the introduction to this chapter.)

Theorem 3.2 (Terada)**.** *Assume that X is non-pseudocompact. If X has a π-base consisting of clopen sets that are homeomorphic to X then X is h-homogeneous.*

The proof of Theorem 3.2 uses the fact that a zero-dimensional non-pseudocompact space can be written as the disjoint union of infinitely many of its non-empty clopen subsets (the converse is also true, trivially). However, that is the only consequence of zero-dimensionality that is actually used (see Appendix B). Therefore such assumption is redundant by the following lemma.

Lemma 3.3. *Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets. If X is non-pseudocompact then X can be written as the disjoint union of infinitely many of its non-empty clopen subsets.*

Proof. Let $f : X \longrightarrow \mathbb{R}$ be an unbounded continuous function. Then the range of f contains a countably infinite subset D that is closed and discrete as a subset of $\mathbb{R}$. Let $D = \{d_n : n \in \omega\}$ be an injective enumeration. Choose pairwise disjoint open sets U_n such that $d_n \in U_n$ for each $n \in \omega$. Then choose open sets V_n such that $d_n \in V_n \subseteq \mathrm{cl}(V_n) \subseteq U_n$ and $\mathrm{diam}(V_n) \leq 2^{-n}$ for each $n \in \omega$.

Next, we will show that $V = \bigcup_{n\in\omega} \mathrm{cl}(V_n)$ is closed. Pick $y \notin V$. Choose $N \in \omega$ big enough so that $2^{-N} < d(y, D)$ and let W be the open ball around y of radius $2^{-(N+1)}$. We claim that $W \cap V_n = \varnothing$ for every $n \geq N+1$. Otherwise, for an element z of such an intersection, we would have

$$d(y, d_n) \leq d(y, z) + d(z, d_n) \leq 2^{-(N+1)} + 2^{-(N+1)} = 2^{-N} < d(y, D),$$

which is a contradiction. So $W \setminus (\mathrm{cl}(V_0) \cup \cdots \cup \mathrm{cl}(V_N))$ is an open neighborhood of y that is disjoint from V.

Finally, fix $B_n \in \mathcal{B}$ such that $B_n \subseteq f^{-1}[V_n]$ for each $n \in \omega$. We will show that $B = \bigcup_{n\in\omega} B_n$ is clopen, concluding the proof. Pick $x \notin B$. If $x \in f^{-1}[U_n]$ for some $n \in \omega$, then $f^{-1}[U_n] \setminus B_n$ is an open neighborhood of x that is disjoint from B. Now assume that $x \notin \bigcup_{n\in\omega} f^{-1}[U_n]$. Then $y = f(x) \notin V$, so we can find an open neighborhood W of y that is disjoint from V. It is clear that $f^{-1}[W]$ is an open neighborhood of x that is disjoint from B. □

Using Theorem 3.2 one can easily prove the following.

Theorem 3.4 (Terada). *Assume that $X = \prod_{i\in I} X_i$ is non-pseudocompact. If X_i is h-homogeneous and it has a π-base consisting of clopen sets for every $i \in I$ then X is h-homogeneous.*

In the next section we will introduce the tools that (together with Glickberg's Theorem 3.20) will ultimately allow us to prove Theorem 3.25, which complements Theorem 3.4. In the process, we will also generalize and exploit some results from Section 2.1, in order to obtain further partial results on products of h-homogeneous spaces (see Corollary 3.17 and Corollary 3.15).

3.2 Strong CLP-rectangularity

Definition 3.5. *A product space $X = \prod_{i \in I} X_i$ is* strongly CLP-rectangular *if every clopen subset of X can be written as the union of a collection of pairwise disjoint clopen rectangles.*

The following theorem will be the key to showing that h-homogeneity is preserved under products in certain cases.

Theorem 3.6. *Let X_i be h-homogeneous for every $i \in I$. If C is a non-empty clopen subset of $X = \prod_{i \in I} X_i$ that can be written as the disjoint union of clopen rectangles then $C \approx X$.*

Proof. Let C be a non-empty clopen subset of X such that $C = \bigcup \mathcal{R}$, where $\mathcal{R}$ is a collection of pairwise disjoint non-empty clopen rectangles. Observe that $R \approx X$ for every $R \in \mathcal{R}$. In particular the result is trivial if $|\mathcal{R}| = 1$. If $|\mathcal{R}| > 1$ then X is disconnected, so X_i is disconnected for some $i \in I$. Since X_i is also h-homogeneous, it follows that $X_i \approx n \times X_i$ whenever $1 \leq n < \omega$. Therefore $X \times n \approx X$ whenever $1 \leq n < \omega$. So the theorem is established for $|\mathcal{R}| < \omega$.

Finally, assume that $\kappa = |\mathcal{R}|$ is an infinite cardinal. Then

$$\begin{aligned} C &\approx \kappa \times X \\ &\approx (X \setminus C) \oplus C \oplus (\kappa \times X) \\ &\approx (X \setminus C) \oplus (\kappa \times X) \\ &\approx X, \end{aligned}$$

which concludes the proof. □

Corollary 3.7. *Let X_i be h-homogeneous for every $i \in I$. If $X = \prod_{i \in I} X_i$ is strongly CLP-rectangular then X is h-homogeneous.*

Trivially, every strongly CLP-rectangular product space is CLP-rectangular. The next theorem shows that, in some cases, the reverse implication holds. However, the reverse implication does not hold in general (see Proposition 3.18). We will need the following obvious definition and a cute (even if I say so myself) 'geometric' lemma, that will also be useful in Section 3.3.

Definition 3.8. *A space X is* CLP-Lindelöf *if every cover of X consisting of clopen sets has a countable subcover.*

Lemma 3.9. *Assume that C is a clopen subset of $X \times Y$ that can be written as a boolean combination of finitely many rectangles. Then C can be written as the union of finitely many pairwise disjoint clopen rectangles.*

Proof. For every $x \in X$, let $C_x = \{y \in Y : (x, y) \in C\}$ be the corresponding vertical cross-section. For every $y \in Y$, let $C^y = \{x \in X : (x, y) \in C\}$ be the corresponding horizontal cross-section. Since C is clopen, each cross-section is clopen.

Let $\mathcal{A}$ be the Boolean subalgebra of $\mathrm{Clop}(X)$ generated by $\{C^y : y \in Y\}$. Since $\mathcal{A}$ is finitely generated, it is finite (see Corollary 4.5 in [32]). Hence $\mathcal{A}$ is isomorphic to the power set algebra of a finite set (see Corollary 2.8 in [32]). Let $P_1, \ldots, P_m$ be the atoms of $\mathcal{A}$. Similarly, let $\mathcal{B}$ be the Boolean subalgebra of $\mathrm{Clop}(Y)$ generated by $\{C_x : x \in X\}$, and let $Q_1, \ldots, Q_n$ be the atoms of $\mathcal{B}$.

Observe that the rectangles $P_i \times Q_j$ are clopen and pairwise disjoint. Furthermore, given any i, j, either $P_i \times Q_j \subseteq C$ or $(P_i \times Q_j) \cap C = \varnothing$. Hence C is the union of a (finite) collection of such rectangles. □

Corollary 3.10. *Assume that C is a clopen subset of $\prod_{i \in I} X_i$ that can be written as a boolean combination of finitely many rectangles. Then C can be written as the union of finitely many pairwise disjoint clopen rectangles.*

Proof. The case $|I| < \omega$ can be established by induction starting from Lemma 3.9. The case $|I| \geq \omega$ follows, because any boolean combination of finitely many rectangles only depends on finitely many coordinates. □

Theorem 3.11. *Assume that $X = \prod_{i \in I} X_i$ is CLP-Lindelöf and let C be a clopen subset of X. If C can be written as the union of clopen rectangles them C can be written as the union of a countable collection of pairwise disjoint open rectangles.*

Proof. Assume that $C = \bigcup \mathcal{Q}$, where $\mathcal{Q}$ is a collection of clopen rectangles. Since X is CLP-Lindelöf, we can assume without loss of generality that $\mathcal{Q} = \{Q_n : n \in \omega\}$ is countable.

By Corollary 3.10, given any $n \in \omega$, it is possible to write

$$Q_n \setminus (Q_0 \cup \cdots \cup Q_{n-1}) = \bigcup \mathcal{R}_n,$$

where $\mathcal{R}_n$ is a finite collection of pairwise disjoint clopen rectangles in X. Clearly $\mathcal{R} = \bigcup_{n\in\omega} \mathcal{R}_n$ is the desired countable collection. □

Corollary 3.12. *If $X = \prod_{i\in I} X_i$ is CLP-Lindelöf and CLP-rectangular then X is strongly CLP-rectangular.*

Corollary 3.13. *Let X_i be h-homogeneous for every $i \in I$. If $X = \prod_{i\in I} X_i$ is CLP-Lindelöf and CLP-rectangular then X is h-homogeneous.*

Corollary 3.14. *Let X_i be h-homogeneous for every $i \in I$. If $X = \prod_{i\in I} X_i$ is CLP-compact then X is h-homogeneous.*

Proof. Simply apply Proposition 2.10 and Corollary 3.13. □

Corollary 3.15. *Let $X = X_1 \times \cdots \times X_n$. If each X_i is h-homogeneous, CLP-compact and sequential then X is h-homogeneous.*

Proof. Notice that X is CLP-compact by Theorem 2.2. □

For example, it follows from Corollary 3.15 that every finite power of the Knaster-Kuratowski K fan is h-homogeneous (see also Question 4.16).

