ORDER-THEORETIC INVARIANTS IN SET-THEORETIC TOPOLOGY

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Abstract

We present several results related to van Douwen's Problem, which asks whether there is homogeneous compactum with cellularity exceeding $\mathfrak{c}$, the cardinality of the reals. For example, just as all known homogeneous compacta have cellularity at most $\mathfrak{c}$, they satisfy similar upper bounds in terms of Peregudov's Noetherian type and related cardinal functions defined by order-theoretic base properties. Also, assuming GCH, every point in a homogeneous compactum X has a local base in which every element has fewer supersets than the cellularity of X.

Our primary technique is the analysis of order-theoretic base properties. This analysis yields many results of independent interest beyond the study of homogeneous compacta, including many independence results about the Noetherian type of the Stone-Čech remainder of the natural numbers. For example, the Noetherian type of this space is at least the splitting number, but it can consistenly be less than the additivity of the meager ideal, strictly between the unbounding number and the dominating number, equal to $\mathfrak{c}$ and greater than the dominating number, or equal to the successor of $\mathfrak{c}$. We also prove several consistency results about Tukey classes of ultrafilters on the natural numbers ordered by almost containment. We also characterize the spectrum of Noetherian types of ordered compacta and mostly characterize the spectrum of Noetherian types of dyadic compacta. Also, we show that if every point in a compactum has a well-quasiordered local base, then some point has a countable local π-base.

Our secondary technique is an amalgam, a new quotient space construction that allows us to transform any homogeneous compactum into a path connected homogeneous compactum without reducing its cellularity, as well as construct the first ZFC example of homogeneous compactum that is not homeomorphic to a product of dyadic compacta and first countable compacta. We also use amalgams to prove results of independent interest about connectifications. For instance, every countably infinite product of infinite sums of metric spaces has a metrizable connectification.

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Contents

Chapter 1

Introduction

1.1 Van Douwen's Problem

The original motivation for this entire dissertation was Van Douwen's Problem, an open problem in set-theoretic topology.

Definition 1.1.1. A *homeomorphism* is a continuous bijection with continuous inverse. Given a topological space X, let $\mathrm{Aut}(X)$ denote the group of autohomeomorphisms of X. A space X is *homogeneous* if for every $p, q \in X$, there exists $h \in \mathrm{Aut}(X)$ such that $h(p) = q$.

Definition 1.1.2. A *compactum* is a compact Hausdorff space.

Question 1.1.3 (Van Douwen's Problem). Is there a homogeneous compactum X and a family $\mathcal{F}$ of pairwise disjoint open subsets of X such that $\mathcal{F}$ has greater cardinality than $\mathbb{R}$?

This problem has been open (in all models of ZFC) for over thirty years [47]. To get an idea of why problems about homogeneous compacta can be so hard, ask, given an arbitrary list $\langle X_i \rangle_{i \in I}$ of homogeneous compacta, what can we do with them to produce a bigger homogeneous compacta? In general, all we know how to do is form products like $\prod_{i \in I} X_i$. (Actually, Chapter 2 describes a method for producing homogeneous quotients of certain products of homogeneous compacta, but this method does not help us build an X solving Van Douwen's Problem, as we shall see in Chapter 3.)

This dissertation is mostly self-contained in the following sense. Before it starts using a definition, lemma, or theorem well-known amongst those who study set theory, general topology, and cardinal functions in topology, but perhaps not well-known amongst mathematicians in general, that definition, lemma, or theorem is usually explicitly stated. However, this first, introductory chapter is necessarily concise. For a full introduction to the above three topics, see Kunen [40], Engelking [20], and Juhász [35], respectively.

Four of the chapters of this dissertation have already been published as journal articles. Excepting very minor modifications, Chapter 2 is [48], Chapter 3 is [49], Chapter 5 is [50], and Chapter 6 is [51].

1.2 Ordinals and cardinals

Definition 1.2.1. A set x is *transitive* if $z \in y \in x$ always implies $z \in x$.

Definition 1.2.2. A *well-ordering* is a linear ordering with no strictly descending infinite sequences. Equivalently, a set is well-ordered if every subset has a minimum. An *ordinal* is a transitive set that is well-ordered by the membership relation $\in$. Let $\mathrm{O}n$ denote the class of ordinals. (We use "class" to denote collections of sets that might be "too big" to be sets themselves. The ordinals are indeed "too big" to be a set.)

The class of ordinals is itself well-ordered by inclusion, and strict inclusion is equivalent to membership, so an ordinal is the set of its predecessors. Every well-ordered set is isomorphic to an ordinal.

We identify the natural numbers $\mathbb{N}$ with ω, which denotes the least infinite ordinal, which is also the set of all finite ordinals.

Definition 1.2.3. A *cardinal* is an ordinal from which there is no bijection to a lesser ordinal. For every set A there is a unique cardinal $|A|$ from which there are bijections to A. This cardinal is also called the *cardinality* or the *size* of A. A set A is *countable* if $|A| \leq \omega$.

The cardinals inherit the well-ordering of the ordinals. Moreover, there is a unique order isomorphism from $\mathrm{O}n$ to the class of infinite cardinals; we denote it by $\alpha \mapsto \omega_\alpha$. In particular, ω_0 is ω and ω_1 is the least cardinal greater than ω (and ω_2 is the least cardinal greater than ω_1, and...).

Given sets A and B, we let B^A denote the set of all maps from A to B. However, when A and B are cardinals, we also abbreviate $\left|B^A\right|$ by B^A. If α is an ordinal, then $B^{<\alpha}$ denotes $\bigcup_{\beta<\alpha} B^\beta$. We analogously define $B^{\leq\alpha}$. However, for cardinals κ and λ, we abbreviate $\left|\kappa^{<\lambda}\right|$ by $\kappa^{<\lambda}$ when there is no danger of confusion. If κ is a cardinal, then $[B]^\kappa$ denotes $\{E \subseteq B : |E| = \kappa\}$ and $[B]^{<\kappa}$ denotes $\bigcup_{\lambda<\kappa}[B]^\lambda$. We analogously define $[B]^{\leq\kappa}$.

Unless otherwise indicated, ordinals are given the order topology. Also, given a space X and a set A, the set X^A is given the product topology (equivalently, the topology of pointwise convergence).

Definition 1.2.4. Let $\mathfrak{c}$ denote the cardinality of the real line, which is also the cardinality of the Cantor space 2^ω.

Definition 1.2.5. The *cofinality* $\operatorname{cf}\alpha$ of an ordinal α is the least ordinal β such that there is a map $f\colon \beta \to \alpha$ such that for every $\gamma < \alpha$ there exists $\delta < \beta$ such that $\gamma \leq f(\delta)$.

An ordinal α is *regular* if $\alpha = \operatorname{cf}\alpha$. Non-regular ordinals are said to be *singular.*

All regular ordinals are cardinals.

Definition 1.2.6. Given an ordinal α, let $\alpha + 1$ and α^+ respectively denote the least ordinal greater than α and the least cardinal greater than α. (In particular, $\omega_\beta^+ = \omega_{\beta+1}$ for all $\beta \in \mathrm{O}n$.) Ordinals of the form $\alpha + 1$ and α^+ are respectively called *successor* ordinals and successor cardinals. A nonzero, non-successor ordinal is called a *limit* ordinal. A nonzero, non-successor cardinal is called a limit cardinal.

For all infinite cardinals κ, we have $\kappa = \kappa^{<\omega} < \kappa^+ \leq \kappa^{\operatorname{cf}\kappa} \leq \kappa^\kappa = 2^\kappa$. The least limit cardinal is ω; the least singular limit cardinal is ω_ω. Every infinite successor cardinal is regular.

Definition 1.2.7. A *weakly inaccessible* cardinal is an uncountable regular limit cardinal. A cardinal κ is a *strong limit* cardinal if $2^{<\kappa} = \kappa$. An *inaccessible* cardinal is a regular uncountable strong limit cardinal.

Definition 1.2.8. Given $\alpha, \beta \in \mathrm{O}n$, let $\alpha + \beta$ denote the unique ordinal isomorphic to the lexicographic ordering of $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$; let $\alpha\beta$ denote the unique ordinal isomorphic to the lexicographic ordering of $\beta \times \alpha$. When there is no danger of confusion, we abbreviate $|\kappa\lambda|$ by $\kappa\lambda$ when κ and λ are cardinals.

For all infinite cardinals κ and all cardinals $\lambda > 0$, we have $|\kappa+\lambda| = |\kappa\lambda| = \max\{\kappa, \lambda\}$.

Definition 1.2.9. Given a linear order I and a linear order J_i for each $i \in I$, let $\sum_{i\in I} J_i$ denote $\bigcup_{i\in I}\{i\} \times J_i$ with the lexicographic ordering. Given a sequence $\langle\kappa_a\rangle_{a\in A}$ of cardinals, we will let $\sum_{a\in A} \kappa_a$ denote $\left|\bigcup_{a\in A}\{a\} \times \kappa_a\right|$ when there is no ambiguity.

1.3 General topology

Definition 1.3.1. The *closure* $\overline{A}$ of a subset A of a space X is the minimal closed superset of A; the *interior* $\operatorname{int} A$ of A is the maximal open subset of A; the *boundary* ∂A of A is $\overline{A} \setminus \operatorname{int} A$. A subset R of a space X is *regular open* if $\operatorname{int} \overline{R} = R$ and *regular closed* if $\overline{\operatorname{int} R} = R$. A *neighborhood* of a subset E of a space X is a set $N \subseteq X$ such that $E \subseteq \operatorname{int} N$. A neighborhood of a point $p \in X$ is a neighborhood of $\{p\}$.

Definition 1.3.2. A *local base* (*local π-base*) at a point in a space is a family of open neighborhoods of that point (family of nonempty open subsets) such that every neighborhood of the point contains an element of the family; a *base* (*π-base*) of a space is a family of open sets that contains local bases (local π-bases) at every point.

A base characterizes a topology because a set is open if and only if it is a union of basic sets.

Example 1.3.3. If $\mathcal{B}$ is a base of X and $Y \subseteq X$, then $\{U \cap Y : U \in \mathcal{B}\}$ is a base of Y (where Y is given the subspace topology).

Definition 1.3.4. A space X is:

- T_0 if for all $p, q \in X$ there is an open $U \subseteq X$ such that $|U \cap \{p, q\}| = 1$;
- T_1 if for all distinct $p, q \in X$ there is an open $U \subseteq X$ such that $p \in U$ and $q \notin U$;
- T_2, or *Hausdorff*, if for all distinct $p, q \in X$ there are disjoint neighborhoods of p and q;
- *Urysohn* if for all distinct $p, q \in X$ there are disjoint closed neighborhoods of p and q;

- *regular* if for all closed $C \subseteq X$ and $p \in X \setminus C$, there are disjoint neighborhoods of p and C;
- T_3 if X is regular and T_1.

Every regular space has a base consisting only of regular open sets.

Definition 1.3.5. We will need terms for several kinds of maps between spaces.

- A map between spaces is *continuous* if all preimages of open sets are open and *open* if all images of open sets are open.
- A *homeomorphism* is a continuous open bijection.
- Spaces X and Y are homeomorphic, or $X \cong Y$, if there is a homeomorphism from X to Y.
- A (topological) *embedding* of X into Y is a homeomorphism from X to a subspace of Y.
- A continuous surjection is *irreducible* if all images of closed proper subsets of the domain are proper subsets of the codomain.
- A continuous surjection is a *quotient* map if all preimages of non-open sets are not open. A space Y is a quotient of a space X if there is a quotient map from X to Y.

Definition 1.3.6. Given a space X and an equivalence relation E on X, the *quotient topology* of the set X/E of E-equivalence classes is defined by declaring subsets A of X/E to be open if (and only if) $\bigcup A$ is open in X.

Given X and E as above, the quotient topology is the unique topology for which the map defined by $x \mapsto x/E$ is a quotient map.

Definition 1.3.7. Given spaces X and Y, let $C(X, Y)$ denote the set of continuous maps from X to Y; let $C(X)$ denote $C(X, \mathbb{R})$.

Definition 1.3.8. A space X is:

- *completely regular* if for all closed $C \subseteq X$ and $p \in X \setminus C$, there is an $f \in C(X)$ such that $f(p) = 0$ and $f[C] = \{1\}$;
- $T_{3.5}$ if X is completely regular and T_1;
- *normal* if for all disjoint closed subsets $A, B \subseteq X$ there are disjoint neighborhoods of A and B;
- T_4 if X is normal and T_1.

The Urysohn Theorem states that for all normal spaces X, if A and B are disjoint closed subsets of X, then there is an $f \in C(X)$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

Definition 1.3.9. A space X is:

- κ-*compact* if every open cover of X has a subcover of size less than κ;
- *Lindelöf* if X is ω_1-compact;
- *compact* if X is ω-compact;
- a *compactum* if X is compact and T_2;
- a *compactification* of a $T_{3.5}$ space Y if X is a compactum with a dense subspace homeomorphic to Y;

- *locally κ-compact* if for every open $U \subseteq X$ and $p \in U$, there is a κ-compact neighborhood V of p such that $V \subseteq U$;

- *locally compact* if X is locally ω-compact.

All compacta are T_4. Continuous images of κ-compact spaces are κ-compact.

Definition 1.3.10. Given two topologies $\mathcal{S}$ and $\mathcal{T}$ on a set X, we say $\mathcal{S}$ is *finer* than $\mathcal{T}$, or $\mathcal{T}$ is *coarser* than $\mathcal{S}$, if $\mathcal{S} \supseteq \mathcal{T}$ or, equivalently, if the identity map from $\langle X, \mathcal{S}\rangle$ to $\langle X, \mathcal{T}\rangle$ is continuous.

Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism. In particular, if a compact topology is finer than a Hausdorff topology, then the topologies are identical.

Given a $T_{3.5}$ space X, there is a subset $\mathcal{F}$ of $C(X, [0,1])$ that separates points and closed sets, *i.e.*, for every closed $C \subseteq X$ and $p \in X \setminus C$, we have $f(p) \notin \overline{f[C]}$ for some $f \in \mathcal{F}$. Given any $\mathcal{F} \subseteq C(X)$ that separates points and closed sets, there is a topological embedding $\Delta_{\mathcal{F}}\colon X \to \mathbb{R}^{\mathcal{F}}$ given by $\Delta_{\mathcal{F}}(x)(f) = f(x)$. Given any two $\mathcal{F}, \mathcal{G} \subseteq C(X, [0,1])$ that separate points and closed sets, the closures of $\Delta_{\mathcal{F}}[X]$ of $\Delta_{\mathcal{G}}[X]$ in $[0,1]^{\mathcal{F}}$ and $[0,1]^{\mathcal{G}}$ are homeomorphic and each may be called *the Čech-Stone compactification* βX of X, which is, up to homeomorphism, the unique compactification B of X such that every continuous map from X to a compactum Y extends to a continuous map from B to Y.

Definition 1.3.11. A space X is:

- *connected* if its only clopen subsets are $\emptyset$ and X;

- *path-connected* if for all $p, q \in X$ there exists $f \in C(X, [0,1])$ such that $f(0) = p$ and $f(1) = q$;

- *totally disconnected* if all connected subspaces of X are singletons;

- *zero-dimensional* if every open cover $\mathcal{U}$ has a pairwise disjoint open *refinement*, meaning there is a pairwise disjoint open cover $\mathcal{V}$ such that every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$.

A T_3 space X has *small inductive dimension* 0, or $\operatorname{ind} X = 0$, if it has a base consisting only of clopen sets; X has small inductive dimension $\leq n+1$, or $\operatorname{ind} X \leq n+1$, if X has a base $\mathcal{A}$ such that every $U \in \mathcal{A}$ satisfies $\operatorname{ind} \partial U \leq n$; X has small inductive dimension $n+1$, or $\operatorname{ind} X = n+1$, if $\operatorname{ind} X \leq n+1$ and $\operatorname{ind} X \nleq n$.

Continuous images of (path-)connected spaces are (path-)connected. A compactum is totally disconnected if and only if $\operatorname{ind} X = 0$ if and only if it is zero-dimensional.

Definition 1.3.12. Let $\langle B, \leq, 0, 1, \wedge, \vee, ' \rangle$ be a boolean algebra (where $a \leq b$ if and only if $a \vee b = b$).

- A subset S of B is a *semifilter* of B if $0 \notin S$ and $\forall s \in S\ \forall t \geq s\ \ t \in S$.

- A semifilter F of B is a *filter* of B if $\forall \sigma \in [F]^{<\omega}\ \ \bigwedge \sigma \in F$.

- A filter U of B is an *ultrafilter of* B if $\forall b \in B$ ($b \in U$ or $b' \in U$).

- A subset $\mathcal{U}$ of the power set algebra $\mathcal{P}(I)$ of a set I is an *ultrafilter on* I if $\mathcal{U}$ is an ultrafilter of $\mathcal{P}(I)$.

- An ultrafilter $\mathcal{U}$ on a set I is *nonprincipal* if $\{i\} \notin \mathcal{U}$ for all $i \in I$.

A filter is an ultrafilter if and only if it is a maximal filter. An ultrafilter on a set is nonprincipal if and only if all its elements are infinite.

The category of boolean algebras is dual to the category of totally disconnected compacta. This is *Stone duality*; let us spell out what it means. Given a totally disconnected compactum X, let $\mathrm{Clop}(X)$ denote the algebra of clopen subsets of X. Given a boolean algebra B, let $\mathrm{Ult}(B)$ denote the set of ultrafilters of B topologized by declaring $\{\{U \in \mathrm{Ult}(B) : a \in U\} : a \in B\}$ to be a base of $\mathrm{Ult}(B)$. The space $\mathrm{Ult}(B)$ is always a totally disconnected compactum. Moreover, X is homeomorphic to $\mathrm{Ult}(\mathrm{Clop}(X))$ and B is isomorphic to $\mathrm{Clop}(\mathrm{Ult}(B))$. Given a continuous map $f\colon X \to Y$ between compacta, there is a homomorphism from $\mathrm{Clop}(Y)$ to $\mathrm{Clop}(X)$ given by $U \mapsto f^{-1}U$. Given a homomorphism $g\colon A \to B$ between boolean algebras, there is a continuous map from $\mathrm{Ult}(B)$ to $\mathrm{Ult}(A)$ given by $U \mapsto g^{-1}U$.

In particular, $\mathrm{Clop}(\beta\omega)$ is isomorphic to $\mathcal{P}(\omega)$, so we declare $\beta\omega$ to be the space of ultrafilters on ω. We then identify each $n < \omega$ with the principal ultrafilter $\{E \subseteq \omega : n \in E\}$. This makes $\beta\omega \setminus \omega$ the compact subspace of nonprincipal ultrafilters on ω, which is naturally homeomorphic to the space of ultrafilters of the quotient algebra $\mathcal{P}(\omega)/[\omega]^{<\omega}$. We will sometimes abbreviate $\beta\omega \setminus \omega$ by ω^*.

Definition 1.3.13. An ultrafilter $\mathcal{U}$ on a set I is *uniform* if $|A| = |I|$ for all $A \in \mathcal{U}$. For all infinite cardinals κ, let $\beta\kappa$ denote $\mathrm{Ult}(\mathcal{P}(\kappa))$, which is a Čech-Stone compactification of κ with the discrete topology. Let $u(\kappa)$ denote the subspace of uniform ultrafilters on κ.

Definition 1.3.14. A set is *nowhere dense* if it is contained in the complement of a dense open subset. A set is *meager* if it is contained in a countable union of nowhere dense sets.

1.4 Cardinal functions in topology

Definition 1.4.1. Given a space X, let the *weight* of X, or $w(X)$, be the least $\kappa \geq \omega$ such that X has a base of size at most κ. Given $p \in X$, let the *character* of p, or $\chi(p, X)$, be the least $\kappa \geq \omega$ such that there is a local base at p of size at most κ. Let the character of X, or $\chi(X)$, be the supremum of the characters of its points. Analogously define π*-weight* and π*-character*, respectively denoting them using π and $\pi\chi$.

A space is *first countable* if $\chi(X) = \omega$. A space is *second countable* if $w(X) = \omega$.

Example 1.4.2. We have $w(X) = \max\{\omega, |\operatorname{Clop} X|\}$ for all totally disconnected compacta X.

Arhangel′skiĭ's Theorem states that every compactum X has size at most $2^{\chi(X)}$. In particular, first countable compacta have size at most $\mathfrak{c}$. The Čech-Pospišil Theorem states that if X is a compactum and $\min_{p \in X} \chi(p, X) \geq \kappa \geq \omega$, then $X \geq 2^\kappa$. Therefore, if X is an infinite homogeneous compactum, then $|X| = 2^{\chi(X)}$.

For every $T_{3.5}$ space X, the weight of X is also the least $\kappa \geq \omega$ such that X embeds into $[0, 1]^\kappa$ and the least $\kappa \geq \omega$ such that some $\mathcal{F} \in [C(X)]^\kappa$ separates points and closed sets.

A space X is *metrizable* if it has a metric d such that $\{\{q : d(p, q) < 2^{-n}\} : n < \omega\}$ is a local base at p, for all $p \in X$. A compactum is metrizable if and only if it is second countable. The Nagata-Smrinov metrization theorem states that a T_3 space X is metrizable if and only if it has a base that is σ*-locally finite*, meaning X has a base $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ such that for all $n < \omega$ and $p \in X$, there is a neighborhood of p that intersects only finitely many elements of $\mathcal{B}_n$.

If X is a compactum, $\mathcal{A}$ is a base of X, and $\mathcal{B}$ is a family of open subsets of X, then

$\mathcal{B}$ is a base of X if and only if, for all $U, V \in \mathcal{A}$, if $\overline{U} \subseteq V$, then there is a finite $\mathcal{F} \subseteq \mathcal{B}$ such that $\overline{U} \subseteq \bigcup \mathcal{F} \subseteq V$. This fact will be used repeatedly in Chapter 3 and will be used in the proof of the following theorem.

Theorem 1.4.3. *If $g\colon X \to Y$ is a continuous surjection and X and Y are compacta, then $w(X) \geq w(Y)$.*

Proof. Let $\mathcal{A}$ be a base of X and let $\mathcal{B}$ be a base of Y. For every pair $U, V \in \mathcal{B}$ such that $\overline{U} \subseteq V$, there is, by compactness, a finite $\mathcal{F}_{U,V} \subseteq \mathcal{A}$ such that $g^{-1}\overline{U} \subseteq \bigcup \mathcal{F}_{U,V} \subseteq g^{-1}V$. Let $\mathcal{C}$ be the set of all sets of the form $\operatorname{int} g\left[\bigcup \mathcal{F}_{U,V}\right]$. Then $\mathcal{C}$ is a base of Y and $|\mathcal{C}| \leq |\mathcal{A}^{<\omega}| \leq \max\{|\mathcal{A}|, \omega\} \leq w(X)$. □

If X is a product space $\prod_{i\in I} X_i$, then $w(X) = \sum_{i\in I} w(X_i)$, $\pi(X) = \sum_{i\in I} \pi(X_i)$, $\chi(X) = \sum_{i\in I} \chi(X_i)$, and $\pi\chi(X) = \sum_{i\in I} \pi\chi(X_i)$.

The following are two weakenings of the notion of base.

Definition 1.4.4. A family $\mathscr{S}$ of open subsets of a space X is a *subbase* of X if the set of finite intersections of elements of $\mathscr{S}$ is a base of X. A family of subsets $\mathcal{N}$ of a space X is a *network* of X if for every open $U \subseteq X$ and $p \in U$, there exists $N \in \mathcal{N}$ such that $p \in N \subseteq U$.

A map $f\colon X \to Y$ is continuous if and only if there is a subbase $\mathscr{S}$ of Y such that $f^{-1}S$ is open for all $S \in \mathscr{S}$.

Example 1.4.5. Given a product space $\prod_{i\in I} X_i$ and $\mathscr{S}_i$ is a subbase of X_i for each $i \in I$, the set of sets of the form $\pi_i^{-1}S = \left\{p \in \prod_{i\in I} : p(i) \in S\right\}$ for $i \in I$ and $S \in \mathscr{S}_i$ is a subbase of $\prod_{i\in I} X_i$.

The notion of a local base at a point naturally generalizes to neighborhood bases of sets.

Definition 1.4.6. A *neighborhood base* of a subset E of a space X is a family of open neighborhoods of E such that every neighborhood of E contains an element of the family. The character $\chi(E,X)$ of E in X is the least $\kappa \geq \omega$ such that E has a neighborhood base of size at most κ.

The following two cardinal functions are close relatives of character.

Definition 1.4.7. The *pseudocharacter* $\psi(E,X)$ of a subset E of a T_1 space X is least $\kappa \geq \omega$ such that E is the intersection of a family of at most κ-many open sets. We say E is G_δ if $\psi(E,X) = \omega$. The pseudocharacter $\psi(p,X)$ of a point $p \in X$ is $\psi(\{p\},X)$. The pseudocharacter $\psi(X)$ of X is $\sup_{p\in X} \psi(p,X)$.

The *tightness* $t(p,X)$ of a point $p \in X$ is the least $\kappa \geq \omega$ such that for every $A \subseteq X$, if $p \in \overline{A}$, then $p \in \overline{B}$ for some $B \in [A]^{\leq\kappa}$. The tightness $t(X)$ of X is $\sup_{p\in X} t(p,X)$.

We always have $\psi(E,X) \leq \chi(E,X)$ and $t(p,X) \leq \chi(p,X)$. Moreover, if X is a compactum and E is closed, then $\psi(E,X) = \chi(E,X)$.

Definition 1.4.8. A *zero* subset of a space X is a set of the form $f^{-1}\{0\}$ for some $f \in C(X)$. A subset of X is *cozero* if it is the complement of a zero set.

Every zero set is a closed G_δ set. Moreover, if X is a compactum, then closed G_δ subsets of X are zero sets.

Definition 1.4.9. The *cellularity* $c(X)$ of a space X is the least $\kappa \geq \omega$ such that every pairwise disjoint family of open subsets of X has size at most κ. A space is *ccc* if its cellularity is ω.

A regular cardinal κ is a *caliber* of a space X if for every κ-sequence $\langle U_\alpha \rangle_{\alpha<\kappa}$ of open subsets of X there exists $I \in [\kappa]^\kappa$ such that $\bigcap_{\alpha\in I} U_\alpha$ is not empty.

The density $d(X)$ of a space X is the least $\kappa \geq \omega$ such that X has a dense subset of size at most κ. A space X is *separable* if $d(X) \leq \omega$.

Clearly, $c(X) \leq d(X) \leq \pi(X) \leq w(X)$ and $c(X) < \kappa$ for all calibers κ of X. Moreover, $d(X)^+$, $\pi(X)^+$, and $w(X)^+$ are all calibers of X. Also notice that if $f\colon X \to Y$ is a continuous surjection, then $c(Y) \leq c(X)$, $d(Y) \leq d(X)$, and every caliber of X is a caliber of Y.

If $c(\prod_{i\in\sigma} X_i) \leq \lambda$ for all $\sigma \in [I]^{<\omega}$, then $c(\prod_{i\in I} X_i) \leq \lambda$. If κ is a caliber of X_i for all $i \in I$, then κ is a caliber of $\prod_{i\in I} X_i$. Every known homogeneous compacta H is a continuous image of a product of compacta each with weight at most $\mathfrak{c}$; hence, $\mathfrak{c}^+$ is a caliber of H. This allows us to uniformly bound the cellularities of all known homogeneous compacta by $\mathfrak{c}$.

1.5 Order theory

This dissertation investigates several cardinal functions defined by order-theoretic base properties. Just like cellularity, these functions have uniform upper bounds when restricted to the class of known homogeneous compacta.

Definition 1.5.1. A *quasiorder* is a set with a transitive reflexive binary relation (denoted by $\leq$ unless otherwise indicated). A *directed set* is a quasiorder in which every finite set has an upper bound; a κ-directed set is a quasiorder in which every set of size less than κ has an upper bound. A *partially ordered set*, or *poset*, is a quasiorder such that $p \leq q \leq p$ always implies $p = q$.

Definition 1.5.2. Given a subset E of a quasiorder Q, let $\upharpoonleft_Q E = \{q \in Q : \exists e \in E\ e \leq q$ and $\downharpoonleft_Q E = \{q \in Q : \exists e \in E\ q \leq e$. Given $q \in Q$, let $\upharpoonleft_Q q = \upharpoonleft_Q \{q\}$ and $\downharpoonleft_Q q = \downharpoonleft_Q \{q\}$. A subset C of a quasiorder Q is *cofinal* if $Q = \downharpoonleft_Q C$. The *cofinality* of Q is the smallest cardinal κ such that Q has a cofinal subset of size κ.

Notice that this definition agrees with Definition 1.2.5 for ordinals.

Definition 1.5.3. A quasiorder is *well-founded* if every subset contains a minimal element. A *well-founded* quasiorder is *well-quasiordered* if it does not contain an infinite set of pairwise incomparable elements.

Every quasiorder has a well-founded cofinal subset.

Definition 1.5.4. Given a quasiorder $\langle Q, \leq \rangle$, let Q^{op} denote $\langle Q, \geq \rangle$. A subset D of a quasiorder Q is *dense* if D is cofinal in Q^{op}.

Definition 1.5.5. Given a cardinal κ, define a poset to be *κ-like* (*κ^{op}-like*) if no element is above (below) κ-many elements. Define a poset to be *almost κ^{op}-like* if it has a κ^{op}-like dense subset.

In the context of families of subsets of a topological space, we always implicitly order by inclusion. Consider the following order-theoretic cardinal functions.

Definition 1.5.6. Given a space X, let the *Noetherian type* of X, or $Nt(X)$, be the least $\kappa \geq \omega$ such that X has a base that is κ^{op}-like. Analogously define *Noetherian π-type* in terms of π-bases and denote it by $\pi Nt(X)$. Given a subset E of X, let the *local Noetherian type* of E in X, or $\chi Nt(E, X)$, be the least $\kappa \geq \omega$ such that there is a κ^{op}-like neighborhood base of E. Given $p \in X$, let the local Noetherian type of p, or $\chi Nt(p, X)$, be $\chi Nt(\{p\}, X)$. Let the local Noetherian type of X, or $\chi Nt(X)$, be the supremum of the local Noetherian types of its points. Let the *compact Noetherian type* of X, or $\chi_K Nt(X)$, be the supremum of the local Noetherian types of its compact subsets. We call Nt, πNt, χNt, and $\chi_K Nt$ *Noetherian cardinal functions.*

Noetherian type and Noetherian π-type were introduced by Peregudov [54]. Preceding this introduction are several papers by Peregudov, Šapirovskiĭ and Malykhin [42, 52, 53, 55] about $\min\{Nt(\cdot), \omega_2\}$ and $\min\{\pi Nt(\cdot), \omega_2\}$ (using different terminologies). Also, Dow and Zhou [16] showed that $\beta\omega \setminus \omega$ has a point with local Noetherian type ω. (An easier construction of such a point will be given in the proof of Theorem 3.5.15, which is a generalization of a construction of Isbell [32].)

It is also reasonable to define an order-theoretic analog of π-character.

Definition 1.5.7. Let the *local Noetherian π-type* $\pi\chi Nt(p, X)$ of a point p in a space X denote the least $\kappa \geq \omega$ such that p has a κ^{op}-like local π-base. Let the local Noetherian π-type $\pi\chi Nt(X)$ of X denote the supremum of the local Noetherian π-types.

However, it is not known whether there is a space X such that $\pi\chi Nt(X) \neq \omega$.

1.6 Models of set theory

Definition 1.6.1. We will need some model theory.

- A *language* is a set of constant symbols, n-ary function symbols, and n-ary relation symbols (for each $n < \omega$).

- Given a language $\mathcal{L}$, a (first order) $\mathcal{L}$-*structure* or $\mathcal{L}$-*model* $\mathcal{M}$ is a list consisting of set M, the *universe* of $\mathcal{M}$, and *interpretations* of each symbol in $\mathcal{L}$: an element $c^{\mathcal{M}} \in M$ for each constant symbol $c \in \mathcal{L}$, a relation $R^{\mathcal{M}} \subseteq M^n$ for each n-ary relation symbol $R \in \mathcal{L}$, and a function $F^{\mathcal{M}}\colon M^n \to M$ for each n-ary function symbol $F \in \mathcal{L}$. We will often abbreviate $\mathcal{M}$ by M.

- Given languages $\mathcal{L} \subseteq \mathcal{L}'$, an $\mathcal{L}'$-structure $\mathcal{M}'$ is an *expansion* of an $\mathcal{L}$-structure $\mathcal{M}$ if $M = M'$ and M and M' agree on their interpretations of symbols in $\mathcal{L}$.

- Given $\mathcal{L}$-models $\mathcal{M}$ and $\mathcal{N}$, we say $\mathcal{M}$ is a *submodel* of $\mathcal{N}$ if $M \subseteq N$, $c^{\mathcal{M}} = c^{\mathcal{N}}$, $F^{\mathcal{M}} = F^{\mathcal{N}} \upharpoonright M^n$, and $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^n$ for all constant, function, and relation symbols $c, F, R \in \mathcal{L}$.

- An $\mathcal{L}$*-term* with parameters from a set A is an expression built using function symbols in $\mathcal{L}$, constant symbols in $\mathcal{L}$, elements of A acting as additional constant symbols, and variable symbols.

- An *atomic* $\mathcal{L}$*-formula* with parameters from a set A is a formula of the form $R(t_0, \ldots, t_{n-1})$ or $t_0 = t_1$ where R is an n-ary relation symbol in $\mathcal{L}$ and $t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms with parameters from A.

- An $\mathcal{L}$*-formula* with parameters from a set A is a logical formula built using existential quantifiers, universal quantifiers, conjunctions, disjunctions, implications, negations, bi-implications, and $\mathcal{L}$-terms with parameters from A.

- An $\mathcal{L}$-structure $\mathcal{M}$ interprets an $\mathcal{L}$-formula by using its interpretations of all symbols in $\mathcal{L}$ and interpreting quantifications $\exists x$ and $\forall x$ by $\exists x \in M$ and $\forall x \in M$, respectively.

- An $\mathcal{L}$-structure $\mathcal{M}$ *satisfies* an $\mathcal{L}$-formula, or $\mathcal{M} \models \mathcal{L}$, if its interpretation of that formula is a true statement.

- An subset S of M^n is *definable* from a subset E of M if M satisfies a formula $\varphi(v_0, \ldots, v_{n-1})$ with variables $v_0, \ldots, v_{n-1}$ and parameters from E such that for all $a_0, \ldots, a_{n-1} \in M$, we have $M \models \varphi(a_0, \ldots, a_{n-1})$ if and only if $\langle a_i \rangle_{i<n} \in S$.

- An element a of M is definable from E if $\{a\}$ is definable from E.

- A submodel $\mathcal{M}$ of an $\mathcal{L}$-model $\mathcal{N}$ is an *elementary submodel* of $\mathcal{N}$, or $\mathcal{M} \prec \mathcal{N}$, if $\mathcal{M}$ and $\mathcal{N}$ satisfy the same $\mathcal{L}$-formulae with parameters from M.

The downward Lowenheim-Skolem Theorem states that for every $\mathcal{L}$-model M and $A \subseteq M$, there is an elementary submodel N such that $A \subseteq N$ and $|N| \leq |A||\mathcal{L}|\omega$.

The language of set theory is just a single binary relation symbol: $\{\in\}$. The standard list of axioms of set theory is denoted by ZFC. This list is infinite, but has a finite description. (There is a simple computer algorithm that can decide whether an arbitrary $\{\in\}$-formula is one of the ZFC axioms.) The exact contents of ZFC are not important here, but it should be noted that these axioms are strong enough for the formalization of almost all of mathematics in the language of set theory.

Definition 1.6.2. Given a regular infinite cardinal θ, let H_θ denote the class of all sets x hereditarily smaller than θ, *i.e.*, those x for which $|x| < \theta$, $|y| < \theta$ for all $y \in x$, $|z| < \theta$ for all $z \in y \in x$, $|w| < \theta$ for all $w \in z \in y \in x \ldots$

The class H_θ is actually a set of size $2^{<\theta}$. Moreover, if θ is regular and uncountable, then $\langle H_\theta, \in \rangle$ satisfies every axiom of ZFC except possibly the power set axiom, which asserts that for every set A, there is a set $\mathcal{P}(A) = \{B : B \subseteq A\}$. ($H_\theta$ satisfies all of ZFC if and only if θ is inaccessible.) However, proofs of statements about a fixed object A almost always talk only about sets of size at most $2^{|A|}$ or $2^{2^{|A|}}$ (or occasionally some other upper bound). Such proofs are valid in H_θ for sufficiently large regular θ. Henceforth, θ will denote a sufficiently large regular cardinal.

We will use elementary submodels of the $\{\in\}$-structure H_θ (with the symbol "$\in$" interpreted as actual membership) to greatly simplify and shorten "closing off" arguments that appear in many of our proofs. Sometimes arbitrary elementary submodels of H_θ will not be sufficiently closed off for our purposes. One easy fix is to add constant symbols for a small number of objects that we care about. For example, it sometimes suffices simply to expand $\langle H_\theta, \in\rangle$ to $\langle H_\theta, \in, C(X)\rangle$ for some space X that we want our elementary substructures to "know" about. When this trick does not suffice, we will use elementary chains.

Definition 1.6.3. A sequence of models $\langle M_\alpha\rangle_{\alpha<\eta}$ such that $M_\alpha \prec M_\beta$ for all $\alpha < \beta < \eta$ is an *elementary chain*. An elementary chain $\langle M_\alpha\rangle_{\alpha<\eta}$ is *continuous* if $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit $\alpha < \eta$. A continuous elementary chain of $\{\in\}$-models is a *continuous* $\in$*-chain* if $M_\alpha \in M_\beta$ for all $\alpha < \beta < \eta$.

Given an elementary chain $\langle M_\alpha\rangle_{\alpha<\eta}$, we have $M_\alpha \prec \bigcup_{\beta<\eta} M_\beta$ for all $\alpha < \eta$. If we also have $M_\alpha \prec N$ for all $\alpha < \eta$, then $\bigcup_{\alpha<\eta} M_\alpha \prec N$. If $\langle M_\alpha\rangle_{\alpha<\eta}$ is a continuous $\in$-chain of elementary submodels of H_θ, then $\eta \subseteq \bigcup_{\alpha<\eta} M_\alpha$. If $A \in M \prec H_\theta$ and $|A| \subseteq M$, then $A \subseteq M$.

Using elementary chains, one can prove that if $\kappa = \operatorname{cf}\kappa > \omega$ and A is a set of size less than κ, then there exists $M \prec H_\theta$ such that $|M| < \kappa$, $A \subseteq M$, and $M \cap \kappa \in \kappa$. The last relation is equivalent to the more useful $M \cap [H_\theta]^{<\kappa} = M \cap [M]^{<\kappa}$, which says that if $B \in M$ and $|B| < \kappa$, then $B \subseteq M$.

1.7 Forcing

Definition 1.7.1. The *Continuum Hypothesis*, or CH, is the assertion that $2^\omega = \omega_1$. The *Generalized Continuum Hypothesis*, or GCH, is the assertion that $2^\kappa = \kappa^+$ for all infinite cardinals κ.

Gödel proved that ZFC does not refute GCH. Cohen invented the technique of forcing to prove that ZFC also does not prove CH. In other words, GCH and $\neg$CH are both consistent with ZFC (but not with each other). Since then a flood of consistency results have been proven using forcing. In Chapter 5, we will extensively use forcing to prove that many (often mutually inconsistent) statements about the values of Noetherian cardinal functions for $\beta\omega \setminus \omega$ are consistent with ZFC.

Definition 1.7.2. A *maximum* of a quasiorder Q is an element $q \in Q$ such that $p \leq q$ for all $p \in Q$. A *forcing* is a quasiorder with a distinguished maximum. This maximum is typically denoted by $\mathbb{1}$.

In the context of forcing, a boolean algebra B refers to the forcing $B \setminus \{0\}$.

Given any finite list of ZFC axioms, ZFC proves that there is a countable transitive set M such that $\langle M, \in\rangle$ satisfies them. This is all that one needs to prove all of our consistency results, but for simplicity we posit the existence of a countable transitive model $\langle M, \in\rangle$ of all of ZFC. There is no danger in doing so because every ZFC proof, being finite, uses only a finite part of ZFC. (In any case, to get an actual countable transitive model of ZFC, one need only assume the existence of an inaccessible cardinal. This is a very mild assumption, the weakest in a grand hierarchy of "large cardinal axioms.")

Definition 1.7.3. Given a subset E of a quasiorder Q, let $\upharpoonleft_Q E$ denote the set of $q \in Q$ for which q has a lower bound in E. A subset F of a quasiorder Q is a *filter* if $F = \upharpoonleft_Q F$ and every finite subset of F has a lower bound in F. A filter G of a quasiorder Q is *Q-generic* over a class M if G intersects every dense subset D of Q for which $D \in M$.

For every quasiorder Q and countable set M, one can easily show that there is a Q-generic filter over M. If M is also a transitive model of ZFC, then one can say much more.

Definition 1.7.4. Given a transitive model $\langle M, \in \rangle$ of ZFC and a set E, let $M[E]$ denote the intersection of all transitive models $\langle N, \in \rangle$ of ZFC for which $N \supseteq M \cup \{E\}$.

Theorem 1.7.5. *Let $\langle M, \in \rangle$ be a countable transitive model of ZFC, $\mathbb{P}$ a forcing such that $\mathbb{P} \in M$, and G a $\mathbb{P}$-generic filter over M. Then $\langle M[G], \in \rangle$ is a countable transitive model of ZFC with the same ordinals as M. We call $M[G]$ a* $\mathbb{P}$-generic extension *of M.*

We can more usefully describe $M[G]$ through names.

Definition 1.7.6. Given M, $\mathbb{P}$, and G as above, the set $M^{\mathbb{P}}$ of $\mathbb{P}$-*names* in M is defined by $\in$-recursion as follows. If $\sigma \in M$ and all elements of σ are pairs of the form $\langle \tau, p \rangle$ where $p \in \mathbb{P}$ and τ is a $\mathbb{P}$-name in M, then σ is a $\mathbb{P}$-name in M. The *interpretation* σ_G of a $\mathbb{P}$-name σ by G is recursively defined as $\{\tau_G : \langle \tau, p \rangle \in \sigma \text{ and } p \in G\}$. Every $x \in M$ has a canonical name $\check{x}$ recursively defined as $\{\langle \check{y}, \mathbb{1} \rangle : y \in x\}$; hence, $\check{x}_G = x$. The $\mathbb{P}$-*forcing language* in M is the set of all $\{\in\}$-formulae with parameters from $M^{\mathbb{P}}$.

Theorem 1.7.7 (The Forcing Theorem)**.** *Given M, $\mathbb{P}$, and G as above, $M[G] = \{\sigma_G : \sigma \in M^{\mathbb{P}}\}$. Moreover, there is a binary relation $\Vdash$ that is definable in M, has domain $\mathbb{P}$, has codomain consisting of the $\mathbb{P}$-forcing language in M, and has the following properties.*

- *$M[G] \models \varphi(\sigma_G^{(0)}, \dots, \sigma_G^{(n-1)})$ if and only if $p \Vdash \varphi(\sigma^{(0)}, \dots, \sigma^{(n-1)})$ for some $p \in G$.*

- *$\mathbb{1} \Vdash \varphi$ if φ is a theorem of ZFC.*

- *If $p \Vdash \varphi$ and $p \Vdash \varphi \to \psi$, then $p \Vdash \psi$.*

- *$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.*

- *If $q \leq p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.*

- *$p \Vdash \neg\varphi$ if and only if $q \nVdash \varphi$ for all $q \leq p$.*

- *$p \Vdash \exists x\ \varphi(\sigma^{(0)}, \dots, \sigma^{(n-1)}, x)$ if and only if $p \Vdash \varphi(\sigma^{(0)}, \dots, \sigma^{(n-1)}, \tau)$ for some τ.*

We call $\Vdash$ the forcing relation.

In Chapter 5 and sometimes in this section, instead of talking about generic extensions of countable models, we will use a convenient shorthand. In set theory, V denotes the class of all sets. However, in the context of forcing we will implicitly use V, also referred to as the *ground model*, to denote a countable transitive model of ZFC. (Among the advantages of this shorthand is that we can speak directly about an uncountable forcing $\mathbb{P}$, as opposed to the interpretation of a definition of $\mathbb{P}$ by some countable transitive model M.) The justification for this convention is that all of the implications in our theorems and proofs are ZFC implications, and as such they are valid in any model of ZFC.

Definition 1.7.8. We say elements p and q of a forcing $\mathbb{P}$ are *incompatible* and write $p \perp q$ if there is no $r \in \mathbb{P}$ such that $p \geq r \leq q$. We say a subset A of $\mathbb{P}$ is an *antichain* if $p \perp q$ for all distinct $p, q \in A$. We say $\mathbb{P}$ is *ccc* if all its antichains are countable. We say a subset L of $\mathbb{P}$ is *linked* if no two elements of L are incompatible. We say $\mathbb{P}$ has *property (K)* if every uncountable subset of $\mathbb{P}$ contains an uncountable linked set. We say a subset C of $\mathbb{P}$ is *centered* if every finite subset of C has a lower bound in $\mathbb{P}$. We say $\mathbb{P}$ is *σ-centered* if it is the union of some countable family of centered sets.

Every σ-centered forcing has property (K); every forcing with property (K) is ccc. If $\mathbb{P}$ is a ccc forcing and G is a $\mathbb{P}$-generic over V, then $V[G]$ preserves cardinals and cofinalities, meaning that if $\alpha \in \mathrm{O}n$, then the V-interpretation and $V[G]$-interpretation of $|\alpha|$ and $\mathrm{cf}\,\alpha$ are identical. We symbolically denote these identities by writing $|\alpha|^{V[G]} = |\alpha|$ and $(\mathrm{cf}\,\alpha)^{V[G]} = \mathrm{cf}\,\alpha$. Moreover, if $A \in V[G]$ and A is an infinite subset of V, then there is a set $B \in V$ such that $A \subseteq B$ and $|A| = |B|$. If $\mathbb{P}$ also has a dense subset of size κ, then $|\lambda^\mu|^{V[G]} \leq (\kappa\lambda)^\mu$ for all cardinals λ and infinite cardinals μ. This is because if $\mathbb{P}$ is ccc and $D \subseteq \mathbb{P}$ is dense, then $p \Vdash \sigma \subseteq \check{B}$ implies that $p \Vdash \sigma = \tau$ for some $\tau = \{\{\check{b}\} \times A_b : b \in B\}$ where each A_b is a countable antichain contained in D.

Definition 1.7.9. *Martin's Axiom*, or MA, says that for every ccc forcing $\mathbb{P}$ and every family $\mathcal{D}$ of fewer than $\mathfrak{c}$-many dense subsets of $\mathbb{P}$, there is already in V a filter of $\mathbb{P}$ that meets every dense set in $\mathcal{D}$.

CH implies that $\mathcal{D}$ as above must be countable, so CH implies MA. Morever, Solovay and Tennenbaum proved that if $\omega < \kappa = \kappa^{<\kappa}$, then $V[G] \models \mathrm{MA} + \mathfrak{c} = \kappa$ for some ccc generic extension $V[G]$, so MA does not imply CH.

Definition 1.7.10. A map $f\colon \mathbb{P} \to \mathbb{Q}$ between forcings is:

- *order preserving* if $p \leq q$ implies $f(p) \leq f(q)$;
- an *order embedding* if $p \leq q$ is equivalent to $f(p) \leq f(q)$;
- *incompatibility preserving* if $p \perp q$ implies $f(p) \perp f(q)$;

- a *reduction* of a map $g\colon \mathbb{Q} \to \mathbb{P}$ if $\forall p \in \mathbb{P}\ \forall q \le f(p)\ g(q) \not\perp p$;
- a *complete embedding* if it is order preserving, is incompatibility preserving, and has a reduction;
- a *dense embedding* if it is order preserving, is incompatibility preserving, and has dense range.

Every order embedding with dense range is a dense embedding; every dense embedding is a complete embedding. Every complete embedding $j\colon \mathbb{P} \to \mathbb{Q}$ induces an embedding of names, which in turn induces an embedding of forcing languages. If we call all these embeddings j, then, for every $p \in \mathbb{P}$ and every atomic φ in the $\mathbb{P}$-forcing language, $p \Vdash \varphi$ if and only if $j(p) \Vdash j(\varphi)$. Moreover, if H is $\mathbb{Q}$-generic, then $j^{-1}H$ is $\mathbb{P}$-generic and $V[j^{-1}H] \subseteq V[H]$. If j is a dense embedding, then $V[j^{-1}H] = V[H]$.

Definition 1.7.11. Let $\mathrm{Fn}(A, B, \kappa)$ denote the set of partial functions f from A to B such that $|\mathrm{dom}\, f| < \kappa$. Let $\mathrm{Fn}(A, B)$ denote $\mathrm{Fn}(A, B, \omega)$. Unless otherwise indicated, sets of this form are ordered by $\supseteq$.

Definition 1.7.12. A subset of c of ω is *Cohen* over V if the indicator function χ_c of c in 2^ω is the union of a generic filter of $\mathrm{Fn}(\omega, 2)$; such a c is also called a *Cohen real.*

There is a dense embedding from $\mathrm{Fn}(\omega, 2)$ to $\mathcal{B}/\mathcal{M}$ where $\mathcal{B}$ is the Borel algebra of 2^ω and $\mathcal{M}$ is the meager ideal, *i.e.*, the set of meager elements of $\mathcal{B}$. It follows that c as above is Cohen over V if and only if χ_c avoids every meager set in V. Moreover, for every $\kappa \ge \omega$ there is a dense embedding from $\mathrm{Fn}(\kappa, 2)$ to $\mathcal{B}_\kappa/\mathcal{M}_\kappa$ where $\mathcal{B}_\kappa$ the Borel algebra of 2^κ and $\mathcal{M}_\kappa$ is its meager ideal. (Cohen proved that if G is $\mathrm{Fn}(\omega_2, 2)$-generic, then $V[G] \models \neg CH$.) It is also useful to know that $\mathrm{Fn}(\kappa, 2)$ has property (K) and that if $I \subseteq J \subseteq \kappa$, then the identity map is a complete embedding of $\mathrm{Fn}(I, 2)$ into $\mathrm{Fn}(J, 2)$.

Definition 1.7.13. Given $A \subseteq [\omega]^\omega$ with the SFIP, define the *Booth forcing for* A to be $[\omega]^{<\omega} \times [A]^{<\omega}$ ordered by $\langle \sigma_0, F_0 \rangle \leq \langle \sigma_1, F_1 \rangle$ if and only if $F_0 \supseteq F_1$ and $\sigma_1 \subseteq \sigma_0 \subseteq \sigma_1 \cup \bigcap F_1$. Define a *generic pseudointersection* of A to be $\bigcup_{\langle \sigma, F \rangle \in G} \sigma$ where G is a generic filter of $[\omega]^{<\omega} \times [A]^{<\omega}$.

If $\sigma_0 = \sigma_1$, then $\langle \sigma_0, F_0 \rangle \geq \langle \sigma_0, F_0 \cup F_1 \rangle \leq \langle \sigma_1, F_1 \rangle$, so Booth forcing is always σ-centered.

Definition 1.7.14. *Hechler forcing*, which is denoted by $\mathbb{D}$, consists of pairs of the form $\langle s, f \rangle \in \mathrm{Fn}(\omega, \omega) \times \omega^\omega$ where $\langle s', f' \rangle \leq \langle s, f \rangle$ if $s' \supseteq s$, $f'(n) \geq f(n)$ for all $n < \omega$, and $s'(n) \geq f(n)$ for all $n \in \mathrm{dom}(s' \setminus s)$. If G is a generic filter of $\mathbb{D}$, then the *generic dominating real* or *Hechler real* $g = \bigcup_{\langle s, f \rangle \in G} s \in \omega^\omega$ dominates $\omega^\omega \cap V$, meaning that every $f \in \omega^\omega \cap V$ is eventually dominated by g.

If $s = s'$, then $\langle s, f \rangle \geq \langle s, \max\{f, f'\} \rangle \leq \langle s', f' \rangle$, so $\mathbb{D}$ is σ-centered.

Definition 1.7.15. A subset r of ω is *random* over V if its indicator function χ_r avoids every $E \in V$ such that E is a Borel subset of 2^ω with Haar measure zero; a random r is also called a *random real*.

If $\mathcal{B}$ is the Borel algebra of 2^ω and $\mathcal{N}$ is the so-called null ideal consisting of zero-measure elements of $\mathcal{B}$, then every $(\mathcal{B} \setminus \mathcal{N})$-generic filter G is such that $\bigcap G = \{x\}$ for some random real x. (There is a natural dense embedding from $\mathcal{B} \setminus \mathcal{N}$ to $\mathcal{B}/\mathcal{N}$.) Since 2^ω has finite Haar measure, it cannot contain uncountably many pairwise disjoint Borel sets each with positive measure. Hence, $\mathcal{B} \setminus \mathcal{N}$ is ccc.

In contrast with Hechler forcing, every element of ω^ω in a $(\mathcal{B} \setminus \mathcal{N})$-generic extension of V is eventually dominated by some element of $\omega^\omega \cap V$.

Definition 1.7.16. The product $P \times Q$ of two quasiorders P and Q is defined by $\langle p_0, q_0\rangle \leq \langle p_1, q_1\rangle$ iff $p_0 \leq p_1$ and $q_0 \leq q_1$.

Given forcings $\mathbb{P}, \mathbb{Q} \in V$, there are complete embeddings i and j from $\mathbb{P}$ and $\mathbb{Q}$ to $\mathbb{P} \times \mathbb{Q}$ given by $i(p) = \langle p, \mathbb{1}_{\mathbb{Q}}\rangle$ and $j(q) = \langle \mathbb{1}_{\mathbb{P}}, q\rangle$. Moreover, if G is a $(\mathbb{P} \times \mathbb{Q})$-generic filter, then $G = i^{-1}G \times j^{-1}G$, $i^{-1}G$ is $\mathbb{P}$-generic over $V[j^{-1}G]$, $j^{-1}G$ is $\mathbb{Q}$-generic over $V[i^{-1}G]$, and $V[i^{-1}G][j^{-1}G] = V[j^{-1}G][i^{-1}G] = V[G]$. Furthermore, if $\mathbb{P}$ and $\mathbb{Q}$ both have property (K), then so does $\mathbb{P} \times \mathbb{Q}$.