Combining Kunen's result from Section 2.1 with the above theorem, we will obtain a further partial result on products of h-homogeneous spaces. Notice that the following is an improvement of Corollary 2.12, where we weaken the assumption of σ-compactness to paracompactness, and we strengthen the conclusion of CLP-rectangularity to strong CLP-rectangularity.

Corollary 3.16. *Let $X = X_1 \times \cdots \times X_n$. If each X_i is locally compact and paracompact then X is strongly CLP-rectangular.*

Proof. By the proof of Theorem 5.1.27 in [19], each X_i is the disjoint sum of σ-compact clopen sets. So X will be the disjoint sum of clopen sets in the form $S = S_1 \times \cdots \times S_n$, where each $S_i \subseteq X_i$ is locally compact and σ-compact. Fix one such product S. Notice that S is CLP-rectangular by Corollary 2.12. Furthermore, since S is σ-compact, it is Lindelöf, hence CLP-Lindelöf. So S is strongly CLP-rectangular by Corollary 3.12, which concludes the proof. $\square$

Corollary 3.17. *Let $X = X_1 \times \cdots \times X_n$. If each X_i is h-homogeneous, locally compact and paracompact then X is h-homogeneous.*

We conclude this section with the promised counterexample. For all the set-theoretic notions, see [34].

Proposition 3.18 (Kunen)**.** *There exists a product space $X \times Y$ that is CLP-rectangular but not strongly CLP-rectangular.*

Proof. Let κ be a regular uncountable cardinal with the order topology. Let X and Y be disjoint stationary subsets of κ. One can easily check that the set

$$C = \{(x, y) \in X \times Y : x < y\} = (X \times Y) \cap \{(x, y) \in \kappa^2 : x \leq y\}$$

consisting of the points above the diagonal is clopen. Since X and Y are both zero-dimensional, the set C can be written as the union of clopen rectangles.

Assume, in order to get a contradiction, that $C = \bigcup \mathcal{R}$, where $\mathcal{R}$ is a collection of pairwise disjoint clopen rectangles in $X \times Y$. Whenever $(x, y) \in C$, we will denote by $R_{(x,y)}$ the unique element of $\mathcal{R}$ that contains (x, y). Fix $x \in X$. For every $y \in Y$ such that $y > x$, choose $\beta_{(x,y)} < y$ such that $\{x\} \times ((\beta_{(x,y)}, y] \cap Y) \subseteq R_{(x,y)}$. By the Pressing Down Lemma, there must be a stationary subset $T_x \subseteq Y$ of κ and $\beta_x \in \kappa$ such that

$\beta_{(x,y)} = \beta_x$ for every $y \in T_x$. Now let y_x be the least element of T_x and let $R_x = R_{(x,y_x)}$. Since the elements of $\mathcal{R}$ are pairwise disjoint, it follows that

$$\{x\} \times ((\beta_x, \kappa) \cap Y) \subseteq R_x$$

for every $x \in X$.

For every x in X let $\alpha_x < x$ be such that $((\alpha_x, x] \cap X) \times ((\beta_x, \kappa) \cap Y) \subseteq R_x$. By the Pressing Down Lemma, there must be a stationary subset $S \subseteq X$ of κ and $\alpha \in \kappa$ such that $\alpha_x = \alpha$ for every $x \in S$.

Finally, fix $x_0 \in S$ and pick $x_1 \in S$ such that $x_0 < y_{x_0} \leq x_1 < y_{x_1}$. Observe that $(x_0, y_{x_1}) \in R_{x_0}$ because $\beta_{x_0} < y_{x_0} \leq y_{x_1}$. On the other hand, $(x_0, y_{x_1}) \in R_{x_1}$ because $\alpha_{x_1} = \alpha_{x_0} < x_0 \leq x_1$ and $\beta_{x_1} < y_{x_1}$. Therefore R_{x_0} and R_{x_1} are the same rectangle R. Since $(x_0, y_{x_0}) \in R_{x_0}$ and $(x_1, y_{x_1}) \in R_{x_1}$, it follows that $(x_1, y_{x_0}) \in R \subseteq C$, which contradicts the definition of C. $\square$

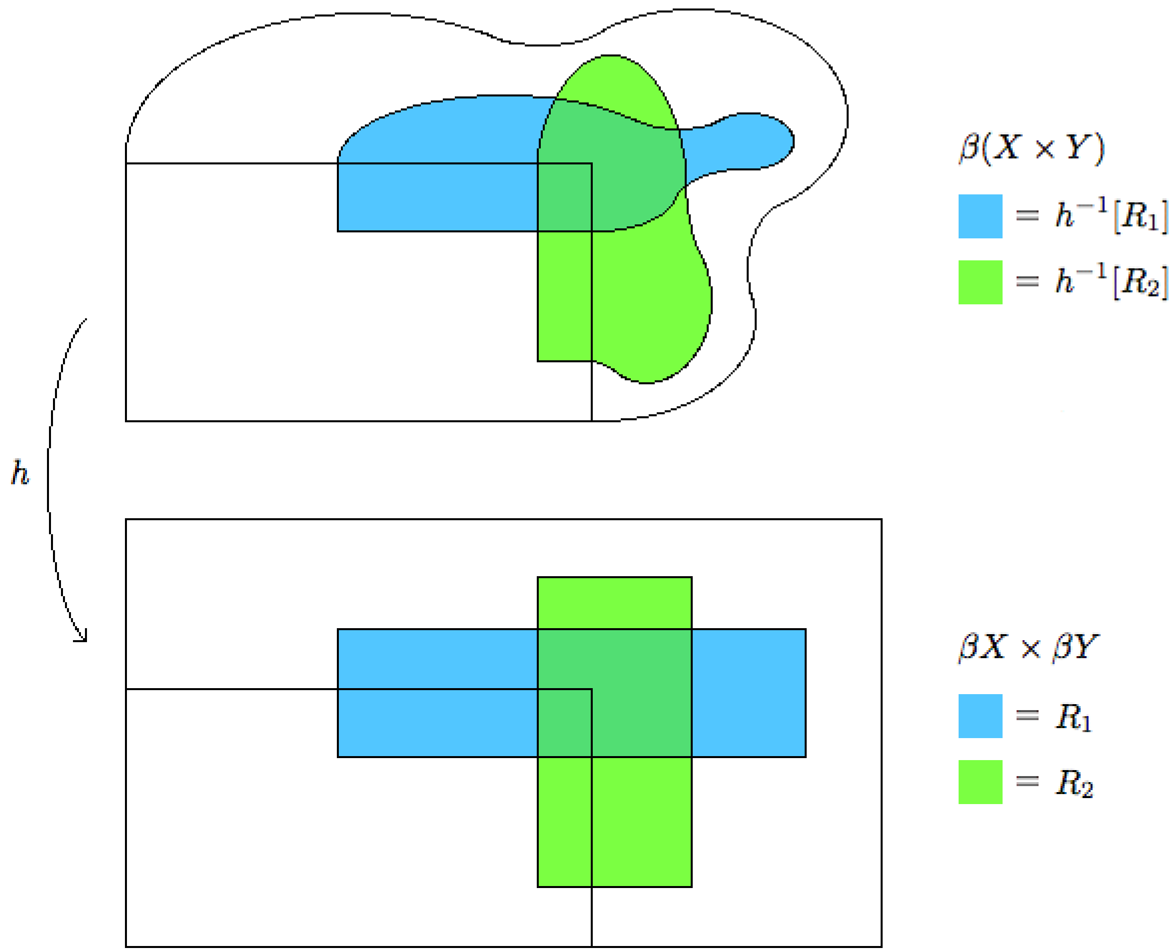
3.3 The productivity of h-homogeneity, part II

For a nice introduction to βX, see [50]. Given any open subset U of X, define $\mathrm{Ex}(U) = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U)$. For proofs of the following facts, see Appendix C.

- $\mathrm{Ex}(U)$ is the biggest open subset of βX such that its intersection with X is U.
- If C is a clopen subset of X then $\mathrm{Ex}(C) = \mathrm{cl}_{\beta X}(C)$, hence $\mathrm{Ex}(C)$ is clopen in βX.
- The collection $\{\mathrm{Ex}(U) : U \text{ is open in } X\}$ is a base for βX.

We remark that it is not true that βX is zero-dimensional whenever X is zero-dimensional (see Example 6.2.20 in [19] or Example 3.39 in [72]). If βX is zero-dimensional then X is called *strongly zero-dimensional*.

Figure 3.19.



A picture proof of Proposition 3.21 in the case $n = 2$*. Given a clopen set* $C \subseteq X \times Y$*, consider its clopen extension* $\mathrm{Ex}(C) \subseteq \beta(X \times Y)$*. Notice that it does not make sense to talk about rectangles in* $\beta(X \times Y)$*. However, Glicksberg's theorem allows us to work in* $\beta X \times \beta Y$ *instead. By compactness,* $h[\mathrm{Ex}(C)] = R_1 \cup R_2$*. The fact that* h *is the identity on* $X \times Y$ *concludes the proof.*

We will need the following theorem (see Theorem 8.25 in [72]), which originally appeared in [24]. Recall that a subspace Y of X is C^*-*embedded* in X if every bounded continuous function $f : Y \longrightarrow \mathbb{R}$ admits a continuous extension to X.

Theorem 3.20 (Glicksberg)**.** *Assume that $X = \prod_{i\in I} X_i$ is pseudocompact. Then X is C^*-embedded in $\prod_{i\in I} \beta X_i$.*

The reverse implication is also true, under the additional assumption that $\prod_{j\neq i} X_j$ is infinite for every $i \in I$. Such assumption is clearly not needed in the above statement (see Proposition 8.2 in [72]).

Proposition 3.21. *Assume that $X \times Y$ is pseudocompact. If C is a clopen subset of $X \times Y$ then C can be written as the union of finitely many open rectangles.*

Proof. First observe that $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$ by Theorem 3.20. Therefore, by the universal property of the Čech-Stone compactification (see Corollary 3.6.3 in [19]), there exists a homeomorphism $h : \beta(X \times Y) \longrightarrow \beta X \times \beta Y$ such that $h \restriction (X \times Y)$ is the identity.

Let C be a clopen subset of $X \times Y$. Notice that $h[\mathrm{Ex}(C)]$ will be a clopen subset of $\beta X \times \beta Y$. Since $\{\mathrm{Ex}(U) : U \text{ is open in } X\}$ is a base for βX and $\{\mathrm{Ex}(V) : V \text{ is open in } Y\}$ is a base for βY, the collection

$$\mathcal{B} = \{\mathrm{Ex}(U) \times \mathrm{Ex}(V) : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a base for $\beta X \times \beta Y$. Therefore, by compactness, it is possible to write $h[\mathrm{Ex}(C)] = R_1 \cup \cdots \cup R_n$ for some $n \in \omega$, where $R_i = \mathrm{Ex}(U_i) \times \mathrm{Ex}(V_i) \in \mathcal{B}$ for each i.

Finally, since $h^{-1} \restriction (X \times Y)$ is the identity, we get

$$\begin{aligned} C &= \mathrm{Ex}(C) \cap (X \times Y) \\ &= h^{-1}[R_1 \cup \cdots \cup R_n] \cap (X \times Y) \\ &= (R_1 \cup \cdots \cup R_n) \cap (X \times Y) \end{aligned}$$

$$\begin{aligned} &= (R_1 \cap (X \times Y)) \cup \cdots \cup (R_n \cap (X \times Y)) \\ &= (U_1 \times V_1) \cup \cdots \cup (U_n \times V_n), \end{aligned}$$

which concludes the proof. □

An obvious modification of the above proof yields the following.

Proposition 3.22. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If C is a clopen subset of X then C can be written as the union of finitely many open rectangles.*

Corollary 3.23. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If C is a clopen subset of X then C depends on finitely many coordinates.*

We remark that the zero-dimensional case of Corollary 3.23 is a trivial consequence of a result by Broverman (see Theorem 2.6 in [7]).

Corollary 3.24. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. Then X is strongly CLP-rectangular.*

Proof. Apply Proposition 3.22 and Corollary 3.10. □

Theorem 3.25. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If X_i is h-homogeneous for every $i \in I$ then X is h-homogeneous.*

Proof. Apply Corollary 3.24 and Corollary 3.7. □

Theorem 3.26. *If X_i is h-homogeneous and it has a π-base consisting of clopen sets for every $i \in I$ then $X = \prod_{i \in I} X_i$ is h-homogeneous.*

Proof. If X is pseudocompact, apply Theorem 3.25; if X is non-pseudocompact, apply Theorem 3.4. □

Corollary 3.27. *If X_i is h-homogeneous and zero-dimensional for every $i \in I$ then $\prod_{i \in I} X_i$ is h-homogeneous.*

A natural problem that remains open is whether the zero-dimensionality requirement can be dropped in the above corollary (see Question 4.15).

3.4 Some applications

The compact case of the following result was essentially proved by Motorov (see Theorem 0.2(9) in [52] and Theorem 2 in [51]).

Theorem 3.28. *Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets. Then $Y = (X \times 2 \times \prod \mathcal{B})^\kappa$ is h-homogeneous for every infinite cardinal κ.*

Proof. One can easily check that Y has a π-base consisting of clopen sets that are homeomorphic to Y. Therefore, if Y is non-pseudocompact, the result follows from Theorem 3.2.

On the other hand, an analysis of Motorov's proof shows that the only consequence of the compactness of Y that is used is the fact that clopen sets in Y depend on finitely many coordinates. Therefore the same proof works if Y is pseudocompact by Corollary 3.23. We reproduce such proof for the convenience of the reader.

Assume that Y is pseudocompact and let C be a non-empty clopen subset of Y. The fact that C depends on finitely many coordinates (see Corollary 3.23) implies that $C \approx Y \times C$. So it will be enough to show that $Y \times C \approx Y$.

Let B be a clopen subset of C that is homeomorphic to Y. Let $D = C \setminus B$ and $E = (Y \setminus C) \oplus B$. Observe that $Y \approx Y^2 \approx (Y \times D) \oplus (Y \times E)$ and that $Y \oplus Y \approx 2 \times Y \approx Y$.

Therefore

$$\begin{aligned} Y \times C &\approx (Y \times D) \oplus (Y \times B) \\ &\approx (Y \times D) \oplus Y^2 \\ &\approx (Y \times D) \oplus ((Y \times D) \oplus (Y \times E)) \\ &\approx ((Y \oplus Y) \times D) \oplus (Y \times E) \\ &\approx (Y \times D) \oplus (Y \times E) \\ &\approx Y, \end{aligned}$$

which concludes the proof. □

Corollary 3.29. *For every non-empty zero-dimensional space X there exists a non-empty zero-dimensional space Y such that $X \times Y$ is h-homogeneous. Furthermore, if X is compact, then Y can be chosen to be compact.*

In [70], using a very brief and elegant argument, Uspenskiĭ proved that for every non-empty space X there exists a non-empty space Y such that $X \times Y$ is homogeneous. However, it is not true that Y can be chosen to be compact whenever X is compact: Motorov proved that the closure in the plane of $\{(x, \sin(1/x)) : x \in (0, 1]\}$ is not the retract of any compact homogeneous space (see Section 3 in [2] for a proof). Corollary 3.29 is an h-homogeneous analog of Uspenskiĭ's result. We do not know whether it holds in the non-zero-dimensional case (see Question 4.18).

The following was proved by Matveev (see Proposition 3 in [37]) under the additional assumption that X is zero-dimensional, even though such assumption is not actually used in the proof (see Appendix B). Recall that a sequence $(A_n : n \in \omega)$ of subsets of a space X *converges to a point* x if for every neighborhood U of x there exists $N \in \omega$ such that

$A_n \subseteq U$ for each $n \geq N$.

Theorem 3.30 (Matveev)**.** *Assume that X has a π-base consisting of clopen sets that are homeomorphic to X. If there exists a sequence $(U_n : n \in \omega)$ of non-empty open subsets of X that converges to a point then X is h-homogeneous.*

The case $\kappa = \omega$ of the following result is an easy consequence of Theorem 3.30. Motorov first proved it under the additional assumption that X is a zero-dimensional first-countable compact space (see Theorem 0.2(2) in [52] and Theorem 1 in [51]). Terada proved it for an arbitrary infinite κ, under the additional assumption that X is zero-dimensional and non-pseudocompact (see Corollary 3.2 in [69]).

Theorem 3.31. *Assume that X is a space such that the isolated points are dense. Then X^κ is h-homogeneous for every infinite cardinal κ.*

Proof. We will show that X^ω is h-homogeneous and it has a π-base consisting of clopen sets. Since $X^\kappa \approx (X^\omega)^\kappa$ for every infinite cardinal κ, an application of Theorem 3.26 will conclude the proof.

Let D be the set of isolated points of X and let $\mathrm{Fn}(\omega, D)$ be the set of finite partial functions from ω to D. Given $s \in \mathrm{Fn}(\omega, D)$, define $U_s = \{f \in X^\omega : f \supseteq s\}$. Now fix $d \in D$ and let $g \in X^\omega$ be the constant function with value d. It is easy to see that $(U_{g \restriction n} : n \in \omega)$ is a sequence of open sets in X^ω that converges to g. Furthermore $\mathcal{B} = \{U_s : s \in \mathrm{Fn}(\omega, D)\}$ is a π-base for X^ω consisting of clopen sets that are homeomorphic to X^ω. So X^ω is h-homogeneous by Theorem 3.30. $\square$

3.5 Infinite powers of zero-dimensional first-countable spaces

The following proposition is folklore, and it explains why h-homogeneous spaces are sometimes called 'strongly homogeneous'. Recall that a space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Proposition 3.32. *Assume that X is a zero-dimensional first-countable space. If X is h-homogeneous then X is homogeneous.*

Proof. Assume that X is h-homogeneous. Fix $x, y \in X$. Choose local bases $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ at x and y respectively so that the following conditions are satisfied.