Definition 1.7.17. Given a forcing $\mathbb{P}$ and $\mathbb{P}$-names $\mathbb{Q}$, $\leq_{\mathbb{Q}}$, $\mathbb{1}_{\mathbb{Q}}$ such that $\mathbb{1}_{\mathbb{Q}} \in \operatorname{dom}\mathbb{Q}$ and $\mathbb{1}_{\mathbb{P}}$ forces $\mathbb{Q}$, $\leq_{\mathbb{Q}}$, and $\mathbb{1}_{\mathbb{Q}}$ to form a forcing, define the *two-step iterated forcing* $\mathbb{P} * \mathbb{Q}$ as the set of all pairs $\langle p, q\rangle \in \mathbb{P} \times \operatorname{dom}\mathbb{Q}$ for which $p \Vdash q \in \mathbb{Q}$, with the ordering given by $\langle p', q'\rangle \leq \langle p, q\rangle$ if $p' \leq p$ and $p' \Vdash q' \leq q$.

Given $\mathbb{P}$ and $\mathbb{Q}$ as above, there is a complete embedding $i\colon \mathbb{P} \to \mathbb{P} * \mathbb{Q}$ given by $i(p) = \langle p, \mathbb{1}_{\mathbb{Q}}\rangle$. Moreover, if K is $(\mathbb{P} * \mathbb{Q})$-generic over V, then $G = i^{-1}K$ is $\mathbb{P}$-generic over V and $H = \{q_G : \exists p \in \mathbb{P}\ \langle p, q\rangle \in K\}$ is $\mathbb{Q}_G$-generic over $V[G]$, and $V[G][H] = V[K]$.

Next, we define a transfinite generalization of the two-step iteration.

Definition 1.7.18. *Finite support iterations* are recursively defined as follows. Given a successor ordinal $\alpha + 1$ and a finite support iteration $\langle \mathbb{P}_\beta\rangle_{\beta\leq\alpha}$, we say that$\langle \mathbb{P}_\beta\rangle_{\beta\leq\alpha+1}$ is a finite support iteration if $\mathbb{P}_{\alpha+1}$ is a quasiordered set of functions all with domain $\alpha + 1$ and there is an order isomorphism from some two-step iteration $\mathbb{P}_\alpha * \mathbb{Q}_\alpha$ to $\mathbb{P}_{\alpha+1}$ given by $\langle p, q\rangle \mapsto p \cup \{\langle \alpha, q\rangle\}$. Given a limit ordinal η and a sequence $\langle \mathbb{P}_\beta\rangle_{\beta<\eta}$ such that $\langle \mathbb{P}_\gamma\rangle_{\gamma\leq\beta}$ is a finite support iteration for all $\beta < \eta$, we say that $\langle \mathbb{P}_\beta\rangle_{\beta\leq\eta}$ is a finite support iteration if $\mathbb{P}_\eta$ is a quasiordered set of functions all with domain η and there is a bijection h from $\bigcup_{\beta<\eta} \mathbb{P}_\beta$ to $\mathbb{P}_\eta$ given by $\mathbb{P}_\beta \ni p \mapsto p \cup \langle \mathbb{1}_\zeta \rangle_{\beta \leq \zeta < \eta}$, such that $h \restriction P_\beta$ is an order embedding for all $\beta < \eta$.

If $\langle \mathbb{P}_\delta \rangle_{\delta \leq \gamma}$ is a finite support iteration as above, then every $p \in \mathbb{P}_\gamma$ satisfies $p(\delta) = \mathbb{1}_\delta$ for all but finitely many $\delta < \gamma$. We call the set of these finitely many $\delta < \gamma$ the *support* of p, or $\operatorname{supp}(p)$. Also, for all $\zeta < \delta \leq \gamma$ there is a complete embedding from $\mathbb{P}_\zeta$ to $\mathbb{P}_\delta$ given by $p \mapsto p \cup \langle \mathbb{1}_\nu \rangle_{\zeta \leq \nu < \delta}$. Indeed, this map has a natural reduction given by $q \mapsto q \restriction \zeta$.

If $\langle \mathbb{P}_\delta \rangle_{\delta \leq \gamma}$ is a finite support iteration as above and $\mathbb{1}_{\mathbb{P}_\delta}$ forces $\mathbb{Q}_\delta$ to be ccc (have property (K)) for all $\delta < \gamma$, then $\mathbb{P}_\gamma$ is ccc (has property (K)). Conversely, if $\mathbb{P}_\gamma$ is ccc, then $\mathbb{1}_{\mathbb{P}_\delta}$ forces $\mathbb{Q}_\delta$ to be ccc for all $\delta < \gamma$.

1.8 Combinatorial set theory

Lemma 1.8.1 (Pigeonhole Principle). *If $f \colon A \to B$ and $\max\{|B|, \kappa\} < |A|$, then f is constant on a set of size κ^+. If $f \colon A \to B$ and $|B| < \operatorname{cf}|A|$, then f is constant on a set of size $|A|$.*

Definition 1.8.2. A subset C of a limit ordinal η is *closed unbounded*, or *club*, if it is closed (in the order topology) and is a cofinal subset of η. A subset S of a limit ordinal η is *stationary* if it intersects every club subset of η. A subset $\mathcal{E}$ of a set of the form $[A]^\omega$ is closed unbounded, or club, if $\mathcal{E}$ is cofinal in $\langle [A]^\omega, \subseteq \rangle$ and every increasing ω-sequence $\langle E_n \rangle_{n<\omega} \in \mathcal{E}^\omega$ has union in $\mathcal{E}$. A subset $\mathcal{S}$ of a set of the form $[A]^\omega$ is stationary if it intersects every club subset of $[A]^\omega$.

For all regular infinite cardinals $\kappa < \lambda$, the set $\{\alpha < \lambda : \operatorname{cf} \alpha = \kappa\}$ is a stationary subset of λ, the set $\{\sup(M \cap \lambda) : M \prec H_\theta \wedge |M| < \lambda\}$ is a club subset of λ, and the set $\{M : M \prec H_\theta \wedge |M| = \omega\}$ is a club subset of $[H_\theta]^\omega$.

If S is a stationary subset of a regular uncountable cardinal κ, then S can be partitioned into κ-many disjoint stationary subsets of κ. If $\mathcal{C}$ is a family of fewer than κ-many club subsets of a regular uncountable cardinal κ, then $\bigcap \mathcal{C}$ is a club subset of κ. If $\mathcal{C}$ is a countable family of club subsets of $[A]^\omega$ for some A, then $\bigcap \mathcal{C}$ is a club subset of $[A]^\omega$.

Lemma 1.8.3 (Pressing Down Lemma)**.** *If S is a stationary subset of a regular uncountable cardinal κ, and $f\colon S \to \kappa$ is regressive,* i.e., *$f(\alpha) < \alpha$ for all $\alpha \in S$, then there is a stationary $T \subseteq \kappa$ such that $T \subseteq S$ and $f \upharpoonright T$ is constant.*

Definition 1.8.4. A set E is a Δ-*system* if there is some r such that $a \cap b = r$ for all distinct $a, b \in E$. Such an r is called the *root* of E.

Lemma 1.8.5 (Δ-System Lemma)**.** *If E is a set of finite sets and $|E| \geq \kappa = \operatorname{cf} \kappa > \omega$, then there exists a Δ-system $D \in [E]^\kappa$.*

Definition 1.8.6. Given a stationary subset S of a regular uncountable cardinal κ, let $\diamondsuit(S)$ denote the statement that there is a sequence $\langle \Xi_\alpha \rangle_{\alpha \in S}$ such that for every $A \subseteq \kappa$ there is a stationary $T \subseteq \kappa$ such that $T \subseteq S$ and $A \cap \alpha = \Xi_\alpha$ for all $\alpha \in T$. Let $\diamondsuit$ denote $\diamondsuit(\omega_1)$.

Jensen first defined $\diamondsuit$ and proved its consistency with ZFC. ZFC proves that $\diamondsuit$ implies CH, but does not prove the converse.

Definition 1.8.7. A *Suslin line* is a linear order that is ccc with respect to the order topology yet is not separable.

ZFC+GCH neither proves nor refutes the existence of Suslin lines. ZFC+MA+$\neg$CH refutes the existence of Suslin lines. ZFC + $\diamondsuit$ implies the existence of Suslin lines.

Definition 1.8.8. A *pseudointersection* of a subset A of $[\omega]^\omega$ is an $x \in [\omega]^\omega$ such that $|x \setminus a| < \omega$ for all $a \in A$. A subset A of $[\omega]^\omega$ has the *strong finite intersection property*, or SFIP, if $|\bigcap \sigma| = \omega$ for all $\sigma \in [A]^{<\omega}$.

Every countable A as above has a pseudointersection if it has the SFIP. MA implies that this implication is also true for all $A \in [[\omega]^\omega]^{<\mathfrak{c}}$.

Definition 1.8.9. A forcing $\mathbb{P}$ is *proper* if for every uncountable $A \in V$ and for every stationary $\mathcal{S} \subseteq [A]^\omega$ such that $\mathcal{S} \in V$, we have that $\mathcal{S}$ remains a stationary subset of $[A]^\omega$ in $V[G]$ for every $\mathbb{P}$-generic filter G. The *Proper Forcing Axiom*, or PFA, asserts that for every proper forcing $\mathbb{P}$ and every family $\mathcal{D}$ of ω_1-many dense subsets of $\mathbb{P}$, there is already in V a filter of $\mathbb{P}$ that meets every dense set in $\mathcal{D}$.

ZFC proves that PFA implies $MA + \mathfrak{c} = \omega_2$, but does not prove the converse. Assuming sufficiently strong large cardinal axioms, PFA is consistent with ZFC.

Chapter 2

Amalgams

2.1 Introduction

M. A. Maurice [44] constructed a family of homogeneous compact ordered spaces with cellularity $\mathfrak{c}$. All these spaces are zero-dimensional. Indeed, it is easy to see that no compact ordered space with uncountable cellularity can be path-connected. The cone over any of Maurice's spaces is path-connected but not homogeneous or ordered. However, there is a path-connected homogeneous compactum with cellularity $\mathfrak{c}$ which, though not an ordered space, has small inductive dimension 1; we construct such a space by gluing copies of powers of one of Maurice's spaces together in a uniform way. Moreover, this space is not homeomorphic to a product of dyadic compacta and first countable compacta. To the best of the author's knowledge, there is only one other known construction, due to van Mill [46] (and generalized by Hart and Ridderbos [27]), of a homogeneous compactum not homeomorphic to such a product, and the homogeneity all spaces so constructed is independent of ZFC.

The above amalgamation technique also can be used to construct new connectifications, where a connected (path-connected) space Y is a *connectification (pathwise connectification)* of a space X if X can be densely embedded in Y, and the connectification is *proper* if the embedding can be chosen not to be surjective. Whether a space has a connectification is uninteresting unless we restrict to connectifications that are at least T_2. For a broad survey of connectification results, see Wilson [71]. Our focus will be on which T_2 (T_3, $T_{3.5}$, metric) spaces have T_2 (T_3, $T_{3.5}$, metric) connectifications or pathwise connectifications. Only partial characterizations are known. For example, Watson and Wilson [70] showed that a countable T_2 space has a T_2 connectification if and only if it has no isolated points. Emeryk and Kulpa [19] proved that the Sorgenfrey line has a T_2 connectification, but no T_3 connectification. Alas *et al* [1] showed that every separable metric space without nonempty open compact subsets has a metric connectification. Gruenhage, Kulesza, and Le Donne [26] showed that every nowhere locally compact metric space has a metric connectification.

There are only a handful of results about pathwise connectifications. For example, Fedeli and Le Donne [22] showed that a nonsingleton countable first countable T_2 space has a T_2 pathwise connectication if and only if it has no isolated points. Druzhinina and Wilson [17] showed that a metric space has a metric pathwise connectification if its path components are open and not locally compact; similarly, a first countable T_2 (T_3) space has a T_2 (T_3) connectification if its path components are open and not locally feebly compact. See also Costantini, Fedeli, and Le Donne [13] for some results about pathwise connectifications of spaces adjoined with a free open filter.

Suppose $i \in \{1, 2, 3, 3.5\}$ and X has a proper T_i connectification. Then $X \times Z$ has a proper T_i connectification for all T_i spaces Z. Thus, given one proper connectification, this product closure property gives us a new connectification. We omit the easy proof of this fact here because we shall prove much stronger amalgam closure properties, which in many cases are also valid for pathwise connectifications. The reals are a pathwise connectification of the Baire space ω^ω because $\omega^\omega \cong \mathbb{R}\setminus\mathbb{Q}$. By applying amalgam closure properties to this particular connectification, we shall prove the following theorem.

Theorem 2.1.1. *If $i \in \{1, 2, 3, 3.5\}$, then every infinite product of infinite topological sums of T_i spaces has a T_i pathwise connectification. Every countably infinite product of infinite topological sums of metrizable spaces has a metrizable pathwise connectification.*

The previously known result most similar to Theorem 2.1.1 is due to Fedeli and Le Donne [21]: a product of T_2 spaces with open components has a T_2 connectification if and only if it does not contain a nonempty proper open subset that is H-closed.

2.2 Amalgams

Definition 2.2.1. Given a topological space X, let $\mathbb{S}(X)$ denote the set of all subbases of X that do not include $\emptyset$.

Let X be a nonempty T_0 space and let $\mathscr{S} \in \mathbb{S}(X)$. For each $S \in \mathscr{S}$, let Y_S be a nonempty topological space. The *amalgam* of $\langle Y_S : S \in \mathscr{S}\rangle$ is the set Y defined by

$$Y = \bigcup_{p\in X} \prod_{p\in S\in\mathscr{S}} Y_S.$$

We say that X is the *base space* of Y. For each $S \in \mathscr{S}$, we say that Y_S is a *factor* of Y. Every amalgam has a natural projection π to its base space: because X is T_0, we may define $\pi\colon Y \to X$ by $\pi^{-1}\{p\} = \prod_{p\in S\in\mathscr{S}} Y_S$ for all $p \in X$. Amalgams also have natural partial projections to their factors: for each $S \in \mathscr{S}$, define $\pi_S\colon \pi^{-1}S \to Y_S$ by $y \mapsto y(S)$.

Consider sets of the form $\pi_S^{-1}U$ where $S \in \mathscr{S}$ and U open in Y_S. We say such sets are *subbasic* and finite intersections of such sets are *basic*. We topologize Y by declaring these basic sets to be a base of open sets. Let us list some easy consequences of this topologization.

- For all $S \in \mathscr{S}$, the map π_S is continuous and open and has open domain.
- The map π is continuous and open.
- If $|Y_S| = 1$ for all $S \in \mathscr{S}$, then $Y \cong X$.
- For each $p \in X$, the product topology of $\prod_{p \in S \in \mathscr{S}} Y_S$ is also the subspace topology inherited from Y.
- Suppose, for each $S \in \mathscr{S}$, that Z_S is a subspace of Y_S. Then the topology of the amalgam of $\langle Z_S : S \in \mathscr{S} \rangle$ is also the subspace topology inherited from Y.
- Suppose, for each $S \in \mathscr{S}$, that $\mathscr{S}_S$ is a subbase of Y_S. Then the set

  $$\{\pi_S^{-1} T : S \in \mathscr{S} \text{ and } T \in \mathscr{S}_S\}$$

  is a subbase of Y.

Throughout this chapter, X, $\mathscr{S}$, and $\langle Y_S \rangle_{S \in \mathscr{S}}$ will vary, but Y will always denote the amalgam of $\langle Y_S \rangle_{S \in \mathscr{S}}$.

Up to homeomorphism, an amalgam is a quotient of the product of its base space and its factors. Specifically, the map from $X \times \prod_{S \in \mathscr{S}} Y_S$ to Y given by

$$\langle x, y \rangle \mapsto y \restriction \{S \in \mathscr{S} : x \in S\}$$

is easily verified to be a quotient map.

We say that a class $\mathcal{A}$ of nonempty T_0 spaces is *amalgamative* if an amalgam is always in $\mathcal{A}$ if its base space and all its factors are in $\mathcal{A}$. Therefore, any class of nonempty T_0 spaces closed with respect to products and quotients is amalgamative. In particular, amalgams preserve compactness, connectedness, and path-connectedness. The next theorem says that several other well-known productive classes are also amalgamative.

Theorem 2.2.2. *The classes listed below are amalgamative provided we exclude the empty space. Conversely, if an amalgam is in one of these classes, then its base space and all its factors are also in that class.*

1. T_0 *spaces*
2. T_1 *spaces*
3. T_2 *spaces*
4. T_3 *spaces*
5. $T_{3.5}$ *spaces*
6. *totally disconnected* T_0 *spaces*
7. T_0 *spaces with small inductive dimension* 0

Proof. For (1)-(3), suppose y_0 and y_1 are distinct elements of Y. If $\pi(y_0) = \pi(y_1)$, then there exists $S \in \operatorname{dom} y_0 = \operatorname{dom} y_1$ such that $y_0(S) \neq y_1(S)$; whence, if U_0 and U_1 are neighborhoods of $y_0(S)$ and $y_1(S)$ witnessing the relevant separation axiom for $y_0(S)$ and $y_1(S)$, then $\pi_S^{-1}U_0$ and $\pi_S^{-1}U_1$ witness the the same separation axiom for y_0 and y_1. If $\pi(y_0) \neq \pi(y_1)$, then let U_0 and U_1 be neighborhoods of $\pi(y_0)$ and $\pi(y_1)$ witnessing the relevant separation axiom for $\pi(y_0)$ and $\pi(y_1)$. Then $\pi^{-1}U_0$ and $\pi^{-1}U_1$ witness the same separation axiom for y_0 and y_1.

For (4) and (5), suppose C is a closed subset of Y and $y \in Y \setminus C$. Then there exist $n < \omega$ and $\langle S_i\rangle_{i<n} \in (\operatorname{dom} y)^n$ and $\langle U_i\rangle_{i<n}$ such that U_i is an open neighborhood of $y(S_i)$ for all $i < n$ and $\bigcap_{i<n} \pi_{S_i}^{-1}U_i$ is disjoint from C. For each $i < n$, let V_i be an open neighborhood of $y(S_i)$ such that $\overline{V_i} \subseteq U_i$. Let U be an open neighborhood of $\pi(y)$ such that $\overline{U} \subseteq \bigcap_{i<n} S_i$. Set $V = \pi^{-1}U \cap \bigcap_{i<n} \pi_{S_i}^{-1} V_i$. Then V is an open neighborhood of y and we have

$$\overline{V} \subseteq \bigcap_{i<n} \pi^{-1} S_i \cap \bigcap_{i<n} \pi_{S_i}^{-1} U_i = \bigcap_{i<n} \pi_{S_i}^{-1} U_i;$$

whence, $\overline{V}$ is disjoint from C.

Now suppose there is a continuous map $f\colon X \to [0,1]$ such that $f(\pi(y)) = 1$ and $f[X \setminus U] = \{0\}$. For each $i < n$, likewise suppose there is a continuous map $f_i\colon Y_{S_i} \to [0,1]$ such that $f_i(y(S_i)) = 1$ and $f[Y_{S_i} \setminus U_i] = \{0\}$. Define $g\colon \bigcap_{i<n} \pi^{-1} S_i \to [0,1]$ by $z \mapsto f(\pi(z)) f_0(z(S_0)) \cdots f_{n-1}(z(S_{n-1}))$. Define $h : \pi^{-1}\big(X \setminus \overline{U}\big) \to [0,1]$ by $z \mapsto 0$. By the pasting lemma, $g \cup h$ is continuous and separates y and C.

For (6), suppose C is a nonempty connected subset of Y and X and Y_S are totally disconnected for all $S \in \mathscr{S}$. Then $\pi[C]$ is connected; whence, $\pi[C] = \{p\}$ for some $p \in X$. For each $S \in \mathscr{S}$, if $p \in S$, then $\pi_S[C]$ is connected; whence, $|\pi_S[C]| = 1$. Thus, $|C| = 1$.

For (7), suppose $S \in \mathscr{S}$ and U open in Y_S and $y \in \pi_S^{-1}U$. Let V be a clopen neighborhood of $y(S)$ contained in U. Then $\pi_S^{-1}V$ is clopen in $\pi^{-1}S$. Let W be a clopen neighborhood of $\pi(y)$ contained in S. Then $\pi^{-1}W \cap \pi_S^{-1}V$ is a clopen neighborhood of y contained in $\pi_S^{-1}U$.

For the converse, first note that each of the classes (1)-(7) is closed with respect to subspaces. Second, Y_S can be embedded in Y for all $S \in \mathscr{S}$ because $\prod_{p\in S\in\mathscr{S}} Y_S$ is a subspace of Y for all $p \in X$. Finally, X can be embedded in Y because the amalgam of $\langle\{f(S)\}\rangle_{S\in\mathscr{S}}$ is homeomorhpic to X for all $f \in \prod_{S\in\mathscr{S}} Y_S$. $\square$

A countable product of metrizable spaces is metrizable; the next theorem is the analog for amalgams.

Theorem 2.2.3. *Suppose X and Y_S are metrizable for all $S \in \mathscr{S}$ and there is a countable $\mathscr{T} \subseteq \mathscr{S}$ such that $|Y_S| = 1$ for all $S \in \mathscr{S} \setminus \mathscr{T}$. Then Y is metrizable.*

Proof. Since Y is T_3 by Theorem 2.2.2, it suffices to exhibit a σ-locally finite base for Y. For each $T \in \mathscr{T}$, let $\bigcup_{n<\omega} \mathcal{U}_{T,n}$ be a σ-locally finite base for Y_T; let $\bigcup_{n<\omega} \mathcal{U}_n$ be a σ-locally finite base for X. For each $n < \omega$ and $\tau \in \mathrm{Fn}(\mathscr{T}, \omega)$, set $\mathcal{U}_{n,\tau} = \left\{U \in \mathcal{U}_n : \overline{U} \subseteq \bigcap \mathrm{dom}\,\tau\right\}$ and

$$\mathcal{V}_{n,\tau} = \left\{\pi^{-1}U \cap \bigcap_{T \in \mathrm{dom}\,\tau} \pi_T^{-1} U_T : U \in \mathcal{U}_{n,\tau} \text{ and } (\forall T \in \mathrm{dom}\,\tau)(U_T \in \mathcal{U}_{T,\tau(T)})\right\}.$$

Then $\bigcup_{n<\omega} \bigcup_{\tau \in \mathrm{Fn}(\mathscr{T},\omega)} \mathcal{V}_{n,\tau}$ is easily verified to be a σ-locally finite base for Y. $\square$

In general, productiveness is logically incomparable to amalgamativeness: the class of finite T_0 spaces is amalgamative but only finitely productive; the class of powers of 2 is productive but not amalgamative. However, all amalgamative classes are finitely productive because if $X \in \mathscr{S}$ and $|Y_S| = 1$ for all $S \in \mathscr{S} \setminus \{X\}$, then $Y \cong X \times Y_X$.

Given Theorem 2.2.2, it is tempting to conjecture that amalgams are really subspaces of products in disguise. This conjecture is false. To see this, consider the class of nonempty Urysohn spaces. This class is closed with respect to arbitrary products and subspaces, yet, as demonstrated by the following example, this class is not amalgamative.

Example 2.2.4. Let $X = \mathbb{Q}$ with the topology generated by $\{\mathbb{Q} \setminus K\}$ and the order topology of $\mathbb{Q}$ where $K = \{2^{-n} : n < \omega\}$. Then X is Urysohn. Let $\mathbb{Q} \setminus K \in \mathscr{S}$ and, for all $S \in \mathscr{S}$, let $|Y_S| = 1$ if $S \neq \mathbb{Q} \setminus K$. Set $Y_{\mathbb{Q}\setminus K} = 2$ (with the discrete topology). Then all the factors of Y are Urysohn. For each $i < 2$, define $y_i \in Y$ by $\{y_i\} = \pi^{-1}\{0\} \cap \pi_{\mathbb{Q}\setminus K}^{-1}\{i\}$. Suppose U_0 and U_1 are disjoint closed neighborhoods of y_0 and y_1, respectively. Then $\pi[U_0]$ and $\pi[U_1]$ are neighborhoods of 0. Therefore, $2^{-n} \in \overline{\pi[U_0]} \cap \overline{\pi[U_1]}$ for some $n < \omega$.

If $2^{-n} \in S \in \mathscr{S}$, then $|Y_S| = 1$; hence, $\{\pi^{-1}S : 2^{-n} \in S \in \mathscr{S}\}$ is a local subbase for y_2 where $\{y_2\} = \pi^{-1}\{2^{-n}\}$. Since $2^{-n} \in \overline{\pi[U_0]} \cap \overline{\pi[U_1]}$, every finite intersection of elements of this local subbase will intersect U_0 and U_1. Hence, $y_2 \in \overline{U_0} \cap \overline{U_1} = U_0 \cap U_1$, which is absurd. Therefore, Y is not Urysohn.

Question 2.2.5. A space is said to be *realcompact* if it is homeomorphic to a closed subspace of a power of $\mathbb{R}$. Is the class of nonempty realcompact spaces amalgamative?

Despite Example 2.2.4, there is a sense in which Y is almost homeomorphic to a subspace of the product of its factors. For each $S \in \mathscr{S}$, let Z_S be Y_S with an added point q_S whose only neighborhood is Z_S. Then Y is easily seen to be homeomorphic to the set

$$\bigcup_{p\in X}\Big\{z \in \prod_{S\in\mathscr{S}} Z_S : (\forall S \in \mathscr{S})(z(S) = q_S \Leftrightarrow p \notin S)\Big\}$$

with the subspace topology inherited from $\prod_{S\in\mathscr{S}} Z_S$. Moreover, this result still holds if we make q_S isolated for all clopen $S \in \mathscr{S}$.

Let us make some auxiliary definitions relating amalgams to continuous maps and subspaces.

Definition 2.2.6. Suppose, for each $S \in \mathscr{S}$, that Z_S is a nonempty space and $f_S\colon Y_S \to Z_S$. Let Z be the amalgam of $\langle Z_S\rangle_{S\in\mathscr{S}}$. Then the *amalgam* of $\langle f_S\rangle_{S\in\mathscr{S}}$ is the map f defined by

$$f = \bigcup_{p\in X}\prod_{p\in S\in\mathscr{S}} f_S.$$

In the above definition, it is immediate that f is a map from Y to Z. Moreover, if f_S is continuous for each $S \in \mathscr{S}$, then f is a continuous map from Y to Z. Similarly, an amalgam of homeomorphisms is a homeomorphism.

Definition 2.2.7. Suppose W is a subspace of X. The *reduced amalgam* of $\langle Y_S \rangle_{S\in\mathscr{S}}$ over W is the space Z defined as follows. Set $\mathscr{T} = \{S \cap W : S \in \mathscr{S}\} \setminus \{\emptyset\}$. Then $\mathscr{T} \in \mathbb{S}(W)$. Given $S_0, S_1 \in \mathscr{S}$, declare $S_0 \sim S_1$ if $S_0 \cap W = S_1 \cap W$. For each $T \in \mathscr{T}$, let $\varepsilon(T)$ be the unique $\mathscr{E}$ that is an equivalence class of $\sim$ for which $W \cap \bigcap \mathscr{E} = T$. For all $T \in \mathscr{T}$, set $Z_T = \prod_{S\in\varepsilon(T)} Y_S$. Let Z be the amalgam of $\langle Z_T \rangle_{T\in\mathscr{T}}$.

In the above definition, Z is homeomorphic to $\bigcup_{p\in W} \prod_{p\in S\in\mathscr{S}} Y_S$ with the subspace topology inherited from Y.

2.3 Connectifiable amalgams

Theorems 2.2.2 and 2.2.3 demonstrate similarities between products and amalgams. Of course, amalgams would not be very interesting if there were no major differences between them and products. Such differences arise for connectedness: unlike a product, an amalgam can be connected even if all its factors are not; connectedness of the base space is sufficient in most cases. Path-connectedness of an amalgam with a path-connected base space is harder to guarantee, but not by much. Some new positive connectification results fall out as corollaries.

Theorem 2.3.1. *Suppose X is connected (path-connected) and there is a finite $E \subseteq X$ such that for all $S \in \mathscr{S}$ we have $E \not\subseteq S$ or Y_S is connected (path-connected). Then Y is connected (path-connected).*

Proof. Proceed by induction on $|E|$. If $E = \emptyset$, then Y is connected (path-connected) because it is a quotient of the product of its base space and its factors, all of which are connected (path-connected).

Now suppose $E \neq \emptyset$ and the theorem holds for all smaller E. Choose $e \in E$ and set $E' = E \setminus \{e\}$. For each $S \in \mathscr{S}$, set $Z_S = Y_S$ if $e \in S$ and choose $Z_S \in [Y_S]^1$ if $e \notin S$. Hence, if $E' \subseteq S \in \mathscr{S}$, then Z_S is connected (path-connected) because either $E \subseteq S$, which implies Z_S connected (path-connected) by assumption, or $e \notin S$, which implies $|Z_S| = 1$. Let Z be the amalgam of $\langle Z_S \rangle_{S \in \mathscr{S}}$. By the induction hypothesis, Z is a connected (path-connected) subspace of Y. Suppose $y \in Y$ and choose $f \in \prod_{S \in \mathscr{S}} Y_S$ extending y. Let F be the amalgam of $\langle \{f(S)\} \rangle_{S \in \mathscr{S}}$. Then $y \in F \cong X$ and $\langle f(S) \rangle_{e \in S \in \mathscr{S}} \in F \cap Z$; hence, the component (path component) of y contains Z. Since y was chosen arbitrarily, Y is connected (path-connected). $\square$

Example 2.3.2. Suppose $X = [0,1]$ and $\mathscr{S} = \{U \subseteq [0,1] : U \text{ open}\}$ and $|Y_S| = 1$ for all $S \in \mathscr{S} \setminus \{[0,1)\}$. Then Y is homeomorphic to the cone over $Y_{[0,1)}$. If $1 \in S \in \mathscr{S}$, then $|Y_S| = 1$; hence, Theorem 2.3.1 implies Y is path-connected. Thus, Theorem 2.3.1 may be interpreted as constructing a class of generalized cones.

Corollary 2.3.3. *Suppose $i \in \{1, 2, 3, 3.5\}$ and X has a proper T_i connectification $\tilde{X}$ and Y_S is T_i for all $S \in \mathscr{S}$. Then Y has a proper T_i connectification $\tilde{Y}$. Moreover, if $\tilde{X}$ is path-connected, then we may choose $\tilde{Y}$ to be path-connected.*

Proof. Fix $p \in \tilde{X} \setminus X$. For each $S \in \mathscr{S}$, let $\Phi(S)$ be an open subset of $\tilde{X} \setminus \{p\}$ such that $\Phi(S) \cap X = S$. Extend $\Phi[\mathscr{S}]$ to some $\tilde{\mathscr{S}} \in \mathbb{S}(\tilde{X})$. For all $S \in \mathscr{S}$, set $\tilde{Y}_{\Phi(S)} = Y_S$. For all $S \in \tilde{\mathscr{S}} \setminus \Phi[\mathscr{S}]$, set $\tilde{Y}_S = 1$. Let $\tilde{Y}$ be the amalgam of $\langle \tilde{Y}_S \rangle_{S \in \tilde{\mathscr{S}}}$. By Theorem 2.2.2, $\tilde{Y}$ is T_i; by Theorem 2.3.1, $\tilde{Y}$ is connected, for $|\tilde{Y}_S| = 1$ if $p \in S \in \tilde{\mathscr{S}}$. Define $f : Y \to \tilde{Y}$ as follows. Given $y \in Y$, let $\pi(f(y)) = \pi(y)$; set $f(y)(\Phi(S)) = y(S)$ for all $S \in \operatorname{dom} y$; set $f(y)(S) = 0$ for all $S \in \tilde{\mathscr{S}} \setminus \Phi[\operatorname{dom} y]$ such that $\pi(y) \in S$. Then f is an embedding of Y into $\tilde{Y}$ with dense range $\pi^{-1}X$; hence, $\tilde{Y}$ is a proper T_i connectification of Y. Finally, by Theorem 2.3.1, $\tilde{Y}$ is path-connected if $\tilde{X}$ is. $\square$

The previously known result most similar to Corollary 2.3.3 is due to Druzhinina and Wilson [17]: if all the path components of a T_2 (T_3, metric) space are open and have proper pathwise connectifications, then the space has a T_2 (T_3, metric) proper pathwise connectification.

Proof of Theorem 2.1.1. Every infinite product is an infinite product of countably infinite subproducts; every infinite topological sum is a countably infinite topological sum of topological sums. Moreover, products preserve the property of having a T_i pathwise connectification; topological sums preserve the T_i axiom and metrizability. Therefore, we only need to prove the theorem for all countably infinite products of countably infinite topological sums. Set $X = \omega^\omega$ with the product topology. For each $m, n < \omega$, let $Z_{m,n}$ be a nonempty T_i space and let $S_{m,n} = \{p \in X : p(m) = n\}$; set $Y_{S_{m,n}} = Z_{m,n}$. Set $\mathscr{S} = \{S_{m,n} : m, n < \omega\} \in \mathbb{S}(X)$. Then $Y \cong \prod_{m<\omega} \bigoplus_{n<\omega} Z_{m,n}$ is witnessed by the map $\langle\langle y(S_{m,\pi(y)(m)})\rangle_{m<\omega}\rangle_{y\in Y}$. Since $X \cong \mathbb{R} \setminus \mathbb{Q}$, there is a proper metrizable pathwise connectification of X, namely a copy of $\mathbb{R}$. By Corollary 2.3.3, Y has a proper T_i pathwise connectification. For the metrizable case, construct a connectification $\tilde{Y}$ of Y as in the proof of Corollary 2.3.3, with $\tilde{X}$ chosen to be homeomorphic to $\mathbb{R}$. Since $\mathscr{S}$ is countable, the space $\tilde{Y}$ is metrizable by Theorem 2.2.3. $\square$

If we care about connectedness but not path-connectedness, then Theorem 2.3.1 and Corollary 2.3.3 can be considerably strengthened.

Theorem 2.3.4. *Suppose X is connected and either $X \notin \mathscr{S}$ or Y_X is connected. Then Y is connected.*

Proof. Let $y_0, y_1 \in Y$. It suffices to show y_1 is in the closure of the component of y_0. Let U be a basic open neighborhood of y_1. Then there exist $n < \omega$ and $\langle S_i \rangle_{i<n} \in (\operatorname{dom} y_1)^n$ and $\langle U_i \rangle_{i<n}$ such that U_i is an open neighborhood of $y_1(S_i)$ for all $i < n$ and $U = \bigcap_{i<n} \pi_{S_i}^{-1} U_i$. Then there exists $E \subseteq X$ such that E is finite and $E \not\subseteq S$ for all $S \in \{S_i : i < n\} \setminus \{X\}$. Choose $f \in \prod_{S \in \mathscr{S}} Y_S$ extending y_0. For each $S \in \mathscr{S}$, set $Z_S = Y_S$ if Y_S is connected or $S \in \{S_i : i < n\}$; otherwise, set $Z_S = \{f(S)\}$. Let Z be the amalgam of $\langle Z_S \rangle_{S \in \mathscr{S}}$. Then Z is connected by Theorem 2.3.1. Moreover, $y_0 \in Z$ and $Z \cap U \neq \emptyset$. Thus, y_1 is in the closure of the component of y_0. □

Corollary 2.3.5. *Suppose $i \in \{1, 2, 3, 3.5\}$ and X has a T_i connectification and Y_S is T_i for all $S \in \mathscr{S}$. Further suppose X has a proper T_i connectification or $X \notin \mathscr{S}$ or Y_X is connected. Then Y has a T_i connectification.*

Proof. If X has a proper T_i connectification, then so does Y by Corollary 2.3.3. If X is T_i and connected but has no proper T_i connectification, then Y is connected by Theorem 2.3.4. □

2.4 A new homogeneous compactum

Definition 2.4.1. We say that a homogeneous compactum is *exceptional* if it is not homeomorphic to a product of dyadic compacta and first countable compacta.

In the previous section, we constructed a machine for strengthening connectification results. Next, we construct a machine that takes a homogeneous compactum and produces a path-connected homogeneous compactum. Applying this machine to a particular homogeneous compactum with cellularity $\mathfrak{c}$, we get a path-connected homogeneous compactum with cellularity $\mathfrak{c}$. Moreover, more careful analysis of the latter space's connectedness properties shows that it is exceptional.

All compact groups are dyadic (Kuz′minov [41]), and most other known examples of homogeneous compacta are products of first countable compacta (see Kunen [37] and van Mill [46]). Besides the exceptional homogeneous compactum we shall construct, there is, to the best of the author's knowledge, only one known construction of an exceptional homogeneous compactum, and its soundness is independent of ZFC. In [46], van Mill constructed a compactum K satisfying $\pi(K) = \omega$ (where $\pi(\cdot)$ here denotes π-weight) and $\chi(K) = \omega_1$. Clearly, $\chi(Z) = \omega \leq \pi(Z)$ for all first countable spaces Z. Moreover, Efimov [18] and Gerlits [24] independently proved that $\pi\chi(Z) = w(Z)$ for all dyadic compacta Z. Hence, $\chi(Z) \leq \pi(Z)$ for all Z homeomorphic to products of dyadic compacta and first countable compacta; hence, K is not homeomorphic to such a product. Under the assumption $\mathfrak{p} > \omega_1$ (which follows from MA$+\neg$CH), van Mill proved that K is homogeneous. However, van Mill also noted that all homogeneous compacta Z satisfy $2^{\chi(Z)} \leq 2^{\pi(Z)}$ as a corollary of a result of Van Douwen [15]. In particular, if $2^\omega < 2^{\omega_1}$, then K is not homogeneous.

Remark 2.4.2. Hart and Ridderbos' [27] generalization of van Mill's construction produces only compacta that have the properties of K listed above. However, van Mill's K is infinite dimensional, while Hart and Ridderbos produce a zero-dimensional example. It is not clear whether there is a consistently homogeneous compactum Z satisfying $0 < \operatorname{ind} Z < \omega$ and $\pi(Z) < \chi(Z)$. Our machine for producing path-connected homogeneous compacta will get us $0 < \operatorname{ind} Z < \omega$, but it will also be easy to see that it entails $\pi\chi(Z) \geq \mathfrak{c}$.

Definition 2.4.3. Given a group G acting on a set A with element a, let the *stabilizer* of a in G denote $\{g \in G : ga = a\}$.

Definition 2.4.4. Given a topological space Z, let $\mathrm{Aut}(Z)$ denote the group of autohomeomorphisms of Z. Let $\mathrm{Aut}(Z)$ act on Z in the natural way: $gz = g(z)$ for all $z \in Z$ and $g \in \mathrm{Aut}(Z)$. Let $\mathrm{Aut}(Z)$ act on $\mathcal{P}(\mathcal{P}(Z))$ such that $g\mathcal{E} = \{g[E] : E \in \mathcal{E}\}$ for all $\mathcal{E} \subseteq \mathcal{P}(Z)$ and $g \in \mathrm{Aut}(Z)$.

Lemma 2.4.5. *Let G be the stabilizer of $\mathscr{S}$ in $\mathrm{Aut}(X)$. Suppose Z is a homogeneous space and $Y_S = Z$ for all $S \in \mathscr{S}$. Further suppose G acts transitively on X. Then Y is homogeneous.*

Proof. Let $y_0, y_1 \in Y$. Choose $g \in G$ such that $g(\pi(y_0)) = \pi(y_1)$. Define $f\colon Y \to Y$ as follows. Given $y \in Y$, let $\pi(f(y)) = g(\pi(y))$ and $f(y)(gS) = y(S)$ for all $S \in \operatorname{dom} y$. Then $f \in \mathrm{Aut}(Y)$ because $f[\pi_S^{-1}U] = \pi_{gS}^{-1}U$ and $f^{-1}(\pi_S^{-1}U) = \pi_{g^{-1}S}^{-1}U$ for all $S \in \mathscr{S}$ and U open in Z. Since $y_1, f(y_0) \in Z^{\operatorname{dom} y_1}$, there exists $\langle h_S\rangle_{S\in\mathscr{S}} \in \mathrm{Aut}(Z)^{\mathscr{S}}$ such that $\left(\prod_{S\in\operatorname{dom} y_1} h_S\right)(f(y_0)) = y_1$. Let h be the amalgam of $\langle h_S\rangle_{S\in\mathscr{S}}$. Then $h \in \mathrm{Aut}(Y)$ and $h(f(y_0)) = y_1$. Thus, Y is homogeneous. □

Lemma 2.4.6. *Suppose X and Y_S are T_3 and $\operatorname{ind} Y_S = 0$ for all $S \in \mathscr{S}$. Then $\operatorname{ind} Y = \operatorname{ind} X$.*

Proof. Set $n = \operatorname{ind} X$. By (7) of Theorem 2.2.2, we may assume $n > 0$. We may also assume the lemma holds if X is replaced by a T_3 space with small inductive dimension less than n. First, Y is T_3 by Theorem 2.2.2. Next, given any $f \in \prod_{S\in\mathscr{S}} Y_S$, the amalgam of $\langle\{f(S)\}\rangle_{S\in\mathscr{S}}$ is homeomorphic to X; hence, $\operatorname{ind} Y \geq n$. Let $y \in Y$ and let U be an open neighborhood of y. Then $y \in V_0 \subseteq U$ where $V_0 = \bigcap_{i<m} \pi_{S_i}^{-1} U_i$ for some $m < \omega$ and $\langle S_i\rangle_{i<m} \in (\operatorname{dom} y)^m$ and $\langle U_i\rangle_{i<m}$ such that U_i is a clopen neighborhood of

$y(S_i)$ for all $i < m$. Let W be an open neighborhood of $\pi(y)$ such that $W \subseteq \bigcap_{i<m} S_i$ and $\operatorname{ind} \partial W < n$. Set $V_1 = V_0 \cap \pi^{-1} W$.

It suffices to show that $\operatorname{ind} \partial V_1 < n$. Set $V_2 = \pi^{-1}\partial W$. Then $\partial V_1 = V_0 \cap V_2$; hence, it suffices to show that $\operatorname{ind} V_2 < n$. Let Z be the reduced amalgam of $\langle Y_S \rangle_{S \in \mathscr{S}}$ over ∂W. Then $Z \cong V_2$ and $\operatorname{ind} Z = \operatorname{ind} \partial W$ because $\operatorname{ind} \partial W < n$ and every factor of Z, being a product of factors of Y, has small inductive dimension 0. □

Theorem 2.4.7. *There is a path-connected homogeneous compact Hausdorff space Y with cellularity $\mathfrak{c}$, weight $\mathfrak{c}$, and small inductive dimension* 1. *Moreover, Y is not homeomorphic to a product of compacta that all have character less than $\mathfrak{c}$ or have* $\operatorname{cf}(\mathfrak{c})$ *a caliber. In particular, Y is exceptional.*

Proof. Let X be the unit circle $\{\langle x, y \rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $\mathscr{S}$ be the set of open semicircles contained in X. Let γ be an indecomposable ordinal (*i.e.*, not a sum of two lesser ordinals) strictly between ω and ω_1. For each $S \in \mathscr{S}$, let Y_S be 2^γ with the topology induced by its lexicographic ordering. It is easily seen that Y_S is zero-dimensional compact Hausdorff and $w(Y_S) = c(Y_S) = \mathfrak{c}$. Moreover, Y_S is homogeneous [44]. Since $|\mathscr{S}| = \mathfrak{c}$, we have $w(Y) = c(Y) = \mathfrak{c}$. Moreover, Y is compact Hausdorff by Theorem 2.2.2. Since no $S \in \mathscr{S}$ contains a pair of antipodes, Y is path-connected by Theorem 2.3.1. The stabilizer of $\mathscr{S}$ in $\operatorname{Aut}(X)$ contains all the rotations of X and therefore acts transitively on X; hence, Y is homogeneous by Lemma 2.4.5. Also, by Lemma 2.4.6, $\operatorname{ind} Y = \operatorname{ind} X = 1$.

Seeking a contradiction, suppose Y is homeomorphic to a product of compacta that all have character less than $\mathfrak{c}$ or have $\operatorname{cf}(\mathfrak{c})$ a caliber. Then there exist a compactum Z with $\operatorname{cf}(\mathfrak{c})$ a caliber, a sequence of nonsingleton compacta $\langle W_i \rangle_{i \in I}$ all with character less than $\mathfrak{c}$, and a homeomorphism φ from $Z \times \prod_{i \in I} W_i$ to Y. Clearly, W_i is path-connected for all $i \in I$. Choose $p \in X$. Then $\varphi^{-1}\pi^{-1}\{p\}$ is a G_δ-set; hence, there exist a nonempty

$Z_0 \subseteq Z$ and $J \in [I]^{\leq\omega}$ and $q \in \prod_{j\in J} W_j$ such that $Z_0 \times \{q\} \times \prod_{i\in I\setminus J} W_i \subseteq \varphi^{-1}\pi^{-1}\{p\}$. Since $\pi^{-1}\{p\} = \prod_{p\in S\in\mathscr{S}} Y_S$, which is zero-dimensional, $Z_0 \times \{q\} \times \prod_{i\in I\setminus J} W_i$ is also zero-dimensional; hence, $\prod_{i\in I\setminus J} W_i$ is also zero-dimensional. Hence, W_i is not connected for all $i \in I \setminus J$; hence, $I = J$; hence, I is countable. Set $W = \prod_{i\in I} W_i$. Then $\chi(W) < \mathfrak{c}$ because $\mathrm{cf}(\mathfrak{c}) > \omega$.

Let $H \subseteq X$ be an open arc subtending $\pi/2$ radians. Set $\mathscr{T} = \{S \in \mathscr{S} : H \subseteq S\}$. Then $|\mathscr{T}| = \mathfrak{c}$. Choose a nonempty open box $U\times V \subseteq Z\times W$ such that $U\times V \subseteq \varphi^{-1}\pi^{-1}H$ and $U = \bigcup_{n<\omega} U_n$ where U_n is open and $\overline{U}_n \subseteq U_{n+1}$ for all $n < \omega$. Choose $r \in V$ and set $\kappa = \chi(r, W) < \mathfrak{c}$. Let $\langle V_\alpha \rangle_{\alpha<\kappa}$ enumerate a local base at r. By compactness, we may choose, for each $\alpha < \kappa$ and $n < \omega$, a finite set $\sigma_{n,\alpha}$ of basic open subsets of Y such that $\overline{U}_n \times \{r\} \subseteq \varphi^{-1}\bigcup\sigma_{n,\alpha} \subseteq U_{n+1} \times V_\alpha$. Set $G = \bigcup_{n<\omega}\bigcap_{\alpha<\kappa}\bigcup\sigma_{n,\alpha}$. Since $\kappa < \mathfrak{c}$, there exist nonempty $\mathscr{R} \subseteq \mathscr{T}$ and $E \subseteq \bigcup_{x\in H}\prod_{x\in S\in\mathscr{S}\setminus\mathscr{R}} Y_S$ such that $G = E \times \prod_{S\in\mathscr{R}} Y_S$. Hence, $c(G) = \mathfrak{c}$. Since $\varphi^{-1}G = U \times \{r\}$, we have $c(U) = \mathfrak{c}$. Since U is an open subset of Z, we have $c(Z) \geq \mathfrak{c}$, which yields our desired contradiction, for $\mathrm{cf}(\mathfrak{c}) \in \mathrm{cal}(Z)$. $\square$

Remark 2.4.8. If there is a homogeneous compactum with cellularity $\kappa > \mathfrak{c}$ (that is, if Van Douwen's Problem (see Kunen [37]) has a positive solution), then the proof of Theorem 2.4.7 is easily modified to produce a path-connected homogeneous compactum with cellularity κ.

It is also easy to modify the above proof so that the unit circle is replaced with an n-dimensional sphere or torus, thereby producing a Y as in Theorem 2.4.7 except it is n-dimensional. The unit circle can also be replaced by its ωth power so as to produce a Y as in Theorem 2.4.7 except it is infinite dimensional.

Chapter 3

Noetherian types of homogeneous compacta and dyadic compacta

3.1 Introduction

Van Douwen's Problem (see Kunen [37]) asks whether there is a homogeneous compactum of cellularity exceeding $\mathfrak{c}$. A homogeneous compactum of cellularity $\mathfrak{c}$ exists by Maurice [44], but Van Douwen's Problem remains open in all models of ZFC.

By Arhangel′skiĭ's Theorem, first countable compacta have size at most $\mathfrak{c}$; dyadic compacta (such as compact groups [41]) are ccc. Since the cellularity of a product space equals the supremum of the cellularities of its finite subproducts (see p. 107 of [35]), all nonexceptional homogeneous compacta have cellularity at most $\mathfrak{c}$. To the best of the author's knowledge, there are only two classes of examples of exceptional homogeneous compacta; these two kinds of spaces have cellularities ω and $\mathfrak{c}$.

(In particular, Hart and Kunen [28] have observed that by a result of Uspenskii [69], not only is every compact group dyadic, but every space (such as a compact quasigroup) that is acted on continuously and transitively by some ω-bounded group is Dugundji, which is stronger than being dyadic.)

We investigate several cardinal functions defined in terms of order-theoretic base properties. Just like cellularity, these functions have upper bounds when restricted to the class of known homogeneous compacta. Moreover, GCH implies that one of these functions is a lower bound on cellularity when restricted to homogeneous compacta.

Observation 3.1.1. Every known homogeneous compactum X satisfies the following.

1. $Nt(X) \leq \mathfrak{c}^+$.

2. $\pi Nt(X) \leq \omega_1$.

3. $\chi Nt(X) = \omega$.

4. $\chi_K Nt(X) \leq \mathfrak{c}$.

We justify this observation in Section 3.2, except that we postpone the case of homogeneous dyadic compacta to Section 3.3, where we investigate Noetherian cardinal functions on dyadic compacta in general. The results relevant to Observation 3.1.1 are summarized by the following theorem.

Theorem 3.1.2. *Suppose X is a dyadic compactum. Then $\pi Nt(X) = \chi_K Nt(X) = \omega$. Moreover, if X is homogeneous, then $Nt(X) = \omega$.*

Also in Section 3.3, we generalize the above theorem to continuous images of products of compacta with bounded weight; we also prove the following:

Theorem 3.1.3. *The class of Noetherian types of dyadic compacta includes ω, excludes ω_1, includes all singular cardinals, and includes κ^+ for all cardinals κ with uncountable cofinality.*

Section 3.4 generalizes our results about dyadic compacta to the proper superclass of k-adic compacta.

Finally, in Section 3.5, we prove several results about the local Noetherian types of all homogeneous compacta, known and unknown, including the following theorem.

Theorem 3.1.4 (GCH). *If X is a homogeneous compactum, then $\chi Nt(X) \leq c(X)$.*

3.2 Observed upper bounds on Noetherian cardinal functions

First, we note some very basic facts about Noetherian cardinal functions.

Definition 3.2.1. Given a subset E of a product $\prod_{i\in I} X_i$ and $\sigma \in [I]^{<\omega}$, we say that E has *support* σ, or $\mathrm{supp}(E) = \sigma$, if $E = \pi_\sigma^{-1}\pi_\sigma[E]$ and $E \neq \pi_\tau^{-1}\pi_\tau[E]$ for all $\tau \subsetneq \sigma$.

Theorem 3.2.2. *Given a point p and a compact subset K of a product space $X = \prod_{i\in I} X_i$, we have the following relations.*

$$Nt(X) \leq \sup_{i\in I} Nt(X_i)$$

$$\pi Nt(X) \leq \sup_{i\in I} \pi Nt(X_i)$$

$$\chi Nt(p, X) \leq \sup_{i\in I} \chi Nt(p(i), X_i)$$

$$\chi Nt(K, X) \leq \sup_{\sigma\in[I]^{<\omega}} \chi Nt(\pi_\sigma[K], \pi_\sigma[X])$$

Proof. See Peregudov [54] for a proof of the first relation. That proof can be easily modified to demonstrate the next two relations. Let us prove the last relation. For each $\sigma \in [I]^{<\omega}$, set $\kappa_\sigma = \chi Nt(\pi_\sigma[K], \pi_\sigma[X])$ and let $\mathcal{A}_\sigma$ be a $\kappa_\sigma^{\mathrm{op}}$-like neighborhood base of $\pi_\sigma[K]$. For each $\sigma \in [I]^{<\omega}$, let $\mathcal{B}_\sigma$ denote the set of sets of the form $\pi_\sigma^{-1}U$ where $U \in \mathcal{A}_\sigma$ and $\mathrm{supp}(U) = \sigma$. Note that if $U \in \mathcal{A}_\sigma$ and $\mathrm{supp}(U) \subsetneq \sigma$, then there exists

$\tau \subsetneq \sigma$ and $V \in \mathcal{A}_\tau$ such that $\pi_\tau^{-1}V \subseteq \pi_\sigma^{-1}U$. Moreover, for any minimal such τ, we have $\pi_\tau^{-1}V \in \mathcal{B}_\tau$.

Set $\mathcal{B} = \bigcup_{\sigma \in [I]^{<\omega}} \mathcal{B}_\sigma$. By compactness, $\mathcal{B}$ is a neighborhood base of K. Moreover, if $\sigma, \tau \in [I]^{<\omega}$ and $\mathcal{B}_\sigma \ni U \subseteq V \in \mathcal{B}_\tau$, then $\sigma = \operatorname{supp}(U) \supseteq \operatorname{supp}(V) = \tau$; hence, given U, there are at most $(\sup_{\tau \subseteq \sigma} \kappa_\tau)$-many possibilities for V. Thus, $\mathcal{B}$ is $(\sup_{\sigma \in [I]^{<\omega}} \kappa_\sigma)^{\mathrm{op}}$-like as desired. □

Lemma 3.2.3. *Every poset P is almost $|P|^{\mathrm{op}}$-like.*

Proof. Let $\kappa = |P|$ and let $\langle p_\alpha \rangle_{\alpha<\kappa}$ enumerate P. Define a partial map $f \colon \kappa \to P$ as follows. Suppose $\alpha < \kappa$ and we have a partial map $f_\alpha \colon \alpha \to P$. If $\operatorname{ran} f_\alpha$ is dense in P, then set $f_{\alpha+1} = f_\alpha$. Otherwise, set $\beta = \min\{\delta < \kappa : p_\delta \not\geq q \text{ for all } q \in \operatorname{ran} f_\alpha\}$ and let $f_{\alpha+1}$ be the smallest map extending f_α such that $f_{\alpha+1}(\alpha) = p_\beta$. For limit ordinals $\gamma \leq \kappa$, set $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$. Then f_κ is nonincreasing; hence, $\operatorname{ran} f_\kappa$ is κ^{op}-like. Moreover, $\operatorname{ran} f_\kappa$ is dense in P. □

Theorem 3.2.4. *For any space X with point p, we have*

- $\chi Nt(p, X) \leq \chi(p, X)$,
- $\pi Nt(X) \leq \pi(X)$,
- $Nt(X) \leq w(X)^+$, *and*
- $\chi_K Nt(X) \leq w(X)$.

Proof. The first two relations immediately follow from Lemma 3.2.3; the third relation is trivial. For the last relation, note that if K is a compact subset of X, then it has a neighborhood base of size at most $w(X)$; apply Lemma 3.2.3. □

Given Theorem 3.2.2, justifying Observation 3.1.1 for $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$ amounts to justifying it for first countable homogeneous compacta, dyadic homogeneous compacta, and the two known kinds of exceptional homogeneous compacta. The first countable case is the easiest. By Arhangel′skiĭ's Theorem, first countable compacta have weight at most $\mathfrak{c}$, and therefore have Noetherian type at most $\mathfrak{c}^+$. Moreover, every point in a first countable space clearly has an ω^{op}-like local base. The only nontrivial bound is the one on Noetherian π-type. For that, the following theorem suffices.

Definition 3.2.5. Give a space X, let $\pi sw(X)$ denote the least κ such that X has a π-base $\mathcal{A}$ such that $\bigcap \mathcal{B} = \emptyset$ for all $\mathcal{B} \in [\mathcal{A}]^{\kappa^+}$.

Theorem 3.2.6. *If X is a compactum, then $\pi Nt(X) \leq \pi sw(X)^+ \leq t(X)^+ \leq \chi(X)^+$.*

Proof. Only the second relation is nontrivial; it is a theorem of Šapirovskiĭ [60]. □

For dyadic homogeneous compacta, it is trivially seen that Theorem 3.1.2 implies Observation 3.1.1; we will prove this theorem in Section 3.3. Now consider the two known classes of exceptional homogeneous compacta. They are constructed by two techniques, resolutions and amalgams. First we consider the exceptional resolution.

Definition 3.2.7. Suppose X is a space, $\langle Y_p \rangle_{p \in X}$ is a sequence of nonempty spaces, and $\langle f_p \rangle_{p \in X} \in \prod_{p \in X} C(X \setminus \{p\}, Y_p)$. Then the *resolution* Z of X at each point p into Y_p by f_p is defined by setting $Z = \bigcup_{p \in X} (\{p\} \times Y_p)$ and declaring Z to have weakest topology such that, for every $p \in X$, open neighborhood U of p in X, and open $V \subseteq Y_p$, the set $U \otimes V$ is open in Z where

$$U \otimes V = (\{p\} \times V) \cup \bigcup_{q \in U \cap f_p^{-1} V} (\{q\} \times Y_q).$$

The resolution of concern to us in constructed by van Mill [46]. It is a compactum with weight $\mathfrak{c}$, π-weight ω, and character ω_1. Moreover, assuming MA + $\neg$CH (or just $\mathfrak{p} > \omega_1$), this space is homogeneous. (It is not homogeneous if $2^\omega < 2^{\omega_1}$.) Clearly, this space has sufficiently small Noetherian type and π-type. We just need to show that it has local Noetherian type ω. Van Mill's space is a resolution of 2^ω at each point into $\mathbb{T}^{\omega_1}$ where $\mathbb{T}$ is the circle group $\mathbb{R}/\mathbb{Z}$.

Notice that $\mathbb{T}$ is metrizable. The following lemma proves that every metric compactum has Noetherian type ω, along with some results that will be useful in Section 3.3.

Lemma 3.2.8. *Let X be a metric compactum with base $\mathcal{A}$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ satisfying the following.*

1. *$\mathcal{B}$ is a base of X.*
2. *$\mathcal{B}$ is ω^{op}-like.*
3. *If $U, V \in \mathcal{B}$ and $U \subsetneq V$, then $\overline{U} \subseteq V$.*
4. *For all $\Gamma \in [\mathcal{B}]^{<\omega}$, there are only finitely many $U \in \mathcal{B}$ such that Γ contains $\{V \in \mathcal{B} : U \subsetneq V\}$.*

Proof. Construct a sequence $\langle \mathcal{B}_n \rangle_{n<\omega}$ of finite subsets of $\mathcal{A}$ as follows. For each $n < \omega$, let E_n be the union of the set of all singletons in $\bigcup_{m<n} \mathcal{B}_m$. Let $\mathcal{C}_n$ be the set of all $U \in \mathcal{A}$ for which $U \cap E_n = \emptyset$ and

$$2^{-n} \geq \operatorname{diam} U < \min\left\{\operatorname{diam} V : V \in \bigcup_{m<n} \mathcal{B}_m \text{ and } 0 < \operatorname{diam} V\right\}$$

and $\overline{U} \subseteq V$ for all $V \in \bigcup_{m<n} \mathcal{B}_m$ strictly containing U. Then $\bigcup \mathcal{C}_n = X \setminus E_n$. Let $\mathcal{B}_n$ be a minimal finite subcover of $\mathcal{C}_n$. Set $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$. To prove (3), suppose $U \in \mathcal{B}_n$

and $V \in \mathcal{B}_m$ and $U \subsetneq V$. Then $m \neq n$ by minimality of $\mathcal{B}_n$. Also, $0 < \operatorname{diam} V$ because $\emptyset \neq U \subsetneq V$. Hence, if $m > n$, then $\operatorname{diam} V < \operatorname{diam} U$, in contradiction with $U \subsetneq V$. Hence, $m < n$; hence, $\overline{U} \subseteq V$.

For (1), let $p \in X$ and $n < \omega$, and let V be the open ball with radius 2^{-n} and center p. Then we just need to show that there exists $U \in \mathcal{B}$ such that $p \in U \subseteq V$. Hence, we may assume $\{p\} \notin \mathcal{B}$. Hence, $p \notin E_{n+1}$; hence, there exists $U \in \mathcal{B}_{n+1}$ such that $p \in U$. Since $\operatorname{diam} U \leq 2^{-n-1}$, we have $U \subseteq V$.

For (2), let $n < \omega$ and $U \in \mathcal{B}_n$. If U is a singleton, then every superset of U in $\mathcal{B}$ is in $\bigcup_{m \leq n} \mathcal{B}_m$. If U is not a singleton, then U has diamater at least 2^{-m} for some $m < \omega$; whence, every superset of U in $\mathcal{B}$ is in $\bigcup_{l \leq m} \mathcal{B}_l$.

For (4), suppose $\Gamma \in [\mathcal{B}]^{<\omega}$ and there exist infinitely many $U \in \mathcal{B}$ such that $\{V \in \mathcal{B} : U \subsetneq V\} \subseteq \Gamma$. We may assume Γ contains no singletons. Choose an increasing sequence $\langle k_n \rangle_{n<\omega}$ in ω such that, for all $n < \omega$, there exists $U_n \in \mathcal{B}_{k_n}$ such that $\{V \in \mathcal{B} : U_n \subsetneq V\} \subseteq \Gamma$. For each $n < \omega$, choose $p_n \in U_n$. Since $\{U_n : n < \omega\}$ is infinite, we may choose $\langle p_n \rangle_{n<\omega}$ such that $\{p_n : n < \omega\}$ is infinite. Let p be an accumulation point of $\{p_n : n < \omega\}$. Choose $m < \omega$ such that $2^{-m} < \operatorname{diam} V$ for all $V \in \Gamma$. Since p is not an isolated point, there exists $W \in \mathcal{B}_m$ such that $p \in W$. Then $W \notin \Gamma$; hence, W does not strictly contain U_n for any $n < \omega$. Choose $q \in W \setminus \{p\}$ such that W contains $\{x : d(p,x) \leq d(p,q)\}$; set $r = d(p,q)$. Let B be the open ball of radius $r/2$ centered about p. Then there exists $n < \omega$ such that $2^{-k_n} < r/2$ and $p_n \in B$. Hence, $\operatorname{diam} U_n < r/2$ and $U_n \cap B \neq \emptyset$; hence, $U_n \subseteq W$ and $q \notin U_n$; hence, $U_n \subsetneq W$, which is absurd. Therefore, for each $\Gamma \in [\mathcal{B}]^{<\omega}$, there are only finitely many $U \in \mathcal{B}$ such that $\{V \in \mathcal{B} : U \subsetneq V\} \subseteq \Gamma$. $\square$

We have $Nt(2^\omega) = Nt(\mathbb{T}^{\omega_1}) = \omega$ by Lemma 3.2.8 and Theorem 3.2.2. Therefore, the following theorem implies that van Mill's space has local Noetherian type ω.

Lemma 3.2.9 ([46]). *Suppose X, $\langle Y_p\rangle_{p\in X}$, $\langle f_p\rangle_{p\in X}$, and Z are as in Definition 3.2.7. Suppose $\mathcal{U}$ is a local base at a point p in X and $\mathcal{V}$ is a local base at a point y in Y_p. Then $\{U\otimes V : \langle U,V\rangle\in\mathcal{U}\times(\mathcal{V}\cup\{Y_p\})\}$ is a local base at $\langle p,y\rangle$ in Z.*

Theorem 3.2.10. *Suppose X, $\langle Y_p\rangle_{p\in X}$, $\langle f_p\rangle_{p\in X}$, and Z are as in Definition 3.2.7. Then $\chi Nt(\langle p,y\rangle, Z)\leq Nt(X)\chi Nt(y,Y_p)$ for all $\langle p,y\rangle\in Z$.*

Proof. Set $\kappa = Nt(X)\chi Nt(y,Y_p)$. Let $\mathcal{A}$ be a κ^{op}-like base of X and let $\mathcal{B}$ be a κ^{op}-like local base at y in Y_p; we may assume $Y_p\in\mathcal{B}$. Set $\mathcal{C}=\{U\in\mathcal{A} : p\in U\}$. Set $\mathcal{D}=\{U\otimes V : \langle U,V\rangle\in\mathcal{C}\times\mathcal{B}\}$, which is a local base at $\langle p,y\rangle$ in Z by Lemma 3.2.9. If there exists $U\otimes V\in\mathcal{D}$ such that $U\cap f_p^{-1}V=\emptyset$, then $U\otimes V$ is homeomorphic to V; whence, $\chi Nt(\langle p,y\rangle,Z)=\chi Nt(y,Y_p)\leq\kappa$. Hence, we may assume $U\cap f_p^{-1}V\neq\emptyset$ for all $U\otimes V\in\mathcal{D}$.

It suffices to show that $\mathcal{D}$ is κ^{op}-like. Suppose $U_i\otimes V_i\in\mathcal{D}$ for all $i<2$ and $U_0\otimes V_0\subseteq U_1\otimes V_1$. Then $V_0\subseteq V_1$ and $\emptyset\neq U_0\cap f_p^{-1}V_0\subseteq U_1\cap f_p^{-1}V_1$. Since $\mathcal{B}$ is κ^{op}-like, there are fewer than κ-many possibilities for V_1 given V_0. Since $\mathcal{A}$ is a κ^{op}-like base, there are fewer than κ-many possibilities for U_1 given U_0 and V_0. Hence, there are fewer than κ-many possibilities for $U_1\otimes V_1$ given $U_0\otimes V_0$. □

Definition 3.2.11. Let $\mathfrak{p}$ denote the least κ for which some $\mathcal{A}\in[[\omega]^\omega]^\kappa$ has the strong finite intersection property but does not have a nontrivial pseudointersection. By a theorem of Bell [10], $\mathfrak{p}$ is also the least κ for which there exist a σ-centered poset $\mathbb{P}$ and a family $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$ such that $\mathbb{P}$ does not have a filter that meets every set in $\mathcal{D}$.

Definition 3.2.12. Given a space X, let $\mathrm{Aut}(X)$ denote the set of its autohomeomorphisms.

Van Mill's construction has been generalized by Hart and Ridderbos [27]. They show that one can produce an exceptional homogeneous compactum with weight $\mathfrak{c}$ and π-weight ω by carefully resolving each point of 2^ω into a fixed space Y satisfying the following conditions.

1. Y is a homogeneous compactum.
2. $\omega_1 \leq \chi(Y) \leq w(Y) < \mathfrak{p}$.
3. $\exists d \in Y \ \ \exists \eta \in \mathrm{Aut}(Y) \ \ \overline{\{\eta^n(d) : n < \omega\}} = Y$.
4. If $\gamma\omega$ is a compactification of ω and $\gamma\omega \setminus \omega \cong Y$, then Y is a retract of $\gamma\omega$.

By Theorem 3.2.10, to show that such resolutions have local Noetherian type ω, it suffices to show that every such Y has local Noetherian type ω. Theorem 3.2.15 will accomplish this.

Theorem 3.2.13. *Suppose X is a compactum and $\pi\chi(p, X) = \chi(q, X)$ for all $p, q \in X$. Then $\chi Nt(p, X) = \omega$ for some $p \in X$. In particular, if X is a homogeneous compactum and $\pi\chi(X) = \chi(X)$, then $\chi Nt(X) = \omega$.*

The proof of Theorem 3.2.13 will be delayed until Section 3.5.

The following lemma is essentially a generalization of a similar result of Juhász [36].

Lemma 3.2.14. *Suppose X is a compactum and $\omega = d(X) \leq w(X) < \mathfrak{p}$. Then there exists $p \in X$ such that $\chi(p, X) \leq \pi(X)$.*

Proof. Let $\mathcal{A}$ be a base of X of size at most $w(X)$. Let $\mathcal{B}$ be a π-base of X of size at most $\pi(X)$. For each $\langle U,V\rangle \in \mathcal{B}^2$ satisfying $\overline{U} \subseteq V$, choose a closed G_δ-set $\Phi(U,V)$ such that $\overline{U} \subseteq \Phi(U,V) \subseteq V$. Then $\operatorname{ran}\Phi$, ordered by $\subseteq$, is σ-centered because $d(X)=\omega$. Since $|\mathcal{A}| < \mathfrak{p}$, there is a filter $\mathcal{G}$ of $\operatorname{ran}\Phi$ such that for all disjoint $U,V \in \mathcal{A}$ some $K \in \mathcal{G}$ satisfies $U \cap K = \emptyset$ or $V \cap K = \emptyset$. Hence, there exists a unique $p \in \bigcap \mathcal{G}$. Hence, p has pseudocharacter, and therefore character, at most $|\mathcal{G}|$, which is at most $\pi(X)$. □

Theorem 3.2.15. *If X is a homogeneous compactum and $\omega = d(X) \le w(X) < \mathfrak{p}$, then $\chi Nt(X) = \omega$.*

Proof. By Lemma 3.2.14, $\chi(X) \le \pi(X) = \pi\chi(X)d(X) = \pi\chi(X)$. Hence, by Theorem 3.2.13, $\chi Nt(X) = \omega$. □

Recall the most basic definitions and notation for amalgams.

Definition 3.2.16. Suppose X is a T_0 space, $\mathscr{S}$ is a subbase of X such that $\emptyset \notin \mathscr{S}$, and $\langle Y_S\rangle_{S\in\mathscr{S}}$ is a sequence of nonempty spaces. The *amalgam* Y of $\langle Y_S : S \in \mathscr{S}\rangle$ is defined by setting $Y = \bigcup_{p\in X} \prod_{p\in S\in\mathscr{S}} Y_S$ and declaring Y to have the weakest topology such that, for each $S \in \mathscr{S}$ and open $U \subseteq Y_S$, the set $\pi_S^{-1}U$ is open in Y where $\pi_S^{-1}U = \{p \in Y : S \in \operatorname{dom} p \text{ and } p(S) \in U\}$. Define $\pi : Y \to X$ by $\{\pi(p)\} = \bigcap \operatorname{dom} p$ for all $p \in Y$. It is easily verified that π is continuous.

Theorem 3.2.17. *Suppose X, $\mathscr{S}$, $\langle Y_S\rangle_{S\in\mathscr{S}}$, and Y are as in Definition 3.2.16. Then we have the following relations for all $p \in Y$.*

$$Nt(Y) \le Nt(X) \sup_{S\in\mathscr{S}} Nt(Y_S)$$

$$\pi Nt(Y) \le \pi Nt(X) \sup_{S\in\mathscr{S}} \pi Nt(Y_S)$$

$$\chi Nt(p,Y) \le \chi Nt(\pi(p),X) \sup_{S\in\operatorname{dom} p} \chi Nt(p(S),Y_S)$$

Proof. We will only prove the first relation; the proofs of the others are almost identical. Set $\kappa = Nt(X)\sup_{S\in\mathscr{S}} Nt(Y_S)$. Let $\mathcal{A}$ be a κ^{op}-like base of X. For each $S \in \mathscr{S}$, let $\mathcal{B}_S$ be a κ^{op}-like base of Y_S. Set

$$\mathcal{C} = \left\{\pi^{-1}U \cap \bigcap_{S\in\operatorname{dom}\tau} \pi_S^{-1}\tau(S) : \tau \in \bigcup_{\mathcal{F}\in[\mathscr{S}]^{<\omega}} \prod_{S\in\mathcal{F}} \mathcal{B}_S \setminus \{Y_S\} \text{ and } \mathcal{A} \ni U \subseteq \bigcap \operatorname{dom}\tau\right\}.$$

Then $\mathcal{C}$ is clearly a base of Y. Let us show that $\mathcal{C}$ is κ^{op}-like. Suppose $\pi^{-1}U_i \cap \bigcap_{S\in\operatorname{dom}\tau_i} \pi_S^{-1}\tau_i(S) \in \mathcal{C}$ for all $i < 2$ and

$$\pi^{-1}U_0 \cap \bigcap_{S\in\operatorname{dom}\tau_0} \pi_S^{-1}\tau_0(S) \subseteq \pi^{-1}U_1 \cap \bigcap_{S\in\operatorname{dom}\tau_1} \pi_S^{-1}\tau_1(S).$$

Then $U_0 \subseteq U_1$ and $\operatorname{dom}\tau_0 \supseteq \operatorname{dom}\tau_1$ and $\tau_0(S) \subseteq \tau_1(S)$ for all $S \in \operatorname{dom}\tau_1$. Hence, there are fewer than κ-many possibilities for U_1 and τ_1 given U_0 and τ_0. □

An exceptional homogeneous compactum Y is constructed with $X = \mathbb{T}$ and $w(Y_S) = \pi(Y_S) = \mathfrak{c}$ and $\chi(Y_S) = \omega$ for all $S \in \mathscr{S}$. Hence, $Nt(Y_S) \leq \mathfrak{c}^+$ and $\chi Nt(Y_S) = \omega$ for each $S \in \mathscr{S}$. Moreover, each Y_S is 2^γ_{lex} (*i.e.*, 2^γ ordered lexicographically) where γ is a fixed indecomposable ordinal in $\omega_1 \setminus (\omega + 1)$. Since $\operatorname{cf}\gamma = \omega$, it is easy to construct an ω^{op}-like π-base of this space. Hence, by Theorem 3.2.17, $Nt(Y) \leq \mathfrak{c}^+$ and $\pi Nt(Y) = \chi Nt(Y) = \omega$. Thus, Observation 3.1.1 is justified for $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$.

It remains to justify Observation 3.1.1 for $\chi_K Nt(\cdot)$. We first note that all known homogeneous compacta are continuous images of products of compacta each of weight at most $\mathfrak{c}$. (Moreover, any Z as in Definition 3.2.16 is a continuous image of $X \times \prod_{S\in\mathscr{S}} Y_S$.) Therefore, the following theorem will suffice.

Theorem 3.2.18. *Suppose Y is a continuous image of a product $X = \prod_{i\in I} X_i$ of compacta. Then $\chi_K Nt(Y) \leq \sup_{i\in I} w(X_i)$*

Before proving the above theorem, we first prove two lemmas.

Definition 3.2.19. Given subsets P and Q of a common poset, define P and Q to be *mutually dense* if for all $p_0 \in P$ and $q_0 \in Q$ there exist $p_1 \in P$ and $q_1 \in Q$ such that $p_0 \geq q_1$ and $q_0 \geq p_1$.

Lemma 3.2.20. *Let κ be a cardinal and let P and Q be mutually dense subsets of a common poset. Then P is almost κ^{op}-like if and only if Q is.*

Proof. Suppose D is a κ^{op}-like dense subset of P. Then it suffices to construct a κ^{op}-like dense subset of Q. Define a partial map f from $|D|^+$ to Q as follows. Set $f_0 = \emptyset$. Suppose $\alpha < |D|^+$ and we have constructed a partial map f_α from α to Q. Set $E = \{d \in D : d \not\geq q \text{ for all } q \in \operatorname{ran} f_\alpha\}$. If $E = \emptyset$, then set $f_{\alpha+1} = f_\alpha$. Otherwise, choose $q \in Q$ such that $q \leq e$ for some $e \in E$, and let $f_{\alpha+1}$ be the smallest function extending f_α such that $f_{\alpha+1}(\alpha) = q$. For limit ordinals $\gamma \leq |D|^+$, set $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$. Set $f = f_{|D|^+}$.

Let us show that $\operatorname{ran} f$ is κ^{op}-like. Suppose otherwise. Then there exists $q \in \operatorname{ran} f$ and an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa}$ in $\operatorname{dom} f$ such that $q \leq f(\xi_\alpha)$ for all $\alpha < \kappa$. By the way we constructed f, there exists $\langle d_\alpha \rangle_{\alpha<\kappa} \in D^\kappa$ such that $f(\xi_\beta) \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. Then $d \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$, which contradicts that D is κ^{op}-like. Therefore, $\operatorname{ran} f$ is κ^{op}-like.

Finally, let us show that $\operatorname{ran} f$ is a dense subset of Q. Suppose $q \in Q$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. By the way we constructed f, there exists $r \in \operatorname{ran} f$ such that $r \leq d$; hence, $r \leq q$. $\square$

Lemma 3.2.21. *Suppose $f\colon X \to Y$ is a continuous surjection between compacta and C is closed in Y. Then $\chi Nt(f^{-1}C, X) = \chi Nt(C, Y)$.*

Proof. Let $\mathcal{A}$ be a neighborhood base of C. By Lemma 3.2.20, it suffices to show that $\{f^{-1}V : V \in \mathcal{A}\}$ is a neighborhood base of $f^{-1}C$. Suppose U is a neighborhood of $f^{-1}C$. By normality of Y, we have $f^{-1}C = \bigcap_{V\in\mathcal{A}} f^{-1}\overline{V}$. By compactness of X, we have $f^{-1}\overline{V} \subseteq U$ for some $V \in \mathcal{A}$. Thus, $\{f^{-1}V : V \in \mathcal{A}\}$ is a neighborhood base of $f^{-1}C$ as desired. □

Proof of Theorem 3.2.18. By Lemma 3.2.21, we may assume $Y = X$. By Theorem 3.2.2, we may assume I is finite. Apply Theorem 3.2.4. □

How sharp are the bounds of Observation 3.1.1? (3) is trivially sharp as every space has local Noetherian type at least ω. We will show that there is a homogeneous compactum with Noethian type $\mathfrak{c}^+$, namely, the double arrow space. Moreover, we will show that Suslin lines have uncountable Noetherian π-type. It is known to be consistent that there are homogeneous compact Suslin lines, but it is also known to be consistent that there are no Suslin lines. It is not clear whether it is consistent that all homogeneous compacta have Noetherian π-type ω, even if we restrict to the first countable case. Also, it is not clear in any model of ZFC whether all homogeneous compacta have compact Noetherian type ω, even if we restrict to the first countable case.

The following proposition is essentially due to Peregudov [54].

Proposition 3.2.22. *If X is a space and $\pi(X) < \operatorname{cf}\kappa \leq \kappa \leq w(X)$, then $Nt(X) > \kappa$.*

Proof. Suppose $\mathcal{A}$ is a base of X and $\mathcal{B}$ is π-base of X of size $\pi(X)$. Then $|\mathcal{A}| \geq \kappa$; hence, there exist $\mathcal{U} \in [\mathcal{A}]^\kappa$ and $V \in \mathcal{B}$ such that $V \subseteq \bigcap\mathcal{U}$. Hence, there exists $W \in \mathcal{A}$ such that $W \subseteq V \subseteq \bigcap\mathcal{U}$; hence, $\mathcal{A}$ is not κ^{op}-like. □

Example 3.2.23. The double arrow space, defined as $((0,1] \times \{0\}) \cup ([0,1) \times \{1\})$

ordered lexicographically, has π-weight ω and weight $\mathfrak{c}$, and is known to be compact and homogeneous. By Proposition 3.2.22, it has Noetherian type $\mathfrak{c}^+$.

Theorem 3.2.24. *Suppose X is a Suslin line. Then $\pi Nt(X) \geq \omega_1$.*

Proof. Let $\mathcal{A}$ be a π-base of X consisting only of open intervals. By Lemma 3.2.20, it suffices to show that $\mathcal{A}$ is not ω^{op}-like. Construct a sequence $\langle \mathcal{B}_n \rangle_{n<\omega}$ of maximal pairwise disjoint subsets of $\mathcal{A}$ as follows. Choose $\mathcal{B}_0$ arbitrarily. Given $n < \omega$ and $\mathcal{B}_n$, choose $\mathcal{B}_{n+1}$ such that it refines $\mathcal{B}_n$ and $\mathcal{B}_n \cap \mathcal{B}_{n+1} \subseteq [X]^1$.

Let E denote the set of all endpoints of intervals in $\bigcup_{n<\omega} \mathcal{B}_n$. Since X is Suslin, there exists $U \in \mathcal{A} \setminus [X]^1$ such that $U \cap E = \emptyset$. For each $n < \omega$, the set $\bigcup \mathcal{B}_n$ is dense in X by maximality; whence, there exists $V_n \in \mathcal{B}_n$ such that $U \cap V_n \neq \emptyset$. Since $U \cap E = \emptyset$, we have $U \subseteq \bigcap_{n<\omega} V_n$. Thus, $\mathcal{A}$ is not ω^{op}-like. $\square$

MA $+ \neg$CH implies there are no Suslin lines. It is not clear whether it further implies every homogeneous compactum has Noetherian π-type ω. However, the next theorem gives us a partial result. First, we need a lemma very similar to the result that MA$+\neg$CH implies all Aronszajn trees are special.

Lemma 3.2.25. *Assume MA. Suppose Q is an ω_1^{op}-like poset of size less than $\mathfrak{c}$. Then Q is almost ω^{op}-like or Q has an uncountable centered subset.*

Proof. Set $\mathbb{P} = [Q]^{<\omega}$ and order $\mathbb{P}$ such that $\sigma \leq \tau$ if and only if $\sigma \cap \uparrow_Q \tau = \tau$. A sufficiently generic filter G of $\mathbb{P}$ will be such that $\bigcup G$ is a dense ω^{op}-like subset of Q. Hence, if $\mathbb{P}$ is ccc, then Q is almost ω^{op}-like. Hence, we may assume $\mathbb{P}$ has an antichain A of size ω_1. We may assume A is a Δ-system with root ρ. Since Q is ω_1^{op}-like, we may assume $\sigma \cap \uparrow_Q \rho = \rho$ for all $\sigma \in A$. Choose a bijection $\langle a_\alpha \rangle_{\alpha<\omega_1}$ from ω_1 to A.

We may assume there exists an $n < \omega$ such that $|a_\alpha \setminus \rho| = n$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, choose a bijection $\langle a_{\alpha,i}\rangle_{i<n}$ from n to $a_\alpha \setminus \rho$. For each $x \in Q$ and $i < n$, set $E_{x,i} = \{\alpha < \omega_1 : x \leq_Q a_{\alpha,i} \text{ or } a_{\alpha,i} \leq_Q x\}$. For each $\alpha < \omega_1$, since A is an antichain, we have $\bigcup_{i<n} \bigcup_{j<n} E_{a_{\alpha,i},j} = \omega_1$. Choose a uniform ultrafilter $\mathcal{U}$ on ω_1. Then we may choose $B \in [(\bigcup A) \setminus \rho]^{\omega_1}$ and $i < n$ such that $E_{x,i} \in \mathcal{U}$ for all $x \in B$.

It suffices to show that B is centered. Let $\sigma \in [B]^{<\omega}$. Set $E = \bigcap_{x\in\sigma} E_{x,i}$. Then $E \in \mathcal{U}$; hence, $|E| = \omega_1$; hence, we may choose $\alpha \in E \setminus \{\beta < \omega_1 : a_{\beta,i} \in \uparrow_Q \sigma\}$. Then $a_{\alpha,i} <_Q x$ for all $x \in \sigma$. Thus, B is centered. $\square$

Lemma 3.2.26. *Suppose $f\colon X \to Y$ is an irreducible continuous surjection between spaces and X is regular. Then $\pi Nt(X) = \pi Nt(Y)$.*

Proof. Let $\mathcal{A}$ be a $\pi Nt(X)^{\mathrm{op}}$-like π-base of X and let $\mathcal{B}$ be a $\pi Nt(Y)^{\mathrm{op}}$-like π-base of Y. By Lemma 3.2.20, we may assume $\mathcal{A}$ consists only of regular open sets. Set $\mathcal{C} = \{f^{-1}U : U \in \mathcal{B}\}$. Then $\mathcal{C}$ is $\pi Nt(Y)^{\mathrm{op}}$-like. Suppose U is a nonempty open subset of X. Then we may choose $V \in \mathcal{B}$ such that $V \cap f[X \setminus U] = \emptyset$. Then $f^{-1}V \subseteq U$. Thus, $\mathcal{C}$ is a π-base of X; hence, $\pi Nt(X) \leq \pi Nt(Y)$.

Set $\mathcal{D} = \{Y \setminus f[X \setminus U] : U \in \mathcal{A}\}$. Suppose V is a nonempty open subset of Y. Then we may choose $U \in \mathcal{A}$ such that $U \subseteq f^{-1}V$. Then $Y \setminus f[X \setminus U] \subseteq V$. Thus, $\mathcal{D}$ is a π-base of Y. Now suppose $U_0, U_1 \in \mathcal{A}$ and $U_0 \not\subseteq U_1$. Then $U_0 \not\subseteq \overline{U}_1$ by regularity. By irreducibility, we may choose $p \in Y \setminus f[X \setminus (U_0 \setminus \overline{U}_1)]$. Then $p \in f[X \setminus U_1]$ and $p \notin f[X \setminus U_0]$. Hence, $Y \setminus f[X \setminus U_0] \not\subseteq Y \setminus f[X \setminus U_1]$. Thus, $\mathcal{D}$ is $\pi Nt(X)^{\mathrm{op}}$-like; hence, $\pi Nt(Y) \leq \pi Nt(X)$. $\square$

Theorem 3.2.27. *Assume MA. Let X be a compactum such that $t(X) = \omega$ and $\pi(X) < \mathfrak{c}$. Then $\pi Nt(X) = \omega$.*

Proof. We may assume X is a closed subspace of $[0,1]^\kappa$ for some cardinal κ. By a result of Šapirovskiĭ [60], since $t(X) = \omega$, there is an irreducible continuous map f from X onto a subspace of $\bigcup_{I\in[\kappa]^\omega}[0,1]^I \times \{0\}^{\kappa\setminus I}$. Because of Lemma 3.2.26, we may replace our hypothesis of $t(X) = \omega$ with $X \subseteq \bigcup_{I\in[\kappa]^\omega}[0,1]^I \times \{0\}^{\kappa\setminus I}$. Set $\mathcal{F} = \mathrm{Fn}(\kappa, (\mathbb{Q}\cap(0,1])^2)$ and

$$\mathcal{A} = \left\{ X \cap \bigcap_{\alpha\in\operatorname{dom}\sigma} \pi_\alpha^{-1}\left(\sigma(\alpha)(0), \sigma(\alpha)(1)\right) : \sigma \in \mathcal{F} \right\} \setminus \{\emptyset\},$$

which is a π-base of X. Then $\mathcal{A}$ witnesses that $\pi sw(X) = \omega$. Hence, by Theorem 3.2.6 and Lemma 3.2.20, $\mathcal{A}$ contains an ω_1^{op}-like dense subset $\mathcal{B}$, and it suffices to show that $\mathcal{B}$ is almost ω^{op}-like. Seeking a contradiction, suppose $\mathcal{B}$ is not almost ω^{op}-like. By Lemma 3.2.25, $\mathcal{B}$ contains an uncountable centered subset $\mathcal{C}$. Let the map

$$\left\langle X \cap \bigcap_{\alpha\in\operatorname{dom}\sigma_\beta} \pi_\alpha^{-1}(\sigma_\beta(\alpha)(0), \sigma_\beta(\alpha)(1)) \right\rangle_{\beta<\omega_1}$$

be an injection from ω_1 to $\mathcal{C}$. Then $|\bigcup_{\beta<\omega_1} \operatorname{dom}\sigma_\beta| = \omega_1$. By compactness, the set

$$X \cap \bigcap_{\beta<\omega_1} \bigcap_{\alpha\in\operatorname{dom}\sigma_\beta} \pi_\alpha^{-1}[\sigma_\beta(\alpha)(0), \sigma_\beta(\alpha)(1)]$$

is nonempty, in contradiction with $X \subseteq \bigcup_{I\in[\kappa]^\omega}[0,1]^I \times \{0\}^{\kappa\setminus I}$. □

Concerning compact Noetherian type, we note that if there is a homogeneous compactum X for which $\chi_K Nt(X) \geq \omega_1$, then X is not an ordered space.

Definition 3.2.28. A point p in a space X is P_κ*-point* if, for every set $\mathcal{A}$ of fewer than κ-many neighborhoods of p, the set $\bigcap\mathcal{A}$ has p in its interior. A P-point is a P_{ω_1}-point.

Theorem 3.2.29. *If X is a homogeneous ordered compactum, then $\chi_K Nt(X) = \omega$.*

Proof. We may assume X is infinite; hence, X has a point that is not a P-point. By homogeneity, $\min X$ is not a P-point; hence, $\min X$ has countable character. By homogeneity, X is first countable. Let C be closed in X. Then $X\setminus C$ is a disjoint union of open intervals $\bigcup_{i\in I}(a_i, b_i)$ such that $(a_i, b_i) = \bigcup_{n<\omega}[a_{i,n}, b_{i,n}]$ and $\langle a_{i,n}\rangle_{n<\omega}$ is nonincreasing and $\langle b_{i,n}\rangle_{n<\omega}$ is nondecreasing for all $i \in I$. Hence, $\{X \setminus \bigcup_{i\in\operatorname{dom}\sigma}[a_{i,\sigma(i)}, b_{i,\sigma(i)}] : \sigma \in \operatorname{Fn}(I,\, \omega)\}$ is an ω^{op}-like neighborhood base of C. □

It is worth noting that while products do not decrease cellularity, they can decrease $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$, as shown by the following theorem, which trivially generalizes a result of Malykhin [42].

Theorem 3.2.30. *Let $p \in X = \prod_{i\in I} X_i$ where X_i is a nonsingleton T_1 space for all $i \in I$. If $\sup_{i\in I} w(X_i) \leq |I|$, then $Nt(X) = \omega$. If $\sup_{i\in I} \pi(X_i) \leq |I|$, then $\pi Nt(X) = \omega$. If $\sup_{i\in I} \chi(p(i), X_i) \leq |I|$, then $\chi Nt(p, X) = \omega$.*

Proof. Let us prove the first implication; the others are proved very similarly. For each $i \in I$, let $\{U_{i,0}, U_{i,1}\}$ be a nontrivial cover of X by two open sets. Let $\mathcal{A}$ be a base of X of size at most $|I|$. Let $f\colon \mathcal{A} \to I$ be an injection. Let $\mathcal{B}$ denote the set of all nonempty sets of the form $V \cap \pi^{-1}_{f(V)}\left[U_{f(V),j}\right]$ where $V \in \mathcal{A}$ and $j < 2$. Since f is injective, every infinite subset of $\mathcal{B}$ has empty interior. Hence, $\mathcal{B}$ is an ω^{op}-like base of X. □

In constrast, $\chi_K Nt(\cdot)$ is not decreased by products when the factors are compacta. Just as is true of cellularity, the compact Noetherian type of a product of compacta is the supremum of the compact Noetherian types of its finite subproducts.

Theorem 3.2.31. *If $X = \prod_{i\in I} X_i$ is a product of compacta, then*

$$\chi_K Nt(X) = \sup_{\sigma\in[I]^{<\omega}} \chi_K Nt(\prod_{i\in\sigma} X_i).$$

Proof. To prove "$\leq$", apply Theorem 3.2.2. To prove "$\geq$", apply Lemma 3.2.21. □

Though cellularity and compact Noetherian type behave similarly for compacta, they do not coincide, even assuming homogeneity. Given any indecomposable ordinal

γ strictly between ω and ω_1, the space 2^γ_{lex} (*i.e.*, 2^γ ordered lexicographically) is homogeneous and compact and has cellularity $\mathfrak{c}$ by a result of Maurice [44]. However, by Theorem 3.2.29, this space has compact Noetherian type ω.

3.3 Dyadic compacta

In this section, we prove a strengthened version of Theorem 3.1.2 and generalize it to continuous images of products of compacta with bounded weight. We also investigate the spectrum of Noetherian types of dyadic compacta. Our approach is to start with results about subsets of free boolean algebras and then use Stone duality to apply them to families of open subsets of dyadic compacta.

By Lemma 3.2.3, every countable subset of a free boolean algebra is almost ω^{op}-like. We wish to prove this for all subsets of free boolean algebras. We achieve this by approximating free boolean algebras by smaller free subalgebras using elementary submodels. More specifically, we use elementary submodels of H_θ where θ is a regular cardinal and H_θ is the $\{\in\}$-structure of all sets that hereditarily have size less than θ. Whenever we use H_θ in an argument, we implicitly assume that θ is sufficiently large to make the argument valid. As is typical with elementary submodels of H_θ, we need reflection properties. For our purposes, the crucial reflection property of free boolean algebras is given by the following lemma.

Lemma 3.3.1. *Let B be a free boolean algebra and let $\{B, \wedge, \vee\} \subseteq M \prec H_\theta$. Then, for all $q \in B$, there exists $r \in B \cap M$ such that, for all $p \in B \cap M$, we have $p \geq q$ if and only if $p \geq r$. In particular, $r \geq q$.*

Proof. Let $q \in B$. We may assume $q \neq 0$. By elementarity, there exists a map $g \in M$

enumerating a set of mutually independent generators of B. Set $G = \bigcup\{\{g(i), g(i)'\} : i \in \operatorname{dom} g\}$. Then there exists $\eta \in [[G]^{<\omega}]^{<\omega}$ such that $q = \bigvee_{\tau\in\eta}\bigwedge\tau$ and $\bigwedge\tau \neq 0$ for all $\tau \in \eta$. Set $r = \bigvee_{\tau\in\eta}\bigwedge(\tau \cap M)$. Let $p \in B \cap M$; we may assume $p \neq 1$. Then there exists $\zeta \in [[G \cap M]^{<\omega}]^{<\omega}$ such that $p = \bigwedge_{\sigma\in\zeta}\bigvee\sigma$ and $\bigvee\sigma \neq 1$ for all $\sigma \in \zeta$. Hence, $p \geq q$ iff, for all $\sigma \in \zeta$ and $\tau \in \eta$, we have $\bigvee\sigma \geq \bigwedge\tau$, which is equivalent to $\sigma \cap \tau \neq \emptyset$, which is equivalent to $\sigma \cap \tau \cap M \neq \emptyset$. Thus, $p \geq q$ if and only if $p \geq r$. □

The above lemma is not new. Fuchino proved that the conclusion of the above lemma is equivalent to the Freese-Nation property, a property free boolean algebras are known to have. (See section 2.2 and Theorem A.2.1 of [31] for details.)

Theorem 3.3.2. *Every subset of every free boolean algebra is almost ω^{op}-like.*

Proof. Let B be a free boolean algebra; set $\kappa = |B|$. Given $A \subseteq B$, let $\uparrow A$ denote the smallest semifilter of B containing A; if $A = \{a\}$ for some a, then set $\uparrow a = \uparrow A$. Let Q be a subset of B. If Q is a countable, then Q is almost ω^{op}-like by Lemma 3.2.3. Therefore, we may assume that $\kappa > \omega$ and the theorem is true for all free boolean algebras of size less than κ.

We will construct a continuous elementary chain $\langle M_\alpha\rangle_{\alpha<\kappa}$ of elementary submodels of H_θ and a continuous increasing sequence of sets $\langle D_\alpha\rangle_{\alpha<\kappa}$ satisfying the following conditions for all $\alpha < \kappa$.

1. $\alpha \cup \{B, \wedge, \vee, Q\} \subseteq M_\alpha$ and $|M_\alpha| \leq |\alpha| + \omega$.

2. D_α is a dense subset of $Q \cap M_\alpha$.

3. $D_\alpha \cap \uparrow q$ is finite for all $q \in Q \cap M_\alpha$.

4. $D_{\alpha+1} \cap \uparrow q = D_\alpha \cap \uparrow q$ for all $q \in Q \cap M_\alpha$.

Given this construction, set $D = \bigcup_{\alpha<\kappa} D_\alpha$. Then D is a dense subset of Q by (2). Moreover, if $\alpha < \kappa$ and $d \in D_\alpha$, then $d \in Q \cap M_\alpha$ by (2); whence, d is below at most finitely many elements of D by (3) and (4). Hence, Q is almost ω^{op}-like.

For stage 0, choose any $M_0 \prec H_\theta$ satisfying (1). Since $Q \cap M_0 \subseteq B \cap M_0$, we may choose D_0 to be an ω^{op}-like dense subset of $Q \cap M_0$, exactly what (2) and (3) require. At limit stages, (1) and (2) are clearly preserved, and (3) is preserved because of (4).

For a successor stage $\alpha + 1$, choose $M_{\alpha+1}$ such that $M_\alpha \prec M_{\alpha+1} \prec H_\theta$ and (1) holds for stage $\alpha + 1$. Since $Q \cap M_{\alpha+1} \subseteq B \cap M_{\alpha+1}$, there is an ω^{op}-like dense subset E of $Q \cap M_{\alpha+1}$. Set $D_{\alpha+1} = D_\alpha \cup (E \setminus \uparrow(Q \cap M_\alpha))$. Then (4) is easily verified: if $q \in Q \cap M_\alpha$, then

$$D_{\alpha+1} \cap \uparrow q = (D_\alpha \cap \uparrow q) \cup ((E \cap \uparrow q) \setminus \uparrow(Q \cap M_\alpha)) = D_\alpha \cap \uparrow q.$$

Let us verify (2) for stage $\alpha + 1$. Let $q \in Q \cap M_{\alpha+1}$. If $q \in \uparrow(Q \cap M_\alpha)$, then $q \in \uparrow D_\alpha \subseteq \uparrow D_{\alpha+1}$ because of (2) for stage α. Suppose $q \notin \uparrow(Q \cap M_\alpha)$. Choose $e \in E$ such that $e \leq q$. Then $e \notin \uparrow(Q \cap M_\alpha)$; hence, $q \in \uparrow(E \setminus \uparrow(Q \cap M_\alpha)) \subseteq \uparrow D_{\alpha+1}$.

It remains only to verify (3) for stage $\alpha + 1$. Let $q \in Q \cap M_{\alpha+1}$. Then $E \cap \uparrow q$ is finite; hence, by the definition of $D_{\alpha+1}$, it suffices to show that $D_\alpha \cap \uparrow q$ is finite. By Lemma 3.3.1, there exists $r \in B \cap M_\alpha$ such that $r \geq q$ and $M_\alpha \cap \uparrow q = M_\alpha \cap \uparrow r$; hence, $D_\alpha \cap \uparrow q = D_\alpha \cap \uparrow r$. Since $q \in Q$, we have $r \in M_\alpha \cap \uparrow Q$. By elementarity, there exists $p \in Q \cap M_\alpha$ such that $p \leq r$; hence, $D_\alpha \cap \uparrow r \subseteq D_\alpha \cap \uparrow p$. By (2) for stage α, we have $D_\alpha \cap \uparrow p$ is finite; hence, $D_\alpha \cap \uparrow q$ is finite. $\square$

Theorem 3.3.3. *Let X be a dyadic compactum and let $\mathcal{U}$ be a family of subsets of X such that for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost ω^{op}-like.*

Proof. Let $f\colon 2^\kappa \to X$ be a continuous surjection for some cardinal κ. Set $\mathcal{B} = \mathrm{Clop}(2^\kappa)$. Then $\mathcal{B}$ is a free boolean algebra. Set $\mathcal{V} = \{f^{-1}U : U \in \mathcal{U}\}$. Then it suffices to show that $\mathcal{V}$ is almost ω^{op}-like. Let $\mathcal{Q}$ denote the set of all $B \in \mathcal{B}$ such that $V \subseteq B$ for some $V \in \mathcal{V}$. By Theorem 3.3.2, $\mathcal{Q}$ is almost ω^{op}-like. Hence, by Lemma 3.2.20, it suffices to show that $\mathcal{Q}$ and $\mathcal{V}$ are mutually dense. By definition, every $Q \in \mathcal{Q}$ contains some $V \in \mathcal{V}$; hence, it suffices to show that every $V \in \mathcal{V}$ contains some $Q \in \mathcal{Q}$. Suppose $V \in \mathcal{V}$. Choose $U \in \mathcal{U}$ such that $\overline{U} \cap \overline{X \setminus f[V]} = \emptyset$. Then there exists $B \in \mathcal{B}$ such that $f^{-1}\overline{U} \subseteq B \subseteq V$; hence, $V \supseteq B \in \mathcal{Q}$. $\square$

The following corollary is immediate and it implies the first half of Theorem 3.1.2.

Corollary 3.3.4. *Let X be a dyadic compactum. Then, for all closed subsets C of X, every neighborhood base of C contains an ω^{op}-like neighborhood base of C. Moreover, every π-base of X contains an ω^{op}-like π-base of X.*

Remark 3.3.5. The first half of the above corollary can also be proved simply by citing Theorem 3.2.18 and Lemma 3.2.20.

Next we state the natural generalizations of Lemma 3.3.1, Theorem 3.3.2, Theorem 3.3.3, and Corollary 3.3.4 to continuous images of products of compacta with bounded weight. We will only remark briefly about the proofs of these generalizations, for they are easy modifications of the corresponding old proofs.

Lemma 3.3.6. *Let κ be a regular uncountable cardinal and let B be a coproduct $\coprod_{i\in I} B_i$ of boolean algebras all of size less than κ; let $\{B, \wedge, \vee, \langle B_i\rangle_{i\in I}\} \subseteq M \prec H_\theta$ and $M \cap \kappa \in \kappa + 1$. Then, for all $q \in B$, there exists $r \in B \cap M$ such that, for all $p \in B \cap M$, we have $p \geq q$ if and only if $p \geq r$. In particular, $r \geq q$.*

Proof. Note that the subalgebra $B \cap M$ is the subcoproduct $\coprod_{i \in I \cap M} B_i$ naturally embedded in B. Then proceed as in the proof of Lemma 3.3.1 with $\bigcup_{i \in I} B_i$, naturally embedded in B, playing the role of G. $\square$

Theorem 3.3.7. *Let $\kappa \geq \omega$ and B be a coproduct of boolean algebras all of size at most κ. Then every subset of B is almost κ^{op}-like.*

Proof. The proof is essentially the proof of Theorem 3.3.2. Instead of using Lemma 3.3.1, use the instance of Lemma 3.3.6 for the regular uncountable cardinal κ^+. $\square$

Theorem 3.3.8. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta all of weight at most κ; let $\mathcal{U}$ be a family of subsets of X such that, for all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost κ^{op}-like.*

Proof. Let $h \colon \prod_{i \in I} X_i \to X$ be a continuous surjection where each X_i is a compactum with weight at most κ. Each X_i embeds into $[0,1]^\kappa$ and is therefore a continuous image of a closed subspace of 2^κ. Hence, we may assume $\prod_{i \in I} X_i$ is totally disconnected. The rest of the proof is just the proof of Theorem 3.3.3 with Theorem 3.3.7 replacing Theorem 3.3.2. $\square$

The following corollary is immediate.

Corollary 3.3.9. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta all of weight at most κ. Then, for all closed subsets C of X, every neighborhood base of C contains a κ^{op}-like neighborhood base of C. Moreover, every π-base of X contains a κ^{op}-like π-base of X.*

Remark 3.3.10. Again, the first half of the above corollary can also proved simply by citing Theorem 3.2.18 and Lemma 3.2.20.

In contrast to Corollary 3.3.4, not all dyadic compacta have ω^{op}-like bases. The following proposition is essentially due to Peregudov (see Lemma 1 of [54]). It makes it easy to produce examples of dyadic compacta X such that $Nt(X) > \omega$.

Proposition 3.3.11. *Suppose a point p in a space X satisfies $\pi\chi(p, X) < \operatorname{cf}\kappa = \kappa \leq \chi(p, X)$. Then $Nt(X) > \kappa$.*

Proof. Let $\mathcal{A}$ be a base of X. Let $\mathcal{U}_0$ and $\mathcal{V}_0$ be, respectively, a local π-base at p of size at most $\pi\chi(p, X)$ and a local base at p of size $\chi(p, X)$. For each element of $\mathcal{U}_0$, choose a subset in $\mathcal{A}$, thereby producing a local π-base $\mathcal{U}$ at p that is a subset of $\mathcal{A}$ of size at most $\pi\chi(p, X)$. Similarly, for each element of $\mathcal{V}_0$, choose a smaller neighborhood of p in $\mathcal{A}$, thereby producing a local base $\mathcal{V}$ at p that is a subset of $\mathcal{A}$ of size $\chi(p, X)$. Every element of $\mathcal{V}$ contains an element of $\mathcal{U}$. Hence, some element of $\mathcal{U}$ is contained in κ-many elements of $\mathcal{V}$; hence, $\mathcal{A}$ is not κ^{op}-like. $\square$

Example 3.3.12. Let X be the discrete sum of 2^ω and 2^{ω_1}. Let Y be the quotient of X resulting from collapsing a point in 2^ω and a point in 2^{ω_1} to a single point p. Then $\pi\chi(p, Y) = \omega$ and $\chi(p, Y) = \omega_1$; hence, $Nt(Y) > \omega_1$.

As we shall see in Theorem 3.3.21, if we make an additional assumption about a dyadic compactum X, namely, that all its points have π-character equal to its weight, then X has an ω^{op}-like base. Also, we may choose this ω^{op}-like base to be a subset of an arbitrary base of X. To prove this, we approximate such an X by metric compacta. Each such metric compactum is constructed using the following technique due to Bandlow [6].

Definition 3.3.13. Suppose X is a space and $\mathcal{F}$ is a set. For all $p \in X$, let $p/\mathcal{F}$ denote the set of $q \in X$ satisfying $f(p) = f(q)$ for all $f \in \mathcal{F} \cap C(X)$. For each $f \in \mathcal{F}$, define $f/\mathcal{F}\colon X/\mathcal{F} \to \mathbb{R}$ by $(f/\mathcal{F})(p/\mathcal{F}) = f(p)$ for all $p \in X$.

Lemma 3.3.14. *Suppose X is a compactum and $\mathcal{F} \subseteq C(X)$. Then $X/\mathcal{F}$ (with the quotient topology) is a compactum and its topology is the coarsest topology for which $f/\mathcal{F}$ is continuous for all $f \in \mathcal{F}$. Further suppose $\{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$ is a base of X and $\mathcal{F} \in M \prec H_\theta$. Then $\{(X \setminus f^{-1}\{0\})/(\mathcal{F} \cap M) : f \in \mathcal{F} \cap M\}$ is a base of $X/(\mathcal{F} \cap M)$.*

Proof. If $f \in \mathcal{F}$, then $f/\mathcal{F}$ is clearly continuous with respect to the quotient topology of $X/\mathcal{F}$. Therefore, the compact quotient topology on $X/\mathcal{F}$ is finer than the Hausdorff topology induced by $\{f/\mathcal{F} : f \in \mathcal{F}\}$. If a compact topology $\mathcal{T}_0$ is finer than a Hausdorff topology $\mathcal{T}_1$, then $\mathcal{T}_0 = \mathcal{T}_1$. Hence, the quotient topology on $X/\mathcal{F}$ is the topology induced by $\{f/\mathcal{F} : f \in \mathcal{F}\}$.

Set $\mathcal{A} = \{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$. Suppose $\mathcal{A}$ is a base of X and $\mathcal{F} \in M \prec H_\theta$. Let us show that $\{(X \setminus f^{-1}\{0\})/(\mathcal{F} \cap M) : f \in \mathcal{F} \cap M\}$ is a base of $X/(\mathcal{F} \cap M)$. Let $\mathcal{U}$ denote the set of preimages of open rational intervals with respect to elements of $\mathcal{F} \cap M$. Let $\mathcal{V}$ denote the set of nonempty finite intersections of elements of $\mathcal{U}$. Then $\mathcal{V} \subseteq M$ and $\{V/(\mathcal{F} \cap M) : V \in \mathcal{V}\}$ is base of $X/(\mathcal{F} \cap M)$. Suppose $p \in V_0 \in \mathcal{V}$. Then it suffices to find $W \in \mathcal{A} \cap M$ such that $p \in W \subseteq V_0$. Choose $V_1 \in \mathcal{V}$ such that $p \in V_1 \subseteq \overline{V}_1 \subseteq V_0$. Then there exist $n < \omega$ and $W_0, \ldots, W_{n-1} \in \mathcal{A}$ such that $\overline{V}_1 \subseteq \bigcup_{i<n} W_i \subseteq V_0$. By elementarity, we may assume $W_0, \ldots, W_{n-1} \in M$. Hence, there exists $i < n$ such that $p \in W_i \subseteq V_0$ and $W_i \in \mathcal{A} \cap M$. □

Given a suitable dyadic compactum X, we will construct an ω^{op}-like base of X by applying Lemma 3.2.8 to metrizable quotient spaces $X/(\mathcal{F} \cap M)$ where $\mathcal{F} \subseteq C(X)$ and M ranges over a transfinite sequence of countable elementary submodels of H_θ. This sequence is constructed such that, loosely speaking, each submodel in the sequence knows about the preceding submodels.

Definition 3.3.15. Let κ be a regular uncountable cardinal and let $\langle H_\theta, \ldots\rangle$ be an expansion of the $\{\in\}$-structure H_θ to an $\mathcal{L}$-structure for some language $\mathcal{L}$ of size less than κ. Then a κ-*approximation sequence* in $\langle H_\theta, \ldots\rangle$ is an ordinally indexed sequence $\langle M_\alpha\rangle_{\alpha<\eta}$ such that for all $\alpha < \eta$ we have $\{\kappa, \langle M_\beta\rangle_{\beta<\alpha}\} \subseteq M_\alpha \prec \langle H_\theta, \ldots\rangle$ and $|M_\alpha| \subseteq M_\alpha \cap \kappa \in \kappa$.