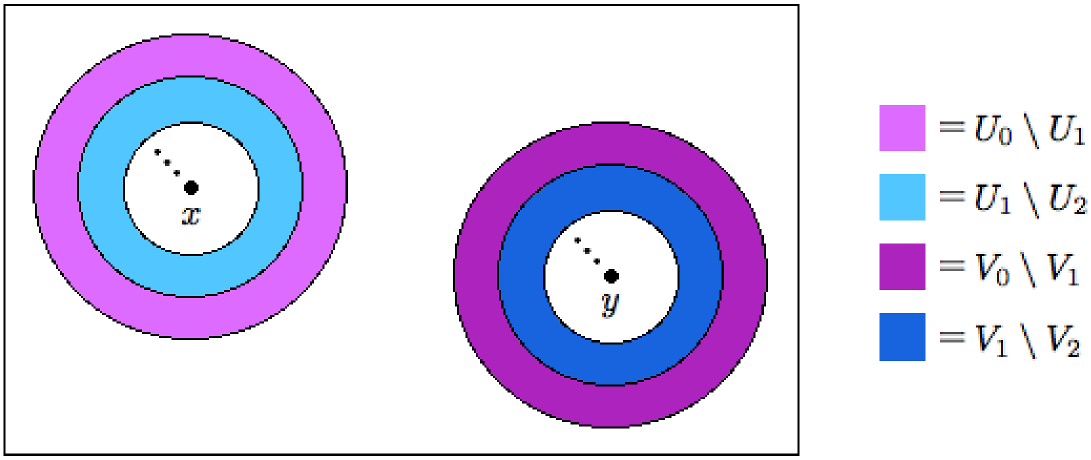
- All U_n and V_n are clopen.
- $U_0 \cap V_0 = \varnothing$.
- $U_{n+1} \subsetneq U_n$ and $V_{n+1} \subsetneq V_n$ for every $n \in \omega$.

By h-homogeneity, for every $n \in \omega$, there exists a homeomorphism

$$h_n : U_n \setminus U_{n+1} \longrightarrow V_n \setminus V_{n+1}.$$

It is easy to see that $\bigcup_{n \in \omega}(h_n \cup h_n^{-1})$ can be extended to a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$ and $h(y) = x$. $\square$

Figure 3.33.



A picture proof of Proposition 3.32. Swapping the bright clopen sets with the dark ones and extending by continuity yields the desired homeomorphism.

As announced by Motorov (see Theorem 0.1 in [52]), the converse of Proposition 3.32 holds for zero-dimensional first-countable compact spaces of uncountable cellularity (see Theorem 2.5 in [61] for a proof). The space $(\omega_1 + 1)^\omega$ is h-homogeneous by Theorem 3.31 but not homogeneous, because it is first-countable at some points but not at others. Another such example is given by $\mathbb{N}^*$, which is not homogeneous because it contains weak P-points (see [33]).

In [15], Van Douwen constructed a zero-dimensional first-countable compact homogeneous space X that is not h-homogeneous (actually, X has no proper subspaces that are homeomorphic to X). In [53], using similar techniques, Motorov constructed a zero-dimensional first-countable compact homogeneous space that is not divisible by 2 (in the sense of Definition 3.35); see also Theorem 7.7 in [47].

In [69], Terada asked whether X^ω is h-homogeneous for every zero-dimensional first-countable space X. In [16], the following remarkable theorem is proved.

Theorem 3.34 (Dow and Pearl). *If X is a zero-dimensional first-countable space then X^ω is homogeneous.*

However, Terada's question remains open, even for separable metric spaces (but see Corollary 3.43 and the remarks before Proposition 3.42). In [51] and [52], Motorov asks whether such an infinite power is always divisible by 2. Using Theorem 3.34, we will show that the two questions are equivalent: actually even weaker conditions suffice (see Proposition 3.38).

Definition 3.35. *A space F is a* factor *of X (or X is* divisible *by F) if there exists Y such that $F \times Y \approx X$. If $F \times X \approx X$ then F is a* strong factor *of X (or X is* strongly divisible *by F).*

We will use the following lemma freely in the rest of this section.

Lemma 3.36. *The following are equivalent.*

1. *F is a factor of X^ω.*
2. *F is a strong factor of X^ω.*
3. *F^ω is a strong factor of X^ω.*

Proof. The implications (2) $\to$ (1) and (3) $\to$ (1) are clear.

Assume that (1) holds. Then there exists Y such that $F \times Y \approx X^\omega$, hence

$$X^\omega \approx (X^\omega)^\omega \approx (F \times Y)^\omega \approx F^\omega \times Y^\omega.$$

Since multiplication by F or by F^ω does not change the right-hand side, it follows that (2) and (3) hold. $\square$

Lemma 3.37. *Assume that Y is a non-empty zero-dimensional first-countable space. Then $X = (Y \oplus 1)^\omega$ is h-homogeneous and $X \approx Y^\omega \times (Y \oplus 1)^\omega \approx 2^\omega \times Y^\omega$.*

Proof. Recall that $1 = \{0\}$ and let $g \in X$ be the constant function with value 0. For each $n \in \omega$, define

$$U_n = \{f \in X : f(i) = 0 \text{ for all } i < n\}.$$

Observe that $\mathcal{B} = \{U_n : n \in \omega\}$ is a local base for X at g consisting of clopen sets that are homeomorphic to X. But X is homogeneous by Theorem 3.34, therefore it has such a local base at every point. In conclusion X has a base (hence a π-base) consisting of clopen sets that are homeomorphic to X. It follows from Theorem 3.30 that X is h-homogeneous.

To prove the second statement, observe that

$$X \approx (Y \oplus 1) \times X \approx (Y \times X) \oplus X,$$

hence $X \approx Y \times X$ by h-homogeneity. It follows that $X \approx Y^\omega \times (Y \oplus 1)^\omega$. Finally,

$$Y^\omega \times (Y \oplus 1)^\omega \approx (Y^\omega \times (Y \oplus 1))^\omega \approx (Y^\omega \oplus Y^\omega)^\omega \approx 2^\omega \times Y^\omega,$$

that concludes the proof. □

Proposition 3.38. *Assume that X is a zero-dimensional first-countable space containing at least two points. Then the following are equivalent.*

1. *$X^\omega \approx (X \oplus 1)^\omega$.*
2. *$X^\omega \approx Y^\omega$ for some space Y with at least one isolated point.*
3. *X^ω is h-homogeneous.*

4. X^ω *has a non-empty clopen subset that is strongly divisible by* 2*.*

5. X^ω *has a proper clopen subset that is homeomorphic to* X^ω*.*

6. X^ω *has a proper clopen subset that is a factor of* X^ω*.*

Proof. The implication (1) $\to$ (2) is trivial; the implication (2) $\to$ (3) follows from Lemma 3.37; the implications (3) $\to$ (4) $\to$ (5) $\to$ (6) are trivial.

Assume that (6) holds. Let C be a proper clopen subset of X^ω that is a factor of X^ω and let $D = X^\omega \setminus C$. Then

$$\begin{aligned} X^\omega &\approx (C \oplus D) \times X^\omega \\ &\approx (C \times X^\omega) \oplus (D \times X^\omega) \\ &\approx X^\omega \oplus (D \times X^\omega) \\ &\approx (1 \oplus D) \times X^\omega, \end{aligned}$$

hence $X^\omega \approx (1 \oplus D)^\omega \times X^\omega$. Since $(1 \oplus D)^\omega \approx 2^\omega \times D^\omega$ by Lemma 3.37, it follows that 2^ω is a factor of X^ω. So 2^ω is a strong factor of X^ω. Therefore (1) holds by Lemma 3.37. $\square$

The next two propositions show that in the pseudocompact case we can say something more.

Proposition 3.39. *Assume that* X *is a zero-dimensional first-countable space such that* X^ω *is pseudocompact. Then* $C^\omega \approx (X \oplus 1)^\omega$ *for every non-empty proper clopen subset* C *of* X^ω*.*

Proof. Let C be a non-empty proper clopen subset of X^ω. It follows from Corollary 3.23 that $C \approx C \times X^\omega$, hence $C^\omega \approx C^\omega \times X^\omega$. Since $C^\omega \times X^\omega$ clearly has a proper clopen subset that is homeomorphic to $C^\omega \times X^\omega$, Proposition 3.38 implies that C^ω is h-homogeneous, hence strongly divisible by 2. So $C^\omega \approx 2^\omega \times C^\omega \approx 2^\omega \times C^\omega \times X^\omega$. Since $2^\omega \times X^\omega \approx (X \oplus 1)^\omega$ by Lemma 3.37, it follows that $C^\omega \approx C^\omega \times (X \oplus 1)^\omega$.

On the other hand, $(X \oplus 1)^\omega \approx X^\omega \times (X \oplus 1)^\omega$ by Lemma 3.37. Hence $(X \oplus 1)^\omega$ has a clopen subset homeomorphic to $C \times (X \oplus 1)^\omega$. But Lemma 3.37 shows that $(X \oplus 1)^\omega$ is h-homogeneous, so $C \times (X \oplus 1)^\omega \approx (X \oplus 1)^\omega$. Therefore $C^\omega \times (X \oplus 1)^\omega \approx (X \oplus 1)^\omega$, that concludes the proof. □

Proposition 3.40. *In addition to the hypotheses of Proposition 3.38, assume that X^ω is pseudocompact. Then the following can be added to the list of equivalent conditions.*

(7) X^ω has a non-empty proper clopen subset that is homeomorphic to Y^ω for some space Y.

Proof. The implication (5) $\to$ (7) is trivial.

Assume that (7) holds. Let C be a non-empty proper clopen subset of X^ω that is homeomorphic to Y^ω for some space Y. Then clearly $C^\omega \approx C$. Therefore $C \approx (X \oplus 1)^\omega$ by Proposition 3.39. Hence C is strongly divisible by 2 by Lemma 3.37, showing that (4) holds. □

Finally, we point out that Proposition 3.38 can be used to give a positive answer to Terada's question for a certain class of spaces. We will need the following definition.