The following lemma is a generalization of a technique of Jackson and Mauldin [34] of approximating a model by a tree of elementary submodels.

Lemma 3.3.16. *If κ and $\langle H_\theta, \ldots\rangle$ are as in Definition 3.3.15, then there exists a $\{\kappa\}$-definable map Ψ that sends every κ-approximation sequence $\langle M_\alpha\rangle_{\alpha<\eta}$ in $\langle H_\theta, \ldots\rangle$ to a sequence $\langle \Sigma_\alpha\rangle_{\alpha\leq\eta}$ such that we have the following for all $\alpha \leq \eta$.*

1. *Σ_α is a finite set.*
2. *$|N| \subseteq N \prec \langle H_\theta, \ldots\rangle$ for all $N \in \Sigma_\alpha$.*
3. *$\bigcup \Sigma_\alpha = \bigcup_{\beta<\alpha} M_\beta$.*
4. *If $\alpha < \eta$, then $\Sigma_\alpha \in M_\alpha$.*
5. *Σ_α is an $\in$-chain.*
6. *If $N_0, N_1 \in \Sigma_\alpha$ and $N_0 \in N_1$, then $|N_0| > |N_1|$.*
7. *$\langle \Sigma_\beta\rangle_{\beta\leq\alpha} = \Psi(\langle M_\beta\rangle_{\beta<\alpha})$.*

Moreover, $|\Sigma_\lambda| = 1$ and $\{\alpha < \lambda : |\Sigma_\alpha| = 1\}$ is closed unbounded in λ for all infinite cardinals $\lambda \leq \eta$.

Proof. Let Ω denote the class of $\langle\gamma_i\rangle_{i<n} \in \mathrm{O}n^{<\omega} \setminus \{\emptyset\}$ for which $\kappa \leq |\gamma_i| > |\gamma_j|$ for all $i < j < n$ and $|\gamma_{n-1}| < \kappa$. Order Ω lexicographically and let Υ be the order isomorphism from $\mathrm{O}n$ to Ω. Given any $\sigma = \langle\gamma_i\rangle_{i<n} \in \mathrm{O}n^{<\omega}$ and $i < n$, set $\phi_i(\sigma) = \langle\gamma_0, \ldots, \gamma_{i-1}, 0\rangle$ and $\phi_n(\sigma) = \sigma$. Let $\langle M_\alpha\rangle_{\alpha<\eta}$ be a κ-approximation sequence in $\langle H_\theta, \ldots\rangle$. For all $\alpha \leq \eta$ and $i \in \operatorname{dom}\Upsilon(\alpha)$, set

$$N_{\alpha,i} = \bigcup\{M_\beta : \phi_i(\Upsilon(\alpha)) \leq \Upsilon(\beta) < \phi_{i+1}(\Upsilon(\alpha))\};$$

set $\Sigma_\alpha = \{N_{\alpha,i} : i \in \operatorname{dom}\Upsilon(\alpha)\} \setminus \{\emptyset\}$. Then Ψ is $\{\kappa\}$-definable and it is easily verified that $|\Sigma_\lambda| = 1$ and $\{\alpha < \lambda : |\Sigma_\alpha| = 1\}$ is closed unbounded in λ for all infinite cardinals $\lambda \leq \eta$. Let us prove (1)-(7). (1), (3), (4), and (7) immediately follow from the relevant definitions. Let $\alpha \leq \eta$ and $\langle\beta_i\rangle_{i<n} = \Upsilon(\alpha)$. We may assume $n > 0$. For all $\sigma \in \Omega$ and $i < n-1$, we have $\phi_i(\Upsilon(\alpha)) \leq \sigma < \phi_{i+1}(\Upsilon(\alpha))$ if and only if σ is the concatenation of $\langle\beta_j\rangle_{j<i}$ and some $\tau \in \Omega$ satisfying $\tau < \langle\beta_i, 0\rangle$. Therefore, $|N_{\alpha,i}| = |\beta_i|$ for all $i < n-1$. For all $\sigma \in \Omega$, we have $\phi_{n-1}(\Upsilon(\alpha)) \leq \sigma < \phi_n(\Upsilon(\alpha))$ if and only if $\sigma = \langle\beta_0, \ldots, \beta_{n-2}, \gamma\rangle$ for some $\gamma < \beta_{n-1}$. Hence, $|N_{\alpha,n-1}| < \kappa$; hence, $|N_{\alpha,i}| > |N_{\alpha,j}|$ for all $i < j < n$. Let $\Upsilon(\alpha_i) = \phi_i(\Upsilon(\alpha))$ for all $i < n$. If $i < j < n$, then $\{N_{\alpha,k} : k < j\} = \Sigma_{\alpha_{j-1}}$; whence, either $N_{\alpha,j} = \emptyset$ or $N_{\alpha,i} \in M_{\alpha_{j-1}} \subseteq N_{\alpha,j}$, depending on whether $\beta_j = 0$. Thus, (5) and (6) hold.

Finally, let us prove (2). Proceed by induction on α. Suppose $\beta_{n-1} > 0$. Since $\{N_{\alpha,i} : i < n-1\} = \Sigma_{\alpha_{n-1}}$ and $\alpha_{n-1} + \beta_{n-1} = \alpha$, it suffices to show that $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots\rangle$. If $\beta_{n-1} \in \mathrm{Lim}$, then $N_{\alpha,n-1}$ is the union of the $\in$-chain $\langle N_{\alpha_{n-1}+\gamma,n-1}\rangle_{\gamma<\beta_{n-1}}$; hence, $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots\rangle$. If $\beta_{n-1} \notin \mathrm{Lim}$, then $N_{\alpha,n-1} = N_{\alpha-1,n-1} \cup M_{\alpha-1} = M_{\alpha-1}$ because $N_{\alpha-1,n-1} \in M_{\alpha-1}$ and $|N_{\alpha-1,n-1}| < \kappa$; hence, $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots\rangle$.

Therefore, we may assume $\beta_{n-1} = 0$. Hence, $\Sigma_\alpha = \{N_{\alpha,i} : i < n-1\}$; hence, we may assume $n > 1$. Since $\{N_{\alpha,i} : i < n-2\} = \Sigma_{\alpha_{n-2}}$ and $\alpha_{n-2} < \alpha$, it suffices to show that $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots\rangle$. If $\beta_{n-2} = \kappa$, then $N_{\alpha,n-2}$ is the union of the $\in$-chain $\langle N_{\alpha_{n-2}+\gamma,n-2}\rangle_{\gamma<\kappa}$; hence, $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots\rangle$. Hence, we may assume $\beta_{n-2} > \kappa$. Let $\Upsilon(\delta_\gamma) = \langle \beta_0, \ldots, \beta_{n-3}, \gamma, 0\rangle$ for all $\gamma \in [\kappa, \beta_{n-2})$. If $\beta_{n-2} \in \mathrm{Lim}$, then $N_{\alpha,n-2}$ is the union of the $\in$-chain $\langle N_{\delta_\gamma,n-2}\rangle_{\kappa\leq\gamma<\beta_{n-2}}$; hence, $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots\rangle$. Hence, we may let $\beta_{n-2} = \varepsilon + 1$. Suppose $|\varepsilon| = \kappa$. Then $N_{\alpha,n-2} = N_{\delta_\varepsilon,n-2} \cup \bigcup_{\gamma<\kappa} M_{\delta_\varepsilon+\gamma}$. If $\gamma < \kappa$, then $\phi_{n-1}(\Upsilon(\delta_\varepsilon + \gamma)) = \Upsilon(\delta_\varepsilon)$; whence, δ_ε and γ are definable from $\delta_\varepsilon + \gamma$ and κ; whence, $\gamma \cup \bigcup_{\rho<\gamma} M_{\delta_\varepsilon+\rho} \subseteq M_{\delta_\varepsilon+\gamma}$. Hence, $|N_{\delta_\varepsilon,n-2}| = \kappa \subseteq \bigcup_{\gamma<\kappa} M_{\delta_\varepsilon+\gamma} \prec \langle H_\theta, \ldots\rangle$. Moreover, since $N_{\delta_\varepsilon,n-2} \in M_{\delta_\varepsilon}$, we have $N_{\delta_\varepsilon,n-2} \subseteq \bigcup_{\gamma<\kappa} M_{\delta_\varepsilon+\gamma}$; hence, $|N_{\alpha,n-2}| = \kappa \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots\rangle$.

Therefore, we may assume $|\varepsilon| > \kappa$. Let $\Upsilon(\zeta_\gamma) = \langle \beta_0, \ldots, \beta_{n-3}, \varepsilon, \kappa + \gamma, 0\rangle$ for all $\gamma < |\varepsilon|$. Then $N_{\alpha,n-2} = N_{\delta_\varepsilon,n-2} \cup \bigcup_{\gamma<|\varepsilon|} N_{\zeta_\gamma,n-1}$. If $\gamma < |\varepsilon|$, then $\Upsilon(\zeta_\gamma)(n-1) = \kappa + \gamma$; whence, $\gamma \in M_{\zeta_\gamma} \subseteq N_{\zeta_{\gamma+1},n-1}$. Hence, $|\varepsilon| \subseteq \bigcup_{\gamma<|\varepsilon|} N_{\zeta_\gamma,n-1} \prec \langle H_\theta, \ldots\rangle$. Since $|N_{\delta_\varepsilon,n-2}| = |\varepsilon|$ and $N_{\delta_\varepsilon,n-2} \in M_{\delta_\varepsilon} \subseteq N_{\zeta_0,n-1}$, we have $N_{\delta_\varepsilon,n-2} \subseteq \bigcup_{\gamma<|\varepsilon|} N_{\zeta_\gamma,n-1}$. Hence, $|N_{\alpha,n-2}| = |\varepsilon| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots\rangle$. □

Proposition 3.3.17. *If X is a topological space, then every base of X contains a base of size at most $w(X)$.*

Proof. Let $\mathcal{A}$ be an arbitrary base of X; let $\mathcal{B}$ be a base of X of size at most $w(X)$. Since X is hereditarily $w(X)^+$-compact, we may choose, for each $U \in \mathcal{B}$, some $\mathcal{A}_U \in [\mathcal{A}]^{\leq w(X)}$ such that $U = \bigcup \mathcal{A}_U$. Then $\bigcup\{\mathcal{A}_U : U \in \mathcal{B}\}$ is a base of X and in $[\mathcal{A}]^{\leq w(X)}$. □

Lemma 3.3.18. *Let X be a dyadic compactum such that $\pi\chi(p, X) = w(X)$ for all $p \in X$. Let $\mathcal{A}$ be a base of X consisting only of cozero sets. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. Set $\kappa = w(X)$; by Proposition 3.3.17, we may assume $|\mathcal{A}| = \kappa$. Choose $\mathcal{F} \subseteq C(X)$ such that $\mathcal{A} = \{X \setminus g^{-1}\{0\} : g \in \mathcal{F}\}$. Let $h\colon 2^\lambda \to X$ be a continuous surjection for some cardinal λ. Let $\mathcal{B}$ be the free boolean algebra $\mathrm{Clop}(2^\lambda)$. By Lemma 3.2.8, we may assume $\kappa > \omega$. Let $\langle M_\alpha \rangle_{\alpha<\kappa}$ be an ω_1-approximation sequence in $\langle H_\theta, \in, \mathcal{F}, h\rangle$; set $\langle \Sigma_\alpha \rangle_{\alpha\leq\kappa} = \Psi(\langle M_\alpha \rangle_{\alpha<\kappa})$ as defined in Lemma 3.3.16.

For each $\alpha < \kappa$, set $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$ and $\mathcal{F}_\alpha = \mathcal{F} \cap M_\alpha$. For every $\mathcal{H} \subseteq \mathcal{A}_\alpha$, let $\mathcal{H}/\mathcal{F}_\alpha$ denote $\{U/\mathcal{F}_\alpha : U \in \mathcal{H}\}$. By Lemma 3.3.14, $\mathcal{A}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$. Since $X/\mathcal{F}_\alpha$ is a metric compactum, there exists $\mathcal{W}_\alpha \subseteq \mathcal{A}_\alpha$ such that $\mathcal{W}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$ satisfying (2), (3), and (4) of Lemma 3.2.8. By (2) of Lemma 3.2.8, we may choose, for each $U \in \mathcal{W}_\alpha$, some $E_{\alpha,U} \in \mathcal{B} \cap M_\alpha$ such that $h^{-1}\overline{U} \subseteq E_{\alpha,U} \subseteq h^{-1}V$ for all $V \in \mathcal{W}_\alpha$ satisfying $\overline{U} \subseteq V$. Set $\mathcal{G}_\alpha = \{E_{\alpha,U} : U \in \mathcal{W}_\alpha\}$.

Suppose $\mathcal{G}_\alpha$ is not ω^{op}-like. Then there exist $U \in \mathcal{W}_\alpha$ and $\langle V_n \rangle_{n<\omega} \in \mathcal{W}_\alpha^\omega$ such that $E_{\alpha,U} \subsetneq E_{\alpha,V_n} \neq E_{\alpha,V_m}$ for all $m < n < \omega$. Set $\Gamma = \{W \in \mathcal{W}_\alpha : U \subsetneq W\}$. By (2) of Lemma 3.2.8, Γ is finite; hence, by (4) of Lemma 3.2.8, there exists $n < \omega$ such that $\{W \in \mathcal{W}_\alpha : V_n \subsetneq W\} \not\subseteq \Gamma$. Hence, there exists $W \in \mathcal{W}_\alpha$ such that W strictly contains V_n but not U. Hence, by (3) of Lemma 3.2.8, $E_{\alpha,V_n} \subseteq h^{-1}W$; hence, $h^{-1}U \subseteq E_{\alpha,U} \subsetneq E_{\alpha,V_n} \subseteq h^{-1}W$; hence, $U \subsetneq W$, which is absurd. Therefore, $\mathcal{G}_\alpha$ is ω^{op}-like.

Let $\mathcal{V}_\alpha$ denote the set of $V \in \mathcal{W}_\alpha$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Sigma_\alpha$. Let us show that $\mathcal{V}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$. If $V \in \mathcal{V}_\alpha$, then $\mathcal{P}(V) \cap \mathcal{W}_\alpha \subseteq \mathcal{V}_\alpha$; hence, it suffices to show that $\mathcal{V}_\alpha$ covers X. Since $|\bigcup \Sigma_\alpha| < \kappa$, every point of X has a neighborhood in $\mathcal{A}$ that does not contain any nonempty open subset of X in $\bigcup \Sigma_\alpha$. By compactness, there is a cover of X by finitely many such neighborhoods, say, $W_0, \ldots, W_{n-1}$. By elementarity, we may assume $W_0, \ldots, W_{n-1} \in \mathcal{A}_\alpha$. Then $\{W_i : i < n\}$ has a refining cover $\mathcal{S} \subseteq \mathcal{W}_\alpha$. Hence, $\mathcal{S} \subseteq \mathcal{V}_\alpha$; hence, $\mathcal{V}_\alpha$ covers X as desired.

Let $\mathcal{U}_\alpha$ denote the set of $U \in \mathcal{V}_\alpha$ such that $\overline{U} \subseteq V$ for some $V \in \mathcal{V}_\alpha$. Then $\mathcal{U}_\alpha/\mathcal{F}_\alpha$ is clearly a base of $X/\mathcal{F}_\alpha$. Set $\mathcal{E}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$. Then $\mathcal{E}_\alpha$ is ω^{op}-like because it is a subset of $\mathcal{G}_\alpha$.

For all $\mathcal{I} \subseteq \mathcal{P}(2^\kappa)$, set $\uparrow\mathcal{I} = \{H \subseteq 2^\kappa : H \supseteq I \text{ for some } I \in \mathcal{I}\}$. For all $H \subseteq 2^\kappa$, set $\uparrow H = \uparrow\{H\}$. Set $\mathcal{U} = \bigcup_{\alpha<\kappa}\mathcal{U}_\alpha$ and $\mathcal{C} = \mathcal{B} \cap \uparrow\{h^{-1}U : U \in \mathcal{U}\}$. For all $\alpha \leq \kappa$, set $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha}\mathcal{E}_\beta$. Then we claim the following for all $\alpha \leq \kappa$.

1. $\mathcal{D}_\alpha$ is a dense subset of $\mathcal{C} \cap \bigcup \Sigma_\alpha$.

2. $\mathcal{D}_\alpha \cap \uparrow H$ is finite for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

3. If $\alpha < \kappa$, then $\mathcal{D}_{\alpha+1} \cap \uparrow H = \mathcal{D}_\alpha \cap \uparrow H$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

We prove this claim by induction. For stage 0, the claim is vacuous. For limit stages, (1) is clearly preserved, and (2) is preserved because of (3). Suppose $\alpha < \kappa$ and (1) and (2) hold for stage α. Then it suffices to prove (3) for stage α and to prove (1) and (2) for stage $\alpha + 1$.

Let us verify (3). Seeking a contradiction, suppose $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$ and $\mathcal{D}_{\alpha+1} \cap \uparrow H \neq \mathcal{D}_\alpha \cap \uparrow H$. Then $\mathcal{E}_\alpha \cap \uparrow H \neq \emptyset$; hence, there exists $U \in \mathcal{U}_\alpha$ such that $H \subseteq E_{\alpha,U}$. By (1), there exist $\beta < \alpha$ and $W \in \mathcal{U}_\beta$ such that $E_{\beta,W} \subseteq H$. By definition, there exists $V \in \mathcal{V}_\alpha$ such that $\overline{U} \subseteq V$. Hence, $h^{-1}W \subseteq E_{\beta,W} \subseteq H \subseteq E_{\alpha,U} \subseteq h^{-1}V$; hence, $W \subseteq V$. Since $W \in M_\beta \subseteq \bigcup \Sigma_\alpha$ and $V \in \mathcal{V}_\alpha$, we have $W \not\subseteq V$, which yields our desired contradiction.

Let us verify (1) for stage $\alpha + 1$. By (1) for stage α, we have

$$\mathcal{D}_{\alpha+1} = \mathcal{D}_\alpha \cup \mathcal{E}_\alpha \subseteq \left(\mathcal{C} \cap \bigcup \Sigma_\alpha\right) \cup (\mathcal{C} \cap M_\alpha) = \mathcal{C} \cap \bigcup \Sigma_{\alpha+1},$$

so we just need to show denseness. Let $H \in \mathcal{C} \cap \bigcup \Sigma_{\alpha+1}$. If $H \in \bigcup \Sigma_\alpha$, then $H \in \uparrow\mathcal{D}_\alpha$, so we may assume $H \in M_\alpha$. By elementarity, there exists $U_0 \in \mathcal{U}_\alpha$ such that $h^{-1}U_0 \subseteq H$.

Choose $U_1 \in \mathcal{U}_\alpha$ such that $\overline{U}_1 \subseteq U_0$. Then $E_{\alpha,U_1} \subseteq h^{-1}U_0$; hence, $E_{\alpha,U_1} \subseteq H$. Hence, $H \in \uparrow\mathcal{D}_{\alpha+1}$.

To complete the proof of the claim, let us verify (2) for stage $\alpha+1$. By (1) for stage $\alpha+1$, it suffices to prove $\mathcal{D}_{\alpha+1} \cap \uparrow H$ is finite for all $H \in \mathcal{D}_{\alpha+1}$. By (3), if $H \in \mathcal{D}_\alpha$, then $\mathcal{D}_{\alpha+1} \cap \uparrow H = \mathcal{D}_\alpha \cap \uparrow H$, which is finite by (1) and (2) for stage α. Hence, we may assume $H \in \mathcal{E}_\alpha$. Since $\mathcal{E}_\alpha$ is ω^{op}-like, it suffices to show that $\mathcal{D}_\alpha \cap \uparrow H$ is finite. Since $\mathcal{D}_\alpha \subseteq \bigcup \Sigma_\alpha$, it suffices to show that $\mathcal{D}_\alpha \cap N \cap \uparrow H$ is finite for all $N \in \Sigma_\alpha$. Let $N \in \Sigma_\alpha$. By Lemma 3.3.1, there exists $G \in \mathcal{B} \cap N$ such that $G \supseteq H$ and $\mathcal{B} \cap N \cap \uparrow H = \mathcal{B} \cap N \cap \uparrow G$; hence, $\mathcal{D}_\alpha \cap N \cap \uparrow H = \mathcal{D}_\alpha \cap N \cap \uparrow G$. Since $G \supseteq H \in \mathcal{C}$, we have $G \in \mathcal{C}$. By (2) for stage α, the set $\mathcal{D}_\alpha \cap N \cap \uparrow G$ is finite; hence, $\mathcal{D}_\alpha \cap N \cap \uparrow H$ is finite.

Since $\mathcal{U} \subseteq \mathcal{A}$, it suffices to prove that $\mathcal{U}$ is an ω^{op}-like base of X. Suppose $p \in V \in \mathcal{A}$. Then there exists $\alpha < \kappa$ such that $V \in \mathcal{A}_\alpha$. Hence, there exists $U \in \mathcal{U}_\alpha$ such that $p/\mathcal{F}_\alpha \in U/\mathcal{F}_\alpha \subseteq V/\mathcal{F}_\alpha$; hence, $p \in U \subseteq V$. Thus, $\mathcal{U}$ is a base of X.

Let us show that $\mathcal{U}$ is ω^{op}-like. Suppose not. Then there exists $\alpha < \kappa$ and $U_0 \in \mathcal{U}_\alpha$ such that there exist infinitely many $V \in \mathcal{U}$ such that $U_0 \subseteq V$. Choose $U_1 \in \mathcal{U}_\alpha$ such that $\overline{U}_1 \subseteq U_0$. Suppose $\beta < \kappa$ and $U_0 \subseteq V \in \mathcal{U}_\beta$. Then $E_{\alpha,U_1} \subseteq h^{-1}U_0 \subseteq h^{-1}V \subseteq E_{\beta,V}$. By (1) and (2), $\mathcal{D}_\kappa$ is ω^{op}-like; hence, there are only finitely many possible values for $E_{\beta,V}$. Therefore, there exist $\langle \gamma_n \rangle_{n<\omega} \in \kappa^\omega$ and $\langle V_n \rangle_{n<\omega} \in \prod_{n<\omega} \mathcal{U}_{\gamma_n}$ such that $V_m \neq V_n$ and $E_{\gamma_m,V_m} = E_{\gamma_n,V_n}$ for all $m < n < \omega$. Suppose that for some $\delta < \kappa$ we have $\gamma_n = \delta$ for all $n < \omega$. Let $i < \omega$ and set $\Gamma = \{W \in \mathcal{W}_\delta : V_i \subsetneq W\}$. By (2) and (4) of Lemma 3.2.8, there exists $j < \omega$ such that $\{W \in \mathcal{W}_\delta : V_j \subsetneq W\} \not\subseteq \Gamma$. Hence, there exists $W \in \mathcal{W}_\delta$ such that W strictly contains V_j but not V_i. By (3) of Lemma 3.2.8, $\overline{V}_j \subseteq W$. Hence, $h^{-1}\overline{V}_i \subseteq E_{\delta,V_i} = E_{\delta,V_j} \subseteq h^{-1}W$. Hence, $\overline{V}_i \subseteq W$. Since W does not strictly contain V_i, we must have $V_i = \overline{V}_i = W$. Hence, $h^{-1}V_i = E_{\delta,V_i} = E_{\delta,V_0}$. Since i was arbitrary chosen, we have $V_m = V_n = h[E_{\delta,V_0}]$ for all $m, n < \omega$, which is absurd. Therefore, our supposed δ does not exist; hence, we may assume $\gamma_0 < \gamma_1$. By definition, there exists $W \in \mathcal{V}_{\gamma_1}$ such that $\overline{V}_1 \subseteq W$. Therefore, $h^{-1}V_0 \subseteq E_{\gamma_0,V_0} = E_{\gamma_1,V_1} \subseteq h^{-1}W$; hence, $V_0 \subseteq W$. Since $V_0 \in M_{\gamma_0} \subseteq \bigcup \Sigma_{\gamma_1}$ and $W \in \mathcal{V}_{\gamma_1}$, we have $V_0 \not\subseteq W$, which is absurd. Therefore, $\mathcal{U}$ is ω^{op}-like. ☐

Let us show that we may remove the requirement that the base $\mathcal{A}$ in Lemma 3.3.18 consist only of cozero sets.

Lemma 3.3.19. *Suppose X is a space with no isolated points and $\chi(p, X) = w(X)$ for all $p \in X$. Further suppose $\kappa = \operatorname{cf}\kappa \leq \min\{Nt(X), w(X)\}$ and X has a network consisting of at most $w(X)$-many κ-compact sets. Then every base of X contains an $Nt(X)^{\mathrm{op}}$-like base of X.*

Proof. Set $\lambda = Nt(X)$ and $\mu = w(X)$. Let $\mathcal{A}$ be an arbitrary base of X; let $\mathcal{B}$ be a λ^{op}-like base of X; let $\mathcal{N}$ be a network of X consisting of at most μ-many κ-compact sets. By Proposition 3.3.17, we may assume $|\mathcal{B}| = \mu$. Let $\langle\langle N_\alpha, B_\alpha\rangle\rangle_{\alpha<\mu}$ enumerate $\{\langle N, B\rangle \in \mathcal{N} \times \mathcal{B} : N \subseteq B\}$. Construct a sequence $\langle \mathcal{G}_\alpha\rangle_{\alpha<\mu}$ as follows. Suppose $\alpha < \mu$ and $\langle \mathcal{G}_\beta\rangle_{\beta<\alpha}$ is a sequence of elements of $[\mathcal{B}]^{<\kappa}$. For each $p \in N_\alpha$, we have $\chi(p, X) = \mu \geq \kappa = \operatorname{cf}\kappa$; hence, we may choose $U_{\alpha,p} \in \mathcal{B}$ such that $p \in U_{\alpha,p} \notin \bigcup_{\beta<\alpha} \mathcal{G}_\beta$. Choose $\sigma_\alpha \in \left[N_\alpha\right]^{<\kappa}$ such that $N_\alpha \subseteq \bigcup_{p\in\sigma_\alpha} U_{\alpha,p}$. Set $\mathcal{G}_\alpha = \{U_{\alpha,p} : p \in \sigma_\alpha\}$.

For each $\alpha < \mu$, choose $\mathcal{F}_\alpha \in [\mathcal{A}]^{<\kappa}$ such that $N_\alpha \subseteq \bigcup \mathcal{F}_\alpha \subseteq B_\alpha$ and $\mathcal{F}_\alpha$ refines $\mathcal{G}_\alpha$. Set $\mathcal{F} = \bigcup_{\alpha<\mu} \mathcal{F}_\alpha$, which is easily seen to be a base of X. Let us show that $\mathcal{F}$ is λ^{op}-like. Suppose not. Then, since $\kappa = \operatorname{cf}\kappa \leq \lambda$, there exist $V \in \mathcal{F}$, $I \in [\mu]^\lambda$, and $\langle W_\alpha\rangle_{\alpha\in I} \in \prod_{\alpha\in I} \mathcal{F}_\alpha$ such that $V \subseteq \bigcap_{\alpha\in I} W_\alpha$. For each $\alpha \in I$, there is a superset of W_α in $\mathcal{G}_\alpha$. By induction, $\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \emptyset$ for all $\alpha < \beta < \mu$; hence, V has λ-many supersets in the λ^{op}-like base $\mathcal{B}$, which is absurd, for V has a subset in $\mathcal{B}$. $\square$

Remark 3.3.20. If X is regular and locally κ-compact and $\kappa \leq w(X)$, then it is easily seen that X has a network consisting of at most $w(X)$-many κ-compact sets.

Theorem 3.3.21. *Let X be a dyadic compactum such that $\pi\chi(p, X) = w(X)$ for all $p \in X$. Then every base $\mathcal{A}$ of X contains an ω^{op}-like base of X.*

Proof. By Lemma 3.3.18, $Nt(X) = \omega$. Since $w(X) = \pi\chi(p, X) \leq \chi(p, X) \leq w(X)$ for all $p \in X$, we may apply Lemma 3.3.19 to get a subset of $\mathcal{A}$ that is an ω^{op}-like base of X. $\square$

Finally, let us prove the second half of Theorem 3.1.2.

Corollary 3.3.22. *Let X be a homogeneous dyadic compactum with base $\mathcal{A}$. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. Efimov [18] and Gerlits [24] independently proved that the π-character of every dyadic compactum is equal to its weight. Since X is homogeneous, $\pi\chi(p, X) = w(X)$ for all $p \in X$. Hence, $\mathcal{A}$ contains an ω^{op}-like base of X by Theorem 3.3.21. $\square$

Note that a compactum is dyadic if and only if it is a continous image of a product of second countable compacta. Let us prove generalizations of Theorem 3.3.21 and Corollary 3.3.22 about continuous images of products of compacta with bounded weight.

Lemma 3.3.23. *Suppose $\kappa = \mathrm{cf}\,\kappa > \omega$ and X is a space such that $\pi\chi(p, X) = w(X) \geq \kappa$ for all $p \in X$. Further suppose X has a network consisting of at most $w(X)$-many κ-compact closed sets. Then every base of X contains a $w(X)^{\mathrm{op}}$-like base of X.*

Proof. Set $\lambda = w(X)$ and let $\mathcal{A}$ be an arbitrary base of X. By Proposition 3.3.17, we may assume $|\mathcal{A}| = \lambda$. Let $\mathcal{N}$ be a network of X consisting of at most λ-many κ-compact sets. Let $\langle M_\alpha \rangle_{\alpha<\lambda}$ be a continuous elementary chain such that for all $\alpha < \lambda$ we have $\mathcal{A}, \mathcal{N}, M_\alpha \in M_{\alpha+1} \prec H_\theta$. We may also require that $M_\alpha \cap \kappa \in \kappa > |M_\alpha|$ for all $\alpha < \kappa$ and $|M_\alpha| = |\kappa + \alpha|$ for all $\alpha \in \lambda \setminus \kappa$. For each $\alpha < \lambda$, set $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$. Set $\mathcal{B} = \bigcup_{\alpha<\lambda} \mathcal{A}_{\alpha+1} \setminus \uparrow\mathcal{A}_\alpha$, which is clearly λ^{op}-like. Let us show that $\mathcal{B}$ is a base of X. Suppose $p \in U \in \mathcal{A}$. Choose $N \in \mathcal{N}$ such that $p \in N \subseteq U$. Choose $\alpha < \lambda$ such that $N, U \in \mathcal{A}_{\alpha+1}$. For each $q \in N$, choose $V_q \in \mathcal{A} \setminus \uparrow\mathcal{A}_\alpha$ such that $q \in V_q \subseteq U$. Then there exists $\sigma \in [N]^{<\kappa}$ such that $N \subseteq \bigcup_{q\in\sigma} V_q$. By elementarity, we may assume $\langle V_q \rangle_{q\in\sigma} \in M_{\alpha+1}$. Choose $q \in \sigma$ such that $p \in V_q$. Then $V_q \in \mathcal{B}$ and $p \in V_q \subseteq U$. Thus, $\mathcal{B}$ is a base of X. □

Theorem 3.3.24. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta each with weight at most κ. Suppose $\pi\chi(p, X) = w(X)$ for all $p \in X$. Then every base of X contains a κ^{op}-like base.*

Proof. Let $h\colon \prod_{i\in I} X_i \to X$ be a continuous surjection where each X_i is a compactum with weight at most κ. Each X_i embeds into $[0,1]^\kappa$ and is therefore a continuous image of a closed subspace of 2^κ. Hence, we may assume $\prod_{i\in I} X_i$ is totally disconnected. Set $\lambda = w(X)$; by Lemmas 3.2.8 and 3.3.23, we may assume $\lambda > \kappa$. By Theorem 3.3.21, we may assume $\kappa > \omega$. Inductively construct a κ^+-approximation sequence $\langle M_\alpha \rangle_{\alpha<\lambda}$ in $\langle H_\theta, \in, C(X), h, \langle \mathrm{Clop}(X_i) \rangle_{i\in I} \rangle$ as follows. For each $\alpha < \lambda$, let $\langle N_{\alpha,\beta} \rangle_{\beta<\kappa}$ be an ω_1-approximation sequence in

$$\langle H_\theta, \in, C(X), h, \kappa, \langle \mathrm{Clop}(X_i) \rangle_{i\in I}, \langle M_\beta \rangle_{\beta<\alpha} \rangle.$$

Set $\langle \Gamma_{\alpha,\beta} \rangle_{\beta\leq\kappa} = \Psi(\langle N_{\alpha,\beta} \rangle_{\beta<\kappa})$ as defined in Lemma 3.3.16; let $\{M_\alpha\} = \Gamma_{\alpha,\kappa}$. Set

$\langle \Sigma_\alpha \rangle_{\alpha \leq \lambda} = \Psi(\langle M_\alpha \rangle_{\alpha < \lambda})$. Set $\mathcal{F} = C(X) \cap \bigcup \Sigma_\lambda$ and $\mathcal{A} = \{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$. Then $\mathcal{A}$ is a base of X. By Lemma 3.3.19, it suffices to construct a subset of $\mathcal{A}$ that is a κ^{op}-like base of X.

For each $\alpha < \lambda$, set $\mathcal{F}_\alpha = \mathcal{F} \cap M_\alpha$. Let $\mathcal{V}_\alpha$ denote the set of $V \in \mathcal{A} \cap M_\alpha$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Sigma_\alpha$. Arguing as in the proof Lemma 3.3.18, $\mathcal{V}_\alpha / \mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$. For each $\beta < \kappa$, let $\mathcal{V}_{\alpha,\beta}$ denote the set of all $V \in \mathcal{V}_\alpha \cap N_{\alpha,\beta}$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Gamma_{\alpha,\beta}$. Let $\mathcal{R}_{\alpha,\beta}$ denote the set of $\langle U, V \rangle \in \mathcal{V}_{\alpha,\beta}^2$ for which $\overline{U} \subseteq V$; set $\mathcal{U}_{\alpha,\beta} = \operatorname{dom} \mathcal{R}_{\alpha,\beta}$; set $\mathcal{U}_\alpha = \bigcup_{\beta < \kappa} \mathcal{U}_{\alpha,\beta}$.

Let us show that $\mathcal{U}_\alpha / \mathcal{F}_\alpha$ is also a base of $X/\mathcal{F}_\alpha$. Suppose $p \in V \in \mathcal{V}_\alpha$. Extend $\{V\}$ to a finite subcover σ of $\mathcal{V}_\alpha$ such that $p \notin \bigcup(\sigma \setminus \{V\})$. Choose $\beta < \kappa$ such that $\sigma \in N_{\alpha,\beta}$. For each $q \in X$, choose $V_{q,0}, V_{q,1} \in \mathcal{A}$ such that $q \in V_{q,0}$ and there exists $W \in \sigma$ such that $U \not\subseteq \overline{V}_{q,0} \subseteq V_{q,1} \subseteq W$ for all nonempty open $U \in \bigcup \Sigma_\alpha \cup \bigcup \Gamma_{\alpha,\beta}$. Choose $\tau \in [X]^{<\omega}$ such that $X = \bigcup_{q \in \tau} V_{q,0}$. By elementarity, we may assume $\langle V_{q,i} \rangle_{\langle q,i \rangle \in \tau \times 2} \in N_{\alpha,\beta}$. Choose $q \in \tau$ such that $p \in V_{q,0}$. Then $V_{q,0} \in \mathcal{U}_{\alpha,\beta}$ and $p \in V_{q,0} \subseteq V$. Thus, $\mathcal{U}_\alpha / \mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$.

Set $\mathcal{B} = \operatorname{Clop}\left(\prod_{i \in I} X_i\right)$. For each $\langle U_0, U_1 \rangle \in \bigcup_{\beta < \kappa} \mathcal{R}_{\alpha,\beta}$, choose $E_\alpha(U_0, U_1) \in \mathcal{B} \cap M_\alpha$ such that $h^{-1}\overline{U}_0 \subseteq E_\alpha(U_0, U_1) \subseteq h^{-1}U_1$. Set $\mathcal{E}_{\alpha,\beta} = E_\alpha[\mathcal{R}_{\alpha,\beta}]$. Set $\mathcal{E}_\alpha = \bigcup_{\beta < \kappa} \mathcal{E}_{\alpha,\beta}$. Let us show that $\mathcal{E}_\alpha$ is κ^{op}-like. Suppose $\beta, \gamma < \kappa$ and $\mathcal{E}_{\alpha,\beta} \ni H \subseteq K \in \mathcal{E}_{\alpha,\gamma}$. Then it suffices to show that $\gamma \leq \beta$. Seeking a contradiction, suppose $\beta < \gamma$. There exist $\langle U_0, U_1 \rangle \in \mathcal{R}_{\alpha,\beta}$ and $\langle V_0, V_1 \rangle \in \mathcal{R}_{\alpha,\gamma}$ such that $H = E_\alpha(U_0, U_1)$ and $K = E_\alpha(V_0, V_1)$. Hence, $\bigcup \Gamma_{\alpha,\gamma} \ni U_0 \subseteq V_1 \in \mathcal{V}_{\alpha,\gamma}$, in contradiction with the definition of $\mathcal{V}_{\alpha,\gamma}$.

Set $\mathcal{U} = \bigcup_{\alpha < \lambda} \mathcal{U}_\alpha$ and $\mathcal{C} = \mathcal{B} \cap \upharpoonleft\{h^{-1}U : U \in \mathcal{U}\}$. For all $\alpha \leq \lambda$, set $\mathcal{D}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$. Then we claim the following for all $\alpha \leq \lambda$.

1. $\mathcal{D}_\alpha$ is a dense subset of $\mathcal{C} \cap \bigcup \Sigma_\alpha$.

2. $|\mathcal{D}_\alpha \cap {\uparrow}H| < \kappa$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

3. If $\alpha < \lambda$, then $\mathcal{D}_{\alpha+1} \cap {\uparrow}H = \mathcal{D}_\alpha \cap {\uparrow}H$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

We prove this claim by induction. For stage 0, the claim is vacuous. For limit stages, (1) is clearly preserved, and (2) is preserved because of (3). Suppose $\alpha < \kappa$ and (1) and (2) hold for stage α. Then it suffices to prove (3) for stage α and to prove (1) and (2) for stage $\alpha + 1$.

Let us verify (3). Seeking a contradiction, suppose $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$ and $\mathcal{D}_{\alpha+1} \cap {\uparrow}H \neq \mathcal{D}_\alpha \cap {\uparrow}H$. Then $\mathcal{E}_\alpha \cap {\uparrow}H \neq \emptyset$; hence, there exists $V \in \mathcal{U}_\alpha$ such that $H \subseteq h^{-1}V$. By (1), there exist $\beta < \alpha$ and $U \in \mathcal{U}_\beta$ and $K \in \mathcal{E}_\beta$ such that $h^{-1}\overline{U} \subseteq K \subseteq H$. Hence, $U \subseteq V$. Since $U \in M_\beta \subseteq \bigcup \Sigma_\alpha$ and $V \in \mathcal{V}_\alpha$, we have $U \not\subseteq V$, which yields our desired contradiction.

Let us verify (1) for stage $\alpha + 1$. By (1) for stage α, we have

$$\mathcal{D}_{\alpha+1} = \mathcal{D}_\alpha \cup \mathcal{E}_\alpha \subseteq \left(\mathcal{C} \cap \bigcup \Sigma_\alpha\right) \cup (\mathcal{C} \cap M_\alpha) = \mathcal{C} \cap \bigcup \Sigma_{\alpha+1},$$

so we just need to show denseness. Let $H \in \mathcal{C} \cap \bigcup \Sigma_{\alpha+1}$. If $H \in \bigcup \Sigma_\alpha$, then $H \in {\uparrow}\mathcal{D}_\alpha$, so we may assume $H \in M_\alpha$. By elementarity, there exists $U \in \mathcal{U}_\alpha$ such that $h^{-1}U \subseteq H$. Choose $\beta < \kappa$ such that $U \in \mathcal{U}_{\alpha,\beta}$; choose $V \in \mathcal{U}_{\alpha,\beta}$ such that $\overline{V} \subseteq U$. Then $E_\alpha(V, U) \subseteq H$; hence, $H \in {\uparrow}\mathcal{D}_{\alpha+1}$.

The proof of the claim is completed by noting that (2) for stage $\alpha + 1$ can be verified just as in the proof of Lemma 3.3.18, except that Lemma 3.3.6 is used in place of Lemma 3.3.1.

Just as in the proof of Lemma 3.3.18, $\mathcal{U}$ is a base of X; hence, it suffices to show that $\mathcal{U}$ is κ^{op}-like. Suppose $\gamma < \lambda$ and $\delta < \kappa$ and $U \in \mathcal{U}_{\gamma,\delta}$ and $\langle\langle\zeta_\alpha, \eta_\alpha\rangle\rangle_{\alpha<\kappa} \in (\lambda \times \kappa)^\kappa$ and $\langle W_\alpha\rangle_{\alpha<\kappa} \in \prod_{\alpha<\kappa} \mathcal{U}_{\zeta_\alpha,\eta_\alpha}$ and $U \subseteq \bigcap_{\alpha<\kappa} W_\alpha$. Then it suffices to show that $W_\alpha = W_\beta$ for some $\alpha < \beta < \kappa$. Choose $V \in \mathcal{U}_{\gamma,\delta}$ such that $\overline{V} \subseteq U$. For each $\alpha < \kappa$, choose $V_\alpha \in \mathcal{V}_{\zeta_\alpha,\eta_\alpha}$ such that $\overline{W}_\alpha \subseteq V_\alpha$; set $H_\alpha = E_{\zeta_\alpha}(W_\alpha, V_\alpha)$. Then $E_\gamma(V,U) \subseteq \bigcap_{\alpha<\kappa} H_\alpha$. By (1) and (2), $\mathcal{D}_\lambda$ is κ^{op}-like; hence, there exists $J \in [\kappa]^{\omega_1}$ such that $H_\alpha = H_\beta$ for all $\alpha, \beta \in J$; hence, $W_\alpha \subseteq V_\beta$ for all $\alpha, \beta \in J$. If $\alpha, \beta \in J$ and $\zeta_\alpha < \zeta_\beta$, then $\bigcup \Sigma_{\zeta_\beta} \ni W_\alpha \subseteq V_\beta$, in contradiction with $V_\beta \in \mathcal{V}_{\zeta_\beta}$. Hence, $\zeta_\alpha = \zeta_\beta$ for all $\alpha, \beta \in J$. If $\alpha, \beta \in J$ and $\eta_\alpha < \eta_\beta$, then $\bigcup \Gamma_{\zeta_\beta,\eta_\beta} \ni W_\alpha \subseteq V_\beta$, in contradiction with $V_\beta \in \mathcal{V}_{\zeta_\beta,\eta_\beta}$. Hence, $\eta_\alpha = \eta_\beta$ for all $\alpha, \beta \in J$. Hence, $\{W_\alpha : \alpha \in J\} \subseteq N_{\zeta_{\min J},\eta_{\min J}}$; hence, $W_\alpha = W_\beta$ for some $\alpha < \beta < \kappa$. $\square$

Lemma 3.3.25. *Let κ be an uncountable regular cardinal; let X be a compactum such that $w(X) \geq \kappa$ and X is a continuous image of a product of compacta each with weight less than κ. Then $\pi(X) = w(X)$.*

Proof. It suffices to prove that $\pi(X) \geq \kappa$. Seeking a contradiction, suppose $\mathcal{A}$ is a π-base of X of size less than κ. Let $\langle X_i \rangle_{i\in I}$ be a sequence of compacta each with weight less than κ and let h be a continuous surjection from $\prod_{i\in I} X_i$ to X. Choose $M \prec H_\theta$ such that $\mathcal{A} \cup \{C(X), h, \langle C(X_i)\rangle_{i\in I}\} \subseteq M$ and $|M| = |\mathcal{A}|$. Choose $p \in M \cap \prod_{i\in I} X_i$ and set $Y = \{q \in \prod_{i\in I} X_i : p \upharpoonright (I \setminus M) = q \upharpoonright (I \setminus M)\}$. Then it suffices to show that $h[Y] = X$, for that implies $\kappa \leq w(X) \leq w(Y) < \kappa$. Seeking a contradiction, suppose $h[Y] \neq X$. Then there exists $U \in \mathcal{A}$ such that $U \cap h[Y] = \emptyset$. By elementarity, there exists $\sigma \in [I \cap M]^{<\omega}$ and $\langle V_i \rangle_{i\in\sigma}$ such that V_i is a nonempty open subset of X_i for all $i \in \sigma$, and $\bigcap_{i\in\sigma} \pi_i^{-1} V_i \subseteq h^{-1} U$. Hence, $Y \cap \bigcap_{i\in\sigma} \pi_i^{-1} V_i \neq \emptyset$, in contradiction with $U \cap h[Y] = \emptyset$. $\square$

Definition 3.3.26. Given any cardinal κ, set $\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Lemma 3.3.27. *Let κ be an uncountable regular cardinal; let X be a compactum such that $w(X) \geq \kappa$ and X is a continuous image of a product of compacta each with weight less than κ. Then $\pi\chi(X) = w(X)$.*

Proof. Let $\langle X_i \rangle_{i\in I}$ be a sequence of compacta each with weight less than κ and let h be a continuous surjection from $\prod_{i\in I} X_i$ to X. For any space Y, we have $\pi(Y) = \pi\chi(Y)d(Y)$. Hence, $w(X) = \pi(X) = \pi\chi(X)d(X)$ by Lemma 3.3.25; hence, we may assume $d(X) = w(X)$. Arguing as in the proof of Lemma 3.3.25, if $\mathcal{A}$ is a π-base of X and $\mathcal{A} \cup \{C(X), h, \langle C(X_i)\rangle_{i\in I}\} \subseteq M \prec H_\theta$, then X is a continuous image of $\prod_{i\in I\cap M} X_i$; hence, we may assume $|I| = \pi(X)$. By 5.5 of [35], $d(X) \leq d(\prod_{i\in I} X_i) \leq \kappa \cdot \log|I|$. By 2.37 of [35], $d(Y) \leq \pi\chi(Y)^{c(Y)}$ for all T_3 non-discrete spaces Y. Since κ is a caliber of X_i for all $i \in I$, it is also a caliber of X; hence, $|I| = \pi(X) = d(X) \leq \pi\chi(X)^\kappa$; hence, $\log|I| \leq \kappa \cdot \pi\chi(X)$. Therefore, $w(X) = d(X) \leq \kappa \cdot \pi\chi(X)$; hence, we may assume $w(X) = \kappa$.

Let $\langle U_\alpha \rangle_{\alpha<\kappa}$ enumerate a base of X. For each $\alpha < \kappa$, choose $p_\alpha \in U_\alpha$. Since $d(X) = w(X) = \kappa$, there is no $\alpha < \kappa$ such that $\{p_\beta : \beta < \alpha\}$ is dense in X. Since κ is a caliber of X, we may choose $p \in X \setminus \bigcup_{\alpha<\kappa} \overline{\{p_\beta : \beta < \alpha\}}$. It suffices to show that $\pi\chi(p, X) = \kappa$. Seeking a contradiction, suppose $\pi\chi(p, X) < \kappa$. Then there exists $\alpha < \kappa$ such that $\{U_\beta : \beta < \alpha\}$ contains a local π-base at p; hence, $p \in \overline{\{p_\beta : \beta < \alpha\}}$, in contradiction with how we chose p. $\square$

Theorem 3.3.28. *Let $\langle X_i \rangle_{i\in I}$ be a sequence of compacta; let X be a homogeneous compactum; let $h\colon \prod_{i\in I} X_i \to X$ be a continuous surjection. If there is a regular cardinal κ such that $w(X_i) < \kappa \leq w(X)$ for all $i \in I$, then every base of X contains a $(\sup_{i\in I} w(X_i))^{\mathrm{op}}$-like base. Otherwise, $w(X) \leq \sup_{i\in I} w(X_i)$ and every base of X trivially contains a $(w(X)^+)^{\mathrm{op}}$-like base.*

Proof. The latter case is a trivial application of Proposition 3.3.17. In the former case, Lemma 3.3.27 implies $\pi\chi(p, X) = w(X)$ for all $p \in X$; apply Theorem 3.3.24. $\square$

Every known homogeneous compactum is a continuous image of a product of compacta each with weight at most $\mathfrak{c}$; hence, Theorem 3.3.28 provides a uniform justification for our observation that all known homogeneous compacta have Noetherian type at most $\mathfrak{c}^+$. Analogously, since every known homogeneous compactum is such a continuous image, it has $\mathfrak{c}^+$ among its calibers; hence, it has cellularity at most $\mathfrak{c}$.

Let us now turn to the spectrum of Noetherian types of dyadic compacta and a proof of Theorem 3.1.3.

Theorem 3.3.29. *Let κ and λ be infinite cardinals such that $\lambda < \kappa$. Let X be the discrete sum of 2^κ and 2^λ. Let Y be the quotient space induced by collapsing $\langle 0\rangle_{\alpha<\kappa}$ and $\langle 0\rangle_{\alpha<\lambda}$ to a single point p. If $\lambda < \operatorname{cf}\kappa$, then $Nt(Y) = \kappa^+$. If $\lambda \geq \operatorname{cf}\kappa$, then $Nt(Y) = \kappa$.*

Proof. Clearly $\chi(p, Y) = \kappa$ and $\pi\chi(p, Y) = \lambda$. Hence, if $\lambda < \operatorname{cf}\kappa$, then $\kappa^+ \leq Nt(Y) \leq w(Y)^+ = \kappa^+$ by Proposition 3.3.11. Suppose $\lambda \geq \operatorname{cf}\kappa$. We still have $\kappa \leq Nt(Y)$ by Proposition 3.3.11, so it suffices to construct a κ^{op}-like base of Y. Let $\sim$ be the equivalence relation such that $Y = X/\sim$. In building a base of Y, we proceed in the canonical way when away from p: for each $\mu \in \{\kappa, \lambda\}$, set

$$\mathcal{A}_\mu = \{\{x \in 2^\mu : \eta \subseteq x\}/\sim\, : \eta \in \operatorname{Fn}(\mu, 2) \text{ and } \eta^{-1}\{1\} \neq \emptyset\}.$$

Choose $f_0 \colon \kappa \to \operatorname{cf}\kappa$ such that for all $\alpha < \operatorname{cf}\kappa$ the preimage $f_0^{-1}\{\alpha\}$ is bounded in κ. Define $f \colon [\kappa]^{<\omega} \to \operatorname{cf}\kappa$ by $f(\sigma) = f_0(\sup\sigma)$ for all $\sigma \in [\kappa]^{<\omega}$. Choose $g_0 \colon \lambda \to \operatorname{cf}\kappa$ such that for all $\alpha < \operatorname{cf}\kappa$ the preimage $g_0^{-1}\{\alpha\}$ is unbounded in λ. Define $g \colon [\lambda]^{<\omega} \to \operatorname{cf}\kappa$ by

$g(\sigma) = g_0(\sup \sigma)$ for all $\sigma \in [\lambda]^{<\omega}$. Set

$$\mathcal{A}_p = \bigcup_{\alpha < \operatorname{cf} \kappa} \Big\{ \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\} \big)/\!\sim \, : \\ \langle \sigma, \tau \rangle \in f^{-1}\{\alpha\} \times g^{-1}\{\alpha\} \Big\}.$$

Set $\mathcal{A} = \mathcal{A}_\kappa \cup \mathcal{A}_\lambda \cup \mathcal{A}_p$. Let us show that $\mathcal{A}$ is a κ^{op}-like base of Y. The only nontrivial aspect of showing that $\mathcal{A}$ is a base of Y is verifying that $\mathcal{A}_p$ is a local base at p. Suppose U is an open neighborhood of p. Then there exist $\sigma \in [\kappa]^{<\omega}$ and $\tau \in [\lambda]^{<\omega}$ such that

$$\big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\} \big)/\!\sim \subseteq U.$$

Choose $\alpha < \lambda$ such that $\sup \tau < \alpha$ and $g_0(\alpha) = f(\sigma)$. Set $\tau' = \tau \cup \{\alpha\}$ and

$$V = \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau'] = \{0\}\} \big)/\!\sim .$$

Then $V \subseteq U$ and $V \in \mathcal{A}_p$ because $f(\sigma) = g(\tau')$. Thus, $\mathcal{A}$ is a base of Y.

Let us show that $\mathcal{A}$ is κ^{op}-like. Suppose $U, V \in \mathcal{A}$ and $U \subseteq V$. If $U \in \mathcal{A}_\kappa$, then, fixing U, there are only finitely many possibilities for V in $\mathcal{A}_\kappa$; the same is true if κ is replaced by λ or p. Hence, we may assume $U \in \mathcal{A}_i$ and $V \in \mathcal{A}_j$ for some $\{i, j\} \in [\{\kappa, \lambda, p\}]^2$. Since no element of $\mathcal{A}_p$ is a subset of an element of $\mathcal{A}_\kappa \cup \mathcal{A}_\lambda$, we have $i \neq p$. Hence, there exists $\eta \in \operatorname{Fn}(i, 2)$ such that $U = \{x \in 2^i : \eta \subseteq x\}/\!\sim$. Since $\bigcup \mathcal{A}_\kappa \cap \bigcup \mathcal{A}_\lambda = \emptyset$, we have $j = p$. Hence, there exist $\sigma \in [\kappa]^{<\omega}$ and $\tau \in [\lambda]^{<\omega}$ such that

$$V = \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\} \big)/\!\sim .$$

If $i = \kappa$, then $\sigma \subseteq \eta^{-1}\{0\}$; hence, fixing U, there are only finitely many possibilities for σ, and at most λ-many possibilities for τ. If $i = \lambda$, then $\tau \subseteq \eta^{-1}\{0\}$; hence, fixing U, there are only finitely many possibilities for τ, and at most $|\sup f_0^{-1}\{g(\tau)\}|^{<\omega}$-many possibilities for σ given τ. Thus, there are fewer than κ-many possibilities for V given U. Thus, $\mathcal{A}$ is κ^{op}-like. $\square$

Corollary 3.3.30. *If κ is a cardinal of uncountable cofinality, then there is a totally disconnected dyadic compactum with Noetherian type κ^+. If κ is a singular cardinal, then there is a totally disconnected dyadic compactum with Noetherian type κ.*

Proof. For the first case, apply Theorem 3.3.29 with $\lambda = \omega$. For the second case, apply Theorem 3.3.29 with $\lambda = \operatorname{cf}\kappa$. $\square$

Combining the above corollary with the following theorem (and a trivial example like $Nt(2^\omega) = \omega$) immediately proves Theorem 3.1.3.