Definition 3.41. *A space X is* ultraparacompact *if every open cover of X has a refinement consisting of pairwise disjoint clopen sets.*

It is easy to see that every ultraparacompact space is zero-dimensional. As noted by Nyikos in [54], a space is ultraparacompact if and only if it is paracompact and strongly zero-dimensional (this is proved like Proposition 1.2 in [17]). A metric space X is ultraparacompact if and only if $\dim X = 0$ (see Theorem 7.2.4 in [19]); see also Theorem 7.3.3 in [19]. For such a metric space X, Van Engelen proved that X^ω is h-homogeneous if X is meager in itself or X has a completely metrizable dense subset (see Theorem 4.2 and Theorem 4.4 in [18]). It follows that X^ω is h-homogeneous if X is a subset with the property of Baire in the restricted sense of some completely metrizable space (see Corollary D.2). In particular, this holds if X is in the σ-algebra generated by the analytic subsets of some completely metrizable space (see Corollary D.4). Similar result were obtained independently by Ostrovskiĭ (see Theorem 8 and Theorem 9 in [56]). See also the discussion that follows Corollary 3.43. However, according to Medvedev, the proof of Theorem 4.3 in [18] contains a gap (which he fixed: see Remark 4 in [43]).

Proposition 3.42. *Assume that X is a (zero-dimensional) first-countable space. If X^ω is ultraparacompact and non-Lindelöf then X^ω is h-homogeneous.*

Proof. Let $\mathcal{U}$ be an open cover of X^ω with no countable subcovers. By ultraparacompactness, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of pairwise disjoint non-empty clopen sets. Let $\mathcal{V} = \{C_\alpha : \alpha \in \kappa\}$ be an enumeration without repetitions, where κ is an uncountable cardinal.

Now fix $x \in X^\omega$ and a local base $\{U_n : n \in \omega\}$ at x consisting of clopen sets. Since X^ω is homogeneous by Theorem 3.34, for each $\alpha < \kappa$ we can find $n(\alpha) \in \omega$ such that a homeomorphic clopen copy D_α of $U_{n(\alpha)}$ is contained in C_α. Since κ is uncountable, there exists an infinite $S \subseteq \kappa$ such that $n(\alpha) = n(\beta)$ for every $\alpha, \beta \in S$. It is easy to check that $\bigcup_{\alpha \in S} D_\alpha$ is a non-empty clopen subset of X^ω that is strongly divisible by 2. Therefore X^ω is h-homogeneous by Proposition 3.38. $\square$

An application of Corollary 4.1.16, Theorem 7.3.2 and Theorem 7.3.16 in [19] immediately yields the following result.

Corollary 3.43. *Assume that X is a metric space such that* $\dim X = 0$*. If X is non-separable then X^ω is h-homogeneous.*

We conclude this chapter with two remarks about Theorem 25 in [43], which was obtained independently by Medvedev and overlaps with the above corollary. The first remark is that the statement of such theorem contains an inaccuracy: to make the proof work, X should be required to have the property of Baire *in the restricted sense* (see Appendix D), not just the property of Baire. In fact, the property of Baire is not enough to obtain the dichotomy given by Proposition D.1 (see the remark preceding it). The second remark is that by Corollary 3.43, in the non-separable case, no additional requirement on X is needed anyway.

Chapter 4

Open problems

4.1 The topology of ultrafilters as subspaces of the Cantor set

In this section, we will follow the same conventions of Chapter 1. Since most of our main constructions need some additional set-theoretic assumption (namely, Martin's Axiom for countable posets), in each case it is natural to ask whether such assumption can be dropped. Questions 4.1, 4.2 and 4.3 first appeared as Questions 2, 3 and 7 respectively in [41]. As we explained in Section 1.8, it follows from a result of Shelah that it is not possible to construct in ZFC an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is completely Baire.

Question 4.1. *Is it possible to construct in* ZFC *an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is not countable dense homogeneous?*[1]

Question 4.2. *Is it possible to construct in* ZFC *an ultrafilter $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous?*

Question 4.3. *Is it possible to construct in* ZFC *an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property whenever A is an analytic subset of 2^ω?*

[1] *Very recently, Repovš, Zdomskyy and Zhang constructed in* ZFC *a non-meager filter $\mathcal{F} \subseteq 2^\omega$ that is not countable dense homogeneous (see [59]).*

The case of countable dense homogeneity seems particularly interesting because at this point it is still conceivable that there exists a model of ZFC in which *every* ultrafilter is countable dense homogeneous and one in which *no* ultrafilter is countable dense homogeneous. Certainly, by Theorem 1.15 and Theorem 1.21, there exists a model of ZFC in which both kinds of ultrafilter exist (take any model of MA(countable)).

Regarding the relation between P-points and the topological properties that we investigated, we remark that the following four questions remain open (see the discussion in Section 1.8). Questions 4.4, 4.6 and 4.7 first appeared as Question 11, the first half of Question 12 and the second half of Question 12 respectively in [41].

Question 4.4. *Is it possible to prove in* ZFC *that every* P*-point* $\mathcal{U} \subseteq 2^\omega$ *is countable dense homogeneous?*[2]

Question 4.5. *Is it possible to prove in* ZFC *that every countable dense homogeneous ultrafilter* $\mathcal{U} \subseteq 2^\omega$ *is a* P*-point?*

Question 4.6. *Is it possible to prove in* ZFC *that if* $\mathcal{U} \subseteq 2^\omega$ *is a* P*-point then* $A \cap \mathcal{U}$ *has the perfect set property whenever* A *is an analytic subset of* 2^ω*?*

Question 4.7. *Is it possible to prove in* ZFC *that if* $\mathcal{U} \subseteq 2^\omega$ *is an ultrafilter such that* $A \cap \mathcal{U}$ *has the perfect set property whenever* A *is an analytic subset of* 2^ω *then* $\mathcal{U}$ *is a* P*-point?*

Observe that Lemma 1.33 might be viewed as a partial answer to Question 4.6.

We conclude this section by mentioning two isolated questions that remain open. The first one aims at 'optimizing the definability' of X in Question 1.23, and it first appeared as Question 6 in [41].

[2] *While this thesis was being written, Hernández-Gutiérrez and Hrušák showed that* $\mathcal{F}$ *and* $\mathcal{F}^\omega$ *are both countable dense homogeneous whenever* $\mathcal{F} \subseteq 2^\omega$ *is a non-meager* P*-filter (see [25]).*

Question 4.8. *Is it consistent that there exists an analytic non-G_δ subset X of 2^ω such that X^ω is countable dense homogeneous? Co-analytic?*

We are only asking for a consistency result because under Projective Determinacy every projective countable dense homogeneous space is completely metrizable (see Corollary 2.7 in [26]).

The second one is about the exact consistency strength of an ultrafilter like the one given by Theorem 1.36, and it first appeared as part of Question 9 in [41].

Question 4.9. *Does the Levy collapse* $\mathrm{Col}(\omega,\kappa)$ *of an inaccessible cardinal κ to ω_1 force the existence an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $A \cap \mathcal{U}$ has the perfect set property for every $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$?*

Notice that Theorem 1.36 implies that such consistency strength is between that of an inaccessible cardinal and a Mahlo cardinal. If the answer to the above question were 'yes', then it would be exactly that of an inaccessible cardinal.

4.2 Products and CLP-compactness

As we mentioned in the introduction to Chapter 2, very little is known about infinite products when it comes to CLP-compactness. Even the following question (which is Question 6.4 in [68]), arguably the 'easiest possible', is still open.

Question 4.10 (Steprāns and Šostak)**.** *Let $X = \prod_{i\in\omega} X_i$. If each X_i is CLP-compact and second-countable, must X be CLP-compact?*

To look at a concrete example: Theorem 2.2 implies that every finite power K^n of the

Knaster-Kuratowski fan is CLP-compact (hence CLP-rectangular by Proposition 2.10), but the following question remains open.

Question 4.11. *For which infinite cardinals κ is K^κ CLP-compact (equivalently, CLP-rectangular)?*

As observed by Glicksberg in [24] (see also Proposition 8.26 in [72]), a product space is pseudocompact if and only if every countable subproduct is pseudocompact. Does the analogous theorem hold for CLP-compactness?

Question 4.12. *Let κ be an uncountable cardinal. If $\{X_\xi : \xi \in \kappa\}$ is a collection of spaces such that $\prod_{\xi\in I} X_\xi$ is CLP-compact for every $I \in [\kappa]^{<\omega_1}$ does it follow that $\prod_{\xi\in\kappa} X_\xi$ is CLP-compact? Does this depend on the size of κ?*

Notice that the above might be viewed as the natural 'sequel' to Question 2.5.

Finally, it is natural to investigate whether the separation properties of our counterexample can be improved.

Question 4.13. *In Corollary 2.22, can 'Hausdorff' be substituted by 'regular'?*

We will conclude this section by showing that our example is indeed not regular. Since for zero-dimensional spaces CLP-compactness is equivalent to compactness, it will be enough to prove that every locally countable regular space is zero-dimensional. This follows from the fact that countable regular spaces are zero-dimensional (see Corollary 6.2.8 in [19]) and the next proposition.

Proposition 4.14. *Assume that X is locally zero-dimensional and regular. Then X is zero-dimensional.*

Proof. Fix $x \in X$. We will show that the clopen neighborhoods of x form a local base of X at x. Let U be an open zero-dimensional neighborhood of x. By regularity, there exists an open subset V of X such that $x \in V \subseteq \mathrm{cl}_X(V) \subseteq U$. By zero-dimensionality, there exists a clopen subset C of U such that $x \in C \subseteq V$.

We claim that C is clopen in X. The fact that C is open in X is obvious. To see that C is closed in X, notice that $\mathrm{cl}_X(C) \subseteq \mathrm{cl}_X(V) \subseteq U$, and therefore $C = \mathrm{cl}_U(C) = \mathrm{cl}_X(C) \cap U = \mathrm{cl}_X(C)$. □

4.3 Products and h-homogeneity

In this section we implicitly assume that all spaces are Tychonoff. All the work done in Chapter 3 notwithstanding, it is still an open problem (even in the case of finite products) whether h-homogeneity is productive. Terada originally posed the problem in [69], but he was including zero-dimensionality in the definition of h-homogeneity. The question in its full generality first appeared in [40].

Question 4.15. *If X_i is h-homogeneous for every $i \in I$, does it follow that $X = \prod_{i \in I} X_i$ is h-homogeneous?*

Observe that any counterexample X must be non-pseudocompact (by Theorem 3.25) and non-zero-dimensional (actually, by Theorem 3.26, it cannot even have a π-base consisting of clopen sets). Also keep in mind that Theorem 3.6, Corollary 3.15 and Corollary 3.17 hold.

Once again, the Knaster-Kuratowski fan K might be an interesting test space. Recall that K^n is h-homogeneous for every $n \in \omega$ by Corollary 3.15. However, we do not know the answer to the following question.

Question 4.16. *For which infinite cardinals κ is K^κ h-homogeneous?*

Recall that $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$ is the Erdős space. It is easy to see that $\mathfrak{E}^n \approx \mathfrak{E}$ for every $n \in \omega$. It is also true, but hard to prove, that $\mathfrak{E}^\omega \approx \mathfrak{E}$ (see Corollary 9.4 in [13]). Therefore $\mathfrak{E}^\kappa$ is h-homogeneous whenever $\kappa \leq \omega$ by Corollary 8.15 in [13]. However, the following question remains open.

Question 4.17. *For which uncountable cardinals κ is $\mathfrak{E}^\kappa$ h-homogeneous?*

Another result which we would like to generalize to the non-zero-dimensional case is Corollary 3.29. The following question first appeared in [40].

Question 4.18. *Is it true that for every non-empty space X there exists a non-empty space Y such that $X \times Y$ is h-homogeneous? If X is compact, can Y be chosen to be compact?*

Together with Question 4.15, we believe the following to be the most interesting open problem on the topic of h-homogeneity, and Section 3.5 is best viewed as a collection of partial results towards its solution.

Question 4.19 (Terada)**.** *Is X^ω h-homogeneous whenever X is a zero-dimensional first-countable space?*

If one drops the 'h', then the answer is 'yes' by Theorem 3.34, which was proved by Dow and Pearl in [16]. Since h-homogeneity implies homogeneity for zero-dimensional first-countable spaces (see Proposition 3.32), a positive answer would give a strenghtening of their result. Proposition 3.38 suggests a strategy for answering 'yes' to Question 4.19: one could try to use the techniques of [16] to show that condition (1) actually holds for *every* zero-dimensional first-countable space X. Also observe that the following question of Motorov (see [51] and [52]) is equivalent to Question 4.19 by Proposition 3.38.

Question 4.20 (Motorov)**.** *Is X^ω divisible by 2 whenever X is a zero-dimensional first-countable space containing at least two points?*

In the same articles, Motorov asked whether the 2 can be dropped from the definition of Y in his version of Theorem 3.28. Inspecting our proof reveals that this is possible if Y is non-pseudocompact, but we do not know the answer in general.

Question 4.21. *Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets. Does it follow that $Y = (X \times \prod \mathcal{B})^\kappa$ is h-homogeneous for every infinite cardinal κ?*

Observe that if the answer were 'yes' then Theorem 3.31 would become an immediate corollary of Theorem 3.28.

Appendix A

The Knaster-Kuratowski fan

In this section, we will discuss a space introduced by Knaster and Kuratowski in [31], which turns out to be an interesting example of h-homogeneous CLP-compact space (see Proposition A.2). This example is also known as *Cantor's teepee* (see Example 129 in [65]). The exposition is based on Exercise 6.3.23 in [19].

We will work in the plane $\mathbb{R}^2$. Identify 2^ω with the usual Cantor set in $[0,1]\times\{0\}$. Fix a countable dense subset Q of 2^ω, and let $P = 2^\omega \setminus Q$. Let $q = (1/2, 1/2)$. Given $c \in 2^\omega$, denote by L_c the half-open line segment connecting c and q, including c but excluding q. Given $c \in 2^\omega$, define

$$K_c = \begin{cases} \{(x,y) \in L_c : y \in \mathbb{Q}\} & \text{if } c \in Q, \\ \{(x,y) \in L_c : y \in \mathbb{R}\setminus\mathbb{Q}\} & \text{if } c \in 2^\omega \setminus Q. \end{cases}$$

Define $K = \bigcup_{c\in 2^\omega} K_c$ and $F = K \cup \{q\}$. We will refer to K as the *Knaster-Kuratowski fan*, even though Engelking uses that name for F, because K is much more interesting than F for our purposes (see Proposition A.3).

Lemma A.1. *Assume that C is a clopen subset of K. Then there exists a clopen set $\widetilde{C}$ in 2^ω such that $C = \bigcup_{c\in\widetilde{C}} K_c$.*

Proof. Let $A = A_0$ and $B = A_1$ be closed subsets of $\mathbb{R}^2$ such that $A \cap K = C$ and $B \cap K = K \setminus C$. Let $\mathbb{Q} \cap [0, 1/2) = \{q_n : n \in \omega\}$ and let $H_n = \{(x, q_n) : x \in \mathbb{R}\}$ for every

$n \in \omega$. Also define

$$M_n = \{c \in 2^\omega : A \cap B \cap L_c \cap H_n \neq \varnothing\} = \{c \in 2^\omega : |A \cap B \cap L_c \cap H_n| = 1\}$$

for every $n \in \omega$. Notice that $M_n \subseteq P$ for every $n \in \omega$, because $A \cap B \cap K = \varnothing$. It is easy to check that each M_n is closed nowhere dense in 2^ω. Since P is dense in 2^ω, it follows that each M_n is closed nowhere dense in P as well. Therefore $P \setminus M$ is dense in P by Baire's category theorem, where $M = \bigcup_{n \in \omega} M_n$. So $P \setminus M$ is dense in 2^ω.

Fix $c \in P \setminus M$. First, we will show that A and B induce a clopen partition of $L_c \approx [0, 1)$. and suppose that $(x, y) \in L_c$. If $y = q_n \in \mathbb{Q}$, then (x, y) cannot belong to both A and B, otherwise we would have $c \in M_n$. If $y \in \mathbb{R} \setminus \mathbb{Q}$ then (x, y) cannot belong to both A and B, otherwise we would have $A \cap B \cap K \neq \varnothing$. On the other hand $L_c \subseteq \mathrm{cl}(K) \subseteq \mathrm{cl}(A \cup B) = A \cup B$. So L_c is the disjoint union of $A \cap L_c$ and $B \cap L_c$.

By connectedness, it follows that for each $c \in P \setminus M$ there exists $\varepsilon(c) \in \{0, 1\}$ such that $K_c \subseteq A_{\varepsilon(c)}$. Since $P \setminus M$ is dense in 2^ω, it follows that the same holds for every $c \in 2^\omega$. In the end, simply let $\widetilde{C} = \{c \in 2^\omega : K_c \subseteq A\}$. $\square$

Proposition A.2. *The Knaster-Kuratowski fan K has the following properties.*

1. *K is hereditarily disconnected.*

2. *K is not totally disconnected.*

3. *K does not have a π-base consisting of clopen sets.*

4. *K is not pseudocompact.*

5. *K is CLP-compact.*

6. *K is h-homogeneous.*

Proof. By Lemma A.1, the quasicomponents of K are in the form K_c. So (2) holds. Also, it follows immediately from Lemma A.1 that (3) holds. Let S be a connected component of K. Since every component is included in some quasicomponent, the above paragraph shows that $S \subseteq K_c$ for some $c \in 2^\omega$. So (1) holds, because each K_c is clearly hereditarily disconnected.

For metric spaces, compactness is equivalent to pseudocompactness (see Proposition 3.10.21 and Theorem 4.1.17 in [19]). So (4) holds. Since 2^ω is CLP-compact, it follows immediately from Lemma A.1 that (5) holds.