Theorem 3.3.31. *Let X be a dyadic compactum with base $\mathcal{A}$ consisting only of cozero sets. If $Nt(X) \leq \omega_1$, then $\mathcal{A}$ contains an ω^{op}-like base of X. Hence, no dyadic compactum has Noetherian type ω_1.*

Proof. Let $\mathcal{Q}$ be an ω_1^{op}-like base of X of size $w(X)$. Import all the notation from the proof of Lemma 3.3.18 verbatim, except require that $\langle M_\alpha \rangle_{\alpha<\kappa}$ be an ω_1-approximation sequence in $\langle H_\theta, \in, \mathcal{F}, h, \mathcal{Q} \rangle$. Then $\mathcal{U}$ is an ω^{op}-like subset of $\mathcal{A}$ as before. On the other hand, $\mathcal{V}_\alpha/\mathcal{F}_\alpha$ is not necessarily a base of $X/\mathcal{F}_\alpha$ for all $\alpha < \kappa$. However, we will show that $\mathcal{U}$ is still a base of X. In doing so, we will repeatedly use the fact that if $U, \mathcal{Q} \in M \prec H_\theta$ and U is a nonempty open subset of X, then all supersets of U in $\mathcal{Q}$ are in M because $\{V \in \mathcal{Q} : U \subseteq V\}$ is a countable element of M.

Suppose $q \in Q \in \mathcal{Q}$. Then it suffices to find $U \in \mathcal{U}$ such that $q \in U \subseteq Q$. Let β be the least $\alpha < \kappa$ such that there exists $A \in \mathcal{A}_\alpha$ satisfying $q \in A \subseteq \overline{A} \subseteq Q$. Fix such an $A \in \mathcal{A}_\beta$. For each $p \in \overline{A}$, choose $\langle A_p, Q_p \rangle \in \mathcal{A} \times \mathcal{Q}$ such that $p \in A_p \subseteq Q_p \subseteq \overline{Q}_p \subseteq Q$.

Since $M_\beta \ni A \subseteq Q \in \mathcal{Q}$, we have $Q \in M_\beta$. Hence, by elementarity, we may assume there exists $\sigma \in \left[\overline{A}\right]^{<\omega}$ such that $\langle\langle A_p, Q_p\rangle\rangle_{p\in\sigma} \in M_\beta$ and $\overline{A} \subseteq \bigcup_{p\in\sigma} A_p$. Choose $p \in \sigma$ such that $q \in A_p$. Suppose $Q_p \notin \bigcup \Sigma_\beta$. Then all nonempty open subsets of Q_p are also not in $\bigcup \Sigma_\beta$; hence, there exist $U \in \mathcal{U}_\beta$ and $V \in \mathcal{V}_\beta$ such that $q/\mathcal{F}_\beta \subseteq U \subseteq V \subseteq A_p \subseteq Q$. Therefore, we may assume $Q_p \in \bigcup \Sigma_\beta$.

Choose $\alpha < \beta$ such that $Q_p \in M_\alpha$. Then $Q \in M_\alpha$ because $Q_p \subseteq Q$. Hence, there exists $\tau \in [\mathcal{A}_\alpha]^{<\omega}$ such that $\overline{Q_p} \subseteq \bigcup \tau \subseteq \overline{\bigcup \tau} \subseteq Q$. Choose $W \in \tau$ such that $q \in W$. Then $q \in W \subseteq \overline{W} \subseteq Q$, in contradiction with the minimality of β. Thus, $\mathcal{U}$ is a base of X. □

We note that the spectrum of Noetherian types of all compacta is trivial.

Theorem 3.3.32. *Let κ be a regular uncountable cardinal. Then there exists a totally disconnected compactum X such that $Nt(X) = \kappa$ and X has a P_κ-point.*

Proof. Let X be the closed subspace of 2^κ consisting of all $f \in 2^\kappa$ for which $f(\alpha) = 0$ or $f[\alpha] = \{1\}$ for all odd $\alpha < \kappa$. First, let us show that X has a κ^{op}-like base. For each $\sigma \in \mathrm{Fn}(\kappa, 2)$, set $U_\sigma = \{f \in X : f \supseteq \sigma\}$. Let E denote the set of $\sigma \in \mathrm{Fn}(\kappa, 2)$ for which $\sup\operatorname{dom}\sigma$ is even and $U_\sigma \neq \emptyset$. Set $\mathcal{A} = \{U_\sigma : \sigma \in E\}$, which is clearly a base of X. Let us show that $\mathcal{A}$ is κ^{op}-like. Suppose $\sigma, \tau \in E$ and $U_\sigma \subseteq U_\tau$. If $\sup\operatorname{dom}\sigma < \sup\operatorname{dom}\tau$, then for each $f \in U_\sigma$ the sequence

$$(f \restriction \sup\operatorname{dom}\tau) \cup \{\langle \sup\operatorname{dom}\tau, 1 - \tau(\sup\operatorname{dom}\tau)\rangle\} \cup \{\langle\beta, 0\rangle : \sup\operatorname{dom}\tau < \beta < \kappa\}$$

is in $U_\sigma \setminus U_\tau$, which is absurd. Hence, $\sup\operatorname{dom}\tau \leq \sup\operatorname{dom}\sigma$; hence, there are fewer than κ-many possibilities for τ given σ. Thus, $\mathcal{A}$ is κ^{op}-like.

Finally, it suffices to show that $\langle 1\rangle_{\alpha<\kappa}$ is a P_κ-point of X, for a P_κ-point must have local Noetherian type at least κ. For each $\alpha < \kappa$, set $\sigma_\alpha = \{\langle 2\alpha + 1, 1\rangle\}$. Then

$\{U_{\sigma_\alpha} : \alpha < \kappa\}$ is a local base at $\langle 1 \rangle_{\alpha<\kappa}$. Moreover, $U_{\sigma_\alpha} \supsetneq U_{\sigma_\beta}$ for all $\alpha < \beta < \kappa$. Since κ is regular, it follows that $\langle 1 \rangle_{\alpha<\kappa}$ is a P_κ-point. □

Corollary 3.3.33. *Every infinite cardinal is the Noetherian type of some totally disconnected compactum.*

Proof. By Lemma 3.2.8, all totally disconnected metric compacta have Noetherian type ω. By Theorem 3.3.32, if κ is a regular uncountable cardinal, then there is a totally disconnected compactum X with Noetherian type κ. If κ is a singular cardinal, then there is a totally disconnected dyadic compactum with Noetherian type κ by Corollary 3.3.30. □

3.4 k-adic compacta

The results of the previous section used reflection properties of free boolean algebras—see Lemma 3.3.1—and more generally coproducts of boolean algebras of bounded size—see Lemma 3.3.6. Let us define a more general family of reflection properties.

Definition 3.4.1. Let B be a boolean algebra and let κ and λ be cardinals. Then we say B has the (κ, λ)-FN if and only if, for every M such that $\{B, \wedge, \vee\} \subseteq M \prec H_\theta$ and $|M| \cap \kappa \subseteq M \cap \kappa \in \kappa + 1$, and for every $b \in B$, there exists $A \in [B \cap M]^{<\lambda}$ such that $M \cap \uparrow b = M \cap \uparrow A$.

Remark 3.4.2. For regular κ, the (κ, κ)-FN and the (κ^+, κ)-FN are both equivalent to the κ-FN as defined by Fuchino, Koppelberg, and Shelah [23]. In particular, the (ω_1, ω)-FN is equivalent to the Freese-Nation property and the (ω_2, ω_1)-FN is equivalent to the weak Freese-Nation property.

The (κ,ω)-FN is equivalent to the $(\kappa,2)$-FN for all κ: if $A \in [B\cap M]^{<\omega}$ and $M\cap{\uparrow}b = M\cap{\uparrow}A$, then $\bigwedge A \in M$ and $M\cap{\uparrow}b = M\cap{\uparrow}\bigwedge A$. Therefore, a boolean algebra has the (ω_1,ω)-FN if and only if it satisfies the conclusion of Lemma 3.3.1. Likewise, a boolean algebra satisfies the conclusion of Lemma 3.3.6 if and only if it has the (κ,ω)-FN.

Theorem 3.4.3. *If $\kappa \geq \omega$ and B has the $(\kappa^+,\operatorname{cf}\kappa)$-FN, then every subset of B is almost κ^{op}-like.*

Proof. Proceed as in the proof of Theorem 3.3.2. The only modifications worth noting happen in the last paragraph. Where Lemma 3.3.1 is used to produce $r \in B\cap M_\alpha$ such that $M_\alpha\cap{\uparrow}q = M_\alpha\cap{\uparrow}r$, instead use the $(\kappa^+,\operatorname{cf}\kappa)$-FN to produce $A\in[B\cap M_\alpha]^{<\operatorname{cf}\kappa}$ such that $M_\alpha\cap{\uparrow}q = M_\alpha\cap{\uparrow}A$. For each $r\in A$, argue as before that there exists $p_r\in Q\cap M_\alpha$ such that $D_\alpha\cap{\uparrow}r\subseteq D_\alpha\cap{\uparrow}p_r$. By an induction hypothesis, $|D_\alpha\cap{\uparrow}p_r|<\kappa$; hence, $|D_\alpha\cap{\uparrow}q|\leq|\bigcup_{r\in A}(D_\alpha\cap{\uparrow}p_r)|<\kappa$. $\square$

Corollary 3.4.4. *It is independent of $\neg CH$ whether every separable compactum X satisfies $\chi Nt(X)\leq\omega_1$.*

Proof. Fuchino, Koppelberg, and Shelah [23] proved that $\mathcal{P}(\omega)$ has the (ω_2,ω_1)-FN in the Cohen model. Arguing as in the proof of Theorem 3.3.3, every separable compactum X, being a continuous image of $\beta\omega$, satisfies $\chi_K Nt(X)\leq\omega_1$ and $\pi Nt(X)\leq\omega_1$ in this model. On the other hand, $\mathfrak{p}=\mathfrak{c}$ implies there is a $P_{\mathfrak{c}}$-point p in $\beta\omega\setminus\omega$. Assuming $\mathfrak{p}=\mathfrak{c}>\omega_1$, let us show that this p does not have an ω_1^{op}-like base in the separable compactum $\beta\omega$. Let $\mathcal{U}$ be a local base at p in $\beta\omega$. Choose $\mathcal{V}\in[\mathcal{U}]^{\omega_1}$ and $U\in\mathcal{U}$ such that $\overline{U}\setminus\omega\subseteq\bigcap\mathcal{V}$. For every $V\in\mathcal{V}$, the compact set $\overline{U}\setminus V$ is contained in ω, so $\overline{U}\setminus V\subseteq n$ for some $n<\omega$. Therefore, there exist $\mathcal{W}\in[\mathcal{V}]^{\omega_1}$ and $n<\omega$ such that $\overline{U}\setminus W\subseteq n$ for all $W \in \mathcal{W}$. Choose $U_0 \in \mathcal{U}$ such that $U_0 \subseteq U \setminus n$. Then $U_0 \subseteq \bigcap \mathcal{W}$; hence, $\mathcal{U}$ is not ω_1^{op}-like. □

Theorem 3.4.5. *Let $\kappa \geq \omega$ and let X be a compactum such that $\pi\chi(p, X) = w(X)$ for all $p \in X$ and such that X is a continuous image of a totally disconnected compactum Y such that* $\mathrm{Clop}(Y)$ *has the* $(\kappa^+, \mathrm{cf}\,\kappa)$*-FN. Then every base of X contains an κ^{op}-like base of X.*

Proof. Proceed as in the proof of Theorem 3.3.24. Modify that proof just as the proof of Theorem 3.3.2 was modified in the above proof of Theorem 3.4.3. □

Ščepin discovered a nice characterization of the Stone spaces of boolean algebras having the (ω_1, ω)-FN.

Definition 3.4.6 (Ščepin [61])**.** Given a space X, let $RC(X)$ denote the set of regular closed subsets of X. A space X is *k-metrizable* if there exists $\rho\colon X \times RC(X) \to [0, \infty)$ such that we have the following for all $C \in RC(X)$.

1. $C = \{x \in X : \rho(x, C) = 0\}$.
2. If $C \supseteq B \in RC(X)$, then $\rho(x, C) \leq \rho(x, B)$ for all $x \in X$.
3. The map $\rho_C\colon X \to \mathbb{R}$ defined by $\rho_C(x) = \rho(x, C)$ is continuous.
4. For each increasing union $\bigcup_{\alpha<\beta} C_\alpha$ of regular closed sets, if $C = \overline{\bigcup_{\alpha<\beta} C_\alpha}$, then $\rho(x, C) = \inf_{\alpha<\beta} \rho(x, C_\alpha)$.

A compactum is *k-adic* if it is a continuous image of k-metrizable compactum.

Remark 3.4.7. Ščepin's notation is "κ-metrizable." Let us use "k-metrizable" for two reasons. First, "κ" has nothing to do with a cardinal κ; it is a Russian abbreviation for canonical. (Canonically closed means regular closed in this context.) Second, for some authors, κ-metrizable means something else, such as having a decreasing uniform base of the form $\{U_\alpha\}_{\alpha<\kappa}$.

The following theorem is implicit in results of Ščepin [61] and more explicit in Heindorf and Šapiro [31]. (See especially Section 2.9 of the latter.)

Theorem 3.4.8. *A totally disconnected compactum X is k-metrizable if and only if* $\mathrm{Clop}(X)$ *has the (ω_1, ω)-FN.*

Lemma 3.4.9 (Ščepin [61])**.** *If X is a k-adic compactum, then $\pi\chi(X) = w(X)$.*

Given the above lemma and the preceding three theorems, it is trivial to generalize our main results from the previous section about the class of dyadic compacta, which are continuous images of powers of 2, to the class of compacta that are continuous images of totally disconnected k-metrizable compacta. Moreover, the next two theorems show that the latter class properly contains the former class.

Theorem 3.4.10 (Ščepin [61])**.** *Metrizable spaces are k-metrizable. Moreover, products and hyperspaces (with the Vietoris topology) preserve k-metrizability. In particular, every power of 2 is k-metrizable.*

Theorem 3.4.11 (Šapiro [59])**.** *If $\kappa \geq \omega_2$, then the hyperspace of 2^κ is not dyadic. Hence, there is a totally disconnected compactum that is k-metrizable but not dyadic.*

With a little more care, we can further generalize our results about dyadic compacta to all k-adic compacta.

Definition 3.4.12. Given a space X and a set M, define $\pi^X_M\colon X \to X/M$ by $\pi^X_M(p) = p/M$.

Lemma 3.4.13. *Let X be a compactum. Then X is k-metrizable if and only if π^X_M is an open map for all M satisfying $C(X) \in M \prec H_\theta$.*

Proof. Ščepin [62] proved that a compactum X is k-metrizable if and only if, for all sufficiently large regular cardinals μ, there is a closed unbounded $C \subseteq [H_\mu]^\omega$ such that $C(X) \in M \prec H_\mu$ and π^X_M is open for all $M \in C$. (Ščepin stated this result in terms of σ-complete inverse systems of metric compacta; the above formulation is due to Bandlow [7].) It follows at once that X is k-metrizable if π^X_M is open for all M satisfying $C(X) \in M \prec H_\theta$. Conversely, suppose X is k-metrizable and $C(X) \in M \prec H_\theta$. Fix μ and C as above. We may assume $\theta > \mu^\omega$; hence, by elementarity, we may assume $C \in M$. Choose a countable $N \prec H_{(2^{<\theta})^+}$ such that $C(X), C, M \in N$. Then $M \cap N \cap H_\mu \in C$, so $\pi^X_{M\cap N\cap H_\mu}$, which is equal to $\pi^X_{M\cap N}$, is open. Suppose $U \subseteq X$ is open and $p \in U$. Since $\pi^X_{M\cap N}$ is open, there exists a cozero $V \subseteq X$ such that $p \in V \in M \cap N$ and $V/(M\cap N) \subseteq U/(M \cap N)$. The last relation is equivalent to the statement that, for all $q \in V$, there exists $r \in U$ such that, for all $f \in C(X) \cap M \cap N$, we have $f(q) = f(r)$. By elementarity, for every open $U \subseteq X$ and $p \in U$, there exists a cozero $V \subseteq X$ such that $p \in V \in M$ and, for all $q \in V$, there exists $r \in U$ such that, for all $f \in C(X) \cap M$, we have $f(q) = f(r)$. Thus, $p/M \in V/M \subseteq U/M$. Since V is cozero and $V \in M$, the set V/M is cozero. Hence, π^X_M is open. □

Theorem 3.4.14. *Let X be a k-metrizable compactum and Q a family of cozero subsets of X such that for every $U \in Q$ there exists $V \in Q$ such that $\overline{V} \subseteq U$. Then Q is almost ω^{op}-like.*

Proof. Proceed by induction on $|Q|$. Argue as in the proof of Theorem 3.3.2 until the verification of (3) for stage $\alpha+1$, where we need a different argument to show that $D_\alpha \cap \uparrow q$

is finite. Let $U = q$ and choose $V \in Q$ such that $\overline{V} \subseteq U$. By Lemma 3.4.13, U/M_α is open; hence, there exists $f \in C(X) \cap M_\alpha$ such that $V/M_\alpha \subseteq (f^{-1}\{0\})/M_\alpha \subseteq U/M_\alpha$. Since $f \in M_\alpha$, we have $V \subseteq f^{-1}\{0\}$. By elementarity, there exists $W \in Q \cap M_\alpha$ such that $W \subseteq f^{-1}\{0\}$. By (3) for stage α, it suffices to show that $D_\alpha \cap \uparrow U \subseteq D_\alpha \cap \uparrow W$. Suppose $Z \in D_\alpha \cap \uparrow U$. Then $W/M_\alpha \subseteq (f^{-1}\{0\})/M_\alpha \subseteq U/M_\alpha \subseteq Z/M_\alpha$. Since $Z \in D_\alpha \subseteq M_\alpha$ and Z is cozero, we have $W \subseteq Z$. Thus, $D_\alpha \cap \uparrow U \subseteq D_\alpha \cap \uparrow W$. □

Corollary 3.4.15. *Let X be a k-adic compactum and $\mathcal{U}$ be a family of subsets of X such that for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost ω^{op}-like. Hence, $\pi Nt(X) = \chi_K Nt(X) = \omega$.*

Proof. Proceed as in the proof of Theorem 3.3.3. Use the above theorem instead of Theorem 3.3.2. □

Theorem 3.4.16. *Let X be a homogeneous k-adic compactum with base $\mathcal{A}$. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. By homogeneity and Lemma 3.4.9, we have $\pi\chi(p, X) = w(X)$ for all $p \in X$. By Lemma 3.3.19, we may assume $\mathcal{A}$ consists only of cozero sets. Proceed as in the proof of Lemma 3.3.18. Replace 2^λ with a k-metrizable compactum Y and replace $\mathcal{B}$ with the set of cozero subsets of Y. For the proof of (2) for stage $\alpha+1$, we need a different argument that, given $H \in \mathcal{E}_\alpha$ and $N \in \Sigma_\alpha$, the set $\mathcal{D}_\alpha \cap N \cap \uparrow H$ is finite.

Choose $U \in \mathcal{U}_\alpha$ such that $H = E_{\alpha,U}$; choose $V \in \mathcal{U}_\alpha$ such that $\overline{V} \subseteq U$. Since π^Y_N is open by Lemma 3.4.13, we have $(h^{-1}V)/N \subseteq (f^{-1}\{0\})/N \subseteq (h^{-1}U)/N$ for some $f \in C(Y) \cap N$. Since $f \in N$, we have $h^{-1}V \subseteq f^{-1}\{0\}$. Choose $\beta < \alpha$ such that $f \in M_\beta$. By elementarity, we may choose $W_0 \in \mathcal{A}_\beta$ such that $h^{-1}W_0 \subseteq f^{-1}\{0\}$. Choose $W_1 \in \mathcal{V}_\beta$ such that $\overline{W}_1 \subseteq W_0$; choose $W_2 \in \mathcal{U}_\beta$ such that $\overline{W}_2 \subseteq W_1$. By (2) for stage α, it suffices to prove $\mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U} \subseteq \uparrow E_{\beta,W_2}$. Suppose $G \in \mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U}$. Then we have

$$(f^{-1}\{0\})/N \subseteq (h^{-1}U)/N \subseteq E_{\alpha,U}/N \subseteq G/N.$$

Since $G \in N$ and G is cozero, we have $f^{-1}\{0\} \subseteq G$. Hence,

$$E_{\beta,W_2} \subseteq h^{-1}W_1 \subseteq h^{-1}W_0 \subseteq f^{-1}\{0\} \subseteq G.$$

Thus, $\mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U} \subseteq \uparrow E_{\beta,W_2}$ as desired. □

Theorem 3.4.17. *Let X be a k-adic compactum. Then $Nt(X) \neq \omega_1$.*

Proof. Proceed as in the proof of Theorem 3.3.31. □

If still greater generality is desired, then one can easily combine the techniques of the proofs of Theorems 3.4.3, 3.4.14, and 3.4.16 to prove the following.

Theorem 3.4.18. *Let κ be an infinite cardinal and let Y be a compactum such that, for all open $U \subseteq Y$ and for all M satisfying $C(Y) \in M \prec H_\theta$ and $\kappa^+ \cap |M| \subseteq \kappa^+ \cap M \in \kappa^+ + 1$, the set U/M is the intersection of fewer than $(\mathrm{cf}\,\kappa)$-many open subsets of Y/M. If X is Hausdorff and a continuous image of Y, then we have the following.*

1. *If $\mathcal{U} \subseteq \mathcal{P}(X)$ and, for all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$, then $\mathcal{U}$ is almost κ^{op}-like. Hence, $\pi Nt(X) \leq \kappa$ and $\chi_K Nt(X) \leq \kappa$.*

2. *If $\pi\chi(p, X) = w(X)$ for all $p \in X$, then every base of X contains a κ^{op}-like base.*

On the other hand, Lemma 3.4.9 cannot be so easily generalized. For example, if X is the Stone space of the interval algrebra generated by $\{[a, b) : a, b \in \mathbb{R}\}$, then $w(X) = \mathfrak{c}$ and $\pi\chi(X) = \pi(X) = \omega$, despite it being shown in [23] that $\mathrm{Clop}(X)$ has the (ω_2, ω_1)-FN.

3.5 More on local Noetherian type

In this section, we find two sufficient conditions for a compactum to have a point with an ω^{op}-like local base. The first of these conditions will be used to prove Theorem 3.1.4. We also present some related results about local bases in terms of Tukey reducibility.

Definition 3.5.1. Given cardinals $\lambda \geq \kappa \geq \omega$ and a subset E in a space X, a *local* $\langle \lambda, \kappa \rangle$*-splitter* at E is a set $\mathcal{U}$ of λ-many open neighborhoods of E such that E is not contained in the interior of $\bigcap \mathcal{V}$ for any $\mathcal{V} \in [\mathcal{U}]^{\kappa}$. If $p \in X$, then we call a local $\langle \lambda, \kappa \rangle$-splitter at $\{p\}$ a local $\langle \lambda, \kappa \rangle$-splitter at p.

Theorem 3.5.2. *Suppose X is a compactum and $\omega_1 \leq \kappa = \min_{p \in X} \pi\chi(p, X)$. Then there is a local $\langle \kappa, \omega \rangle$-splitter at some $p \in X$.*

Proof. Given any map f, let $\prod f$ denote $\{\langle x_i \rangle_{i \in \operatorname{dom} f} : \forall i \in \operatorname{dom} f \;\; x_i \in f(i)\}$. Given any infinite open family $\mathcal{E}$, let $\Phi(\mathcal{E})$ denote the set of $\langle \sigma, \Gamma \rangle \in [\mathcal{E}]^{<\omega} \times ([\mathcal{E}]^{\omega})^{<\omega}$ for which every $\tau \in \prod \Gamma$ satisfies $\bigcap \sigma \subseteq \overline{\bigcup \operatorname{ran} \tau}$. Then $\Phi(\mathcal{E}) = \emptyset$ always implies $\mathcal{E}$ is ω^{op}-like and centered.

Let $\mathcal{R}$ denote the set of nonempty regular open subsets of X. Choose $\langle W_n \rangle_{n<\omega} \in \mathcal{R}^{\omega}$ such that $\overline{W}_{n+1} \subsetneq W_n \neq X$ for all $n < \omega$. Let Ω denote the class of transfinite sequences $\langle \langle U_\alpha, V_\alpha \rangle \rangle_{\alpha<\eta}$ of elements of $\mathcal{R}^2$ satisfying the following.

1. $\eta \geq \omega$ and $\langle \langle U_n, V_n \rangle \rangle_{n<\omega} = \langle \langle W_{n+1}, W_n \rangle \rangle_{n<\omega}$.

2. $\overline{U}_\alpha \subseteq V_\alpha$ for all $\alpha < \eta$.

3. $\mathcal{P}(V_\alpha) \cap \left\{ \bigcap \sigma \setminus \overline{\bigcup \tau} : \sigma, \tau \in \left[\bigcup_{\beta<\alpha} \{U_\beta, V_\beta\} \right]^{<\omega} \right\} \subseteq \{\emptyset\}$ for all $\alpha < \eta$.

4. $\Phi\left(\bigcup_{\alpha<\eta} \{U_\alpha, V_\alpha\}\right) = \emptyset$.

Seeking a contradiction, suppose η is a limit ordinal and $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta} \notin \Omega$, but $\langle\langle U_\beta, V_\beta\rangle\rangle_{\beta<\alpha} \in \Omega$ for all $\alpha < \eta$. Then (1), (2), and (3) hold for $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$, so there exists $\langle\sigma, \Gamma\rangle \in \Phi\big(\bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}\big)$. We may choose $i \in \operatorname{dom}\Gamma$ such that $\Gamma(i) \not\subseteq \bigcup_{\beta<\alpha}\{U_\beta, V_\beta\}$ for all $\alpha < \eta$. Set $\Lambda = \Gamma \upharpoonright (\operatorname{dom}\Gamma \setminus \{i\})$. We may assume $\operatorname{dom}\Gamma$ is minimal among its possible values; hence, there exists $\tau \in \prod \Lambda$ such that $\bigcap \sigma \not\subseteq \overline{\bigcup \operatorname{ran}\tau}$. Choose $\alpha < \eta$ and $W \in \Gamma(i)$ such that $\sigma \cup \operatorname{ran}\tau \subseteq \bigcup_{\beta<\alpha}\{U_\beta, V_\beta\}$ and $W \in \{U_\alpha, V_\alpha\}$. Then $\bigcap \sigma \setminus \overline{\bigcup \operatorname{ran}\tau} \not\subseteq W$ by (2) and (3). Since W is regular, $\bigcap \sigma \setminus \overline{\bigcup \operatorname{ran}\tau} \not\subseteq \overline{W}$; hence, $\bigcap \sigma \not\subseteq \overline{W \cup \bigcup \operatorname{ran}\tau}$, in contradiction with $\langle\sigma, \Gamma\rangle \in \Phi\big(\bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}\big)$. Thus, Ω is closed with respect to unions of increasing chains.

It follows from (3) that $\Omega \subseteq (\mathcal{R}^2)^{<|\mathcal{R}|^+}$. Moreover, $\langle\langle W_{n+1}, W_n\rangle\rangle_{n<\omega} \in \Omega$. Hence, by Zorn's Lemma, Ω has a maximal element $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$. Set $\mathcal{B} = \bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}$. Let us show that $\eta \geq \kappa$. Suppose not. For each $x \in X$, choose $Y_x, Z_x \in \mathcal{R}$ such that $x \in Y_x \subseteq \overline{Y}_x \subseteq Z_x$ and Z_x does not contain any nonempty open set of the form $\bigcap \sigma \setminus \overline{\bigcup \tau}$ where $\sigma, \tau \in [\mathcal{B}]^{<\omega}$. Choose $\rho \in [X]^{<\omega}$ such that $\bigcup_{x\in\rho} Y_x = X$. Let us show that $\Phi(\mathcal{B} \cup \{Y_x, Z_x\}) = \emptyset$ for some $x \in \rho$. Seeking a contradiction, suppose $\langle\sigma_x, \Gamma_x\rangle \in \Phi(\mathcal{B} \cup \{Y_x, Z_x\})$ for all $x \in \rho$. We may assume $\bigcup_{x\in\rho}\bigcup \operatorname{ran}\Gamma_x \subseteq \mathcal{B}$. Let Λ be a concatenation of $\{\Gamma_x : x \in \rho\}$ and set $\tau = \mathcal{B} \cap \bigcup_{x\in\rho} \sigma_i$. Then for all $\zeta \in \prod \Lambda$ we have

$$\bigcap \tau = \bigcap_{y\in\rho}\bigcap(\sigma_y \cap \mathcal{B}) = \bigcup_{x\in\rho}\left(Y_x \cap \bigcap_{y\in\rho}\bigcap(\sigma_y \cap \mathcal{B})\right) \subseteq \bigcup_{x\in\rho}\bigcap \sigma_x \subseteq \overline{\bigcup \operatorname{ran}\zeta}.$$

Hence, $\langle\tau, \Lambda\rangle \in \Phi(\mathcal{B})$, in contradiction with (4). Therefore, we may choose $x \in \rho$ such that $\Phi(\mathcal{B} \cup \{Y_x, Z_x\}) = \emptyset$. But then $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta+1} \in \Omega$ if we set $U_\eta = Y_x$ and $V_\eta = Z_x$, in contradiction with the maximality of $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$. Thus, $\eta \geq \kappa$.

Set $\mathcal{A} = \{V_\alpha : \alpha < \eta\}$. By (3), $|\mathcal{A}| = |\eta| \geq \kappa$. Set $K = \bigcap_{\alpha<\eta} \overline{U}_\alpha$. Then it suffices to show that $\mathcal{A}$ is a local $\langle|\eta|, \omega\rangle$-splitter at some $x \in K$. Suppose not. Then each $x \in K$

has an open neighborhood W_x that is a subset of infinitely many elements of $\mathcal{A}$. Hence, $\Phi(\mathcal{B} \cup \{W_x\}) \neq \emptyset$ for all $x \in K$. Choose $\rho \in [K]^{<\omega}$ such that $K \subseteq \bigcup_{x\in\rho} W_x$. Choose an open set W such that $W \cup \bigcup_{x\in\rho} W_x = X$ and $\overline{W} \cap K = \emptyset$. By compactness, $\mathcal{B} \cup \{W\}$ is not centered; hence, $\Phi(\mathcal{B} \cup \{W\}) \neq \emptyset$. Reusing our earlier concatenation argument, we have $\Phi(\mathcal{B}) \neq \emptyset$, in contradiction with (4). Thus, $\mathcal{A}$ is a local $\langle|\eta|, \omega\rangle$-splitter at some $x \in K$. □

Lemma 3.5.3. *Suppose X is a space with a point p at which there is no finite local base. Then $\chi Nt(p, X)$ is the least $\kappa \geq \omega$ for which there is a local $\langle \chi(p, X), \kappa\rangle$-splitter at p. Moreover, if $\lambda > \chi(p, X)$, then p does not have a local $\langle \lambda, \kappa\rangle$-splitter at p for any $\kappa < \lambda$ or $\kappa \leq \operatorname{cf} \lambda$.*

Proof. By Lemma 3.2.3, $\chi(p, X) \geq \chi Nt(p, X)$; hence, a $\chi Nt(p, X)^{\mathrm{op}}$-like local base at p (which necessarily has size $\chi(p, X)$) is a local $\langle \chi(p, X), \chi Nt(p, X)\rangle$-splitter at p. To show the converse, let $\lambda = \chi(p, X)$ and let $\langle U_\alpha\rangle_{\alpha<\lambda}$ be a sequence of open neighborhoods of p. Let $\{V_\alpha : \alpha < \lambda\}$ be a local base at p. For each $\alpha < \lambda$, choose $W_\alpha \in \{V_\beta : \beta < \lambda\}$ such that $W_\alpha \subseteq U_\alpha \cap V_\alpha$. Then $\{W_\alpha : \alpha < \lambda\}$ is a local base at p. Let $\kappa < \chi Nt(p, X)$. Then there exist $\alpha < \lambda$ and $I \in [\lambda]^\kappa$ such that $W_\alpha \subseteq \bigcap_{\beta\in I} W_\beta$. Hence, p is in the interior of $\bigcap_{\beta\in I} U_\beta$. Hence, $\{U_\alpha : \alpha < \lambda\}$ is not a local $\langle \lambda, \kappa\rangle$-splitter at p.

To prove the second half of the lemma, suppose $\lambda > \chi(p, X)$ and $\mathcal{A}$ is a set of λ-many open neighborhoods of p. Let $\mathcal{B}$ be a local base at p of size $\chi(p, X)$. Then, for all $\kappa < \lambda$ and $\kappa \leq \operatorname{cf} \lambda$, there exist $U \in \mathcal{B}$ and $\mathcal{C} \in [\mathcal{A}]^\kappa$ such that $U \subseteq \bigcap \mathcal{C}$. Hence, $\mathcal{A}$ is not a local $\langle \lambda, \kappa\rangle$-splitter at p. □

Proof of Theorem 3.2.13. We may assume $\chi(X) \geq \omega_1$. By Theorem 3.5.2, there is a local $\langle \chi(X), \omega\rangle$-splitter at some $p \in X$. By Lemma 3.5.3, $\chi Nt(p, X) = \omega$. □

Proof of Theorem 3.1.4. Let X be a homogeneous compactum. By a result of Arhangel′skiĭ (see 1.5 of [2]), $|Y| \leq 2^{\pi\chi(Y)c(Y)}$ for all homogeneous T_3 spaces Y. Since $|X| = 2^{\chi(X)}$ by Arhangel′skiĭ's Theorem and the Čech-Pospišil Theorem, we have $\chi(X) \leq \pi\chi(X)c(X)$ by GCH. If $\pi\chi(X) = \chi(X)$, then $\chi Nt(X) = \omega$ by Theorem 3.2.13. Hence, we may assume $\pi\chi(X) < \chi(X)$; hence, $\chi Nt(X) \leq \chi(X) \leq c(X)$ by Theorem 3.2.4. □

Example 3.5.4. Consider $2^{\omega_1}_{\text{lex}}$ (*i.e.*, 2^{ω_1} ordered lexicographically). Every point in this space has character and local Noetherian type ω_1, and some but not all points have π-character ω.

Definition 3.5.5 (Tukey [68])**.** Given two quasiorders P and Q, we say f is a *Tukey* map from P to Q and write $f\colon P \leq_T Q$ if f is a map from P to Q such that all preimages of bounded subsets of Q are bounded in P. We say that P is *Tukey reducible* to Q and write $P \leq_T Q$ if there exists $f\colon P \leq_T Q$. We say that P and Q are *Tukey equivalent* and write $P \equiv_T Q$ if $P \leq_T Q \leq_T P$.

Tukey showed that two directed sets are Tukey equivalent if and only if they embed as cofinal subsets of a common directed set. In particular, any two local bases at a common point in a topological space are Tukey equivalent. Another, easily checked fact is that $P \leq_T \langle [\operatorname{cf} P]^{<\omega}, \subseteq \rangle$ for every directed set P. Also, $[\kappa]^{<\omega} \leq_T [\lambda]^{<\omega}$ if $\kappa \leq \lambda$.

Lemma 3.5.6. *Suppose $\kappa \geq \omega$ and E is a subset of a space X with a local $\langle \kappa, \omega \rangle$-splitter at E. Then $\langle [\kappa]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{A}, \supseteq \rangle$ for every neighborhood base $\mathcal{A}$ of E.*

Proof. Let $\mathcal{U}$ be a local $\langle \kappa, \omega \rangle$-splitter at E. Let $\mathcal{N}$ be the set of open neighborhoods of E. Then $\mathcal{N}$ is Tukey equivalent to every neighborhood base of E (with respect to $\supseteq$), so it suffices to show that $[\mathcal{U}]^{<\omega} \leq_T \langle \mathcal{N}, \supseteq \rangle$. Define $f\colon [\mathcal{U}]^{<\omega} \to \mathcal{N}$ by $f(\sigma) = \bigcap \sigma$

for all $\sigma \in [\mathcal{U}]^{<\omega}$. Then, for all $N \in \mathcal{N}$, we have $|f^{-1}{\uparrow}N| < \omega$ because $\mathcal{U}$ is a local $\langle \kappa, \omega \rangle$-splitter; whence, $f^{-1}{\uparrow}N$ is bounded in $[\mathcal{U}]^{<\omega}$. Thus, $f \colon [\mathcal{U}]^{<\omega} \leq_T \langle \mathcal{N}, \supseteq \rangle$. □

Theorem 3.5.7. *Suppose X is a compactum and $\omega_1 \leq \kappa = \min_{p \in X} \pi\chi(p, X)$. Then, for some $p \in X$, every local base $\mathcal{A}$ at p satisfies $\langle [\kappa]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{A}, \supseteq \rangle$.*

Proof. Combine Theorem 3.5.2 and Lemma 3.5.6. □

Lemma 3.5.8. *Suppose E is a subset of a space X and E has no finite neighborhood base. Then the following are equivalent.*

1. *$\chi Nt(E, X) = \omega$.*
2. *There is a local $\langle \chi(E, X), \omega \rangle$-splitter at E.*
3. *Every neighborhood base $\mathcal{A}$ of E satisfies $\langle [\chi(E, X)]^{<\omega}, \subseteq \rangle \equiv_T \langle \mathcal{A}, \supseteq \rangle$.*

Proof. By Lemma 3.5.3, (1) and (2) are equivalent. Let $\mathcal{B}$ be a neighborhood base of E of size $\chi(E, X)$. By Lemma 3.5.6, (2) implies $[\chi(E, X)]^{<\omega} \leq_T \langle \mathcal{A}, \supseteq \rangle \equiv_T \langle \mathcal{B}, \supseteq \rangle \leq_T [\chi(E, X)]^{<\omega}$ for every neighborhood base $\mathcal{A}$ of E. Thus, (2) implies (3). Finally, suppose $\mathcal{A}$ is a neighborhood base of E and $[\chi(E, X)]^{<\omega} \equiv_T \langle \mathcal{A}, \supseteq \rangle$. Then $[\chi(E, X)]^{<\omega}$ and $\langle \mathcal{A}, \supseteq \rangle$ embed as cofinal subsets of a common directed set. Hence, $\langle \mathcal{A}, \subseteq \rangle$ is almost ω^{op}-like by Lemma 3.2.20. Hence, $\mathcal{A}$ contains an ω^{op}-like neighborhood base of E. Thus, (3) implies (1). □

Theorem 3.5.9. *Suppose X is an infinite homogeneous compactum and $\pi\chi(X) = \chi(X)$. Then, for all $p \in X$ and for all local bases $\mathcal{A}$ at p, we have $\langle \mathcal{A}, \supseteq \rangle \equiv_T \langle [\chi(X)]^{<\omega}, \subseteq \rangle$.*

Proof. Combine Theorem 3.2.13 and Lemma 3.5.8. □

Definition 3.5.10. Given $n < \omega$ and ordinals $\alpha, \beta_0, \ldots, \beta_n$, let $\alpha \to (\beta_0, \ldots, \beta_n)$ denote the proposition that for all $f\colon [\alpha]^2 \to n+1$ there exist $i \leq n$ and $H \subseteq \alpha$ such that $f[[H]^2] = \{i\}$ and H has order type β_i.

Lemma 3.5.11. *Suppose $\kappa = \operatorname{cf} \kappa > \omega$ and P is a directed set such that $[\kappa]^{<\omega} \leq_T P$. Then P contains a set of κ-many pairwise incomparable elements.*

Proof. Let Q be a well-founded, cofinal subset of P. Then $P \equiv_T Q$; let $f\colon [\kappa]^{<\omega} \leq_T Q$. Define $g\colon [\kappa]^2 \to 3$ by $g(\{\alpha < \beta\}) = 0$ if $f(\{\alpha\}) \not\leq f(\{\beta\}) \not\leq f(\{\alpha\})$ and $g(\{\alpha < \beta\}) = 1$ if $f(\{\alpha\}) > f(\{\beta\})$ and $g(\{\alpha < \beta\}) = 2$ if $f(\{\alpha\}) \leq f(\{\beta\})$. By the Erdös-Dushnik-Miller Theorem, $\kappa \to (\kappa, \omega+1, \omega+1)$. Since Q is well-founded, there is no $H \in [\kappa]^\omega$ such that $g[[H]^2] = \{1\}$. Since f is Tukey and all infinite subsets of $[\kappa]^{<\omega}$ are unbounded, there is no $H \subseteq \kappa$ of order type $\omega + 1$ such that $g[[H]^2] = \{2\}$. Hence, there exists $H \in [\kappa]^\kappa$ such that $g[[H]^2] = \{0\}$; whence, $f[[H]^1]$ is a κ-sized, pairwise incomparable subset of P. □

Theorem 3.5.12. *Suppose $\kappa = \operatorname{cf} \kappa > \omega$ and X is a compactum such that every point has a local base not containing a set of κ-many pairwise incomparable elements. Then some point in X has π-character less than κ.*

Proof. Combine Theorem 3.5.7 and Lemma 3.5.11 to prove the contrapositive of the theorem. □

Corollary 3.5.13. *Suppose X is a compactum such that every point has a local base that is well-quasiordered with respect to $\supseteq$. Then some point in X has countable π-character.*

Finally, let us present a few results about local Noetherian type and topological embeddings.

Lemma 3.5.14. *Suppose X is a space, $Y \subseteq X$, and $p \in Y$ satisfies $\chi(p, Y) = \chi(p, X)$. Then $\chi Nt(p, X) \leq \chi Nt(p, Y)$.*

Proof. Set $\lambda = \chi(p, Y)$ and $\kappa = \chi Nt(p, Y)$; we may assume $\lambda > \omega$ by Theorem 3.2.4. By Lemma 3.5.3, we may choose a local $\langle \lambda, \kappa \rangle$-splitter $\mathcal{A}$ at p in Y. For each $U \in \mathcal{A}$, choose an open subset $f(U)$ of X such that $f(U) \cap Y = U$. Set $\mathcal{B} = f[\mathcal{A}]$. Then $|\mathcal{B}| = \lambda$ because f is bijective. Suppose $\mathcal{C} \in [\mathcal{B}]^\kappa$ and p is in the interior of $\bigcap \mathcal{C}$ with respect to X. Then p is in the interior of $Y \cap \bigcap \mathcal{C}$ with respect to Y, in contradiction with how we chose $\mathcal{A}$. Thus, $\mathcal{B}$ is a local $\langle \lambda, \kappa \rangle$-splitter at p in X. By Lemma 3.5.3, $\chi Nt(p, X) \leq \kappa$. □

Theorem 3.5.15. *For each $\kappa \geq \omega$, there exists $p \in u(\kappa)$ such that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$.*

Proof. Generalizing an argument of Isbell [32] about $\beta\omega$, let A be an independent family of subsets of κ of size 2^κ. Set $B = \bigcup_{F \in [A]^\omega} \{x \subseteq \kappa : \forall y \in F \quad |x \setminus y| < \kappa\}$. Since A is independent, we may extend A to an ultrafilter p on κ such that $p \cap B = \emptyset$. For each $x \subseteq \kappa$, set $x^* = \{q \in u(\kappa) : x \in q\}$. Then $\{x^* : x \in A\}$ is a local $\langle 2^\kappa, \omega \rangle$-splitter at p. Since $\chi(p, u(\kappa)) \leq 2^\kappa$, it follows from Lemma 3.5.3 that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$. □

Theorem 3.5.16. *Suppose $\kappa \geq \omega$ and X is a space such that $\chi(X) = 2^\kappa$ and $u(\kappa)$ embeds in X. Then there is an ω^{op}-like local base at some point in X. Hence, $\chi Nt(X) = \omega$ if X is homogeneous.*

Proof. Let j embed $u(\kappa)$ into X. By Theorem 3.5.15, there exists $p \in u(\kappa)$ such that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$. By Lemma 3.5.14, $\chi Nt(j(p), X) = \omega$. □

Theorem 3.5.17. *Suppose p is a point in a dense subspace Y of a T_3 space X. Then $\chi Nt(p, X) \geq \chi Nt(p, Y)$.*

Proof. Set $\kappa = \chi Nt(p, Y)$ and let $\mathcal{A}$ be a κ^{op}-like local base at p in X. By Lemma 3.2.20, we may assume $\mathcal{A}$ consists only of regular open sets. Set $\mathcal{B} = \{U \cap Y : U \in \mathcal{A}\}$. Given any $U, V \in \mathcal{A}$ such that $U \not\subseteq V$, we have $U \setminus \overline{V} \neq \emptyset$; whence, $U \cap Y \setminus \overline{V} \neq \emptyset$; whence, $U \cap Y \not\subseteq V \cap Y$. Therefore, $\mathcal{B}$ is κ^{op}-like; hence, $\chi Nt(p, Y) \leq \chi Nt(p, X)$. $\square$

Example 3.5.18. Consider the sequential fan Y with ω-many spines. More explicitly, Y is the space $\omega^2 \cup \{p\}$ obtained by taking $\omega \times (\omega+1)$ and collapsing the subspace $\omega \times \{\omega\}$ to a point p. It is easily checked that Y is $T_{3.5}$. Choose a compactification X of Y. Then $c(X) = c(Y) = \omega$ and X is not homogeneous because it has isolated points. We will show $\chi Nt(p, X) \geq \omega_1$, thereby demonstrating that homogeneity cannot be removed from the hypothesis of Theorem 3.1.4. It suffices to show that $\chi Nt(p, Y) \geq \omega_1$, for we can then apply Theorem 3.5.17. Given $f \in \omega^\omega$, set $U_f = \{p\} \cup \{\langle m, n\rangle \in \omega^2 : n \geq f(m)\}$. Set $\mathcal{A} = \{U_f : f \in \omega^\omega\}$, which is a local base at p in Y. Suppose $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}$ is a local base at p. Then it suffices to show that $\mathcal{B}$ is not ω^{op}-like. By an easy diagonalization argument, no local base at p is countable. Choose $\mathcal{B}_0 \in [\mathcal{B}]^{\omega_1}$. Given $n < \omega$, $\mathcal{B}_n \in [\mathcal{B}]^{\omega_1}$, and $U_{f_0}, \ldots, U_{f_{n-1}} \in \mathcal{B}$, choose $\mathcal{B}_{n+1} \in [\mathcal{B}_n]^{\omega_1}$ such that $g(n) = h(n)$ for all $U_g, U_h \in \mathcal{B}_{n+1}$. Then choose $U_{f_n} \in \mathcal{B}_{n+1} \setminus \{U_{f_0}, \ldots, U_{f_{n-1}}\}$. For each $n < \omega$, set $g(n) = \max\{f_0(n), \ldots, f_n(n)\}$ Then $U_g \subseteq U_{f_n}$ for all $n < \omega$; hence, $\mathcal{B}$ is not ω^{op}-like.

3.6 Questions

Question 3.6.1. Do there exist spaces X and Y such that $\chi_K Nt(X \times Y)$ exceeds $\chi_K Nt(X)\chi_K Nt(Y)$? [In very recent unpublished work, Santi Spadaro has shown that there is a $T_{3.5}$ space X such that such that $Nt(\omega_1) = \omega_2 > \omega_1 = Nt(X \times \omega_1)$.]

Question 3.6.2. Does ZFC prove there is a homogeneous compactum X such that

$\pi Nt(X) \geq \omega_1$?

Question 3.6.3. Suppose X is a compactum and $\chi_K Nt(X) \geq \omega_1$. Can X be homogeneous? first countable? both?

Question 3.6.4. Is there a dyadic compactum X such that $\pi\chi(p, X) = \chi(p, X)$ for all $p \in X$ but X has no ω^{op}-like base? In particular, if Y is as in Example 3.3.12 and Z is the discrete sum of Y and 2^{ω_2}, then does Z^{ω_1} have an ω^{op}-like base?

Question 3.6.5. If κ is a singular cardinal with cofinality ω, then is there a dyadic compactum with Noetherian type κ^+? Is there a dyadic compactum with weakly inaccessible Noetherian type?

Question 3.6.6. Is every k-adic compactum a continuous image of a totally disconnected k-metrizable compactum?

Question 3.6.7. Is there a homogeneous compactum with a local base Tukey equivalent to $\omega \times \omega_1$? For each $n < \omega$, all the local bases in $\prod_{i \leq n} 2^{\omega_i}_{\mathrm{lex}}$ are Tukey equivalent to $\prod_{i \leq n} \omega_i$, but these spaces are not homogeneous for $n \geq 1$ because some but not all points have countable π-character. By Theorem 3.1.4, if a homogeneous compactum X in a model of GCH has a local base Tukey equivalent to $\prod_{i \leq n} \omega_i$ for some $n \geq 2$, then $c(X) > \mathfrak{c}$ in that model.

Chapter 4

More about Noetherian type

4.1 Subsets of bases

Given a space X, does every base of X contain an $Nt(X)^{\mathrm{op}}$-like base of X? There is no known counterexample, and Lemma 3.3.19 says the answer is yes for the wide class of spaces X satisfying $\chi(p, X) = w(X)$ for all $p \in X$. We will present some further partial answers to this question. In particular, it is consistent that the answer is yes for all homogeneous compacta.

Proposition 4.1.1. *If X is a space and $\mathcal{A}$ is a $(w(X)^+)^{\mathrm{op}}$-like base of X, then $|\mathcal{A}| \leq w(X)$.*

Proof. Seeking a contradiction, suppose $|\mathcal{A}| > w(X)$. Let $\mathcal{B}$ be a base of X of size $w(X)$. Then every element of $\mathcal{A}$ contains an element of $\mathcal{B}$. Hence, some $U \in \mathcal{B}$ is contained in $w(X)^+$-many elements of $\mathcal{A}$. Clearly U contains some $V \in \mathcal{A}$, so $\mathcal{A}$ is not $(w(X)^+)^{\mathrm{op}}$-like. $\square$

Lemma 4.1.2. *If X is a compactum and $\pi\chi(p, X) < \operatorname{cf} \kappa = \kappa \leq w(X)$ for all $p \in X$, then $Nt(X) > \kappa$.*

Proof. Let $\mathcal{A}$ be a base of X. By Misčenko's Lemma, there exist $p \in X$ and $\mathcal{B} \in [\mathcal{A}]^{\kappa}$ such that $p \in \bigcap \mathcal{B}$. Let $\mathcal{C} \in [\mathcal{A}]^{<\kappa}$ be a local π-base at p. Then some element of $\mathcal{C}$ is contained in κ-many elements of $\mathcal{B}$. Hence, $\mathcal{A}$ is not κ^{op}-like. $\square$

Theorem 4.1.3. *If X is a homogeneous compactum with regular weight, then every base of X contains an $Nt(X)^{\mathrm{op}}$-like base.*

Proof. If $\chi(X) = w(X)$, then just apply Lemma 3.3.19. If $\chi(X) < w(X)$, then $Nt(X) = w(X)^+$ by Lemma 4.1.2; whence, by Proposition 3.3.17, every base $\mathcal{A}$ of X contains a base $\mathcal{B}$ that is $Nt(X)^{\mathrm{op}}$-like simply because $|\mathcal{B}| < Nt(X)$. □

We can exchange the above requirement that $w(X)$ be regular for a weak form of GCH.

Corollary 4.1.4. *Suppose every limit cardinal is strong limit. Then, for every homogeneous compactum X, every base of X contains an $Nt(X)^{\mathrm{op}}$-like base.*

Proof. By Arhangel′skiĭ's Theorem, $\chi(X) \leq w(X) \leq 2^{\chi(X)}$. If $\chi(X) < w(X)$, then $w(X)$ is a successor cardinal; apply Theorem 4.1.3. If $\chi(X) = w(X)$, then apply Lemma 3.3.19. □

Without assuming homogeneity, we still can get some weak results. Lemma 3.3.23 says that for a broad class of spaces X, if $\pi\chi(p, X) = w(X)$ for all $p \in X$, then $Nt(X) \leq w(X)$. We also have the following.

Theorem 4.1.5. *Suppose κ is a regular cardinal and X is a locally κ-compact T_3 space such that $Nt(X) \leq w(X) = \kappa$. Then every base of X contains a κ^{op}-like base of X.*

Proof. Let $\mathcal{A}$ be a base of X and let $\mathcal{B}$ be a κ^{op}-like base of X. By Proposition 3.3.17, we may assume $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Suppose $\kappa = \omega$. Then X is a metrizable; fix a compatible metric. Moreover, there is a sequence $\langle U_n \rangle_{n<\omega}$ of open subsets of X such that $X = \bigcup_{n<\omega} U_n$ and $\overline{U}_n$ is compact for all $n < \omega$. For each $\langle m, n \rangle \in \omega^2$, let $\mathcal{A}_{m,n}$ be a finite cover of $\overline{U}_m$ by elements of $\mathcal{A}$ with diameter less than 2^{-n}. Set $\mathcal{A}' = \bigcup_{m<n<\omega} \mathcal{A}_{m,n}$.

Then $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{A}'$ is a base of X. Suppose $V, W \in \mathcal{A}'$ and $V \subsetneq W$. Then V contains a ball of radius 2^{-n} for some $n < \omega$; hence, $\operatorname{diam} W \geq 2^{-n}$; hence, $W \in \bigcup_{l<m<n} \mathcal{A}_{l,m}$. Thus, $\mathcal{A}'$ is ω^{op}-like.

Suppose $\kappa > \omega$. Let $\langle M_\alpha \rangle_{\alpha \leq \kappa}$ be a continuous elementary chain such that $\{M_\beta : \beta < \alpha\} \cup \{\mathcal{A}, \mathcal{B}\} \subseteq M_\alpha \prec H_\theta$ and $|M_\alpha| < \kappa$ and $M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. Then $\mathcal{A} \cup \mathcal{B} \subseteq M_\kappa$. For each $\alpha < \kappa$, let $\mathcal{U}_\alpha$ denote the set of all $U \in \mathcal{A} \cap M_{\alpha+1}$ for which U has a superset in $\mathcal{B} \setminus M_\alpha$. Set $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha \subseteq \mathcal{A}$. First, let us show that $\mathcal{U}$ is κ^{op}-like. Suppose $\alpha < \kappa$ and $\mathcal{U}_\alpha \ni U \subseteq V \in \mathcal{U}$. Then there exist $\beta < \kappa$ and $B \in \mathcal{B} \setminus M_\beta$ such that $B \supseteq V \in M_{\beta+1}$. Hence, $U \subseteq B$; hence, $B \in \{W \in \mathcal{B} : U \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$; hence, $B \in M_{\alpha+1}$; hence, $\beta \leq \alpha$; hence, $V \in M_{\alpha+1}$. Thus, $\mathcal{U}$ is κ^{op}-like.

Finally, let us show that $\mathcal{U}$ is a base of X. Suppose $p \in B \in \mathcal{B}$ and $\overline{B}$ is κ-compact. Then it suffices to find $U \in \mathcal{U}$ such that $p \in U \subseteq B$. Let β be the least $\alpha < \kappa$ such that there exists $A \in \mathcal{A} \cap M_{\alpha+1}$ satisfying $p \in A \subseteq \overline{A} \subseteq B$. Fix such an A. If $B \notin M_\beta$, then $A \in \mathcal{U}_\beta$ and $p \in A \subseteq B$. Hence, we may assume $B \in M_\beta$. For each $q \in \overline{A}$, choose $\langle A_q, B_q \rangle \in \mathcal{A} \times \mathcal{B}$ such that $q \in A_q \subseteq B_q \subseteq \overline{B}_q \subseteq B$. Then there exists $\sigma \in \left[\overline{A}\right]^{<\kappa}$ such that $\overline{A} \subseteq \bigcup_{q\in\sigma} A_q$. By elementarity, we may assume $\langle\langle A_q, B_q \rangle\rangle_{q\in\sigma} \in M_{\beta+1}$; hence, $A_q, B_q \in M_{\beta+1}$ for all $q \in \sigma$. Choose $q \in \sigma$ such that $p \in A_q$. If $B_q \notin M_\beta$, then $A_q \in \mathcal{U}_\beta$ and $p \in A_q \subseteq B$. Hence, we may assume $B_q \in M_\beta$; hence, we may choose $\alpha < \beta$ such that $B_q \in M_{\alpha+1}$. Then $B \in \{W \in \mathcal{B} : B_q \subseteq W\} \in M_{\alpha+1} \cap [\mathcal{B}]^{<\kappa}$; hence, $B \in M_{\alpha+1}$. For each $r \in \overline{B_q}$, choose $W_r \in \mathcal{A}$ such that $r \in W_r \subseteq \overline{W}_r \subseteq B$. Then there exists $\tau \in \left[\overline{B}_q\right]^{<\kappa}$ such that $\overline{B}_q \subseteq \bigcup_{r\in\tau} W_r$. By elementarity, we may assume $\langle W_r \rangle_{r\in\tau} \in M_{\alpha+1}$. Choose $r \in \tau$ such that $p \in W_r$. Then $W_r \in \mathcal{A} \cap M_{\alpha+1}$ and $p \in W_r \subseteq \overline{W_r} \subseteq B$, in contradiction with the minimality of β. Thus, $\mathcal{U}$ is a base of X. $\square$

Question 4.1.6. Is there a space X with a base that does not contain an $Nt(X)^{\mathrm{op}}$-like base of X? Is there such a metric space? Can $X = \omega^\omega$?

4.2 Power homogeneous compacta

This section is joint work of Guit-Jan Ridderbos and myself.

Definition 4.2.1. A space is *power homogeneous* if some power of it is homogeneous.

The following theorem is due to van Mill [45].

Theorem 4.2.2. *Every power homogeneous compactum X satisfies $|X| \leq 2^{\pi\chi(X)c(X)}$.*

Arhangel′skiĭ's identical result for homogeneous T_3 spaces was used in the proof of Theorem 3.1.4. Given the above extension of Arhangel′skiĭ's result, it is natural to ask to what extent Theorem 3.1.4 is true of power homogeneous compacta. Specifically, assuming GCH, do all power homogeneous compacta X satisfy $\chi Nt(X) \leq c(X)$, or at least $\chi Nt(X) \leq d(X)$? This section presents a partial positive answer to the last question. We show that if $d(X) < \mathrm{cf}\,\chi(X) = \max_{p\in X} \chi(p, X)$, then there is a nonempty open $U \subseteq X$ such that $\chi Nt(p, X) = \omega$ for all $p \in U$. (Note that $\chi Nt(X) \leq \chi(X)$.)

Definition 4.2.3. A sequence $\langle U_i \rangle_{i\in I}$ of neighborhoods of a point p in a space X is λ-*splitting* at p if, for all $J \in [I]^\lambda$, we have $p \notin \mathrm{int} \bigcap_{j\in J} U_j$.

Given an infinite cardinal κ and a point p in a space X, let $\mathrm{split}_\kappa(p, X)$ denote the least λ such that there exists a λ-splitting sequence $\langle U_\alpha \rangle_{\alpha<\kappa}$ of neighborhoods of p. Set $\mathrm{split}_\kappa(X) = \sup_{p\in X} \mathrm{split}_\kappa(p, X)$.

Remark 4.2.4. Note that if $\kappa = \chi(p, X)$, then $\chi Nt(p, X) = \mathrm{split}_\kappa(p, X)$. If $\kappa > \chi(p, X)$, then $\mathrm{split}_\kappa(p, X) = \kappa^+$. Also, if $\kappa \leq \chi(p, X)$, then $\mathrm{split}_\kappa(p, X) \leq \chi Nt(p, X) \leq \chi(p, X)$.

Definition 4.2.5. Given I and p, let $\Delta_I(p)$ denote the constant function $\langle p\rangle_{i\in I}$.

Definition 4.2.6. Given a subset E of a product $\prod_{i\in I} X_i$ and a subset J of I, we say that E is supported on J, or $\mathrm{supp}(E) \subseteq J$, if $E = (\pi^I_J)^{-1}\left[\pi^I_J[E]\right]$. If there is a least set J for which E is supported on J, then we may write $\mathrm{supp}(E) = J$.

Remark 4.2.7. We always have that $\mathrm{supp}(E) \subseteq A$ and $\mathrm{supp}(E) \subseteq B$ together imply $\mathrm{supp}(E) \subseteq A\cap B$. If a subset E of a product space is open, closed, or finitely supported, then there exists J such that $\mathrm{supp}(E) = J$, so we may unambiguously speak of $\mathrm{supp}(E)$.

Definition 4.2.8. A map f from a space X to a space Y is *open* at a point p in X if $f(p) \in \mathrm{int}\, f[N]$ for every neighborhood N of p (where $\mathrm{int}\, f[N]$ denotes the interior of $f[N]$ in Y).

Lemma 4.2.9. *Suppose $f\colon X \to Y$ and $p \in X$ and f is continuous at p and open at p. Then* $\mathrm{split}_\kappa(p, X) \leq \mathrm{split}_\kappa(f(p), Y)$ *for all κ.*

Proof. Set $\lambda = \mathrm{split}_\kappa(f(p), Y)$ and let $\langle V_\alpha\rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $f(p)$. For each $\alpha < \kappa$, choose a neighborhood U_α of p such that $U_\alpha \subseteq f^{-1}[V_\alpha]$. Suppose $I \in [\kappa]^\lambda$. Then $f(p) \notin \mathrm{int} \bigcap_{\alpha\in I} V_\alpha$. If $p \in \mathrm{int} \bigcap_{\alpha\in I} U_\alpha$, then $f(p) \in \mathrm{int}\, f\left[\bigcap_{\alpha\in I} U_\alpha\right] \subseteq \mathrm{int} \bigcap_{\alpha\in I} V_\alpha$, which is absurd. Thus, $p \notin \mathrm{int} \bigcap_{\alpha\in I} U_\alpha$, so $\mathrm{split}_\kappa(p, X) \leq \lambda$. □

Lemma 4.2.10. *Suppose p is a point in a space X and $n < \omega$. Then* $\mathrm{split}_\kappa(\Delta_n(p), X^n) = \mathrm{split}_\kappa(p, X)$ *for all κ.*

Proof. By Lemma 4.2.9, it suffices to show that $\mathrm{split}_\kappa(\Delta_n(p), X^n) \geq \mathrm{split}_\kappa(p, X)$. Set $\lambda = \mathrm{split}_\kappa(\Delta_n(p), X^n)$ and let $\langle V_\alpha\rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $\Delta_n(p)$. We may shrink each V_α to a smaller neighborhood of $\Delta_n(p)$ while preserving λ-splitting, so we may assume that each V_α is a finite product $\prod_{i<n} V_{\alpha,i}$ of open sets.

Set $U_\alpha = \bigcap_{i<n} V_{\alpha,i}$ for all α. Suppose $I \in [\kappa]^\lambda$. Then $\Delta_n(p) \notin \operatorname{int} \bigcap_{\alpha\in I} V_\alpha$. If $p \in \operatorname{int} \bigcap_{\alpha\in I} U_\alpha$, then $\Delta_n(p) \in \left(\operatorname{int} \bigcap_{\alpha\in I} U_\alpha\right)^n \subseteq \operatorname{int} \bigcap_{\alpha\in I} V_\alpha$, which is absurd. Thus, $p \notin \operatorname{int} \bigcap_{\alpha\in I} U_\alpha$, so $\operatorname{split}_\kappa(p, X) \leq \lambda$. $\square$

Lemma 4.2.11. *Suppose p is a point in a space X and $\gamma < \operatorname{cf} \kappa \geq \omega$. Then*

$$\operatorname{split}_\kappa(\Delta_\gamma(p), X^\gamma) = \operatorname{split}_\kappa(p, X).$$

Proof. By Lemma 4.2.9, it suffices to show that $\operatorname{split}_\kappa(\Delta_\gamma(p), X^\gamma) \geq \operatorname{split}_\kappa(p, X)$. Set $\lambda = \operatorname{split}_\kappa(\Delta_\gamma(p), X^\gamma)$ and let $\langle V_\alpha \rangle_{\alpha<\kappa}$ be a λ-splitting sequence of neighborhoods of $\Delta_\gamma(p)$. We may assume each V_α has finite support and therefore choose $\sigma_\alpha \in \operatorname{Fn}(\gamma, \{U \subseteq X : U \text{ open}\})$ such that $V_\alpha = \bigcap_{\langle \beta, U\rangle \in \sigma_\alpha} \pi_\beta^{-1} U$. Since $|[\gamma]^{<\omega}| < \operatorname{cf} \kappa$, we may assume there is some $s \in [\gamma]^{<\omega}$ such that $\operatorname{dom} \sigma_\alpha = s$ for all $\alpha < \kappa$. But then $\langle \pi_s^\gamma[V_\alpha] \rangle_{\alpha<\kappa}$ is λ-splitting at $\Delta_s(p)$ in X^s. Thus, $\operatorname{split}_\kappa(\Delta_\gamma(p), X^\gamma) \geq \operatorname{split}_\kappa(\Delta_s(p), X^s)$. Apply Lemma 4.2.10. $\square$

Definition 4.2.12. Let U be an open neighborhood of a set K in a product space. We say that U is a *simple* neighborhood of K if, for every open V satisfying $K \subseteq V \subseteq U$, we have $\operatorname{supp}(U) \subseteq \operatorname{supp}(V)$.

Lemma 4.2.13. *If K is a compact subset of a compact product space $X = \prod_{i\in I} X_i$ and U is an open neighborhood of K, then K has a finitely supported simple neighborhood that is contained in U.*

Proof. Set $\sigma = \operatorname{supp}(U)$. By compactness of K, we may shrink U such that σ is finite. Hence, we may further shrink U until it is minimal in the sense that if V is open and $K \subseteq V \subseteq U$, then $\operatorname{supp}(V)$ is not a proper subset of σ. Suppose V is open and $K \subseteq V \subseteq U$; set $\tau = \operatorname{supp}(V)$. Then it suffices to show that $\sigma \subseteq \tau$. Suppose $p \in K$ and $q \in X$ and $\pi^I_{\sigma\cap\tau}(p) = \pi^I_{\sigma\cap\tau}(q)$. Set $r = (p \upharpoonright \tau) \cup q \upharpoonright (I \setminus \tau)$. Then $\pi^I_\tau(r) = \pi^I_\tau(p)$, so

$r \in V \subseteq U$. Moreover, $\pi^I_\sigma(q) = \pi^I_\sigma(r)$, so $q \in U$. Thus, $(\pi^I_{\sigma\cap\tau})^{-1}\left[\pi^I_{\sigma\cap\tau}[K]\right] \subseteq U$. By the Tube Lemma, there is an open W such that $K \subseteq W \subseteq U$ and $\mathrm{supp}(W) \subseteq \sigma \cap \tau$. By minimality of U, the set $\sigma \cap \tau$ is not a proper subset of σ; hence, $\sigma \subseteq \tau$. $\square$

Lemma 4.2.14. *Suppose κ is a regular uncountable cardinal and I is a set and $X = \prod_{i\in I} X_i$ is a compactum and $p \in X$ and $h \in \mathrm{Aut}(X)$ and $\mathrm{split}_\kappa(p(i), X_i) \geq \omega_1$ for all $i \in I$. Further suppose $\{C(X), p, h\} \subseteq M \prec H_\theta$ and $\kappa \cap M \in \kappa + 1$. Then we have*

$$\mathrm{supp}(h\left[(\pi^I_{I\cap M})^{-1}\left[\left\{\pi^I_{I\cap M}(p)\right\}\right]\right]) \subseteq M.$$

Proof. For each $i \in I$, let $\mathcal{U}_i$ denote the set of open neighborhoods of $p(i)$. For each $U \in \mathcal{U}_i$, let $V(U, i)$ be a finitely supported simple neighborhood of $h\left[\pi_i^{-1}[\{p(i)\}]\right]$ that is contained in $h\left[\pi_i^{-1}[U]\right]$ (using Lemma 4.2.13); set $\sigma(U, i) = \mathrm{supp}(V(U, i))$. By elementarity, we may assume the map V is in M, so $\sigma \in M$ too. Let $W(U, i)$ be an open neighborhood of $p(i)$ with such that $\pi_i^{-1}[W(U, i)] \subseteq h^{-1}[V(U, i)]$.

Fix $j \in I$. Suppose $\left|\bigcup_{U\in\mathcal{U}_j} \sigma(U, j)\right| \geq \kappa$. Then there exists $\langle U_\alpha\rangle_{\alpha<\kappa} \in \mathcal{U}_j^\kappa$ such that $\sigma(U_\alpha, j) \not\subseteq \sigma(U_\beta, j)$ for all $\beta < \alpha < \kappa$. Fix $E \in [\kappa]^\omega$ and an open neighborhood H of $h\left[\pi_j^{-1}[\{p(j)\}]\right]$ with finite support τ. Choose $\alpha \in E$ such that $\sigma(U_\alpha, j) \not\subseteq \tau$. By simplicity, $H \not\subseteq V(U_\alpha, j)$. Thus, $h\left[\pi_j^{-1}[\{p(j)\}]\right] \not\subseteq \mathrm{int}\bigcap_{\alpha\in E} V(U_\alpha, j)$; hence,

$$\pi_j^{-1}[\{p(j)\}] \not\subseteq \mathrm{int}\bigcap_{\alpha\in E} h^{-1}\left[V(U_\alpha, j)\right] \supseteq \mathrm{int}\bigcap_{\alpha\in E} \pi_j^{-1}[W(U_\alpha, j)];$$

hence, $p(j) \notin \mathrm{int}\bigcap_{\alpha\in E} W(U_\alpha, j)$. Since E was arbitrary, $\{W(U_\alpha, j) : \alpha < \kappa\}$ is ω-splitting at $p(j)$, in contradiction with $\mathrm{split}_\kappa(p(j), X_j) \geq \omega_1$. Thus, $\left|\bigcup_{U\in\mathcal{U}_j} \sigma(U, j)\right| < \kappa$.

Hence, for each $i \in I \cap M$, we have $\bigcup_{U\in\mathcal{U}_i} \sigma(U, i) \in [I]^{<\kappa} \cap M \subseteq \mathcal{P}(M)$; hence,

$$\mathrm{supp}(h\left[(\pi^I_{I\cap M})^{-1}\left[\left\{\pi^I_{I\cap M}(p)\right\}\right]\right]) \subseteq \bigcup_{i\in I\cap M}\bigcup_{U\in\mathcal{U}_i} \sigma(U, i) \subseteq M.$$

□

Corollary 4.2.15. *Let X be a compactum and κ an infinite cardinal. Suppose F is a closed subset of X and $\chi(F, X) < \kappa$ and $\pi\chi(p, X) \geq \kappa$ for all $p \in F$. Then $\mathrm{split}_\kappa(p, X) = \omega$ for some $p \in F$.*

Proof. Since $\pi\chi(p, X) \leq \pi\chi(p, F)\chi(F, X)$ for all $p \in F$, we have $\pi\chi(p, F) \geq \kappa$ for all $p \in F$. Apply Theorem 3.5.2 to F. □

The following theorem is an easy generalization of Ridderbos' Lemma 2.2 in [57].

Theorem 4.2.16. *Suppose X is a power homogeneous Hausdorff space, κ is a regular uncountable cardinal, and D is a dense subset of X such that $\pi\chi(d, X) < \kappa$ for all $d \in D$. Then $\pi\chi(p, X) < \kappa$ for all $p \in X$.*

Theorem 4.2.17. *Let κ be a regular uncountable cardinal, X be a power homogeneous compactum, and D be a dense subset of X of size less than κ. Suppose $\mathrm{split}_\kappa(d, X) \geq \omega_1$ for all $d \in D$. Then $\mathrm{split}_\kappa(p, X) = \mathrm{split}_\kappa(q, X)$ for all $p, q \in X$. Moreover, $\pi(X) < \kappa$.*

Proof. Let us first show that $\mathrm{split}_\kappa(p, X) = \mathrm{split}_\kappa(q, X)$ for all $p, q \in X$. Fix $p, q \in X$ such that $\mathrm{split}_\kappa(p, X) \geq \omega_1$ and $\mathrm{split}_\kappa(q, X) = \min_{x \in X} \mathrm{split}_\kappa(x, X)$. Then it suffices to show that that $\mathrm{split}_\kappa(p, X) = \mathrm{split}_\kappa(q, X)$. By Lemmas 4.2.9 and 4.2.11, it suffices to show that there exist $A \in [I]^{<\kappa}$ and $f \colon X^A \to X^A$ such that $f(\Delta_A(p)) = \Delta_A(q)$ and f is continuous at $\Delta_A(p)$ and open at $\Delta_A(p)$. Choose I and $h \in \mathrm{Aut}(X^I)$ such that $h(\Delta_I(p)) = h(\Delta_I(q))$. Fix $M \prec H_\theta$ such that $|M| < \kappa$ and $\kappa \cap M \in \kappa$ and $\{C(X), D, h, p\} \subseteq M$. Set $A = I \cap M$ and $Y = X^A \times \{p\}^{I \setminus A} \cong X^A$. Set $f = \pi^I_A \circ (h \restriction Y)$, which is continuous. Since $f(\Delta_I(p)) = \Delta_A(q)$, it suffices to show that f is open at $\Delta_I(p)$.

Fix a closed neighborhood $C \times \{p\}^{I \setminus A}$ of $\Delta_I(p)$ in Y. By the Tube Lemma and Lemma 4.2.14, there is an open neighborhood U of $\Delta_A(q)$ in X^A such that $(\pi^I_A)^{-1}U \subseteq h\left[(\pi^I_A)^{-1}[C]\right]$. Hence, it suffices to show that $U \subseteq f\left[C \times \{p\}^{I \setminus A}\right]$. Set

$$E = \bigcup \left\{D^\sigma \times \{p\}^{I \setminus \sigma} : \sigma \in [I]^{<\omega}\right\}$$

and $Z = \pi^I_A[E] \times \{p\}^{I \setminus A} = E \cap M$. Then $\pi^I_A[Z]$ is dense in X^A. Fix $z \in \pi^I_A[Z] \cap U$. By Lemma 4.2.14 applied to h^{-1} and $z \cup \Delta_{I \setminus A}(p)$, we have $\operatorname{supp}(h^{-1}\left[(\pi^I_A)^{-1}[\{z\}]\right]) \subseteq A$; hence, for all $x \in \pi^I_A\left[h^{-1}\left[(\pi^I_A)^{-1}[\{z\}]\right]\right] \subseteq C$, we have $f(x \cup \Delta_{I \setminus A}(p)) = z$. Thus, $\pi^I_A[Z] \cap U \subseteq f\left[C \times \{p\}^{I \setminus A}\right]$. Hence, $U \subseteq \overline{f\left[C \times \{p\}^{I \setminus A}\right]} = f\left[C \times \{p\}^{I \setminus A}\right]$.

Thus, $\operatorname{split}_\kappa(p, X) = \operatorname{split}_\kappa(q, X) \geq \omega_1$ for all $p, q \in X$. By Corollary 4.2.15, X has no closed G_δ subset K for which $\pi\chi(p, X) \geq \kappa$ for all $p \in K$. Hence, X has no open subset U for which $\pi\chi(p, X) \geq \kappa$ for all $p \in U$. By Theorem 4.2.16, $\pi\chi(p, X) < \kappa$ for all $p \in X$. Hence, $\pi(X) \leq \sum_{d \in D} \pi\chi(d, X) < \kappa$. □

Corollary 4.2.18. *Let D be a dense subset of a power homogeneous compactum X and let κ be a regular uncountable cardinal. Suppose* $\max_{p \in X} \chi(p, X) = \kappa$ *and* $|D| < \kappa$ *and* $\chi Nt(d, X) \geq \omega_1$ *for all* $d \in D$*. Then* $\pi(X) < \chi(p, X) = \kappa$ *and* $\chi Nt(p, X) = \chi Nt(X)$ *for all* $p \in X$*.*

Proof. Every $d \in D$ either has character κ, in which case $\operatorname{split}_\kappa(d, X) = \chi Nt(d, X) \geq \omega_1$, or has character less than κ, in which case $\operatorname{split}_\kappa(d, X) = \kappa^+ \geq \omega_1$. By Theorem 4.2.17, $\operatorname{split}_\kappa(p, X) = \operatorname{split}_\kappa(q, X)$ for all $p, q \in X$ and $\pi(X) < \kappa$. If $\operatorname{split}_\kappa(X) = \kappa^+$, then no point of X has character κ, which is absurd. Hence, $\operatorname{split}_\kappa(X) \leq \kappa$; hence, every point of Z has character at least κ; hence, every point has character κ; hence, $\chi Nt(p, X) = \operatorname{split}_\kappa(X)$ for all $p \in X$. □

Corollary 4.2.19 (GCH)**.** *There do not exist X, D, and κ as in the previous corollary. Hence, if X is a power homogeneous compactum and $\max_{p\in X}\chi(p,X) = \operatorname{cf}\chi(X) > d(X)$, then there is a nonempty open $U \subsetneq X$ such that $\chi Nt(p,X) = \omega$ for all $p \in U$.*

Proof. Seeking a contradiction, suppose X, D, and κ are as in the previous corollary. By Arhangel′skiĭ's Theorem and the Čech-Pospišil Theorem, $|X| = 2^\kappa$. Hence, by GCH and Theorem 4.2.2, $\kappa \leq \pi\chi(X)c(X)$. Since, $\pi\chi(X) \leq \pi(X) < \kappa$, it follows that $\kappa \leq c(X)$. Hence, $\kappa \leq c(X) \leq \pi(X) < \kappa$, which is absurd. □

4.3 Noetherian types of ordered Lindelöf spaces

We will show that a Lindelöf linearly ordered topological space has an ω^{op}-like base if and only if it is metric. Moreover, a compact linearly ordered topological space has an ω_1^{op}-like base if and only if it is metric.

Theorem 4.3.1. *Every metric space has an ω^{op}-like base.*

Proof. Let X be a metric space. For each $n < \omega$, let $\mathcal{A}_n$ be a locally finite open refinement of the balls of radius 2^{-n} in X. Set $\mathcal{A} = \bigcup_{n<\omega}\mathcal{A}_n$. Then $\mathcal{A}$ is a base of X because if $p \in X$ and $n < \omega$, then there exists $U \in \mathcal{A}_{n+1}$ such that $p \in U$ and U is contained in the ball of radius 2^{-n} with center p. Let us show that $\mathcal{A}$ is ω^{op}-like. Suppose $m < \omega$ and $U \in \mathcal{A}$ and $V \in \mathcal{A}_m$ and $U \subsetneq V$. Then there exist $p \in U$ and $\epsilon_0 > \epsilon_1 > 0$ such that the ϵ_0-ball with center p is contained in U and the ϵ_1-ball with center p intersects only finitely many elements of $\mathcal{A}_n$ for all $n < \omega$ satisfying $2^{-n} > \epsilon_0/2$. If $2^{-m} \leq \epsilon_0/2$, then V is contained in the ϵ_0-ball with center p, in contradiction with $U \subsetneq V$. Hence, $2^{-m} > \epsilon_0/2$; hence, there are only finitely many possibilities for m and V given U, for V intersects the ϵ_1-ball with center p. □

Lemma 4.3.2. *Let X be a Lindelöf linearly ordered topological space with open cover $\mathcal{A}$. Then $\mathcal{A}$ has a countable, locally finite refinement consisting only of countable unions of open intervals.*

Proof. Let $\{A_n : n < \omega\}$ be a countable refinement of $\mathcal{A}$ consisting only of open intervals. For each $n < \omega$, set $B_n = A_n \setminus \bigcup_{m<n} A_m$; set $\mathcal{B} = \{B_n : n < \omega\}$. Then $\mathcal{B}$ is a locally finite refinement of $\mathcal{A}$. Let $\mathcal{C}$ be the set of open intervals of X which intersect only finitely many elements of $\mathcal{B}$. Let $\mathcal{D}$ be the set of $U \in \mathcal{C}$ satisfying $\overline{U} \subseteq V$ for some $V \in \mathcal{C}$. Let $\{D_n : n < \omega\}$ be a countable subcover of $\mathcal{D}$. For each $n < \omega$, set $E_n = \overline{D_n \setminus \bigcup_{m<n} D_m}$; set $\mathcal{E} = \{E_n : n < \omega\}$. Then $\mathcal{E}$ is a locally finite refinement of $\mathcal{C}$. For each $n < \omega$, set $F_n = A_n \setminus \bigcup\{E \in \mathcal{E} : B_n \cap E = \emptyset\}$, which is a countable union of intervals; set $\mathcal{F} = \{F_n : n < \omega\}$. Since $\mathcal{E}$ is locally finite, each F_n is open. Hence, each F_n is a countable union of open intervals. Moreover, $B_n \subseteq F_n \subseteq A_n$ for all $n < \omega$; hence, $\mathcal{F}$ is a refinement of $\mathcal{A}$.

Thus, it suffices to show that $\mathcal{F}$ is locally finite. Since $\mathcal{E}$ is a locally finite cover of X, it suffices to show that each element of $\mathcal{E}$ only intersects finitely many elements of $\mathcal{F}$. Let $i < \omega$ and choose $V \in \mathcal{C}$ such that $E_i \subseteq V$. Suppose $j < \omega$ and $E_i \cap F_j \neq \emptyset$. Then $E_i \cap B_j \neq \emptyset$ by definition of F_j. Hence, $V \cap B_j \neq \emptyset$; hence, there are only finitely possibilities for B_j; hence, there are only finitely many possibilities for F_j. □

Lemma 4.3.3. *Let X be a nonseparable, Lindelöf, linearly ordered topological space. Then X does not have an ω^{op}-like base.*

Proof. Let $\mathcal{A}$ be a base of X. Let us show that $\mathcal{A}$ is not ω^{op}-like. First, let us construct sequences of open sets $\langle A_{n,k}\rangle_{n,k<\omega}$ and $\langle B_{n,k}\rangle_{n,k<\omega}$. Our requirements are that $B_{n,i} \subseteq A_{n,i} \in \mathcal{A}$, that $B_{n,i}$ is a countable union of open intervals, that $\{B_{n,k} : k < \omega\}$ is a locally finite cover of X and pairwise $\subsetneq$-incomparable, and that $\{A_{i,k} : k < \omega\} \cap \{A_{j,k} : k < \omega\} \subseteq [X]^1$ for all $i < j < \omega$ and $n < \omega$.

Suppose $n < \omega$ and we are given $\langle A_{m,k}\rangle_{k<\omega}$ and $\langle B_{m,k}\rangle_{k<\omega}$ for all $m < n$ and they meet our requirements. Let $p \in X$. Set $V_p = \bigcap\{B_{m,k} : m < n$ and $k < \omega$ and $p \in B_{m,k}\}$. Then V_p is open. If $|V_p| = 1$, then set $U_p = V_p$. If $|V_p| > 1$, then choose $U_p \in \mathcal{A}$ such that $p \in U_p \subsetneq V_p$. Set $\mathcal{U} = \{U_p : p \in X\}$. By Lemma 4.3.2, there exists a countable, locally finite refinement $\mathcal{B}_n$ of $\mathcal{U}$ consisting only of countable unions of open intervals. Since $\mathcal{B}_n$ is locally finite, it has no infinite ascending chains; hence, we may assume $\mathcal{B}_n$ is pairwise $\subsetneq$-incomparable because we may shrink $\mathcal{B}_n$ to its maximal elements. Let $\{B_{n,k} : k < \omega\} = \mathcal{B}_n$. For each $k < \omega$, set $A_{n,k} = U_p$ for some $p \in X$ satisfying $B_{n,k} \subseteq U_p$. Suppose $m < n$ and $i, j < \omega$ and $A_{m,i} = A_{n,j} \notin [X]^1$. Choose $p \in X$ such that $A_{n,j} = U_p$; choose $k < \omega$ such that $p \in B_{m,k}$. Then $B_{m,i} \subseteq A_{m,i} = U_p \subsetneq V_p \subseteq B_{m,k}$, in contradiction with the pairwise $\subsetneq$-incomparability of $\{B_{m,l} : l < \omega\}$. Thus, $\{A_{m,l} : l < \omega\} \cap \{A_{n,l} : l < \omega\} \subseteq [X]^1$ for all $m < n$. By induction, $\langle A_{n,k}\rangle_{n,k<\omega}$ and $\langle B_{n,k}\rangle_{n,k<\omega}$ meet our requirements.

Let $\{X, \leq, \mathcal{A}\} \subseteq M \prec H_\theta$ and $|M| = \omega$. Choose $x \in X \setminus \overline{X \cap M}$. Then there exists $y, z \in X$ such that $y < x < z$ and (y, z) does not intersect M. Choose $U \in \mathcal{A}$ such that $U \subseteq (y, z)$. By elementarity, we may assume that $A_{n,k}, B_{n,k} \in M$ for all $n, k < \omega$. For each $n < \omega$, choose $i_n < \omega$ such that $x \in B_{n,i_n}$. Fix $n < \omega$. Since $x \notin M$, we cannot have $A_{n,i_n} = \{x\}$; hence, $A_{n,i_n} \neq A_{m,i_m}$ for all $m < n$. Hence, it suffices to show that $U \subseteq A_{n,i_n}$. There exist $\langle u_j\rangle_{j<\omega}, \langle v_j\rangle_{j<\omega} \in (X \cup \{\infty, -\infty\})^\omega \cap M$ such that $B_{n,i_n} = \bigcup_{j<\omega}(u_j, v_j)$. Hence, there exists $j < \omega$ such that $u_j < x < v_j$. Since $x \in (y, z) \cap (u_j, v_j)$ and (y, z) does not intersect M, we have $(y, z) \subseteq (u_j, v_j)$; hence, $U \subseteq A_{n,i_n}$. $\square$

Theorem 4.3.4. *Let X be a Lindelöf linearly ordered topological space. Then the following are equivalent.*

1. *X is metric.*

2. *X has an ω^{op}-like base.*

3. *X is separable and has an ω_1^{op}-like base.*

Proof. By Theorem 4.3.1, (1) implies (2). By Lemma 4.3.3, (2) implies (3). Hence, it suffices to show that (3) implies (1). Suppose X has a countable dense subset D and an ω_1^{op}-like base. Then $\pi(X) = \omega$; hence, by Proposition 3.2.22, $w(X) = \omega$; hence, X is metric. □

For compact linearly ordered topological spaces, Theorem 4.3.4 can be strengthened.

Lemma 4.3.5. *Suppose κ is a regular uncountable cardinal and X is a linearly ordered compactum such that $Nt(X) \leq \kappa$. Then $d(X) < \kappa$.*

Proof. Suppose $d(X) \geq \kappa$ and $\mathcal{A}$ is a κ^{op}-like base of X. Let $\{X, \leq, \mathcal{A}\} \in M \prec H_\theta$ and $|M| < \kappa$ and $M \cap \kappa \in \kappa$. By compactness, X contains a nonempty open interval (x, y) that is maximal among the open convex subsets of X that are disjoint from M. If $x, y \in M$, then $(x, y) \cap M$ is nonempty by elementarity; hence, we may assume $x \notin M$. Therefore, by maximality of (x, y), we have $x = \sup([\min X, x) \cap M)$. Choose $z \in (x, y)$; choose $U \in \mathcal{A}$ such that $x \in U \subseteq [\min X, z)$. Then there exist $u, v \in X$ such that $x \in (u, v) \subseteq U$. Hence, there exist $p_0, p_1, p_2 \in M$ such that $u < p_0 < p_1 < p_2 < x$. Choose $V \in \mathcal{A}$ such that $p_1 \in V \subseteq (p_0, p_2)$; by elementarity, we may assume $V \in M$. Set $\mathcal{B} = \{W \in \mathcal{A} : V \subseteq W\}$. Then $U \in \mathcal{B} \in M$ and $|\mathcal{B}| < \kappa$; hence, $U \in M$. Set

$w = \min([p_1, \max X] \setminus U)$. Then $w \in M$ and $v \leq w \leq z$; hence, $w \in (x, y) \cap M$, which is absurd. Thus, $d(X) < \kappa$. □

Theorem 4.3.6. *Let X be a linearly ordered compactum. Then the following are equivalent.*

1. *X is metric.*

2. *X has an ω^{op}-like base.*

3. *X has an ω_1^{op}-like base.*

4. *X is separable and has an ω_1^{op}-like base.*

Proof. By Theorem 4.3.4, (1), (2), and (4) are equivalent. Moreover, (2) trivially implies (3). By Lemma 4.3.5, (3) implies (4). □

Example 4.3.7. Theorem 4.3.6 fails for Lindelöf linearly ordered topological spaces. Let X be $(\omega_1 \times \mathbb{Z}) \cup (\{\omega_1\} \times \{0\})$ ordered lexicographically. Then X is Lindelöf and nonseparable and $\{\{\langle \alpha, n \rangle\} : \alpha < \omega_1 \text{ and } n \in \mathbb{Z}\} \cup \{X \setminus (\alpha \times \mathbb{Z}) : \alpha < \omega_1\}$ is an ω_1^{op}-like base of X.

4.4 The Noetherian spectrum of ordered compacta

Theorem 4.3.6 implies that no linearly ordered compactum has Noetherian type ω_1. What is the class of Noetherian types of linearly ordered compacta? We shall prove that an infinite cardinal κ is the Noetherian type of a linearly ordered compactum if and only if $\kappa \neq \omega_1$ and κ is not weakly inaccessible.

Theorem 4.4.1. *Let κ be an uncountable cardinal and give $\kappa+1$ the order topology. If κ is regular, then $Nt(\kappa+1)=\kappa^+$; otherwise, $Nt(\kappa+1)=\kappa$.*

Proof. Let $\mathcal{A}$ be a base of $\kappa+1$ and let λ be a regular cardinal $\leq\kappa$. Let us show that $\mathcal{A}$ is not λ^{op}-like. For every limit ordinal $\alpha<\lambda$, choose $U_\alpha\in\mathcal{A}$ such that $\alpha=\max U_\alpha$; choose $\eta(\alpha)<\alpha$ such that $[\eta(\alpha),\alpha]\subseteq U_\alpha$. By the Pressing Down Lemma, η is constant on a stationary subset S of λ. Hence, $\mathcal{A}\ni\{\eta(\min S)+1\}\subseteq U_\alpha$ for all $\alpha\in S$; hence, $\mathcal{A}$ is not λ^{op}-like. Hence, $Nt(\kappa+1)\geq\kappa$ and $Nt(\kappa+1)>\operatorname{cf}\kappa$. Moreover, $Nt(\kappa+1)\leq w(\kappa+1)^+=\kappa^+$. Hence, it suffices to show that $\kappa+1$ has a κ^{op}-like base if κ is singular. Suppose $E\in[\kappa]^{<\kappa}$ is unbounded in κ. Let F be the set of limit points of E in $\kappa+1$. Define $\mathcal{B}$ by

$$\mathcal{B}=\{(\beta,\alpha]:E\ni\beta<\alpha\in F\text{ or }\sup(E\cap\alpha)\leq\beta<\alpha\in\kappa\setminus F\}.$$

Then $\mathcal{B}$ is a κ^{op}-like base of $\kappa+1$. □

Definition 4.4.2. Given a poset P with ordering $\leq$, let P^{op} denote the set P with ordering $\geq$.

Theorem 4.4.3. *Suppose κ is a singular cardinal. Then there is a linearly ordered compactum with Noetherian type κ^+.*

Proof. Set $\lambda=\operatorname{cf}\kappa$ and $X=\lambda^++1$. Partition the set of limit ordinals in λ^+ into λ-many stationary sets $\langle S_\alpha\rangle_{\alpha<\lambda}$. Let $\langle\kappa_\alpha\rangle_{\alpha<\lambda}$ be an increasing sequence of regular cardinals with supremum κ. For each $\alpha<\lambda$ and $\beta\in S_\alpha$, set $Y_\beta=(\kappa_\alpha+1)^{\mathrm{op}}$. For each $\alpha\in X\setminus\bigcup_{\beta<\lambda}S_\beta$, set $Y_\alpha=1$. Set $Y=\bigcup_{\alpha\in X}\{\alpha\}\times Y_\alpha$ ordered lexicographically. Then $Nt(Y)\leq w(Y)^+\leq|Y|^+=\kappa^+$. Hence, it suffices to show that Y has no κ^{op}-like base.

Seeking a contradiction, suppose $\mathcal{A}$ is a κ^{op}-like base of Y. For each $\alpha<\lambda$, let $\mathcal{U}_\alpha$ be the set of all $U\in\mathcal{A}$ that have at least κ_α-many supersets in $\mathcal{A}$. Then, for all isolated points p of Y, there exists $\alpha < \lambda$ such that $\{p\} \notin \mathcal{U}_\alpha$; whence, $p \notin \bigcup \mathcal{U}_\alpha$. Since $\langle \alpha + 1, 0 \rangle$ is isolated for all $\alpha < \lambda^+$, there exist $\beta < \lambda$ and a set E of successor ordinals in λ^+ such that $|E| = \lambda^+$ and $(E \times 1) \cap \bigcup \mathcal{U}_\beta = \emptyset$. Let C be the closure of E in λ^+. Then C is closed unbounded; hence, there exists $\gamma \in C \cap S_{\beta+1}$. Set $q = \langle \gamma, \kappa_{\beta+1} \rangle$. Then $q \in \overline{E \times 1}$; hence, $q \notin \bigcup \mathcal{U}_\beta$. Since q has coinitiality $\kappa_{\beta+1}$, any local base $\mathcal{B}$ at q will contain an element U such that U has κ_β-many supersets in $\mathcal{B}$. Hence, there exists $U \in \mathcal{U}_\beta$ such that $q \in U$; hence, $q \in \bigcup \mathcal{U}_\beta$, which yields our desired contradiction. $\square$

Theorem 4.4.4. *No linearly ordered compactum has weakly inaccessible Noetherian type.*

Proof. Suppose κ is weakly inaccessible and X is a linearly ordered compactum satisfying $Nt(X) \leq \kappa$. Then it suffices to prove $Nt(X) < \kappa$. By Lemma 4.3.5, we have $\pi(X) = d(X) < \kappa$. If $w(X) \geq \kappa$, then $Nt(X) > \kappa$ by Proposition 3.2.22, in contradiction with our assumptions about X. Hence, $w(X) < \kappa$; hence, $Nt(X) \leq w(X)^+ < \kappa$. $\square$

Chapter 5

Splitting families and the Noetherian type of $\beta\omega \setminus \omega$

5.1 Introduction

Let ω^* denote the space of nonprincipal ultrafilters on ω. Malykhin [42] proved that MA implies $\pi Nt(\omega^*) = \mathfrak{c}$ and CH implies $Nt(\omega^*) = \mathfrak{c}$. We extend these results by investigating $Nt(\omega^*)$, $\pi Nt(\omega^*)$, $\chi Nt(\omega^*)$, and $\pi\chi Nt(\omega^*)$ as cardinal characteristics of the continuum. For background on such cardinals, see Blass [11]. We also examine the sequence $\langle Nt((\omega^*)^{1+\alpha})\rangle_{\alpha\in\mathrm{O}n}$.

Definition 5.1.1. Let $\mathfrak{b}$ denote the minimum of $|\mathcal{F}|$ where $\mathcal{F}$ ranges over the subsets of ω^ω that have no upper bound in $\langle\omega^\omega\rangle$, where $\leq^*$ denotes eventual domination.

Definition 5.1.2. A *tree π-base* of a space X is a π-base that is a tree when ordered by containment. Let $\mathfrak{h}$ be the minimum of the set of heights of tree π-bases of ω^*.

Balcar, Pelant, and Simon [3] proved that tree π-bases of ω^* exist, and that $\mathfrak{h} \leq \min\{\mathfrak{b}, \operatorname{cf}\mathfrak{c}\}$. They also proved that the above definition of $\mathfrak{h}$ is equivalent to the more common definition of $\mathfrak{h}$ as the distributivity number of $[\omega]^\omega$ ordered by $\subseteq^*$.

Definition 5.1.3. Given $x, y \in [\omega]^\omega$, we say that x *splits* y if $|y \cap x| = |y \setminus x| = \omega$. Let $\mathfrak{r}$ be the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ such that no

$x \in [\omega]^\omega$ splits every $y \in A$. Let $\mathfrak{s}$ be the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ such that every $x \in [\omega]^\omega$ is split by some $y \in A$.

It is known that $\mathfrak{b} \leq \mathfrak{r}$ and $\mathfrak{h} \leq \mathfrak{s}$. (See Theorems 3.8 and 6.9 of [11].)

Clearly, $Nt(\omega^*) \leq w(\omega^*)^+ = \mathfrak{c}^+$. We will show that also $\pi\chi Nt(\omega^*) = \omega$ and $\pi Nt(\omega^*) = \mathfrak{h}$ and $\mathfrak{s} \leq Nt(\omega^*)$. Furthermore, $Nt(\omega^*)$ can consistently be $\mathfrak{c}$, $\mathfrak{c}^+$, or any regular κ satisfying $2^{<\kappa} = \mathfrak{c}$. Also, $Nt(\omega^*) = \omega_1$ is relatively consistent with any values of $\mathfrak{b}$ and $\mathfrak{c}$. The relations $\omega_1 < \mathfrak{b} = \mathfrak{s} = Nt(\omega^*) < \mathfrak{c}$ and $\omega_1 = \mathfrak{b} = \mathfrak{s} < Nt(\omega^*) < \mathfrak{c}$ are also each consistent. We also prove some relations between $\mathfrak{r}$ and $Nt(\omega^*)$, as well as some consistency results about the local Noetherian type of points in ω^*.

5.2 Basic results

Definition 5.2.1. For all $x \in [\omega]^\omega$, set $x^* = \{p \in \omega^* : p \in x\}$.

Theorem 5.2.2. *It is relatively consistent with any value of* $\mathfrak{c}$ *satisfying* $\operatorname{cf} \mathfrak{c} > \omega_1$ *that* $Nt(\omega^*) = \mathfrak{c}^+$.

Proof. We may assume $\operatorname{cf} \mathfrak{c} > \omega_1$. By Exercise A10 on p. 289 of Kunen [40], there is a ccc generic extension $V[G]$ such that $\check{\mathfrak{c}} = \mathfrak{c}^{V[G]}$ and, in $V[G]$, there exists $p \in \omega^*$ such that $\chi(p, \omega^*) = \omega_1$. Henceforth work in $V[G]$. Let φ be a bijection from ω^2 to ω. Define $\psi\colon \omega^* \to \omega^*$ by

$$x \mapsto \{E \subseteq \omega : \{m < \omega : \{n < \omega : \varphi(m, n) \in E\} \in p\} \in x\}.$$

Since $\pi\chi(p, \omega^*) \leq \chi(p, \omega^*) = \omega_1$, there exists $\langle E_\alpha \rangle_{\alpha < \omega_1} \in ([\omega]^\omega)^{\omega_1}$ such that every neighborhood of p contains E_α^* for some $\alpha < \omega_1$. Hence, for all $x \in \omega^*$, every neighborhood of $\psi(x)$ contains $(\varphi[\{m\} \times E_\alpha])^*$ for some $m < \omega$ and $\alpha < \omega_1$; whence, $\pi\chi(\psi(x), \omega^*) = \omega_1$.

Since ψ is easily verified to be a topological embedding, $\chi(x,\omega^*) \leq \chi(\psi(x),\omega^*)$ for all $x \in \omega^*$. By a result of Pospišil [56], there exists $q \in \omega^*$ such that $\chi(q,\omega^*) = \mathfrak{c}$. Hence, $\pi\chi(\psi(q),\omega^*) = \omega_1$ and $\chi(\psi(q),\omega^*) = \mathfrak{c}$. By Proposition 3.3.11, $Nt(\omega^*) > \chi(\psi(q),\omega^*) = \mathfrak{c}$. □

Definition 5.2.3. Given $n < \omega$, let $\mathfrak{ss}_n$ ($\mathfrak{ss}_\omega$) denote the least cardinal κ for which there exists a sequence $\langle f_\alpha \rangle_{\alpha<\mathfrak{c}}$ of functions on ω each with range contained in n (each with finite range) such that for all $I \in [\mathfrak{c}]^\kappa$ and $x \in [\omega]^\omega$ there exists $\alpha \in I$ such that f_α is not eventually constant on x. (The notation $\mathfrak{ss}$ was chosen with the phrase "supersplitting number" in mind.) Note that if such an $\langle f_\alpha \rangle_{\alpha<\mathfrak{c}}$ does not exist for any $\kappa \leq \mathfrak{c}$, then $\mathfrak{ss}_n$ ($\mathfrak{ss}_\omega$) is by definition equal to $\mathfrak{c}^+$.

Clearly $\mathfrak{ss}_n \geq \mathfrak{ss}_{n+1} \geq \mathfrak{ss}_\omega$ for all $n < \omega$. Moreover, since $\operatorname{cf}\mathfrak{c} > \omega$, we have $\mathfrak{ss}_\omega = \mathfrak{ss}_n$ for some $n < \omega$. However, for any particular $n \in \omega \setminus 2$, it is not clear whether ZFC proves $\mathfrak{ss}_\omega = \mathfrak{ss}_n$.

Definition 5.2.4. Given $\lambda \geq \kappa \geq \omega$ and a space X, a $\langle \lambda, \kappa \rangle$-*splitter* of X is a sequence $\langle \mathcal{F}_\alpha \rangle_{\alpha<\lambda}$ of finite open covers of X such that, for all $I \in [\lambda]^\kappa$ and $\langle U_\alpha \rangle_{\alpha\in I} \in \prod_{\alpha\in I} \mathcal{F}_\alpha$, the interior of $\bigcap_{\alpha\in I} U_\alpha$ is empty.

Lemma 5.2.5. *Suppose X is a compact space with a base $\mathcal{A}$ of size at most $w(X)$ such that $U \cap V \in \mathcal{A} \cup \{\emptyset\}$ for all $U, V \in \mathcal{A}$. If $\kappa \leq w(X)$ and X has a $\langle w(X), \kappa \rangle$-splitter, then $\mathcal{A}$ contains a κ^{op}-like base of X. Hence, $Nt(\omega^*) \leq \mathfrak{ss}_\omega$.*

Proof. Set $\lambda = w(X)$ and let $\langle \mathcal{F}_\alpha \rangle_{\alpha<\lambda}$ be a $\langle \lambda, \kappa \rangle$-splitter of X. For each $\alpha < \lambda$, the cover $\mathcal{F}_\alpha$ is refined by a finite subcover of $\mathcal{A}$; hence, we may assume $\mathcal{F}_\alpha \subseteq \mathcal{A}$. Let $\mathcal{A} = \{U_\alpha : \alpha < \lambda\}$. For each $\alpha < \lambda$, set $\mathcal{B}_\alpha = \{U_\alpha \cap V : V \in \mathcal{F}_\alpha\}$. Set $\mathcal{B} = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha \setminus \{\emptyset\}$. Then $\mathcal{B}$ is easily seen to be a base of X and a κ^{op}-like subset of $\mathcal{A}$. □

Lemma 5.2.6. *Let X be a compact space without isolated points and let $\omega \leq \kappa \leq \lambda \leq \min_{p \in X} \chi(p, X)$. If X has no $\langle \lambda, \kappa \rangle$-splitter, then $Nt(X) > \kappa$.*

Proof. Let $\mathcal{A}$ be a base of X. Construct a sequence $\langle \mathcal{F}_\alpha \rangle_{\alpha < \lambda}$ of finite subcovers of $\mathcal{A}$ as follows. Suppose we have $\alpha < \lambda$ and $\langle \mathcal{F}_\beta \rangle_{\beta < \alpha}$. For each $p \in X$, choose $V_p \in \mathcal{A}$ such that $p \in V_p \notin \bigcup_{\beta < \alpha} \mathcal{F}_\beta$. Let $\mathcal{F}_\alpha$ be a finite subcover of $\{V_p : p \in X\}$. Then $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for all $\alpha < \beta < \lambda$. Suppose X has no $\langle \lambda, \kappa \rangle$-splitter. Then choose $I \in [\lambda]^\kappa$ and $\langle U_\alpha \rangle_{\alpha \in I} \in \prod_{\alpha \in I} \mathcal{F}_\alpha$ such that $\bigcap_{\alpha \in I} U_\alpha$ has nonempty interior. Then there exists $W \in \mathcal{A}$ such that $W \subseteq \bigcap_{\alpha \in I} U_\alpha$. Thus, $\mathcal{A}$ is not κ^{op}-like. $\square$

Definition 5.2.7. Let $\mathfrak{u}$ denote the minimum of the set of characters of points in ω^*. Let $\pi\mathfrak{u}$ denote the minimum of the set of π-characters of points in ω^*.

By a theorem of Balcar and Simon [4], $\pi\mathfrak{u} = \mathfrak{r}$.

Theorem 5.2.8. *Suppose $\mathfrak{u} = \mathfrak{c}$. Then $Nt(\omega^*) = \mathfrak{ss}_\omega$.*

Proof. By Lemma 5.2.5, $Nt(\omega^*) \leq \mathfrak{ss}_\omega$. Suppose $\kappa \leq \mathfrak{c}$. Since every finite open cover of ω^* is refined by a finite, pairwise disjoint, clopen cover, ω^* has a $\langle \mathfrak{c}, \kappa \rangle$-splitter if and only if $\mathfrak{ss}_\omega \leq \kappa$. Hence, $Nt(\omega^*) \geq \mathfrak{ss}_\omega$ by Lemma 5.2.6. $\square$

Lemma 5.2.9. *Suppose $\mathfrak{r} = \mathfrak{c}$. Then $\mathfrak{ss}_2 \leq \mathfrak{c}$.*

Proof. Let $\langle x_\alpha \rangle_{\alpha < \mathfrak{c}}$ enumerate $[\omega]^\omega$. Construct $\langle y_\alpha \rangle_{\alpha < \mathfrak{c}} \in ([\omega]^\omega)^\mathfrak{c}$ as follows. Given $\alpha < \mathfrak{c}$ and $\langle y_\beta \rangle_{\beta < \alpha}$, choose y_α such that y_α splits every element of $\{x_\alpha\} \cup \{y_\beta : \beta < \alpha\}$. Suppose $I \in [\mathfrak{c}]^\mathfrak{c}$ and $\alpha < \mathfrak{c}$. Then x_α is split by y_β for all $\beta \in I \setminus \alpha$. Thus, $\langle \{y_\alpha, \omega \setminus y_\alpha\} \rangle_{\alpha < \mathfrak{c}}$ witnesses $\mathfrak{ss}_2 \leq \mathfrak{c}$. $\square$

Theorem 5.2.10. *The cardinals $\mathfrak{r}$ and $Nt(\omega^*)$ are related as follows.*

1. *If* $\mathfrak{r} = \mathfrak{c}$*, then* $Nt(\omega^*) = \mathfrak{ss}_\omega \leq \mathfrak{c}$*.*

2. *If* $\mathfrak{r} < \mathfrak{c}$*, then* $Nt(\omega^*) \geq \mathfrak{c}$*.*

3. *If* $\mathfrak{r} < \operatorname{cf} \mathfrak{c}$*, then* $Nt(\omega^*) = \mathfrak{c}^+$*.*

Proof. Statement (1) follows from Lemma 5.2.9, Theorem 5.2.8, and $\pi\mathfrak{u} = \mathfrak{r}$. The proof of Theorem 5.2.2 shows how to construct $p \in \omega^*$ such that $\pi\chi(p, \omega^*) = \pi\mathfrak{u} = \mathfrak{r}$ and $\chi(p, \omega^*) = \mathfrak{c}$. Hence, (2) and (3) follow from Proposition 3.3.11. □

Definition 5.2.11. Let $\mathfrak{d} = \operatorname{cf}(\langle \omega^\omega, \leq^* \rangle)$. Given a regular cardinal κ, κ*-scale* is a cofinal subset of $\langle, \omega^\omega, \leq^* \rangle$ that has order type κ.

A κ-scale exists if and only if $\mathfrak{b} = \mathfrak{d} = \kappa$.

Theorem 5.2.12. *For all cardinals* κ *satisfying* $\kappa > \operatorname{cf} \kappa > \omega$*, it is consistent that* $\mathfrak{r} = \mathfrak{u} = \mathfrak{b} = \mathfrak{d} = \operatorname{cf} \kappa$ *and* $Nt(\omega^*) = \mathfrak{ss}_2 = \mathfrak{c} = \kappa$*.*

Proof. Assuming GCH in the ground model, construct a finite support iteration $\langle \mathbb{P}_\alpha \rangle_{\alpha \leq \kappa}$ as follows. First choose some $U_0 \in \omega^*$. Then suppose we have $\alpha < \kappa$ and $\mathbb{P}_\alpha$ and $\Vdash_\alpha U_\alpha \in \omega^*$. Let $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * (\mathbb{Q}_\alpha \times \mathbb{D}_\alpha)$ where $\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for the Booth forcing for U_α and $\mathbb{D}_\alpha$ is the $\mathbb{P}_\alpha$-name for Hechler forcing. Let x_α be a $\mathbb{P}_{\alpha+1}$-name for the generic pseudointersection of U_α added by $\mathbb{Q}_\alpha$; let $U_{\alpha+1}$ be a $\mathbb{P}_{\alpha+1}$-name for an element of ω^* containing $U_\alpha \cup \{x_\alpha\}$. Let f_α be a $\mathbb{P}_{\alpha+1}$-name for the generic dominating function added by $\mathbb{D}_\alpha$. For limit $\alpha < \kappa$, let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$.