To show that (6) holds, let C be non-empty clopen subsets of K. Let $\widetilde{C}$ be given by Lemma A.1. Then $\widetilde{C}$ is a non-empty clopen subset of 2^ω containing $\widetilde{C} \cap Q$ as a countable dense subset. Since 2^ω is h-homogeneous and countable dense homogeneous (see Theorem 1.6.9 in [48]), there exists a homeomorphism $h : \widetilde{C} \longrightarrow 2^\omega$ such that $h[\widetilde{C} \cap Q] = Q$. It is easy to see that h induces a homeomorphism between C and K. $\square$

Proposition A.3. *The space F is connected.*

Proof. Let C be a clopen subset of F. Then $D_1 = C \setminus \{q\}$ and $D_2 = K \setminus C$ are clopen subset of K. By Lemma A.1, there exist clopen subsets $\widetilde{D_1}$ and $\widetilde{D_2}$ of 2^ω such that each $D_i = \bigcup_{c \in \widetilde{D_i}} K_c$. So, if D_1 and D_2 were both non-empty, we would have $q \in C \cap (F \setminus C)$. It follows that $C = \varnothing$ or $C = F$. $\square$

Appendix B

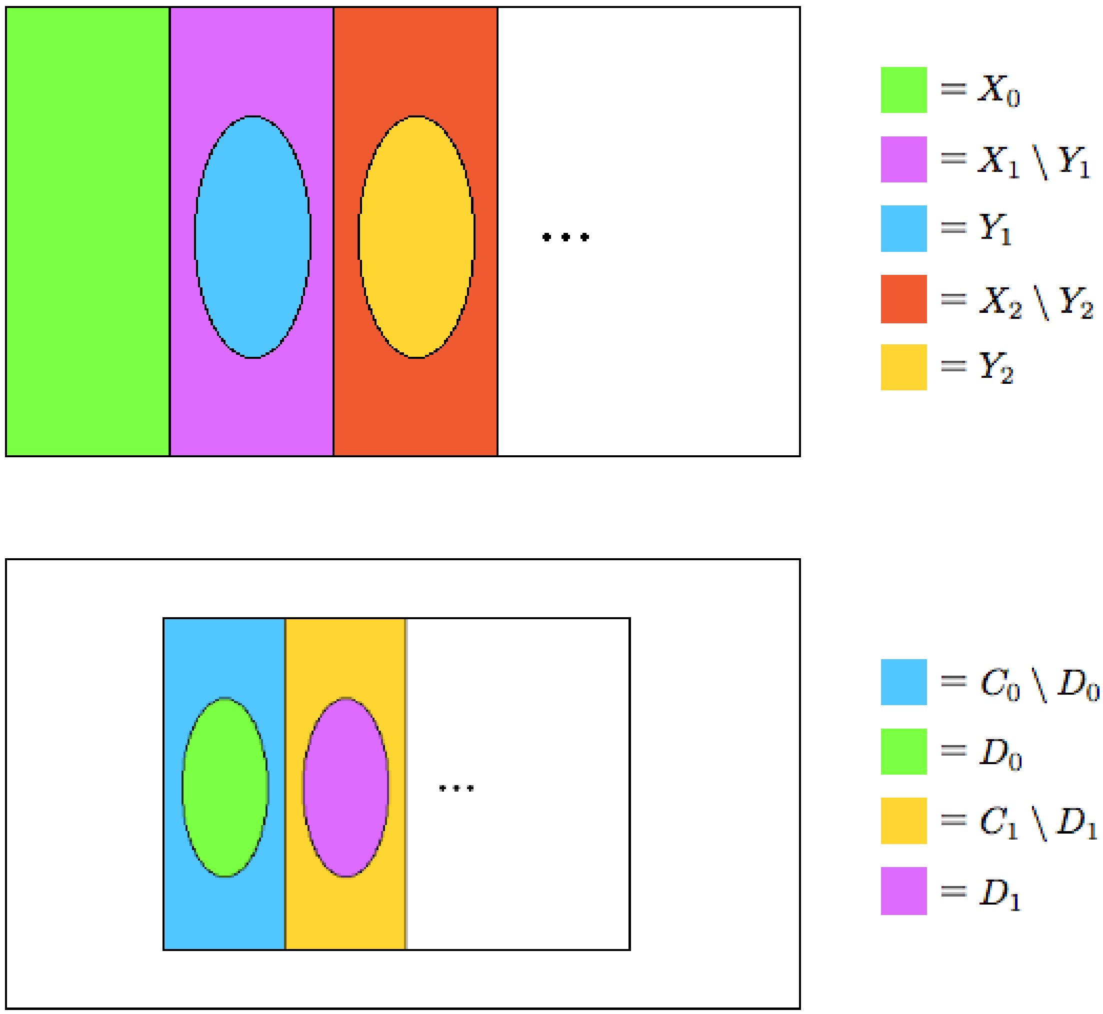
Proofs of the results by Terada and Matveev

In this section we will present a somewhat unified approach to the proofs of Theorem 3.2 and Theorem 3.30. Notice that zero-dimensionality is never needed.

Proof of Theorem 3.2. Assume that X has a π-base consisting of clopen sets that are homeomorphic to X. By Lemma 3.3, we can fix a collection $\{X_n : n \in \omega\}$ consisting of pairwise disjoint non-empty clopen subsets of X such that $X = \bigcup_{n\in\omega} X_n$. Let C be a non-empty clopen subset of X. Since C contains a clopen subset that is homeomorphic to X, we can fix a collection $\{C_n : n \in \omega\}$ consisting of pairwise disjoint non-empty clopen subsets of C such that $C = \bigcup_{n\in\omega} C_n$.

We will recursively construct clopen sets $Y_n \subseteq X_n$ and $D_n \subseteq C_n$, together with partial homeomorphisms h_n and k_n for every $n \in \omega$. In the end, setting $h = \bigcup_{n\in\omega}(h_n \cup k_n)$ will yield the desired homeomorphism. Start by setting $Y_0 = \varnothing$ and $h_0 = \varnothing$. Then, let $D_0 \subseteq C_0$ be a clopen set that is homeomorphic to $X_0 \setminus Y_0$ and fix a homeomorphism $k_0 : X_0 \setminus Y_0 \longrightarrow D_0$. Now assume that clopen sets $D_n \subseteq C_n$ and $Y_n \subseteq X_n$ have been defined. Let $Y_{n+1} \subseteq X_{n+1}$ be a clopen set that is homeomorphic to $C_n \setminus D_n$ and fix a homeomorphism $h_{n+1} : Y_{n+1} \longrightarrow C_n \setminus D_n$. Then, let $D_{n+1} \subseteq C_{n+1}$ be a clopen set that is homeomorphic to $X_{n+1} \setminus Y_{n+1}$ and fix a homeomorphism $k_{n+1} : X_{n+1} \setminus Y_{n+1} \longrightarrow D_{n+1}$. $\square$

Figure B.1.

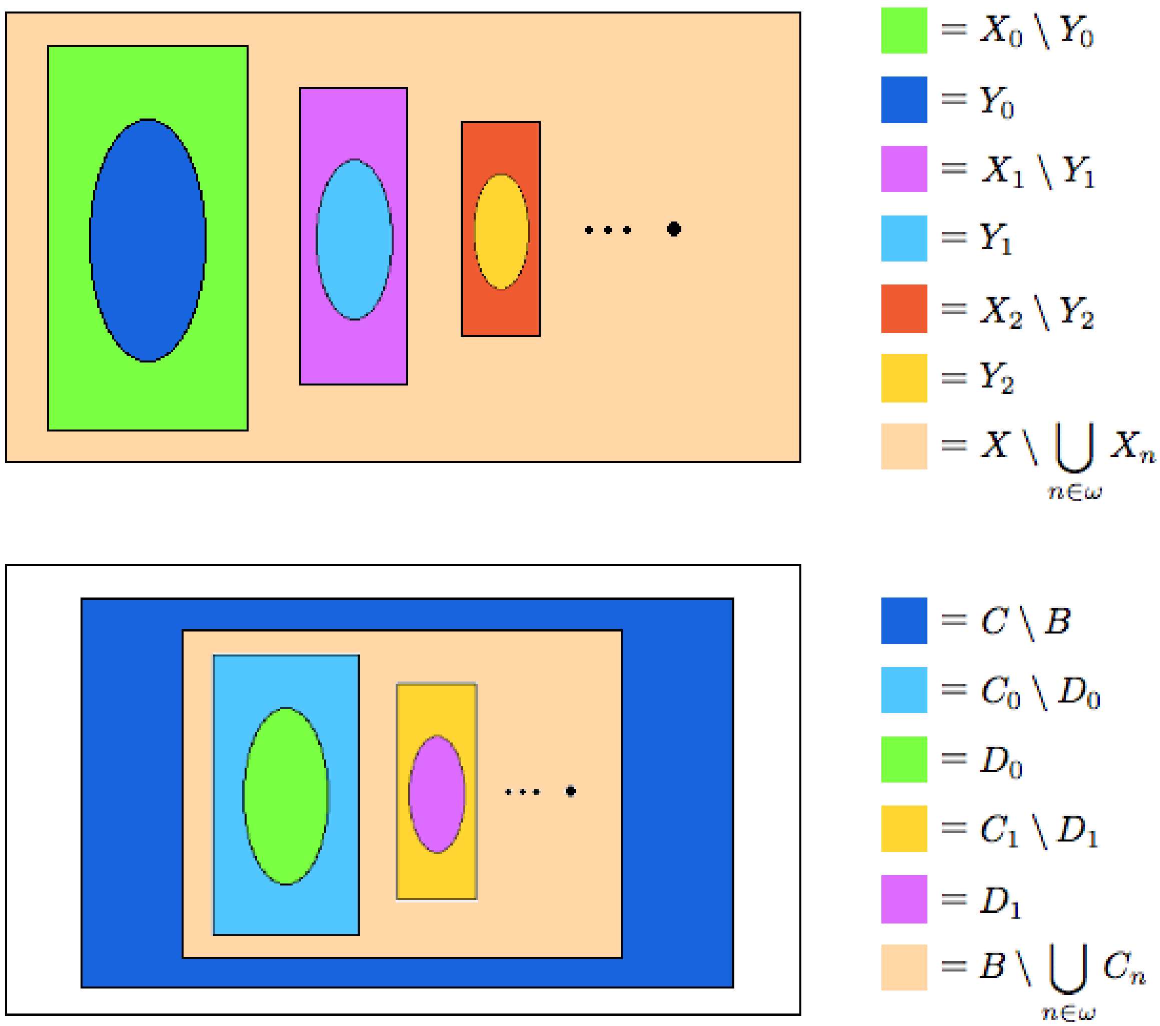


A picture proof of Theorem 3.2. Clopen sets of the same color are homeomorphic.

Proof of Theorem 3.30. Let $(U_n : n \in \omega)$ be a sequence of non-empty open subsets of X that converges to a point x. One can easily obtain a sequence $(X_n : n \in \omega)$ of pairwise disjoint non-empty clopen sets that converges to x, such that $x \notin X_n$ for each $n \in \omega$. Let C be a non-empty clopen subset of X. Let B be a clopen subset of C that is homeomorphic to X. Fix a homeomorphism $f : X \longrightarrow B$ and let $C_n = f[X_n]$ for each $n \in \omega$.

Now define clopen sets $Y_n \subseteq X_n$ and $D_n \subseteq C_n$ for each $n \in \omega$ and a (partial) homeomorphism h as in the proof of Theorem 3.2, but start by choosing Y_0 homeomorphic to $C \setminus B$ and fixing a homeomorphism $h_0 : Y_0 \longrightarrow C \setminus B$. Finally, extend h by setting $h(x) = f(x)$ for every $x \in X \setminus \bigcup_{n\in\omega} X_n$. It is easy to check that this yields the desired homeomorphism. $\square$

Figure B.2.



A picture proof of Theorem 3.30. Sets of the same color are homeomorphic.

Appendix C

Extensions of topological spaces

In this section, we will assume that Y is Tychonoff and X is a dense subspace of Y. In this case we say that Y is an *extension* of X. In all our applications $Y = \beta X$. The exposition is based on Section 11 in [50].

Whenever U is an open set in X, define

$$\mathrm{Ex}(U) = Y \setminus (\mathrm{cl}_Y(X \setminus U)).$$

It is clear that $\mathrm{cl}_Y(X \setminus U)$ is the smallest closed subset of Y such that its intersection with X is $X \setminus U$. Therefore $\mathrm{Ex}(U)$ is the largest open subset of Y such that its intersection with X is U.

Proposition C.1. *The collection*

$$\{\mathrm{Ex}(U) : U \text{ open in } X\}$$

is a base of Y*.*

Proof. Let O be an open subset of Y and pick $y \in O$. By regularity, there exists an open subset W of Y such that $y \in W \subseteq \mathrm{cl}(W) \subseteq O$. Let $U = W \cap X$. We claim that $y \in \mathrm{Ex}(U) \subseteq O$, which would conclude the proof.

It will be enough to show that $W \subseteq \mathrm{Ex}(U) \subseteq \mathrm{cl}(W)$. The first inclusion follows from the fact that W is open in Y and $W \cap X = U$. In order to prove the second inclusion, fix $z \in \mathrm{Ex}(U)$ and let N be an open neighborhood of z in Y. Notice that

$$N \cap U = N \cap \mathrm{Ex}(U) \cap X \neq \varnothing,$$

because $N \cap \mathrm{Ex}(U)$ is a non-empty open subset of Y and X is dense in Y. But $N \cap U \subseteq W$, hence $N \cap W \neq \varnothing$. This shows that $z \in \mathrm{cl}(W)$. □

Proposition C.2. *If U and V are open subsets of X then* $\mathrm{Ex}(U \cap V) = \mathrm{Ex}(U) \cap \mathrm{Ex}(V)$.

Proof.

$$\begin{aligned}
\mathrm{Ex}(U \cap V) &= Y \setminus \mathrm{cl}_Y(X \setminus (U \cap V)) \\
&= Y \setminus \mathrm{cl}_Y((X \setminus U) \cup (X \setminus V)) \\
&= Y \setminus (\mathrm{cl}_Y(X \setminus U) \cup \mathrm{cl}_Y(X \setminus V)) \\
&= (Y \setminus \mathrm{cl}_Y(X \setminus U)) \cap (Y \setminus \mathrm{cl}_Y(X \setminus U)) \\
&= \mathrm{Ex}(U) \cap \mathrm{Ex}(V).
\end{aligned}$$

□

Proposition C.3. *Assume that $Y = \beta X$. If C is a clopen subset of X then* $\mathrm{Ex}(C) = \mathrm{cl}_Y(C)$*, hence* $\mathrm{Ex}(C)$ *is clopen in Y.*

Proof. Let $f : X \longrightarrow [0,1]$ be the characteristic function of C. Since C is clopen, the function f is continuous. Let $g = \beta f : Y \longrightarrow [0,1]$ be the continuous extension of f. By continuity, it is clear that $g[Y] \subseteq \{0,1\}$. So $Y = g^{-1}(0) \cup g^{-1}(1)$.

But $g^{-1}(1) = g^{-1}[(0,1]]$ is an open subset of Y whose intersection with X is C, therefore $g^{-1}(1) \subseteq \mathrm{Ex}(C)$. Similarly $g^{-1}(0) \subseteq \mathrm{Ex}(X \setminus C)$. Since $\mathrm{Ex}(C) \cap \mathrm{Ex}(X \setminus C) = \varnothing$ by Proposition C.2, it follows that $\mathrm{Ex}(C) = Y \setminus \mathrm{Ex}(X \setminus C) = \mathrm{cl}_Y(C)$. □

Appendix D

Some descriptive set theory

The following results seem to be folklore, but we could not find satisfactory references. Notice that a large part of our discussion holds for arbitrary topological spaces.

Given a space Z, we will denote by $\mathcal{B}(Z)$ be the collection of subsets of Z that have the property of Baire (see Section 8.F in [30]). Recall that $\mathcal{B}(Z)$ is the smallest σ-algebra of subsets of Z containing all open sets and all meager sets. Also recall that $X \in \mathcal{B}(Z)$ if and only if there exist a G_δ subset G of Z and a meager subset M of Z such that $X = G \cup M$.

A subset X of a space Z has *property of Baire in the restricted sense* if $X \cap S \in \mathcal{B}(S)$ for every $S \subseteq Z$ (see Subsection VI of Section 11 in [35]). We will denote by $\mathcal{B}_r(Z)$ the collection of subsets of Z that have the property of Baire in the restricted sense. Using the fact that $\mathcal{B}(Z)$ is a σ-algebra, it is easy to check that $\mathcal{B}_r(Z)$ is a σ-algebra.

The inclusion $\mathcal{B}_r(Z) \subseteq \mathcal{B}(Z)$ is obvious. To see that the reverse inclusion need not hold, let $Z = 2^\omega \times 2^\omega$ and consider a Bernstein subset X of $P = \{x\} \times 2^\omega$ for some $x \in 2^\omega$. Since X is meager in Z, it has the property of Baire. However, $X \cap P = X$ cannot have the property of Baire in P, otherwise it would contain a copy of 2^ω.

Notice that the same example X shows that the property of Baire would not be a sufficient hypothesis in Proposition D.1. In fact, if X had a completely metrizable dense subset D, then D would have to be uncountable, hence it would contain a copy of 2^ω.

On the other hand, X is completely Baire because it is a Bernstein set, so in particular it is Baire.

Proposition D.1. *Let Z be a completely metrizable space and assume that $X \in \mathcal{B}_r(Z)$. Then either X has a completely metrizable dense subset or X is not Baire.*

Proof. Since $X \in \mathcal{B}(\mathrm{cl}(X))$, we can write $X = G \cup M$, where G is a G_δ subset of $\mathrm{cl}(X)$ and M is meager in $\mathrm{cl}(X)$.

Since G is a G_δ subset of the completely metrizable space $\mathrm{cl}(X)$, it is completely metrizable (see Theorem 3.11 in [30]). Since X is dense in $\mathrm{cl}(X)$, the set M is meager in X as well. In conclusion, if G is dense in X then the first alternative in the statement of the proposition will hold, otherwise the second alternative will hold. □

Corollary D.2. *Let Z be a completely metrizable space and assume that $X \in \mathcal{B}_r(Z)$. Then either X^ω has a completely metrizable dense subset or X^ω is meager in itself.*

Proof. If X has a completely metrizable dense subset D then D^ω is a completely metrizable dense subset of X^ω.

So assume that X has a non-empty meager open subset U. Observe that $M_n = \{f \in X^\omega : f(n) \in U\}$ is meager in X^ω for every $n \in \omega$. Also, it is clear that $(X \setminus U)^\omega$ is closed nowhere dense in X^ω. It follows that X^ω is meager in itself. □

Finally, we will point out a significant class of sets that have the property of Baire in the restricted sense. We will say that a subset A of a space Z is *analytic* if A can be obtained by applying operation $\mathcal{A}$ to a a system $\langle F_s : s \in {}^{<\omega}\omega \rangle$ of closed sets (see Section 25.C in [30]). We will denote by σ-$\mathcal{A}(Z)$ the σ-algebra of subsets of Z generated by the analytic sets.

Proposition D.3. *Let Z be a space. Then σ-$\mathcal{A}(Z) \subseteq \mathcal{B}_r(Z)$.*

Proof. Since, as we already observed, $\mathcal{B}_r(Z)$ is a σ-algebra, it will be enough to show that every analytic set has the property of Baire in the restricted sense.

Trivially, closed sets have the property of Baire in the restricted sense. So, by our definition of analytic set, it will be enough to show that the property of Baire in the restricted sense is preserved by operation $\mathcal{A}$. But this is a straightforward corollary of the classical fact that the property of Baire is preserved by operation $\mathcal{A}$ (see Corollary 29.14 in [30]). $\square$

Corollary D.4. *Let Z be a completely metrizable space and assume that $X \in \sigma$-$\mathcal{A}(Z)$. Then either X^ω has a completely metrizable dense subset or X^ω is meager in itself.*

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