Let $\langle \eta_\alpha \rangle_{\alpha < \operatorname{cf} \kappa}$ be an increasing sequence of ordinals with supremum κ. The sequence $\langle f_{\eta_\alpha} \rangle_{\alpha < \operatorname{cf} \kappa}$ is forced to be a $\operatorname{cf} \kappa$-scale, so $\Vdash_\kappa \mathfrak{b} = \mathfrak{d} = \operatorname{cf} \kappa$. Moreover, $\{x_{\eta_\alpha} : \alpha < \operatorname{cf} \kappa\}$ is forced to generate an ultrafilter in the $\mathbb{P}_\kappa$-generic extension of V. Hence, $\Vdash_\kappa \mathfrak{r} \leq \mathfrak{u} \leq \operatorname{cf} \kappa < \kappa = \mathfrak{c}$. Therefore, by Lemma 5.2.5 and Theorem 5.2.10, it suffices to show that

$\Vdash_\kappa \mathfrak{ss}_2 \leq \kappa$. Every nontrivial finite support iteration of infinite length adds a Cohen real. Hence, we may choose for each $\alpha < \kappa$ a $\mathbb{P}_{\omega(\alpha+1)}$-name y_α for an element of $[\omega]^\omega$ that is Cohen over the $\mathbb{P}_{\omega\alpha}$-generic extension of V. Then every name S for the range of a cofinal subsequence of $\langle y_\alpha \rangle_{\alpha<\kappa}$ is such that

$$\Vdash_\kappa \forall z \in [\omega]^\omega \;\; \exists w \in S \;\; w \text{ splits } z.$$

Hence, $\langle y_\alpha \rangle_{\alpha<\kappa}$ witnesses that $\Vdash_\kappa \mathfrak{ss}_2 \leq \kappa$. □

Theorem 5.2.13. $Nt(\omega^*) \geq \mathfrak{s}$.

Proof. Suppose $Nt(\omega^*) = \kappa < \mathfrak{s}$. Since $Nt(\omega^*) < \mathfrak{c}$, we have $\mathfrak{r} = \mathfrak{c}$ by Theorem 5.2.10. Hence, $\mathfrak{u} = \mathfrak{c}$. By Theorem 5.2.8, it suffices to show that $\mathfrak{ss}_\omega > \kappa$. Suppose $\langle f_\alpha \rangle_{\alpha<\mathfrak{c}}$ is a sequence of functions on ω with finite range and $I \in [\mathfrak{c}]^\kappa$. Since $\kappa < \mathfrak{s}$, there exists $x \in [\omega]^\omega$ such that f_α is eventually constant on x for all $\alpha \in I$. Thus, $\mathfrak{ss}_\omega > \kappa$. □

Theorem 5.2.14. $\pi Nt(\omega^*) = \mathfrak{h}$.

Proof. First, we show that $\pi Nt(\omega^*) \leq \mathfrak{h}$. Let $\mathcal{A}$ be a tree π-base of ω^* such that $\mathcal{A}$ has height $\mathfrak{h}$ with respect to containment. Then $\mathcal{A}$ is clearly $\mathfrak{h}^{\mathrm{op}}$-like. To show that $\mathfrak{h} \leq \pi Nt(\omega^*)$, let $\mathcal{A}$ be as above and let $\mathcal{B}$ be a $\pi Nt(\omega^*)^{\mathrm{op}}$-like π-base of ω^*. Then $\mathcal{A}$ and $\mathcal{B}$ are mutually dense; hence, by Lemma 3.2.20, $\mathcal{A}$ contains a $\pi Nt(\omega^*)^{\mathrm{op}}$-like π-base $\mathcal{C}$ of ω^*. Since $\mathcal{C}$ is also a tree π-base, it has height at most $\pi Nt(\omega^*)$. Hence, $\mathfrak{h} \leq \pi Nt(\omega^*)$. □

Corollary 5.2.15. *If* $\mathfrak{h} = \mathfrak{c}$*, then* $\pi Nt(\omega^*) = Nt(\omega^*) = \mathfrak{ss}_2 = \mathfrak{c}$.

Proof. Suppose $\mathfrak{h} = \mathfrak{c}$. Then $\mathfrak{r} = \mathfrak{c}$ because $\mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{c}$. Hence, by Theorem 5.2.14, Theorem 5.2.10, and Lemma 5.2.9, $\mathfrak{c} \leq \pi Nt(\omega^*) \leq Nt(\omega^*) = \mathfrak{ss}_\omega \leq \mathfrak{ss}_2 \leq \mathfrak{c}$. □

5.3 Models of $Nt(\beta\omega \setminus \omega) = \omega_1$

Adding $\mathfrak{c}$-many Cohen reals collapses $\mathfrak{ss}_2$ to ω_1. By Lemma 5.2.5, it therefore also collapses $Nt(\omega^*)$ to ω_1. The same result holds for random reals and Hechler reals.

Theorem 5.3.1. *Suppose $\kappa^\omega = \kappa$ and $\mathbb{P} = \mathcal{B}(2^\kappa)/\mathcal{I}$ where $\mathcal{B}(2^\kappa)$ is the Borel alegebra of the product space 2^κ and $\mathcal{I}$ is either the meager ideal or the null ideal (with respect to the product measure). (In other words, $\mathbb{P}$ adds κ-many Cohen reals or κ-many random reals in the usual way.) Then $\mathbb{1}_\mathbb{P} \Vdash \omega_1 = \mathfrak{ss}_2$.*

Proof. Working in a $\mathbb{P}$-generic extension $V[G]$, we have $\kappa = \mathfrak{c}$ and a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ in $[\omega]^\omega$ such that $V[G] = V[\langle x_\alpha \rangle_{\alpha<\kappa}]$ and, if $E \in \mathcal{P}(\kappa) \cap V$ and $\alpha \in \kappa \setminus E$, then x_α is Cohen or random over $V[\langle x_\beta \rangle_{\beta\in E}]$. (See [38] for a proof.) Suppose $I \in [\kappa]^{\omega_1}$ and $y \in [\omega]^\omega$. Then $y \in V[\langle x_\alpha \rangle_{\alpha\in J}]$ for some $J \in [\kappa]^\omega \cap V$; hence, x_α splits y for all $\alpha \in I \setminus J$. Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha<\kappa}$ witnesses $\mathfrak{ss}_2 = \omega_1$. $\square$

Corollary 5.3.2. *Every transitive model of ZFC has a ccc forcing extension that preserves $\mathfrak{b}$, $\mathfrak{d}$, and $\mathfrak{c}$, and collapses $\mathfrak{ss}_2$ to ω_1.*

Proof. Add $\mathfrak{c}$-many random reals to the ground model. Then every element of ω^ω in the extension is eventually dominated by an element of ω^ω in the ground model; hence, $\mathfrak{b}$, $\mathfrak{d}$, and $\mathfrak{c}$ are preserved by this forcing, while $\mathfrak{ss}_2$ becomes ω_1. $\square$

Definition 5.3.3. We say that a transfinite sequence $\langle x_\alpha \rangle_{\alpha<\eta}$ of subsets of ω is *eventually splitting* if for all $y \in [\omega]^\omega$ there exists $\alpha < \eta$ such that for all $\beta \in \eta \setminus \alpha$ the set x_β splits y.

Theorem 5.3.4. *Let $\kappa = \kappa^\omega$. Then $\mathfrak{ss}_2 = \omega_1$ is forced by the κ-long finite support iteration of Hechler forcing.*

Proof. Let $\mathbb{P}$ be the κ-long finite support iteration of Hechler forcing. Let G be a generic filter of $\mathbb{P}$. For each $\alpha < \kappa$, let g_α be the generic dominating function added at stage α; set $x_\alpha = \{n < \omega : g_\alpha(n) \text{ is even}\}$. Suppose $p \in G$ and I and y are names such that p forces $I \in [\kappa]^{\omega_1}$ and $y \in [\omega]^\omega$. Choose $q \in G$ and a name h such that $q \leq p$ and q forces h to be an increasing map from ω_1 to I. For each $\alpha < \omega_1$, set $E_\alpha = \{\beta < \kappa : q \nVdash h(\alpha) \neq \check{\beta}\}$; let k_α be a surjection from ω to E_α. Let $q \geq r \in G$ and $n < \omega$ and $\gamma \leq \kappa$ and J be a name such that r forces $J \in [\omega_1]^{\omega_1}$ and $\sup \operatorname{ran} h = \check{\gamma}$ and $h(\alpha) = \check{k_\alpha(n)}$ for all $\alpha \in J$. Set $F = \{k_\alpha(n) : \alpha < \omega_1\} \cap \gamma$; let j be the order isomorphism from some ordinal η to F. Then $\operatorname{cf} \eta = \operatorname{cf} \gamma = \omega_1$. For all $\alpha < \kappa$, the set x_α is Cohen over $V[\langle g_\beta \rangle_{\beta<\alpha}]$; hence, $\langle x_{j(\alpha)} \rangle_{\alpha<\eta}$ is eventually splitting in $V[\langle g_\alpha \rangle_{\alpha<\gamma}]$. By a result of Baumgartner and Dordal [9], $\langle x_{j(\alpha)} \rangle_{\alpha<\eta}$ is also eventually splitting in $V[G]$. Choose $\beta < \eta$ such that $x_{j(\alpha)}$ splits y_G for all $\alpha \in \eta \setminus \beta$. Then there exist $s \in G$ and $\alpha \in \gamma \setminus j(\beta)$ such that $r \geq s \Vdash \check{\alpha} \in h[J]$. Hence, $\alpha \in I_G$ and x_α splits y_G. Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha<\kappa}$ witnesses $\mathfrak{ss}_2 = \omega_1$ in $V[G]$. □

Definition 5.3.5. Let $\operatorname{add}(\mathcal{B})$ denote the additivity of the ideal of meager sets of reals.

It is known that $\operatorname{add}(\mathcal{B}) \leq \mathfrak{b}$ and that it is consistent that $\operatorname{add}(\mathcal{B}) < \mathfrak{b}$. (See 5.4 and 11.7 of [11] and 7.3.D of [8]).

Corollary 5.3.6. *If $\kappa = \operatorname{cf} \kappa > \omega$, then it is consistent that $\mathfrak{ss}_2 = \omega_1$ and $\operatorname{add}(\mathcal{B}) = \mathfrak{c} = \kappa$.*

Proof. Starting with GCH in the ground model, perform a κ-long finite support iteration of Hechler forcing. This forces $\operatorname{add}(\mathcal{B}) = \mathfrak{c} = \kappa$ (see 11.6 of [11]). By Theorem 5.3.4, this also forces $\mathfrak{ss}_2 = \omega_1$. □

5.4 Models of $\omega_1 < Nt(\beta\omega \setminus \omega) < \mathfrak{c}$

To prove the consistency of $\omega_1 < Nt(\omega^*) < \mathfrak{c}$, we employ generalized iteration of forcing along posets as defined by Groszek and Jech [25]. We will only use finite support iterations along well-founded posets. For simplicity, we limit our definition of generalized iterations to this special case.

Definition 5.4.1. Suppose X is a well-founded poset and $\mathbb{P}$ a forcing order consisting of functions on X. Given any $x \in X$, partial map f on X, and down-set Y of X, set $\mathbb{P} \upharpoonright Y = \{p \upharpoonright Y : p \in \mathbb{P}\}$, $X \upharpoonright x = \{y \in X : y < x\}$, $X \upharpoonright_{\leq} x = \{y \in X : y \leq x\}$, $\mathbb{P} \upharpoonright x = \mathbb{P} \upharpoonright (X \upharpoonright x)$, $\mathbb{P} \upharpoonright_{\leq} x = \mathbb{P} \upharpoonright (X \upharpoonright_{\leq} x)$, $f \upharpoonright x = f \upharpoonright (X \upharpoonright x)$, and $f \upharpoonright_{\leq} x = f \upharpoonright (X \upharpoonright_{\leq} x)$. Then $\mathbb{P}$ is a *finite support iteration along* X if there exists a sequence $\langle \mathbb{Q}_x \rangle_{x \in X}$ satisfying the following conditions for all $x \in X$ and all $p, q \in \mathbb{P}$.

- $\mathbb{P} \upharpoonright x$ is a finite support iteration along $X \upharpoonright x$.
- $\mathbb{Q}_x$ is a $(\mathbb{P} \upharpoonright x)$-name for a forcing order.
- $\mathbb{P} \upharpoonright_{\leq} x = \{p \cup \{\langle x, q \rangle\} : \langle p, q \rangle \in (\mathbb{P} \upharpoonright x) * \mathbb{Q}_x\}$.
- $\mathbb{1}_{\mathbb{P}} \upharpoonright x \Vdash \mathbb{1}_{\mathbb{P}}(x) = \mathbb{1}_{\mathbb{Q}_x}$.
- $\mathbb{P}$ is the set of functions r on X for which $r \upharpoonright_{\leq} y \in \mathbb{P} \upharpoonright_{\leq} y$ for all $y \in X$ and $\mathbb{1}_{\mathbb{P} \upharpoonright z} \Vdash r(z) = \mathbb{1}_{\mathbb{Q}_z}$ for all but finitely many $z \in X$.
- $p \leq q$ if and only if $p \upharpoonright y \leq q \upharpoonright y$ and $p \upharpoonright y \Vdash p(y) \leq q(y)$ for all $y \in X$.

Given a finite support iteration $\mathbb{P}$ along X and $x \in X$ and a filter G of $\mathbb{P}$, set $G_x = \{p(x) : p \in G\}$, $G \upharpoonright x = \{p \upharpoonright x : p \in G\}$, and $G \upharpoonright_{\leq} x = \{p \upharpoonright_{\leq} x : p \in G\}$. Given any down-set Y of X, set $G \upharpoonright Y = \{p \upharpoonright Y : p \in G\}$.

Remark 5.4.2. If $\mathbb{P}$ is a finite support iteration along a well-founded poset X with down-set Y, then $\mathbb{P} \restriction Y$ is an iteration along Y, and $\mathbb{1}_{\mathbb{P}\restriction Y} = \mathbb{1}_{\mathbb{P}} \restriction Y$.

Definition 5.4.3. Suppose $\mathbb{P}$ is a finite support iteration along a well-founded poset X with down-sets Y and Z such that $Y \subseteq Z$. Then there is a complete embedding $j^Z_Y \colon \mathbb{P} \restriction Y \to \mathbb{P} \restriction Z$ given by $j^Z_Y(p) = p \cup (\mathbb{1}_{\mathbb{P}} \restriction Z \setminus Y)$ for all $p \in \mathbb{P} \restriction Y$. This embedding naturally induces an embedding of the class of $(\mathbb{P} \restriction Y)$-names, which in turn naturally induces an embedding of the class of atomic forumlae in the $(\mathbb{P} \restriction Y)$-forcing language. Let j^Z_Y also denote these embeddings.

Proposition 5.4.4. *Suppose $\mathbb{P}$, Y, and Z are as in the above definition, and φ is an atomic formula in the $(\mathbb{P} \restriction Y)$-forcing language. Then, for all $p \in \mathbb{P} \restriction Z$, we have $p \Vdash j^Z_Y(\varphi)$ if and only if $p \restriction Y \Vdash \varphi$.*

Proof. If $p \restriction Y \Vdash \varphi$, then $p \leq j^Z_Y(p \restriction Y) \Vdash j^Z_Y(\varphi)$. Conversely, suppose $p \restriction Y \nVdash \varphi$. Then we may choose $q \leq p \restriction Y$ such that $q \Vdash \neg\varphi$. Hence, $j^Z_Y(q) \Vdash \neg j^Z_Y(\varphi)$. Set $r = q \cup (p \restriction Z \setminus Y)$. Then $j^Z_Y(q) \geq r \leq p$; hence, $p \nVdash j^Z_Y(\varphi)$. $\square$

Lemma 5.4.5. *Suppose $\mathbb{P}$ is a finite support iteration along a well-founded poset X and x is a maximal element of X. Set $Y = X \setminus \{x\}$. Then there is a dense embedding $\phi \colon \mathbb{P} \to (\mathbb{P} \restriction Y) * j^Y_{X\restriction x}(\mathbb{Q}_x)$ given by $\phi(p) = \langle p \restriction Y, j^Y_{X\restriction x}(p(x))\rangle$. Hence, if G is a $\mathbb{P}$-generic filter, then G_x is $(\mathbb{Q}_x)_{G\restriction x}$-generic over $V[G \restriction Y]$.*

Proof. First, let us show that ϕ is an order embedding. Suppose $r, s \in \mathbb{P}$. Then $r \leq s$ if and only if $r \restriction Y \leq s \restriction Y$ and $r \restriction x \Vdash r(x) \leq s(x)$. Also, $\phi(r) \leq \phi(s)$ if and only if $r \restriction Y \leq s \restriction Y$ and $r \restriction Y \Vdash j^Y_{X\restriction x}(r(x) \leq s(x))$. By Proposition 5.4.4, $r \restriction Y \Vdash j^Y_{X\restriction x}(r(x) \leq s(x))$ if and only if $r \restriction x \Vdash r(x) \leq s(x)$; hence, $r \leq s$ if and only if $\phi(r) \leq \phi(s)$.

Finally, let us show that $\operatorname{ran}\phi$ is dense. Suppose $\langle p, q\rangle \in (\mathbb{P} \upharpoonright Y) * j^Y_{X\upharpoonright x}(\mathbb{Q}_x)$. Then there exist $r \leq p$ and $s \in \operatorname{dom}\big(j^Y_{X\upharpoonright x}(\mathbb{Q}_x)\big)$ such that $r \Vdash s = q \in j^Y_{X\upharpoonright x}(\mathbb{Q}_x)$. Hence, $\langle r, s\rangle \leq \langle p, q\rangle$. Also, s is a $(j^Y_{X\upharpoonright x}[\mathbb{P} \upharpoonright x])$-name; hence, there exists a $(\mathbb{P} \upharpoonright x)$-name t such that $j^Y_{X\upharpoonright x}(t) = s$. Hence, $r \Vdash j^Y_{X\upharpoonright x}(t \in \mathbb{Q}_x)$; hence, $r \upharpoonright x \Vdash t \in \mathbb{Q}_x$. Hence, $r \cup \{\langle x, t\rangle\} \in \mathbb{P}$ and $\phi(r \cup \{\langle x, t\rangle\}) = \langle r, s\rangle$. Thus, $\operatorname{ran}\phi$ is dense. □

Remark 5.4.6. Proposition 5.4.4 and Lemma 5.4.5 and their proofs remain valid for arbitrary iterations along posets as defined in [25].

Lemma 5.4.7. *Let $\mathbb{P}$ be a forcing order, A a subset of $[\omega]^\omega$ with the SFIP, $\mathbb{Q}$ the Booth forcing for A, x a $\mathbb{Q}$-name for a generic pseudointersection of A, and B a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}}$ forces $\check{A} \subseteq B \subseteq [\omega]^\omega$ and forces B to have the SFIP. Let i and j be the canonical embeddings, respectivly, of $\mathbb{P}$-names and $\mathbb{Q}$-names into $(\mathbb{P} * \check{\mathbb{Q}})$-names. Then $\mathbb{1}_{\mathbb{P}*\check{\mathbb{Q}}}$ forces $i(B) \cup \{j(x)\}$ to have the SFIP.*

Proof. Seeking a contradiction, suppose $r_0 = \langle p_0, \langle \sigma, F\rangle\check{}\,\rangle \in \mathbb{P} * \check{\mathbb{Q}}$ and $n < \omega$ and $p_0 \Vdash H \in [B]^{<\omega}$ and $r_0 \Vdash j(x) \cap \bigcap i(H) \subseteq \check{n}$. Then p_0 forces $\check{F} \cup H \subseteq B$, which is forced to have the SFIP; hence, there exist $p_1 \leq p_0$ and $m \in \omega \setminus n$ such that $p_1 \Vdash \check{m} \in \bigcap(\check{F} \cup H)$. Set $r_1 = \langle p_1, \langle \sigma \cup \{m\}, F\rangle\check{}\,\rangle$. Then $r_0 \geq r_1 \Vdash \check{m} \in j(x) \cap \bigcap i(H)$, contradicting how we chose r_0. □

Lemma 5.4.8. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcing orders such that $\mathbb{P}$ is ccc and $\mathbb{Q}$ has property (K). Then $\mathbb{1}_{\mathbb{P}}$ forces $\check{\mathbb{Q}}$ to have property (K).*

Proof. Suppose the lemma fails. Then there exist $p \in \mathbb{P}$ and f such that $p \Vdash f \in \check{\mathbb{Q}}^{\omega_1}$ and $p \Vdash \forall J \in [\omega_1]^{\omega_1}\ \exists \alpha, \beta \in J\ f(\alpha) \perp f(\beta)$. For each $\alpha < \omega_1$, choose $p_\alpha \leq p$ and $q_\alpha \in \mathbb{Q}$ such that $p_\alpha \Vdash f(\alpha) = \check{q}_\alpha$. Then there exists $I \in [\omega_1]^{\omega_1}$ such that $q_\alpha \not\perp q_\beta$ for all $\alpha, \beta \in I$.

Let J be the $\mathbb{P}$-name $\{\langle\check{\alpha}, p_\alpha\rangle : \alpha \in I\}$. Then $p \Vdash \forall\alpha, \beta \in J\ f(\alpha) = \check{q}_\alpha \not\perp \check{q}_\beta = f(\beta)$. Hence, $p \Vdash |J| \leq \omega$. Since $\mathbb{P}$ is ccc, there exists $\alpha \in I$ such that $p \Vdash J \subseteq \check{\alpha}$. But this contradicts $p \geq p_\alpha \Vdash \check{\alpha} \in J$. $\square$

Lemma 5.4.9. *Suppose $\mathbb{P}$ is a finite support iteration along a well-founded poset X and $\mathbb{1}_{\mathbb{P}} \restriction x$ forces $\mathbb{Q}_x$ to have property (K) for all $x \in X$. Then $\mathbb{P}$ has property (K).*

Proof. We may assume the lemma holds whenever X is replaced by a poset of lesser height. Let $I \in [\mathbb{P}]^{\omega_1}$. We may assume $\{\mathrm{supp}(p) : p \in I\}$ is a Δ-system; let σ be its root. Set $Y_0 = \bigcup_{x\in\sigma} X \restriction x$. Then $\mathbb{P} \restriction Y_0$ has property (K). Let $n = |\sigma \setminus Y_0|$ and $\langle x_i\rangle_{i<n}$ biject from n to $\sigma \setminus Y_0$. Set $Y_{i+1} = Y_i \cup \{x_i\}$ for all $i < n$. Suppose $i < n$ and $\mathbb{P} \restriction Y_i$ has property (K). By Lemma 5.4.8, $\mathbb{1}_{\mathbb{P}\restriction Y_i}$ forces $j^{Y_i}_{X\restriction x_i}(\mathbb{Q}_{x_i})$ to have property (K). Hence, $\mathbb{P} \restriction Y_{i+1}$ has property (K), for it densely embeds into $\mathbb{P} \restriction Y_i * j^{Y_i}_{X\restriction x_i}(\mathbb{Q}_{x_i})$ by Lemma 5.4.5. By induction, $\mathbb{P} \restriction Y_n$ has property (K); hence, there exists $J \in [I]^{\omega_1}$ such that $p \restriction Y_n \not\perp q \restriction Y_n$ for all $p, q \in J$. Fix $p, q \in J$ and choose r such that $r \leq p \restriction Y_n$ and $r \leq q \restriction Y_n$. Set $s = r \cup (p \restriction \mathrm{supp}(p) \setminus Y_n) \cup (q \restriction \mathrm{supp}(q) \setminus Y_n)$ and $t = s \cup (\mathbb{1}_{\mathbb{P}} \restriction X \setminus \mathrm{dom}\, s)$. Then $t \leq p, q$. $\square$

Lemma 5.4.10. *Suppose* $\mathrm{cf}\,\kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. *Then there exists a κ-like, κ-directed, well-founded poset Ξ with cofinality and cardinality λ.*

Proof. Let $\{x_\alpha : \alpha < \lambda\}$ biject from λ to $[\lambda]^{<\kappa}$. Construct $\langle y_\alpha\rangle_{\alpha<\lambda} \in ([\lambda]^{<\kappa})^\lambda$ as follows. Given $\alpha < \lambda$ and $\langle y_\beta\rangle_{\beta<\alpha}$, choose $\xi_\alpha \in \lambda \setminus \bigcup_{\beta<\alpha} y_\beta$ and set $y_\alpha = x_\alpha \cup \{\xi_\alpha\}$. Let Ξ be $\{y_\alpha : \alpha < \lambda\}$ ordered by inclusion. Then Ξ is cofinal with $[\lambda]^{<\kappa}$; hence, Ξ is κ-directed and has cofinality λ. Also, Ξ is well-founded because $\langle y_\alpha\rangle_{\alpha<\lambda}$ is nondecreasing. Finally, Ξ is κ-like because for all $I \in [\lambda]^\kappa$ we have $|\bigcup_{\alpha\in I} y_\alpha| \geq |\{\xi_\alpha : \alpha \in I\}| = \kappa$; whence, $\{y_\alpha : \alpha \in I\}$ has no upper bound in $[\lambda]^{<\kappa}$. $\square$

Definition 5.4.11. For all $x, y \subseteq \omega$, define $x \subseteq^* y$ as $|x \setminus y| < \omega$. Let $\mathfrak{p}$ denote the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ that have SFIP yet have no pseudointersection.

Remark 5.4.12. It easily seen that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{h}$.

Theorem 5.4.13. *Suppose $\omega_1 \leq \operatorname{cf}\kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension in which*

$$\mathfrak{p} = \pi Nt(\omega^*) = Nt(\omega^*) = \mathfrak{s}\mathfrak{s}_2 = \mathfrak{b} = \kappa \leq \lambda = \mathfrak{c}.$$

Moreover, in this extension ω^* has P_κ-points; whence, $\max_{q \in \omega^*} \chi Nt(q, \omega^*) = \kappa$.

Proof. Let Ξ be as in Lemma 5.4.10. Let $\langle \sigma_\alpha \rangle_{\alpha<\lambda}$ biject from λ to Ξ. Let $\langle \langle \zeta_\alpha, \eta_\alpha \rangle \rangle_{\alpha<\lambda}$ biject from λ to λ^2. Given $\alpha < \lambda$ and $\langle \tau_{\zeta_\beta,\eta_\beta} \rangle_{\beta<\alpha} \in \Xi^\alpha$, choose $\tau_{\zeta_\alpha,\eta_\alpha} \in \Xi$ such that $\sigma_{\zeta_\alpha} < \tau_{\zeta_\alpha,\eta_\alpha} \not\leq \tau_{\zeta_\beta,\eta_\beta}$ for all $\beta < \alpha$. We may so choose $\tau_{\zeta_\alpha,\eta_\alpha}$ because Ξ is directed and has cofinality λ.

Let us construct a finite support iteration $\mathbb{P}$ along Ξ. Since Ξ is well-founded, we may define $\mathbb{Q}_\sigma$ in terms of $\mathbb{P} \upharpoonright \sigma$ for each $\sigma \in \Xi$. Suppose $\sigma \in \Xi$ and, for all $\tau < \sigma$, we have $|\mathbb{P} \upharpoonright_\leq \tau| < \kappa$ and $\mathbb{1}_{\mathbb{P}\upharpoonright\tau}$ forces $\mathbb{Q}_\tau$ to have property (K). Then $\mathbb{P} \upharpoonright \sigma$ has property (K) by Lemma 5.4.9, and hence is ccc. Moreover, $|\mathbb{P} \upharpoonright \sigma| < \kappa$ because $\mathbb{P} \upharpoonright \sigma$ is a finite support iteration along $\Xi \upharpoonright \sigma$ and $|\Xi \upharpoonright \sigma| < \kappa$. Hence, $\mathbb{1}_{\mathbb{P}\upharpoonright\sigma} \Vdash |\mathfrak{c}^{<\kappa}| \leq ((\kappa^\omega)^{<\kappa})\check{} \leq \lambda$. Let $\mathcal{E}_\sigma$ be a $(\mathbb{P} \upharpoonright \sigma)$-name for the set of all E in the $(\mathbb{P} \upharpoonright \sigma)$-generic extension for which $E \in [[\omega]^\omega]^{<\kappa}$ and E has the SFIP. Then we may choose a $(\mathbb{P} \upharpoonright \sigma)$-name f_σ such that $\mathbb{1}_{\mathbb{P}\upharpoonright\sigma}$ forces f_σ to be a surjection from λ to $\mathcal{E}_\sigma$. We may assume we are given corresponding f_τ for all $\tau < \sigma$. If there exist $\alpha, \beta < \lambda$ such that $\sigma = \tau_{\alpha,\beta}$, then let $\mathbb{Q}_\sigma$ be a $(\mathbb{P} \upharpoonright \sigma)$-name for $\mathbb{Q}'_\sigma \times \mathrm{Fn}(\omega, 2)$ where $\mathbb{Q}'_\sigma$ is a $(\mathbb{P} \upharpoonright \sigma)$-name for the Booth forcing for $f_{\sigma_\alpha}(\beta)$. If there are no such α and β, then let $\mathbb{Q}_\sigma$ be a $(\mathbb{P} \upharpoonright \sigma)$-name for a singleton poset. Then $\mathbb{1}_{\mathbb{P}\upharpoonright\sigma}$ forces $\mathbb{Q}_\sigma$ to have property (K). Also, we may assume $|\mathbb{Q}_\sigma| < \kappa$. Hence, $|\mathbb{P} \upharpoonright_\leq \sigma| < \kappa$.

By induction, $|\mathbb{P} \upharpoonright_\leq \sigma| < \kappa$ and $\mathbb{1}_{\mathbb{P}\upharpoonright\sigma}$ forces $\mathbb{Q}_\sigma$ to have property (K) for all $\sigma \in \Xi$. Hence, $\mathbb{P}$ has property (K) by Lemma 5.4.9, and hence is ccc. Also, since $|\Xi| \leq \lambda$ and $\mathbb{P}$ is a finite support iteration, $|\mathbb{P}| \leq \lambda$. Let G be a $\mathbb{P}$-generic filter. Then $\mathfrak{c}^{V[G]} \leq \lambda^\omega = \lambda$. Moreover, $\mathfrak{c}^{V[G]} \geq \lambda$ because $\mathbb{P}$ adds λ-many Cohen reals.

By Theorem 5.2.14 and Lemma 5.2.5, it suffices to show that $\mathfrak{b}^{V[G]} \leq \kappa \leq \mathfrak{p}^{V[G]}$, that $\mathfrak{ss}_2^{V[G]} \leq \kappa$, and that some $q \in (\omega^*)^{V[G]}$ is a P_κ-point. First, we prove $\kappa \leq \mathfrak{p}^{V[G]}$. Suppose $E \in ([[\omega]^\omega]^{<\kappa})^{V[G]}$ and E has the SFIP. Then there exists $\alpha < \lambda$ such that $E \in V[G \upharpoonright \sigma_\alpha]$ because Ξ is κ-directed. Hence, there exists $\beta < \lambda$ such that $(f_{\sigma_\alpha})_{G\upharpoonright\sigma_\alpha}(\beta) = E$. Hence, E has a pseudointersection in $V[G \upharpoonright_\leq \tau_{\alpha,\beta}]$. Thus, $\kappa \leq \mathfrak{p}^{V[G]}$.

Second, let us show that $\mathfrak{b}^{V[G]} \leq \kappa$. For each $\alpha < \kappa$, let u_α be the increasing enumeration of the Cohen real added by the $\mathrm{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0,\alpha}}$. Then it suffices to show that $\{u_\alpha : \alpha < \kappa\}$ is unbounded in $(\omega^\omega)^{V[G]}$. Suppose $v \in (\omega^\omega)^{V[G]}$. Then there exists $\sigma \in \Xi$ such that $v \in V[G \upharpoonright \sigma]$. Since Ξ is κ-like, there exists $\alpha < \kappa$ such that $\tau_{0,\alpha} \not\leq \sigma$. By Lemma 5.4.5, u_α enumerates a real Cohen generic over $V[G \upharpoonright \sigma]$; hence, u_α is not eventually dominated by v.

Third, let us prove $\mathfrak{ss}_2^{V[G]} \leq \kappa$. For each $\alpha < \lambda$, let x_α be the Cohen real added by the $\mathrm{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0,\alpha}}$. Suppose $I \in ([\lambda]^\kappa)^{V[G]}$ and $y \in ([\omega]^\omega)^{V[G]}$. Then there exists $\sigma \in \Xi$ such that $y \in V[G \upharpoonright \sigma]$. Since Ξ is κ-like, there exists $\alpha \in I$ such that $\tau_{0,\alpha} \not\leq \sigma$. By Lemma 5.4.5, x_α is Cohen generic over $V[G \upharpoonright \sigma]$, and therefore splits y. Thus, $\langle\{x_\alpha, \omega \setminus x_\alpha\}\rangle_{\alpha<\lambda}$ witnesses $\mathfrak{ss}_2^{V[G]} \leq \kappa$.

Finally, let us construct a P_κ-point $q \in (\omega^*)^{V[G]}$. Let $\sqsubseteq$ be an extension of the ordering of Ξ to a well-ordering of Ξ. For each $\sigma \in \Xi$, set $Y_\sigma = \{\tau \in \Xi : \tau \sqsubset \sigma\}$. Set

$\rho = \min_{\sqsubseteq} \Xi$ and choose $U_\rho \in (\omega^*)^V$. Suppose $\tau \in \Xi$ and σ is a final predecessor of τ with respect to $\sqsubseteq$ and $U_\sigma \in (\omega^*)^{V[G\restriction Y_\sigma]}$. If there are no $\alpha, \beta < \lambda$ such that $\sigma = \tau_{\alpha,\beta}$ and $(f_{\sigma_\alpha})_{G\restriction\sigma_\alpha}(\beta) \subseteq U_\sigma$, then choose $U_\tau \in (\omega^*)^{V[G\restriction Y_\tau]}$ such that $U_\tau \supseteq U_\sigma$. Now suppose such α and β exist. Let v_σ be the pseudointersection of $(f_{\sigma_\alpha})_{G\restriction\sigma_\alpha}(\beta)$ added by $\mathbb{Q}'_\sigma$.

By Lemmas 5.4.5 and 5.4.7, $U_\sigma \cup \{v_\sigma\}$ has the SFIP; hence, we may choose $U_\tau \in (\omega^*)^{V[G\restriction Y_\tau]}$ such that $U_\tau \supseteq U_\sigma \cup \{v_\sigma\}$. For $\tau \in \Xi$ that are limit points with respect to $\sqsubseteq$, choose $U_\tau \in (\omega^*)^{V[G\restriction Y_\tau]}$ such that $U_\tau \supseteq \bigcup_{\sigma \sqsubset \tau} U_\sigma$; set $q = \bigcup_{\tau\in\Xi} U_\tau$. Then, arguing as in the proof of $\kappa \leq \mathfrak{p}^{V[G]}$, we have that q is a P_κ-point in $(\omega^*)^{V[G]}$. □

The forcing extension of Theorem 5.4.13 can be modified to satisfy $\mathfrak{b} = \mathfrak{s} < Nt(\omega^*) < \mathfrak{c}$.

Definition 5.4.14. Given a class $\mathcal{J}$ of posets and a cardinal κ, let M$A(\kappa;\, \mathcal{J})$ denote the statement that, given any $\mathbb{P} \in \mathcal{J}$ and fewer than κ-many dense subsets of $\mathbb{P}$, there is a filter of $\mathbb{P}$ intersecting each of these dense sets. We may replace $\mathcal{J}$ with a descriptive term for $\mathcal{J}$ when there is no ambiguity. For example, M$A(\mathfrak{c};$ ccc$)$ is Martin's axiom.

Theorem 5.4.15. *Suppose $\omega_1 < \operatorname{cf}\kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension in which*

$$\omega_1 = \pi Nt(\omega^*) = \mathfrak{b} = \mathfrak{s} < Nt(\omega^*) = \mathfrak{s}\mathfrak{s}_2 = \kappa \leq \lambda = \mathfrak{c}.$$

Proof. Let $\mathbb{P}$ be as in the proof of Theorem 5.4.13. Set $\mathbb{R} = \mathbb{P} \times \mathrm{Fn}(\omega_1, 2)$, which has property (K) because $\mathbb{P}$ does. Let K be a generic filter of $\mathbb{R}$. Let π_0 and π_1 be the natural coordinate projections on $\mathbb{R}$; let π_0 and π_1 also denote their respective natural extensions to the class of $\mathbb{R}$-names. Set $G = \pi_0[K]$ and $H = \pi_1[K]$. Then $\mathfrak{c}^{V[K]} = \lambda$ clearly holds. Adding ω_1-many Cohen reals to any model of ZFC forces $\mathfrak{b} = \mathfrak{s} = \omega_1$, and $\pi Nt(\omega^*) = \mathfrak{h} \leq \mathfrak{b}$, so $\pi Nt(\omega^*)^{V[K]} = \mathfrak{b}^{V[K]} = \mathfrak{s}^{V[K]} = \omega_1$.

For each $\alpha < \lambda$, let x_α be the Cohen real added by the $\mathrm{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0,\alpha}}$. Suppose $I \in ([\lambda]^\kappa)^{V[K]}$ and $y \in ([\omega]^\omega)^{V[K]}$. Then there exists $\sigma \in \Xi$ such that $y \in V[(G \upharpoonright \sigma) \times H]$. Since Ξ is κ-like, there exists $\alpha \in I$ such that $\tau_{0,\alpha} \not\leq \sigma$. By Lemma 5.4.5, x_α is Cohen generic over $V[G \upharpoonright \sigma]$; hence, x_α is Cohen generic over $V[(G \upharpoonright \sigma) \times H]$ and therefore splits y. Thus, $\langle\{x_\alpha, \omega \setminus x_\alpha\}\rangle_{\alpha<\lambda}$witnesses $\mathfrak{ss}_2^{V[K]} \leq \kappa$.

Therefore, it suffices to show that $Nt(\omega^*)^{V[K]} \geq \kappa$. Suppose $\mu < \kappa$ and $\mathcal{A}$ is an $\mathbb{R}$-name for a base of ω^*. Choose an $\mathbb{R}$-name q for an element of ω^* with character λ. Let f be a name for an injection from λ into $\mathcal{A}$ such that $q \in \bigcap \operatorname{ran} f$. Let g be a name for an element of $([\omega]^\omega)^\lambda$ such that $q \in g(\alpha)^* \subseteq f(\alpha)$ for all $\alpha < \lambda$. For each $\alpha < \lambda$, let u_α be a name for $g(\alpha)$ such that $u_\alpha = \{\{\check{n}\} \times A_{\alpha,n} : n < \omega\}$ where each $A_{\alpha,n}$ is a countable antichain of $\mathbb{R}$. Since $\max\{\omega_1, \mu\} < \lambda$, there exist $\xi < \omega_1$ and $J \in [\lambda]^\mu$ such that $\operatorname{ran} \pi_1(u_\alpha) \subseteq \mathrm{Fn}(\xi, 2)$ for all $\alpha \in J$. It suffices to show that $\{(u_\alpha)_K : \alpha \in J\}$ has a pseudointersection in $V[K]$.

For each $\alpha \in J$, set $v_\alpha = \{\langle\check{n}, r\rangle : \langle\check{n}, \langle p, r\rangle\rangle \in u_\alpha$ and $p \in G\}$. Set $H_0 = H \cap \mathrm{Fn}(\xi, 2)$. By Bell's Theorem [10], $\mathrm{M}A(\mathfrak{p}$; σ-centered) is a theorem of ZFC. Hence, $V[G]$ satisfies $\mathrm{M}A(\kappa$; σ-centered). By an argument of Baumgartner and Tall communicated by Roitman [58], adding a single Cohen real preserves $\mathrm{M}A(\kappa$; σ-centered). Since Booth forcing for $\{(v_\alpha)_{H_0} : \alpha \in J\}$ is σ-centered, $\{(v_\alpha)_{H_0} : \alpha \in J\}$, which is equal to $\{(u_\alpha)_K : \alpha \in J\}$, has a pseudointersection in $V[G \times H_0]$. $\square$

5.5 Local Noetherian type and π-type

Dow and Zhou [16] proved that there is a point in ω^* that (along with satisfying some additional properties) has an ω^{op}-like local base. The proof of Theorem 3.5.15 is a simpler construction of an ω^{op}-like local base which also naturally generalizes to every $u(\kappa)$. This construction is essentially due to Isbell [32], who was interested in actual intersections as opposed to pseudointersections.

Definition 5.5.1. Let $\mathfrak{a}$ denote the minimum of the cardinalities of infinite, maximal almost disjoint subfamilies of $[\omega]^{\omega}$. Let $\mathfrak{i}$ denote the minimum of the cardinalities of infinite, maximal independent subfamilies of $[\omega]^{\omega}$.

It is known that $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{r} \leq \mathfrak{i} \geq \mathfrak{d} \geq \mathfrak{s}$. (See 8.4, 8.12, 8.13 and 3.3 of [11].) Because of Kunen's result that $\mathfrak{a} = \omega_1$ in the Cohen model (see VIII.2.3 of [40]), it is consistent that $\mathfrak{a} < \mathfrak{r}$. Also, Shelah [65] has constructed a model of $\mathfrak{r} \leq \mathfrak{u} < \mathfrak{a}$.

In ZFC, the best upper bound of $\chi Nt(\omega^*)$ of which we know is $\mathfrak{c}$ by Lemma 3.2.3. We will next prove Theorem 5.5.5, which implies that, except for $\mathfrak{c}$ and possibly cf $\mathfrak{c}$, all of the cardinal characteristics of the continuum with definitions included in Blass [11] can consistently be simultaneously strictly less than $\chi Nt(\omega^*)$.

Lemma 5.5.2. *Suppose κ, λ, and μ are cardinals and $\kappa \leq \operatorname{cf} \lambda = \lambda > \mu$. Then $(\kappa \times \lambda)^{\mathrm{op}}$ is not almost μ^{op}-like.*

Proof. Let I be a cofinal subset of $\kappa \times \lambda$. Then it suffices to show that I is not μ-like. If $\kappa = \lambda$, then I is not μ-like because it is λ-directed. Suppose $\kappa < \lambda$. Then there exists $\alpha < \kappa$ such that $|I \cap (\{\alpha\} \times \lambda)| = \lambda$; hence, I has an increasing λ-sequence; hence, I is not μ-like. $\square$

Lemma 5.5.3. *Given any infinite independent subfamily I of $[\omega]^{\omega}$, there exists $J \subseteq [\omega]^{\omega}$ such that if x is a generic pseudointersection of J then $I \cup \{x\}$ is independent, but $I \cup \{x, y\}$ is not independent for any $y \in [\omega]^{\omega} \cap V \setminus I$.*

Proof. See Exercise A12 on page 289 of Kunen [40]. □

Definition 5.5.4. We say a P_κ-point in a space is *simple* if it has a local base of order type κ^{op}.

Theorem 5.5.5. *Suppose $\omega_1 \leq \operatorname{cf}\kappa = \kappa \leq \operatorname{cf}\lambda = \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension satisfying $\mathfrak{p} = \mathfrak{a} = \mathfrak{i} = \mathfrak{u} = \kappa \leq \lambda = \chi Nt(\omega^*) = \mathfrak{c}$.*

Proof. We will construct a finite support iteration $\langle \mathbb{P}_\alpha \rangle_{\alpha \leq \lambda\kappa}$ where $\lambda\kappa$ denotes the ordinal product of λ and κ. It suffices to ensure that the iteration is at every stage property (K) and of size at most λ, and that the $\mathbb{P}_{\lambda\kappa}$-generic extension of V satisfies $\max\{\mathfrak{a}, \mathfrak{i}, \mathfrak{u}\} \leq \kappa \leq \mathfrak{p}$ and $\lambda \leq \chi Nt(\omega^*)$. Our strategy is to interleave an iteration of length $\lambda\kappa$ and three iterations of length κ. At every stage below $\lambda\kappa$, add another piece of what will be an ultrafilter base that, ordered by $\supseteq^*$, will be isomorphic to a cofinal subset of $\kappa \times \lambda$. Also, at every stage we will add a pseudointersection, such that the final model satisfies $\mathfrak{p} \geq \kappa$. After each limit stage of cofinality λ, add an element to each of three objects that, when completed, will be a maximal almost disjoint family of size κ, a maximal independent family of size κ, and a base of a simple P_κ-point in ω^*.

Let $\varphi\colon \lambda^2 \to \lambda$ be a bijection such that $\varphi(\alpha, \beta) \geq \alpha$ for all $\alpha, \beta < \lambda$. For each $\langle \alpha, \beta \rangle \in \kappa \times \lambda$, set $E_{\alpha,\beta} = \{\langle \gamma, \delta \rangle \in \kappa \times \lambda : \lambda\gamma + \delta < \lambda\alpha + \beta\}$. Suppose $\langle \alpha, \beta \rangle \in \kappa \times \lambda$ and we have constructed $\langle \mathbb{P}_\gamma \rangle_{\gamma \leq \lambda\alpha+\beta}$ to have property (K) and size at most λ at all of its stages, and a sequence $\langle x_{\gamma,\delta} \rangle_{\langle \gamma,\delta \rangle \in E_{\alpha,\beta}}$ of $\mathbb{P}_{\lambda\alpha+\beta}$-names each forced to be in $[\omega]^\omega$. Set $B = \{x_{\gamma,\delta} : \langle \gamma, \delta \rangle \in E_{\alpha,\beta}\}$. Let $\langle S_\gamma \rangle_{\gamma<\kappa}$ be a partition of λ into κ-many stationary sets such that S_0 contains all successor ordinals. Suppose we have constructed a sequence $\langle \rho_{\gamma,\delta} \rangle_{\langle \gamma,\delta \rangle \in E_{\alpha,\beta}} \in \lambda^{E_{\alpha,\beta}}$ such that we always have $\rho_{\gamma,\delta} \in S_\gamma$ and $\rho_{\gamma,\delta_0} < \rho_{\gamma,\delta_1}$ whenever $\delta_0 < \delta_1$. Set $D_{\alpha,\beta} = \{\langle \gamma, \rho_{\gamma,\delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha,\beta}\}$. Further suppose that $\{\langle \langle \gamma, \rho_{\gamma,\delta} \rangle, x_{\gamma,\delta} \rangle :$

$\langle \gamma, \delta \rangle \in E_{\alpha,\beta}\}$ is forced to be an order embedding of $D_{\alpha,\beta}$ into $\langle [\omega]^\omega, \supseteq^* \rangle$ and that its range B is forced to have the SFIP. Also suppose that we have the following if $\alpha > 0$.

$$\Vdash_{\lambda\alpha+\beta} \forall \sigma \in [B]^{<\omega}\ \exists \delta < \lambda\ \ \bigcap \sigma \not\subseteq^* x_{0,\delta} \tag{5.5.1}$$

For each $\varepsilon < \lambda$, set $A_\varepsilon = \{x_{\gamma,\delta} : \langle \gamma, \delta \rangle \in E_{\alpha,\beta}$ and $\langle \gamma, \rho_{\gamma,\delta} \rangle < \langle \alpha, \varepsilon \rangle\}$.

Let y_β be a $\mathbb{P}_{\lambda\alpha+\beta}$-name for a surjection from λ to $[\omega]^\omega$. We may assume that corresponding y_γ have already been constructed for all $\gamma < \beta$. Let $\varphi(\zeta, \eta) = \beta$.

Claim. *If $\alpha > 0$, then we may choose $z \in \{y_\zeta(\eta),\ \omega \setminus y_\zeta(\eta)\}$ such that*

$$\Vdash_{\lambda\alpha+\beta} \forall \sigma \in [B]^{<\omega}\ \exists \delta < \lambda\ \ z \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}.$$

Proof. Suppose not. Let $\{z_0, z_1\} = \{y_\zeta(\eta),\ \omega \setminus y_\zeta(\eta)\}$. Then, working in a generic extension by $\mathbb{P}_{\lambda\alpha+\beta}$, there exist $\sigma_0, \sigma_1 \in [B]^{<\omega}$ such that $z_i \cap \bigcap \sigma_i \subseteq^* x_{0,\delta}$ for all $i < 2$ and $\delta < \lambda$. Hence, $\bigcap \bigcup_{i<2} \sigma_i \subseteq^* x_{0,\delta}$ for all $\delta < \lambda$, in contradiction with (5.5.1). $\square$

If $\alpha > 0$, then choose z as in the above claim; otherwise, choose z arbitrarily. If $\alpha = 0$, then set $\rho_{\alpha,\beta} = \beta + 1$. Otherwise, we may choose $\rho_{\alpha,\beta} \in S_\alpha$ such that $\rho_{\alpha,\beta} > \rho_{\alpha,\gamma}$ for all $\gamma < \beta$ and

$$\Vdash_{\lambda\alpha+\beta} \forall \sigma \in [A_{\rho_{\alpha,\beta}}]^{<\omega}\ \exists \delta < \rho_{\alpha,\beta}\ \ z \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}.$$

Set $D_{\alpha,\beta+1} = D_{\alpha,\beta} \cup \{\langle \alpha, \rho_{\alpha,\beta} \rangle\}$. Let A' be a $\mathbb{P}_{\lambda\alpha+\beta}$-name forced to satisfy $A' = A_{\rho_{\alpha,\beta}} \cup \{z\}$ if z splits B and $A' = A_{\rho_{\alpha,\beta}}$ otherwise. Let $\mathbb{Q}_0$ be a name for the Booth forcing for $A' \cup \{\omega \setminus n : n < \omega\}$; let $x_{\alpha,\beta}$ be a name for a generic pseudointersection of $A' \cup \{\omega \setminus n : n < \omega\}$. (The purpose of $\{\omega \setminus n : n < \omega\}$ is to ensure that $x_{\alpha,\beta}$ does not almost contain any element of $[\omega]^\omega$ in the $\mathbb{P}_{\lambda\alpha+\beta}$-generic extension of V.)

Let $F_{\lambda\alpha+\beta}$ to be a $\mathbb{P}_{\lambda\alpha+\beta}$-name for a surjection from λ to the elements of $[[\omega]^\omega]^{<\kappa}$ that have the SFIP. We may assume that corresponding F_γ have already been constructed for all $\gamma < \lambda\alpha + \beta$. Let $\mathbb{Q}_1$ be a name for the Booth forcing for $F_{\lambda\alpha+\zeta}(\eta)$.

Further suppose we have constructed sequences $\langle w_\gamma \rangle_{\gamma<\alpha}$ and $\langle U_\gamma \rangle_{\gamma<\alpha}$ of $\mathbb{P}_{\lambda\alpha}$-names such that $\Vdash_{\lambda\gamma} U_\delta \cup \{w_\delta\} \subseteq U_\gamma \in \omega^*$ for all $\delta < \gamma < \alpha$, and such that w_γ is forced to be a pseudointersection of U_γ for all $\gamma < \alpha$. If $\beta \neq 0$, then let $\mathbb{Q}_2$ be a name for the trivial forcing. If $\beta = 0$, then choose U_α such that $\Vdash_{\lambda\alpha} \bigcup_{\gamma<\alpha} U_\gamma \cup \{w_\gamma\} \subseteq U_\alpha \in \omega^*$, let $\mathbb{Q}_2$ be a name for the Booth forcing for U_α, and let w_α be a name for a generic pseudointersection of U_α.

Further suppose we have constructed a sequence $\langle a_\gamma \rangle_{\gamma<\alpha}$ of $\mathbb{P}_{\lambda\alpha}$-names whose range is forced to be an almost disjoint subfamily of $[\omega]^\omega$. If $\beta \neq 0$, then let $\mathbb{Q}_3$ be a name for the trivial forcing. If $\beta = 0$, then let $\mathbb{Q}_3$ be a name for the Booth forcing for $\{\omega \setminus a_\gamma : \gamma < \alpha\}$, and let a_α be a name for a generic pseudointersection of $\{\omega \setminus a_\gamma : \gamma < \alpha\}$.

Further suppose we have constructed a sequence $\langle i_\gamma \rangle_{\gamma<\alpha}$ of $\mathbb{P}_{\lambda\alpha}$-names whose range is forced to be an independent subfamily of $[\omega]^\omega$. If $\beta \neq 0$, then let $\mathbb{Q}_4$ be a name for the trivial forcing. If $\beta = 0$, then set $I = \{i_\gamma : \gamma < \alpha\}$ and let J and x be as in Lemma 5.5.3; let $\mathbb{Q}_4$ be a name for the Booth forcing for J; let i_α be a name for x.

Set $\mathbb{P}_{\lambda\alpha+\beta+1} = \mathbb{P}_{\lambda\alpha+\beta} * \prod_{n<5} \mathbb{Q}_n$. We may assume $|\prod_{n<5} \mathbb{Q}_n| \leq \lambda$; hence, $\mathbb{P}_{\lambda\alpha+\beta+1}$ has property (K) and size at most λ. Also, $B \cup \{x_{\alpha,\beta}\}$ is forced to have the SFIP by $\mathbb{Q}_0$-genericity because for every $b \in B$ we have that $\{b\} \cup A'$ is forced to have the SFIP because $\{b\} \cup A' \subseteq B \cup \{z\}$ if z splits B and $\{b\} \cup A' \subseteq B$ otherwise. Let us also show that (5.5.1) holds if we replace β with $\beta + 1$. We may assume $\alpha > 0$. Let $\sigma \in [B]^{<\omega}$. Then there exists $\delta < \lambda$ such that $\Vdash_{\lambda\alpha+\beta} z \cap \bigcap(\sigma \cup \tau) \not\subseteq^* x_{0,\delta}$ for all $\tau \in [A_{\rho_{\alpha,\beta}}]^{<\omega}$; hence, $\big\{\big(\bigcap \sigma\big) \setminus x_{0,\delta}\big\} \cup A'$ is forced to have the SFIP; hence, $\Vdash_{\lambda\alpha+\beta+1} x_{\alpha,\beta} \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}$ by

$\mathbb{Q}_0$-genericity. Thus, (5.5.1) holds as desired.

To complete our inductive construction of $\langle \mathbb{P}_\gamma \rangle_{\gamma \le \lambda\kappa}$, it suffices to show that the set

$$\{\langle \langle \gamma, \rho_{\gamma,\delta} \rangle, x_{\gamma,\delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha,\beta+1}\}$$

is forced to be an order embedding of $D_{\alpha,\beta+1}$ into $\langle [\omega]^\omega, \supseteq^* \rangle$. Suppose $\langle \gamma, \delta \rangle \in E_{\alpha,\beta}$. Then $\langle \alpha, \rho_{\alpha,\beta} \rangle \not\le \langle \gamma, \rho_{\gamma,\delta} \rangle$ and $\Vdash_{\lambda\alpha+\beta+1} x_{\alpha,\beta} \not\supseteq^* x_{\gamma,\delta}$ by $\mathbb{Q}_0$-genericity. If $\langle \gamma, \rho_{\gamma,\delta} \rangle < \langle \alpha, \rho_{\alpha,\beta} \rangle$, then $x_{\gamma,\delta} \in A'$; whence, $\Vdash_{\lambda\alpha+\beta+1} x_{\gamma,\delta} \supsetneq^* x_{\alpha,\beta}$. Suppose $\langle \gamma, \rho_{\gamma,\delta} \rangle \not< \langle \alpha, \rho_{\alpha,\beta} \rangle$. Then $\rho_{\alpha,\beta} < \rho_{\gamma,\delta}$; hence, $\rho_{\gamma,\delta} \ge \rho_{\alpha,\beta} + 1 = \rho_{0,\rho_{\alpha,\beta}}$; hence, $x_{\gamma,\delta} \subseteq^* x_{0,\rho_{\alpha,\beta}}$. By construction, $A' \cup \{\omega \setminus x_{0,\rho_{\alpha,\beta}}\}$ is forced to have the SFIP; hence, $\Vdash_{\lambda\alpha+\beta+1} x_{\gamma,\delta} \subseteq^* x_{0,\rho_{\alpha,\beta}} \not\supseteq^* x_{\alpha,\beta}$ by $\mathbb{Q}_0$-genericity. Thus, $\{\langle \langle \gamma, \rho_{\gamma,\delta} \rangle, x_{\gamma,\delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha,\beta+1}\}$ is forced to be an embedding as desired.

Let us show that the $\mathbb{P}_{\lambda\kappa}$-generic extension of V satisfies $\lambda \le \chi Nt(\omega^*)$. Let G be a generic filter of $\mathbb{P}_{\lambda\kappa}$ and set $\mathcal{B} = \{(x_{\alpha,\beta})^*_G : \langle \alpha, \beta \rangle \in \kappa \times \lambda\}$. Then $\mathcal{B}$ is a local base at some $p \in (\omega^*)^{V[G]}$ because every element of $([\omega]^\omega)^{V[G]}$ is handled by an appropriate $\mathbb{Q}_0$. By Lemma 3.2.20, $\mathcal{B}$ contains a $\chi Nt(p, \omega^*)^{\mathrm{op}}$-like local base $\{(x_{\alpha,\beta})^*_G : \langle \alpha, \beta \rangle \in I\}$ at p for some $I \subseteq \kappa \times \lambda$. Set $J = \{\langle \alpha, \rho_{\alpha,\beta} \rangle : \langle \alpha, \beta \rangle \in I\}$. Then J is cofinal in $\kappa \times \lambda$; hence, by Lemma 5.5.2, J is not ν-like for any $\nu < \lambda$. Hence, $\chi Nt(\omega^*)^{V[G]} \ge \lambda$.

Finally, let us show that the $\mathbb{P}_{\lambda\kappa}$-generic extension of V satisfies $\max\{\mathfrak{a}, \mathfrak{i}, \mathfrak{u}\} \le \kappa \le \mathfrak{p}$. Working in $V[G]$, notice that $\mathfrak{u} \le \kappa$ because $\bigcup_{\alpha<\kappa} (U_\alpha)_G \in \omega^*$ and $\{(w_\alpha)^*_G : \alpha < \kappa\}$ is a local base at $\bigcup_{\alpha<\kappa} (U_\alpha)_G$. Moreover, $\{(a_\alpha)_G : \alpha < \kappa\}$ and $\{(i_\alpha)_G : \alpha < \kappa\}$ witness that $\mathfrak{a} \le \kappa$ and $\mathfrak{i} \le \kappa$. For $\mathfrak{p} \ge \kappa$, note that very element of $[[\omega]^\omega]^{<\kappa}$ with the SFIP is $(F_{\lambda\alpha+\zeta}(\eta))_G$ for some $\alpha < \kappa$ and $\zeta, \eta < \lambda$. By $\mathbb{Q}_1$-genericity, a pseudointersection of $(F_{\lambda\alpha+\zeta}(\eta))_G$ is added at stage $\lambda\alpha + \varphi(\zeta, \eta)$. $\square$

Theorem 5.5.6. *$\pi\chi Nt(\omega^*) = \omega$.*

Proof. Fix $p \in \omega^*$. By a result of Balcar and Vojtáš [5], there exists $\langle y_x \rangle_{x\in p}$ such that $y_x \in [x]^\omega$ for all $x \in p$ and $\{y_x\}_{x\in p}$ is an almost disjoint family. Clearly, $\{y_x^*\}_{x\in p}$ is a pairwise disjoint—and therefore ω^{op}-like—local π-base at p. $\square$

5.6 Powers of $\beta\omega \setminus \omega$

Definition 5.6.1. A *box* is a subset E of a product space $\prod_{i\in I} X_i$ such that there exist $\sigma \in [I]^{<\omega}$ and $\langle E_i \rangle_{i\in\sigma}$ such that $E = \bigcap_{i\in\sigma} \pi_i^{-1} E_i$. Let $Nt_{\mathrm{box}}(\prod_{i\in I} X_i)$ denote the least infinite κ such that $\prod_{i\in I} X_i$ has a κ^{op}-like base of open boxes.

Lemma 5.6.2 (Peregudov [54])**.** *In any product space $X = \prod_{i\in I} X_i$, we have $Nt(X) \leq Nt_{\mathrm{box}}(X) \leq \sup_{i\in I} Nt(X_i)$.*

Lemma 5.6.3 (Malykhin [42])**.** *Let $X = \prod_{i\in I} X_i$ where each X_i is a nonsingleton T_1 space. If $w(X) \leq |I|$, then $Nt(X) = Nt_{\mathrm{box}}(X) = \omega$.*

Remark 5.6.4. In Lemma 5.6.3, the hypothesis that the factor spaces be nonsingleton and T_1 can be weakened to merely require that each factor space is the union of two nontrivial open sets. Also, the conclusion of Lemma 5.6.3 may be amended with the statement that X has a $\langle |I|, \omega \rangle$-splitter: use $\langle \{\pi_i^{-1} U_i, \pi_i^{-1} V_i\} \rangle_{i\in I}$ where each $\{U_i, V_i\}$ is a nontrivial open cover of X_i.

Theorem 5.6.5. *The sequence $\langle Nt((\omega^*)^{\omega+\alpha}) \rangle_{\alpha\in\mathrm{On}}$ is nonincreasing and $Nt((\omega^*)^{\mathfrak{c}}) = \omega$.*

Proof. Note that if $\omega \leq \alpha \leq \beta$, then $(\omega^*)^\beta \cong ((\omega^*)^\alpha)^\beta$. Then apply Lemmas 5.6.2 and 5.6.3. $\square$

Lemma 5.6.6. *Let $0 < n < \omega$ and X be a space. Then $Nt_{\mathrm{box}}(X^n) = Nt(X)$.*

Proof. Set $\kappa = Nt_{\mathrm{box}}(X^n)$. By Lemma 5.6.2, $\kappa \leq Nt(X)$. Let us show that $Nt(X) \leq \kappa$. Let $\mathcal{A}$ be a κ^{op}-like base of X^n consisting only of boxes. Let $\mathcal{B}$ denote the set of all nonempty open $V \subseteq X$ for which there exists $\prod_{i<n} U_i \in \mathcal{A}$ such that $V = \bigcap_{i<n} U_i$. Then $\mathcal{B}$ is a base of X because if $p \in U$ and U is an open subset of X, then there exists $\prod_{i<n} U_i \in \mathcal{A}$ such that $\langle p \rangle_{i<n} \in \prod_{i<n} U_i \subseteq U^n$; whence, $p \in \bigcap_{i<n} U_i \subseteq U$ and $\bigcap_{i<n} U_i \in \mathcal{B}$.

It suffices to show that $\mathcal{B}$ is κ^{op}-like. Suppose not. Then there exist $\prod_{i<n} U_i \in \mathcal{A}$ and $\langle \prod_{i<n} V_{\alpha,i} \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ such that

$$\emptyset \neq \bigcap_{i<n} U_i \subseteq \bigcap_{i<n} V_{\alpha,i} \neq \bigcap_{i<n} V_{\beta,i}$$

for all $\alpha < \beta < \kappa$. Clearly, $\prod_{i<n} V_{\alpha,i} \neq \prod_{i<n} V_{\beta,i}$ for all $\alpha < \beta < \kappa$. Choose $U \in \mathcal{A}$ such that $U \subseteq (\bigcap_{i<n} U_i)^n$. Then $U \subseteq \prod_{i<n} V_{\alpha,i}$ for all $\alpha < \kappa$, in contradiction with how we chose $\mathcal{A}$. $\square$

Lemma 5.6.7. *If $0 < n < \omega$ and X is a compact space such that $\chi(p, X) = w(X)$ for all $p \in X$, then $Nt(X) = Nt(X^n)$.*

Proof. By Lemma 5.6.6, it suffices to show that $Nt_{\mathrm{box}}(X^n) \leq Nt(X^n)$. By Lemma 5.2.6, either X^n has a $\langle w(X^n), Nt(X^n) \rangle$-splitter, or $Nt(X^n) = w(X^n)^+$. By Lemma 5.2.5, we may conclude $Nt_{\mathrm{box}}(X^n) \leq Nt(X^n)$. $\square$

Theorem 5.6.8. *If $0 < n < \omega$, then $Nt(\omega^*) \geq Nt((\omega^*)^n) \geq \min\{Nt(\omega^*), \mathfrak{c}\}$. Moreover, $\max\{\mathfrak{u}, \mathrm{cf}\,\mathfrak{c}\} = \mathfrak{c}$ implies $Nt(\omega^*) = Nt((\omega^*)^n)$.*

Proof. Lemma 5.6.2 implies $Nt(\omega^*) \geq Nt((\omega^*)^n)$. To prove the rest of the theorem, first consider the case $\mathfrak{r} < \mathfrak{c}$. As in the proof of Theorem 5.2.2, construct a point $p \in \omega^*$ such that $\pi\chi(p, \omega^*) = \mathfrak{r}$ and $\chi(p, \omega^*) = \mathfrak{c}$. Then $\pi\chi(\langle p \rangle_{i<n}, (\omega^*)^n) = \mathfrak{r}$ and

$\chi(\langle p\rangle_{i<n}, (\omega^*)^n) = \mathfrak{c}$; hence, $Nt((\omega^*)^n) \geq \mathfrak{c}$ by Theorem 3.3.11. Moreover, if $\operatorname{cf}\mathfrak{c} = \mathfrak{c}$, then $Nt((\omega^*)^n) = Nt(\omega^*) = \mathfrak{c}^+$. If $\mathfrak{u} = \mathfrak{c}$, then $Nt(\omega^*) = Nt((\omega^*)^n)$ by Lemma 5.6.7. Finally, in the case $\mathfrak{r} = \mathfrak{c}$, we have $\mathfrak{u} = \mathfrak{c}$, which again implies $Nt(\omega^*) = Nt((\omega^*)^n)$. □

Corollary 5.6.9. *Suppose* $\max\{\mathfrak{u}, \operatorname{cf}\mathfrak{c}\} = \mathfrak{c}$. *Then* $\langle Nt((\omega^*)^{1+\alpha})\rangle_{\alpha\in\mathrm{On}}$ *is nonincreasing.*

Proof. By Theorem 5.6.8 and Lemma 5.6.2, $Nt((\omega^*)^n) = Nt(\omega^*) \geq Nt((\omega^*)^\alpha)$ whenever $0 < n < \omega \leq \alpha$. The rest follows from Theorem 5.6.5. □

Theorem 5.6.10. *Suppose* $\mathfrak{u} = \mathfrak{c}$. *Then* $Nt((\omega^*)^{1+\alpha}) = Nt(\omega^*)$ *for all* $\alpha < \operatorname{cf}\mathfrak{c}$.

Proof. Let λ be an arbitrary infinite cardinal less than $Nt(\omega^*)$. By Lemma 5.2.6, it suffices to show that $(\omega^*)^{1+\alpha}$ does not have a $\langle\mathfrak{c}, \lambda\rangle$-splitter. Seeking a contradiction, suppose $\langle\mathcal{F}_\beta\rangle_{\beta<\mathfrak{c}}$ is such a $\langle\mathfrak{c}, \lambda\rangle$-splitter. We may assume $\bigcup_{\beta<\mathfrak{c}}\mathcal{F}_\beta$ consists only of open boxes because we can replace each $\mathcal{F}_\beta$ with a suitable refinement. Since $\alpha < \operatorname{cf}\mathfrak{c}$, there exist $\sigma \in [1+\alpha]^{<\omega}$ and $I \in [\mathfrak{c}]^\mathfrak{c}$ such that, for every $U \in \bigcup_{\beta\in I}\mathcal{F}_\beta$, there exists $\varphi(U) \subseteq (\omega^*)^\sigma$ such that $U = \pi_\sigma^{-1}\varphi(U)$. Let j be a bijection from $\mathfrak{c}$ to I. Then $\langle\varphi[\mathcal{F}_{j(\beta)}]\rangle_{\beta<\mathfrak{c}}$ is a $\langle\mathfrak{c}, \lambda\rangle$-splitter of $(\omega^*)^\sigma$. Hence, $Nt((\omega^*)^\sigma) \leq \lambda < Nt(\omega^*)$ by Lemma 5.2.5. But $Nt((\omega^*)^\sigma) < Nt(\omega^*)$ contradicts Theorem 5.6.8. □

Lemma 5.6.11. *Suppose a space* X *has a* $\langle\operatorname{cf} w(X), \operatorname{cf} w(X)\rangle$*-splitter. Then* $Nt(X) \leq w(X)$.

Proof. Set $\kappa = \operatorname{cf} w(X)$ and $\lambda = w(X)$. Let $\langle\mathcal{F}_\alpha\rangle_{\alpha<\kappa}$ be a $\langle\kappa, \kappa\rangle$-splitter of X. Let $h : \lambda \to \kappa$ satisfy $|h^{-1}\{\alpha\}| < \lambda$ for all $\alpha < \kappa$. Then $\langle\mathcal{F}_{h(\alpha)}\rangle_{\alpha<\lambda}$ is a $\langle\lambda, \lambda\rangle$-splitter because if $I \in [\lambda]^\lambda$, then $h[I] \in [\kappa]^\kappa$. By Lemma 5.2.5, $Nt(X) \leq \lambda$. □

Remark 5.6.12. The proof of the above lemma shows that for any infinite cardinal κ, a space with a $\langle\operatorname{cf}\kappa, \operatorname{cf}\kappa\rangle$-splitter also has a $\langle\kappa, \kappa\rangle$-splitter.

Theorem 5.6.13. *$Nt((\omega^*)^{\operatorname{cf}\mathfrak{c}}) \leq \mathfrak{c}$.*

Proof. The sequence

$$\langle\{\pi_\alpha^{-1}(\{2n : n < \omega\}^*), \pi_\alpha^{-1}(\{2n+1 : n < \omega\}^*)\}\rangle_{\alpha<\operatorname{cf}\mathfrak{c}}$$

is a $\langle\operatorname{cf}\mathfrak{c}, \omega\rangle$-splitter of $(\omega^*)^{\operatorname{cf}\mathfrak{c}}$. Apply Lemma 5.6.11. □

Theorem 5.6.14. *For all cardinals κ satisfying $\kappa > \operatorname{cf}\kappa > \omega_1$, it is consistent that $\mathfrak{c} = \kappa$ and $\mathfrak{r} < \operatorname{cf}\mathfrak{c}$. The last inequality implies $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}^+$ for all $\alpha < \operatorname{cf}\mathfrak{c}$ and $Nt((\omega^*)^\beta) = \mathfrak{c} = \kappa$ for all $\beta \in \mathfrak{c} \setminus \operatorname{cf}\mathfrak{c}$.*

Proof. Starting with $\mathfrak{c} = \kappa$ in the ground model, the proof of Theorem 5.2.2 shows how to force $\mathfrak{r} = \mathfrak{u} = \omega_1$ while preserving $\mathfrak{c}$. Now suppose $\mathfrak{r} < \operatorname{cf}\mathfrak{c}$. Fix $\alpha < \operatorname{cf}\mathfrak{c}$ and $\beta \in \mathfrak{c}\setminus\operatorname{cf}\mathfrak{c}$. By Theorems 5.6.13 and 5.6.5, $Nt((\omega^*)^\beta) \leq \mathfrak{c}$. To see that $Nt((\omega^*)^\beta) \geq \mathfrak{c}$, proceed as in the proof of Theorem 5.6.8, constructing a point with character $\mathfrak{c}$ and π-character $|\beta|$. Similarly prove $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}^+$ by constructing a point with character $\mathfrak{c}$ and π-character $|\mathfrak{r} + \alpha|$. □

Lemma 5.6.15. *Suppose κ, λ, and μ are cardinals and p is a point in a product space $X = \prod_{\alpha<\kappa} X_\alpha$ satisfying the following for all $\alpha < \kappa$.*

1. *$0 < \kappa < w(X)$ and $\omega \leq \lambda \leq w(X)$.*
2. *$\kappa < \operatorname{cf} w(X)$ or $\lambda < w(X)$.*
3. *$\mu < \lambda$ or $\mu = \operatorname{cf}\lambda$.*
4. *$\chi(p(\alpha), X_\alpha) < \lambda$ or the intersection of any μ-many neighborhoods of $p(\alpha)$ has nonempty interior.*

Then $\chi(p, X) < w(X)$ or $Nt(X) > \mu$.

Proof. Let $\mathcal{A}$ be a base of X. Set $\mathcal{B} = \{U \in \mathcal{A} : p \in U\}$. For each $\alpha < \kappa$, let $\mathcal{C}_\alpha$ be a local base at $p(\alpha)$ of size $\chi(p(\alpha), X_\alpha)$. Set $F = \bigcup_{r\in[\kappa]^{<\omega}} \prod_{\alpha\in r} \mathcal{C}_\alpha$. For each $\sigma \in F$, set $U_\sigma = \bigcap_{\alpha\in\operatorname{dom}\sigma} \pi_\alpha^{-1}\sigma(\alpha)$. For each $V \in \mathcal{B}$, choose $\sigma(V) \in F$ such that $p \in U_{\sigma(V)} \subseteq V$. We may assume $\chi(p, X) = w(X)$; hence, by (1) and (2), there exist $r \in [\kappa]^{<\omega}$ and $\mathcal{D} \in [\mathcal{B}]^\lambda$ such that $\operatorname{dom}\sigma(V) = r$ for all $V \in \mathcal{D}$. Set $s = \{\alpha \in r : \chi(p(\alpha), X_\alpha) < \lambda\}$ and $t = r \setminus s$. By (3), there exist $\tau \in \prod_{\alpha\in s} \mathcal{C}_\alpha$ and $\mathcal{E} \in [\mathcal{D}]^\mu$ such that $\sigma(V) \restriction s = \tau$ for all $V \in \mathcal{E}$. By (4), $\bigcap_{V\in\mathcal{E}} \sigma(V)(\alpha)$ has nonempty interior for all $\alpha \in t$. Hence, $\bigcap \mathcal{E}$ has nonempty interior because it contains $U_\tau \cap \bigcap_{\alpha\in t} \pi_\alpha^{-1} \bigcap_{V\in\mathcal{E}} \sigma(V)(\alpha)$. Thus, $Nt(X) > \mu$. □

Theorem 5.6.16. *Suppose $0 < \alpha < \mathfrak{c}$ and $\langle X_\beta\rangle_{\beta<\alpha}$ is a sequence of spaces each with weight at most $\mathfrak{c}$. Set $X = \prod_{\beta<\alpha}(X_\beta \oplus \omega^*)$. Then $Nt(X) \geq \mathfrak{p}$.*

Proof. Let ν be an arbitrary infinite cardinal less than $\mathfrak{p}$. Set $\kappa = |\alpha|$, $\lambda = \nu^+$, and $\mu = \nu$. Choose $q \in \omega^*$ such that $\chi(q, \omega^*) = \mathfrak{c}$; set $p = \langle q\rangle_{\beta<\alpha}$. Then Lemma 5.6.15 applies because if $\kappa \geq \operatorname{cf} w(X) = \operatorname{cf}\mathfrak{c}$, then $\lambda \leq \mathfrak{p} \leq \operatorname{cf}\mathfrak{c} < \mathfrak{c} = w(X)$. Therefore, $Nt(X) > \nu$. □

Corollary 5.6.17. *Suppose $\mathfrak{p} = \mathfrak{c}$. Then $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}$ for all $\alpha < \mathfrak{c}$.*

Proof. By Theorem 5.2.10, $Nt(\omega^*) \leq \mathfrak{c}$. Hence, by Corollary 5.6.9, $Nt((\omega^*)^{1+\alpha}) \leq \mathfrak{c}$ for all $\alpha \in \mathrm{O}n$. By Theorem 5.6.16, $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}$ for all $\alpha < \mathfrak{c}$. □

Corollary 5.6.18. *Suppose $\alpha < \mathfrak{c}$ and $\langle X_\beta\rangle_{\beta<\alpha}$ is a sequence of spaces each with weight at most $\mathfrak{c}$. Then $\prod_{\beta<\alpha}(X_\beta \oplus \omega^*)$ is not homeomorphic to a product of $\mathfrak{c}$-many nonsingleton spaces.*

Proof. Combine Theorem 5.6.16 and Lemma 5.6.3. □

5.7 Questions

Question 5.7.1. Is it consistent that $Nt(\omega^*) = \mathfrak{c} = \operatorname{cf}\mathfrak{c} > \mathfrak{d}$? By Theorem 5.2.10, the above relations imply $\mathfrak{c} = \operatorname{cf}\mathfrak{c} = \mathfrak{r} > \mathfrak{d}$, which can be attained by adding many random reals. However, adding many random reals collapses $Nt(\omega^*)$ to ω_1.

Question 5.7.2. Is it consistent that $Nt(\omega^*) = \mathfrak{c}^+$ and $\mathfrak{r} \geq \operatorname{cf}\mathfrak{c}$?

Question 5.7.3. Is $Nt(\omega^*) < \mathfrak{ss}_\omega$ consistent? This inequality implies $\mathfrak{u} < \mathfrak{c}$. Hence, by Theorem 5.2.10, the inequality further implies

$$\operatorname{cf}\mathfrak{c} \leq \mathfrak{r} \leq \mathfrak{u} < \mathfrak{c} = Nt(\omega^*) < \mathfrak{ss}_\omega = \mathfrak{c}^+.$$

More generally, does any space X have a base that does not contain an $Nt(X)^{\mathrm{op}}$-like base?

Question 5.7.4. Is $\mathfrak{ss}_\omega < \mathfrak{ss}_2$ consistent?

Question 5.7.5. Letting $\mathfrak{g}$ denote the groupwise density number (see 6.26 of [11]), is $Nt(\omega^*) < \mathfrak{g}$ consistent? $\chi Nt(\omega^*) < \mathfrak{g}$? In particular, what are $Nt(\omega^*)$ and $\chi Nt(\omega^*)$ in the Laver model (see 11.7 of [11])?

Question 5.7.6. Is $\operatorname{cf} Nt(\omega^*) < Nt(\omega^*) < \mathfrak{c}$ consistent? $\operatorname{cf} Nt(\omega^*) = \omega$?

Question 5.7.7. Is $\operatorname{cf}\mathfrak{c} < Nt(\omega^*) < \mathfrak{c}$ consistent?

Question 5.7.8. What is $\chi Nt(\omega^*)$ in the forcing extension of the proof of Theorem 5.4.15? More generally, is it consistent that $\chi Nt(\omega^*) < Nt(\omega^*) \leq \mathfrak{c}$?

Question 5.7.9. Is $\chi Nt(\omega^*) = \omega$ consistent? An affirmative answer would be a strengthening of Shelah's result [64] that ω^* consistently has no P-points. If the answer is negative, then which, if any, of $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$, and $\mathfrak{g}$ are lower bounds of $\chi Nt(\omega^*)$ in ZFC?

Question 5.7.10. Is $\operatorname{cf}\mathfrak{c} < \chi Nt(\omega^*)$ consistent? $\operatorname{cf}\mathfrak{c} < \chi Nt(\omega^*) < \mathfrak{c}$?

Question 5.7.11. Does any Hausdorff space have uncountable local Noetherian π-type? (It is easy to construct such T_1 spaces: give $\omega_1 + 1$ the topology $\{(\omega_1 + 1) \setminus (\alpha \cup \sigma) : \alpha < \omega_1$ and $\sigma \in [\omega_1 + 1]^{<\omega}\} \cup \{\emptyset\}$.)

Question 5.7.12. Is it consistent that $Nt((\omega^*)^{1+\alpha}) < \min\{Nt(\omega^*), \mathfrak{c}\}$ for some $\alpha < \mathfrak{c}$? Is it consistent that $Nt((\omega^*)^{1+\alpha}) < Nt(\omega^*)$ for some $\alpha < \operatorname{cf}\mathfrak{c}$?

Chapter 6

Tukey classes of ultrafilters on ω

6.1 Tukey classes

Definition 6.1.1 (Tukey [68])**.** Given directed sets P and Q and a map $f\colon P \to Q$, we say f is a Tukey map, writing $f\colon P \leq_T Q$, if the f-image of every unbounded subset of P is unbounded in Q. We say P is Tukey reducible to Q, writing $P \leq_T Q$, if there is a Tukey map from P to Q. If $P \leq_T Q \leq_T P$, then we say P and Q are Tukey equivalent and write $P \equiv_T Q$.

By the next proposition, the above definition is equivalent to Definition 3.5.5.

Proposition 6.1.2 (Tukey [68])**.** *A map $f\colon P \to Q$ is Tukey if and only the f-preimage of every bounded subset of Q is bounded in P. Moreover, $P \leq_T Q$ if and only if there is a map $g\colon Q \to P$ such that the image of every cofinal subset of Q is cofinal in P.*

Theorem 6.1.3 (Tukey [68])**.** *$P \equiv_T Q$ if and only if P and Q order embed as cofinal subsets of a common third directed set. Moreover, if $P \cap Q = \emptyset$, then we may assume the order embeddings are identity maps onto a quasiordering of $P \cup Q$.*

The following is a list of basic facts about Tukey reducibility.

- $P \leq_T Q \Rightarrow \mathrm{cf}(P) \leq \mathrm{cf}(Q)$.

- For all ordinals α, β, we have $\alpha \leq_T \beta \Leftrightarrow \mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

- $P \leq_T P \times Q$.
- $P \leq_T R \geq_T Q \Rightarrow P \times Q \leq_T R$.
- $P \times P \equiv_T P$.
- $P \leq_T \langle [\mathrm{cf}(P)]^{<\omega}, \subseteq \rangle$.
- For all infinite sets A, B, we have $\langle [A]^{<\omega}, \subseteq \rangle \leq_T \langle [B]^{<\omega}, \subseteq \rangle \Leftrightarrow |A| \leq |B|$.
- Given finitely many ordinals $\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}$, we have
$$\prod_{i<m} \alpha_i \leq_T \prod_{i<n} \beta_i \Leftrightarrow \{\mathrm{cf}(\alpha_i) : i < m\} \subseteq \{\mathrm{cf}(\beta_i) : i < n\}.$$
- Every countable directed set is Tukey equivalent to 1 or ω.

Theorem 6.1.4 (Isbell [32]). *No two of* 1, ω, ω_1, $\omega \times \omega_1$, *and* $\langle [\omega_1]^{<\omega}, \subseteq \rangle$ *are Tukey equivalent.*

Isbell [32] asked if these five Tukey classes encompass all directed sets of size ω_1. In [33], he answered "no" assuming CH. In particular, ω^ω, ordered by domination, is not Tukey equivalent to any of the above five orders. Devlin, Steprāns, and Watson [14] showed that $\diamondsuit$ implies there are 2^{ω_1}-many pairwise Tukey inequivalent directed sets of size ω_1. Todorčević [67] weakened the hypothesis of $\diamondsuit$ to CH and also showed that PFA implies that 1, ω, ω_1, $\omega \times \omega_1$, and $\langle [\omega_1]^{<\omega}, \subseteq \rangle$ represent the only Tukey classes of directed sets of size ω_1.

6.2 Tukey reducibility and topology

Traditionally, Tukey reducibility has mainly been connected to topology by the concept of subnet: we say $\langle x_i \rangle_{i \in I}$ is a subnet of $\langle y_j \rangle_{j \in J}$ if there exists $f\colon I \to J$ such that the image of every cofinal subset of I is cofinal in J, and $x_i = y_{f(i)}$ for all $i \in I$. In contrast, our results are about classifying points in certain spaces by the Tukey classes of their local bases ordered by reverse inclusion. The following theorem, which is of independent interest, implies that the Tukey class of a local base at a point in a space is a topological invariant.

Theorem 6.2.1. *Suppose X and Y are spaces, $p \in X$, $q \in Y$, $\mathcal{A}$ is a local base at p in X, $\mathcal{B}$ is a local base at q in Y, $f\colon X \to Y$ is continuous and open (or just continuous at p and open at p), and $f(p) = q$. Then $\langle \mathcal{B}, \supseteq \rangle \leq_T \langle \mathcal{A}, \supseteq \rangle$.*

Proof. Choose $H\colon \mathcal{A} \to \mathcal{B}$ such that $H(U) \subseteq f[U]$ for all $U \in \mathcal{A}$. (Here we use that f is open.) Suppose $\mathcal{C} \subseteq \mathcal{A}$ is cofinal. For any $U \in \mathcal{B}$, we may choose $V \in \mathcal{A}$ such that $f[V] \subseteq U$ by continuity of f. Then choose $W \in \mathcal{C}$ such that $W \subseteq V$. Hence, $H(W) \subseteq f[W] \subseteq f[V] \subseteq U$. Thus, $H[\mathcal{C}]$ is cofinal. $\square$

Corollary 6.2.2. *In the above theorem, if f is a homeomorphism, then every local base at p is Tukey-equivalent to every local base at q.*

Example 6.2.3. Consider the ordered space $X = \omega_1 + 1 + \omega^{\mathrm{op}}$. It has a point p that is the limit of an ascending ω_1-sequence and a descending ω-sequence. Every local base at p, ordered by $\supseteq$, is Tukey equivalent to $\omega \times \omega_1$.

Next, consider $D_{\omega_1} \cup \{\infty\}$, the one-point compactification of the ω_1-sized discrete space. Glue X and $D_{\omega_1} \cup \{\infty\}$ together into a new space Y by a quotient map that identifies p and ∞. In Y, every local base at p, ordered by $\supseteq$, is Tukey equivalent to $\langle [\omega_1]^{<\omega}, \subseteq \rangle$, which is not Tukey equivalent to $\omega \times \omega_1$.

Thus, we can distinguish p in X from p in Y by their associated Tukey classes, even though other topological properties, such as character and π-character, have not changed. Moreover, since $\omega \times \omega_1 <_T [\omega_1]^{<\omega}$, we may conclude there is no continuous open map from X to Y that sends p to p.

6.3 Ultrafilters

By Stone duality, every ultrafilter $\mathcal{U}$ on ω is such that $\mathcal{U}$ ordered by containment, $\supseteq$, is Tukey-equivalent to every local base of $\mathcal{U}$ in $\beta\omega$. Likewise, $\mathcal{U}$ ordered by almost containment, $\supseteq^*$, is Tukey equivalent to every local base of $\mathcal{U}$ in ω^*. Therefore, let us now restrict our attention to the Tukey classes of nonprincipal ultrafilters on ω, ordered by $\supseteq$ or $\supseteq^*$. Note that the identity map on a $\mathcal{U} \in \omega^*$ is a Tukey map from $\langle \mathcal{U}, \supseteq^* \rangle$ to $\langle \mathcal{U}, \supseteq \rangle$. Moreover, since $\langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$ is Tukey-maximal among the directed sets of cofinality at most $\mathfrak{c}$, if $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$, then $\langle \mathcal{U}, \supseteq \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.

Note that a given $\mathcal{U} \in \omega^*$ is a P_κ-point if and only if $\langle \mathcal{U}, \supseteq^* \rangle$ is κ-directed. Also note that $\mathfrak{u}$ is the least κ such that there exists $\mathcal{U} \in \omega^*$ such that $\operatorname{cf}(\langle \mathcal{U}, \supseteq^* \rangle) = \kappa$; moreover, $\operatorname{cf}(\langle \mathcal{U}, \supseteq \rangle) = \operatorname{cf}(\langle \mathcal{U}, \supseteq^* \rangle)$ always holds.

Isbell [32], using an independent family of sets, showed that there is always some $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$. Moreover, his proof also implicitly shows that $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.

Definition 6.3.1. We say $\mathcal{I} \subseteq [\omega]^\omega$ is independent if for all disjoint $\sigma, \tau \in [\mathcal{I}]^{<\omega}$ we have $\bigcap \sigma \not\subseteq^* \bigcup \tau$.

Lemma 6.3.2 (Hausdorff [29]). *There exists an independent $\mathcal{I} \in [[\omega]^\omega]^\mathfrak{c}$.*

Theorem 6.3.3 (Isbell [32]). *There exists $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.*

Proof. It suffices to show that there exists $f \colon \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{U}, \supseteq^* \rangle$. Let $\mathcal{I} \in [[\omega]^\omega]^\mathfrak{c}$ be independent. Let $\mathcal{F}$ be the filter generated by $\mathcal{I}$. Let $\mathcal{J}$ be the ideal generated by the set of pseudointersections of infinite subsets of $\mathcal{I}$. Extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ disjoint from $\mathcal{J}$. Define $f\colon [\mathfrak{c}]^{<\omega} \to \mathcal{U}$ by $\sigma \mapsto \bigcap_{\alpha\in\sigma} I_\alpha$. Then f is Tukey as desired. $\square$

There are also known constructions of various $\mathcal{U} \in \omega^*$ that satisfy $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$ and some additional property. See, for example, Dow and Zhou [16]. Also, Kunen [39] proved that there exists a non-P-point $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is $\mathfrak{c}$-OK, and the next proposition shows that such a point must satisfy $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.

Definition 6.3.4 (Kunen [39])**.** We say $\mathcal{U} \in \omega^*$ is κ-OK if for every $\langle A_n \rangle_{n<\omega} \in \mathcal{U}^\omega$ there exists $\langle B_\alpha \rangle_{\alpha<\kappa} \in \mathcal{U}^\kappa$ such that for all nonempty $\sigma \in [\kappa]^{<\omega}$ we have $\bigcap_{\alpha\in\sigma} B_\alpha \subseteq^* A_{|\sigma|}$. (Therefore, Keisler's notion of κ^+-good implies κ-OK.)

Proposition 6.3.5. *If $\mathcal{U}$ is a $\mathfrak{c}$-OK non-P-point in ω^*, then $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$.*

Proof. It suffices to show that there exists $f\colon \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{U}, \supseteq^* \rangle$. Choose $\langle A_n \rangle_{n<\omega} \in \mathcal{U}^\omega$ such that $\{A_n : n < \omega\}$ has no pseudointersection in $\mathcal{U}$. Then choose $\langle B_\alpha \rangle_{\alpha<\mathfrak{c}} \in \mathcal{U}^{\mathfrak{c}}$ as in Definition 6.3.4. Define $f\colon [\mathfrak{c}]^{<\omega} \to \mathcal{U}$ by $\sigma \mapsto \bigcap_{\alpha\in\sigma} B_\alpha$. Then every infinite subset of $[\mathfrak{c}]^{<\omega}$ has unbounded f-image; hence, f is Tukey as desired. $\square$

Isbell [32] asked if every $\mathcal{U} \in \omega^*$ satisfies $\langle \mathcal{U}, \supseteq \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$. It is now well-known that it is consistent with $\neg$CH that $\mathfrak{u} < \mathfrak{c}$, which implies the existence of $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq \rangle \leq_T \langle [\mathfrak{u}]^{<\omega}, \subseteq \rangle <_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$. To keep Isbell's question interesting, let us restrict our attention to models of $\mathfrak{u} = \mathfrak{c}$. (Another way to keep Isbell's question interesting to demand a ZFC proof of the existence of $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq \rangle \not\equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq \rangle$. This is an open problem.)

Definition 6.3.6. Given cardinals κ and λ, let E^κ_λ denote $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$.

Theorem 6.3.7. *Assume $\diamondsuit(E^{\mathfrak{c}}_{\omega})$ and $\mathfrak{p} = \mathfrak{c}$. Then there exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is not a P-point and $\mathfrak{c} <_T \langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle <_T [\mathfrak{c}]^{<\omega}$.*

Proof. To simplify notation, we construct $\mathcal{U}$ as an ultrafilter on ω^2. Indeed, we construct $P_{\mathfrak{c}}$-points $\mathcal{V}, \mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \ldots \in \omega^*$ and set $\mathcal{U} = \{E \subseteq \omega^2 : \mathcal{V} \ni \{i : \mathcal{W}_i \ni \{j : \langle i, j \rangle \in E\}\}\}$. This immediately implies that $\{(\omega \setminus n) \times \omega : n < \omega\}$ is a countable subset of $\mathcal{U}$ with no pseudointersection in $\mathcal{U}$; whence, $\mathcal{U}$ is not a P-point. Our construction proceeds in $\mathfrak{c}$ stages such that, for each $n < \omega$, the sequences $\langle \mathcal{V}_\alpha \rangle_{\alpha < \mathfrak{c}}$ and $\langle \mathcal{W}_{n,\alpha} \rangle_{\alpha < \mathfrak{c}}$ are continuous increasing chains of filters such that $\mathcal{V} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{V}_\alpha$ and $\mathcal{W}_n = \bigcup_{\alpha < \mathfrak{c}} \mathcal{W}_{n,\alpha}$. Set $\mathcal{U}_\alpha = \{E \subseteq \omega^2 : \mathcal{V}_\alpha \ni \{i : \mathcal{W}_{i,\alpha} \ni \{j : \langle i, j \rangle \in E\}\}\}$ for all $\alpha < \mathfrak{c}$.

Let $\langle \Xi_\alpha \rangle_{\alpha \in E^{\mathfrak{c}}_{\omega}}$ be a $\diamondsuit$-sequence. Let $\zeta \colon \mathfrak{c} \leftrightarrow [\omega]^\omega$ and $\eta \colon \mathfrak{c} \leftrightarrow \left[\omega^2\right]^\omega$. Set $\mathcal{V}_0 = \mathcal{W}_{n,0} = \{\omega \setminus \sigma : \sigma \in [\omega]^{<\omega}\}$ for all $n < \omega$. Suppose $\alpha < \mathfrak{c}$ and we've constructed $\langle \mathcal{V}_\beta \rangle_{\beta < \alpha}$ and $\langle \mathcal{W}_{n,\beta} \rangle_{\langle n,\beta \rangle \in \omega \times \alpha}$ such that, for all $\beta < \alpha$ and $n < \omega$, $\mathcal{V}_\beta$ and $\mathcal{W}_{n,\beta}$ are filters on ω; if $\mathrm{cf}(\beta) \neq \omega$ and $\beta + 1 < \alpha$, then further suppose that $\mathcal{V}_\beta$ and $\mathcal{W}_{n,\beta}$ have pseudointersections in $\mathcal{V}_{\beta+1}$ and $\mathcal{W}_{n,\beta+1}$, respectively. If α is a limit ordinal, then set $\mathcal{V}_\alpha = \bigcup_{\beta < \alpha} \mathcal{V}_\beta$ and $\mathcal{W}_{n,\alpha} = \bigcup_{\beta < \alpha} \mathcal{W}_{n,\beta}$ for each $n < \omega$. If α is the successor of an ordinal with cofinality other than ω, then we use stage α as follows to help our filters become ultrafilters that are $P_{\mathfrak{c}}$-points. Choose the least $\beta < \mathfrak{c}$ such that $\zeta(\beta), \omega \setminus \zeta(\beta) \notin \mathcal{V}_{\alpha-1}$. Choose $E \in \{\zeta(\beta),\ \omega \setminus \zeta(\beta)\}$ such that $\{E\} \cup \mathcal{V}_{\alpha-1}$ has the SFIP and let $\mathcal{V}_\alpha$ be a filter generated by $\mathcal{V}_{\alpha-1}$ and a pseudointersection of $\{E\} \cup \mathcal{V}_{\alpha-1}$. Likewise, for each $n < \omega$, choose the least $\beta < \mathfrak{c}$ such that $\zeta(\beta), \omega \setminus \zeta(\beta) \notin \mathcal{W}_{n,\alpha-1}$. Choose $E \in \{\zeta(\beta),\ \omega \setminus \zeta(\beta)\}$ such that $\{E\} \cup \mathcal{W}_{n,\alpha-1}$ has the SFIP and let $\mathcal{W}_{n,\alpha}$ be a filter generated by $\mathcal{W}_{n,\alpha-1}$ and a pseudointersection of $\{E\} \cup \mathcal{W}_{n,\alpha-1}$.

Finally, suppose α is the successor of an ordinal with cofinality ω. Then we use stage α to kill a potential witness to $\langle \mathcal{U}, \supseteq \rangle \equiv_T [\mathfrak{c}]^{<\omega}$. Choose, if it exists, the least

$\beta < \mathfrak{c}$ for which $\eta(\beta)$ is contained in the intersection of an infinite subset of $\eta[\Xi_\alpha]$ and $\{\eta(\beta)\} \cup \mathcal{U}_{\alpha-1}$ has the SFIP. Let $\mathcal{V}_\alpha$ be the filter generated by $\{F\} \cup \mathcal{V}_{\alpha-1}$ where $F = \{i : \mathcal{W}_{i,\alpha-1} \not\ni \omega \setminus \{j : \langle i, j\rangle \in \eta(\beta)\}\}$; for each $i \in F$, let $\mathcal{W}_{i,\alpha}$ be the filter generated by $\{\{j : \langle i, j\rangle \in \eta(\beta)\}\} \cup \mathcal{W}_{i,\alpha-1}$; for each $i \in \omega \setminus F$, set $\mathcal{W}_{i,\alpha} = \mathcal{W}_{i,\alpha-1}$. Note that this implies $\eta(\beta) \in \mathcal{U}_\alpha$. If no such β exists, then set $\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1}$ and $\mathcal{W}_{n,\alpha} = \mathcal{W}_{n,\alpha-1}$ for all $n < \omega$. This completes the construction.

Clearly, $\mathfrak{c} \leq_T \langle \mathcal{V}, \supseteq^*\rangle \leq_T \langle \mathcal{U}, \supseteq^*\rangle$. Since $\mathcal{U}$ is not a P-point, $\mathfrak{c} \not\equiv_T \langle \mathcal{U}, \supseteq^*\rangle$. Therefore, it remains only to show that $\langle \mathcal{U}, \supseteq\rangle \not\equiv_T [\mathfrak{c}]^{<\omega}$. Suppose $\mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}}$. Then it suffices to show that the intersection of an infinite subset of $\mathcal{A}$ is in $\mathcal{U}$. By $\Diamond(E^{\mathfrak{c}}_\omega)$, there exists $M \prec H_{\mathfrak{c}^+}$ such that $|M| = \omega$ and $M \supseteq \{\mathcal{A}, \langle \mathcal{V}_\alpha\rangle_{\alpha<\mathfrak{c}}, \langle \mathcal{W}_{n,\alpha}\rangle_{\langle n,\alpha\rangle\in\omega\times\mathfrak{c}}\}$ and $\eta[\Xi_\delta] = \mathcal{A} \cap M$ where $\delta = \sup(\mathfrak{c} \cap M)$. Hence, it suffices to show that the intersection E of some infinite subset of $\mathcal{A} \cap M$ is such that $\{E\} \cup \mathcal{U}_\delta$ has the SFIP.

Let $\{V_n : n < \omega\} \subseteq M$ generate of the filter $\mathcal{V}_\delta$; for each $i < \omega$, let $\{W_{i,j} : j < \omega\} \subseteq M$ generate the filter $\mathcal{W}_{i,\delta}$. Set $\mathcal{B}_0 = \mathcal{A}$. Suppose $k < \omega$ and, for all $l < k$, we have $A_l \in \mathcal{B}_{l+1} \in [\mathcal{B}_l]^{\mathfrak{c}}$ and $n_l < \omega$ and $\mathcal{W}_{n_l} \ni \left\{j : \langle n_l, j\rangle \in B \cap \bigcap_{h<l} A_h\right\}$ for all $B \in \mathcal{B}_{l+1}$. Since $\mathrm{cf}(\mathfrak{c}) > \omega$, there exist $\mathcal{B}_{k+1} \in [\mathcal{B}_k]^{\mathfrak{c}}$ and $n_k \in \bigcap_{h<k}(V_h \setminus \{n_h\})$ and $\sigma_k \colon \{n_l : l < k\} \to \omega$ such that, for all $l < k$ and $B \in \mathcal{B}_{k+1}$, we have $\mathcal{W}_{n_k} \ni \left\{j : \langle n_k, j\rangle \in B \cap \bigcap_{h<k} A_h\right\}$ and $\sigma_k(n_l) \in \bigcap_{h<k} W_{n_l,h}$ and $\sigma_k \subseteq B \cap \bigcap_{h<k} A_h$. Choose any $A_k \in \mathcal{B}_{k+1} \setminus \{A_h : h < k\}$. By induction, we can repeat the above for all $k < \omega$. Moreover, we may carry out any finite initial segment of the construction in M. Hence, we may assume $\{A_i : i < \omega\} \subseteq M$. Finally, $\bigcup_{i<\omega} \sigma_i \subseteq \bigcap_{i<\omega} A_i$ and $\{\bigcup_{i<\omega} \sigma_i\} \cup \mathcal{U}_\delta$ has the SFIP. $\square$

Note that $\Diamond(E^{\mathfrak{c}}_\omega)$ is equivalent to $\Diamond$ under CH. Furthermore, a recent result of Shelah [63] is that if κ is an uncountable cardinal and $2^\kappa = \kappa^+$, then $\Diamond(S)$ holds for every stationary S disjoint from $E^{\kappa^+}_{\mathrm{cf}(\kappa)}$. Hence, we could drop the hypothesis $\Diamond(E^{\mathfrak{c}}_\omega)$ under the assumption that $\mathfrak{c} = \kappa^+$ for some cardinal κ of uncountable cofinality. (We would have $2^\kappa = \kappa^+$ because $\mathfrak{c}^{<\mathfrak{p}} = \mathfrak{c}$. (See Martin and Solovay [43].))

[In very recent unpublished work, Stevo Todorčević has deduced Theorem 6.3.7 from the mere existence of a P-point, which is known to follow from $\mathfrak{d} = \mathfrak{c}$, which is a strictly weaker hypothesis than $\mathfrak{p} = \mathfrak{c}$. Thus, the need for $\Diamond(E^{\mathfrak{c}}_\omega)$ has been eliminated altogether.]

Remark 6.3.8. When thinking about Tukey classes of ultrafilters, one may be reminded of Hechler's result [30] that any ω_1-directed set without a maximum can be forced to be isomorphic to a cofinal subset of ω^ω ordered by eventual domination. Similarly, Brendle and Shelah [12] have implicitly shown that, for a fixed regular uncountable κ and set R of regular cardinals exceeding κ, there is a model of ZFC in which, for each $\lambda \in R$, some $\mathcal{U} \in \omega^*$, when ordered by $\supseteq^*$, has a cofinal subset isomorphic to $\kappa \times \lambda$. It is not clear whether an arbitrary ω_1-directed set can be forced to be isomorphic to a cofinal subset of an ultrafilter ordered by $\supseteq^*$. In constructing non-P-points, which are not ω_1-directed when ordered by $\supseteq^*$, order-theoretic results seem to come even less easily.

It is worth noting another relationship between the Tukey classes arising from ultrafilters ordered by $\supseteq^*$ and those ordered by $\supseteq$.

Proposition 6.3.9. *Suppose $\mathcal{U}$ is a non-P-point in ω^*. Then there exists $\mathcal{V} \in \omega^*$ such that $\langle \mathcal{V}, \supseteq \rangle \leq_T \langle \mathcal{U}, \supseteq^* \rangle$.*

Proof. Choose $\langle x_n \rangle_{n<\omega} \in \mathcal{U}^\omega$ such that $x_n \supseteq x_{n+1} \not\supseteq^* x_n$ for all $n < \omega$, that $\bigcap_{n<\omega} x_n = \emptyset$, and that $\{x_n : n < \omega\}$ has no pseudointersection in $\mathcal{U}$. For each $n < \omega$, set $y_n = x_n \setminus x_{n+1}$. Set $\mathcal{V} = \left\{E \subseteq \omega : \bigcup_{n \in E} y_n \in \mathcal{U}\right\}$. Then $\mathcal{V} \in \omega^*$ and the map from $\langle \mathcal{V}, \supseteq \rangle$ to $\langle \mathcal{U}, \supseteq^* \rangle$ defined by $E \mapsto \bigcup_{n \in E} y_n$ is Tukey. $\square$

Next, we have a pair of negative ZFC results.

Theorem 6.3.10. *Let Q be a directed set that is a countable union of ω_1-directed sets. Then $\langle \mathcal{U}, \supseteq^* \rangle \not\equiv_T \omega \times Q$ for all $\mathcal{U} \in \omega^*$.*

Proof. Seeking a contradiction, suppose $\mathcal{U} \in \omega^*$ and $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \omega \times Q$. Then there is a quasiordering $\sqsubseteq$ on $\mathcal{U} \cup (\omega \times Q)$ such that $\langle \mathcal{U}, \supseteq^* \rangle$ and $\langle \omega \times Q, \leq_{\omega\times Q} \rangle$ are cofinal suborders. Let $Q = \bigcup_{n<\omega} Q_n$ where Q_n is ω_1-directed for all $n < \omega$. Fix $p \in Q$. Fix $\eta \in \omega^\omega$ such that $\eta^{-1}\{n\}$ is unbounded and $\eta(4n) = \eta(4n+1) = \eta(4n+2) = \eta(4n+3)$ for all $n < \omega$. For all $n < \omega$ and $q \in Q$, choose $x_{n,q} \in \mathcal{U}$ such that $\langle n, q \rangle \sqsubseteq x_{n,q}$. We may assume that $x_{i,p} \sqsubseteq x_{j,q}$ for all $i \leq j < \omega$ and $q \in Q$.

Construct $\zeta \in \omega^\omega$ as follows. Suppose we are given $n < \omega$ and $\zeta \upharpoonright n$. Then, for all $q \in Q$, the set $\{x_{\zeta(m),q} : m < n\}$ has a $\sqsubseteq$-upper bound $\langle k, r \rangle$ for some $k < \omega$ and $r \in Q$. Since $Q_{\eta(n)}$ is ω_1-directed, every countable partition of $Q_{\eta(n)}$ includes a cofinal subset. Hence, there exist $k < \omega$ and a cofinal subset S_n of $Q_{\eta(n)}$ such that for all $q \in S_n$ there exists $r \in Q$ such that $\{x_{\zeta(m),q} : m < n\} \sqsubseteq \langle k, r \rangle$. We may assume $k > \zeta(m)$ for all $m < n$. Set $\zeta(n) = k$.

Since ω^* is an F-space (or, more directly, by an easy diagonalization argument), there exists $z \subseteq \omega$ such that $x_{\zeta(4n),p} \setminus x_{\zeta(4n+2),p} \subseteq^* z$ and $x_{\zeta(4n+2),p} \setminus x_{\zeta(4n+4),p} \subseteq^* \omega \setminus z$ for all $n < \omega$. Suppose $z \in \mathcal{U}$. Then there exist $m < \omega$ and $\langle l, r \rangle \in \omega \times Q_m$ such that $\langle l, r \rangle \sqsupseteq z$. Choose $n < \omega$ such that $\eta(4n+3) = m$ and $\zeta(4n+2) \geq l$. Then choose $q \in S_{4n+3}$ such that $q \geq r$. Then $\langle \zeta(4n+2), q \rangle \sqsupseteq z$. Hence, $x_{\zeta(4n+2),q} \sqsupseteq z \cap x_{\zeta(4n+2),p} \sqsupseteq x_{\zeta(4n+4),p} \sqsupseteq \langle \zeta(4n+4), p \rangle$. Hence, $\langle \zeta(4n+4), p \rangle \sqsubseteq x_{\zeta(4n+2),q} \sqsubseteq \langle \zeta(4n+3), s \rangle$ for some $s \in Q$, which is absurd because ζ is strictly increasing. By symmetry, we can also derive an absurdity from $\omega \setminus z \in \mathcal{U}$. Thus, $\mathcal{U}$ is not an ultrafilter on ω, which yields our desired contradiction. □

The above result is optimal in the following sense. As noted before, it is not hard to show that, for a fixed regular uncountable κ and set R of regular cardinals exceeding κ, a construction of Brendle and Shelah [12] can be trivially modified to yield a model of ZFC in which, for each $\lambda \in R$, some $\mathcal{U} \in \omega^*$ satisfies $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \kappa \times \lambda$ for each λ in an arbitrary set of regular cardinals exceeding κ.

Lemma 6.3.11. *Given a quasiorder Q with an unbounded cofinal subset C, there exists a cofinal subset A of C such that A is $|C|$-like.*

Proof. Let $\langle c_\alpha \rangle_{\alpha < |C|} \colon |C| \leftrightarrow C$. For each $\alpha < |C|$, let $a_\alpha = c_\beta$ where β is the least $\gamma < |C|$ such that c_γ has no upper bound in $\{a_\delta : \delta < \alpha\}$, provided such a γ exists. If no such γ exists, then $\alpha > 0$, so we may set $a_\alpha = a_0$. Then $A = \{a_\alpha : \alpha < |C|\}$ is as desired. $\square$

Theorem 6.3.12. *Suppose Q is a directed set that is a countable union of ω_1-directed sets. Then $\langle \mathcal{U}, \supseteq \rangle \not\leq_T Q$ for all $\mathcal{U} \in \omega^*$.*

Proof. Seeking a contradiction, suppose $\mathcal{U} \in \omega^*$ and $f : \langle \mathcal{U}, \supseteq \rangle \leq_T Q$. By a result of Brendle and Shelah [12],

$$\mathrm{cf}(\mathrm{cf}(\langle \mathcal{U}, \supseteq \rangle)) = \mathrm{cf}(\mathrm{cf}(\langle \mathcal{U}, \supseteq^* \rangle)) > \omega.$$

By Lemma 6.3.11, $\mathcal{U}$ has a cofinal subset $\mathcal{A}$ that is $\mathrm{cf}(\langle \mathcal{U}, \supseteq \rangle)$-like. Since $\mathcal{A}$ is cofinal, $f \restriction \mathcal{A}$ is a Tukey map and $|\mathcal{A}| = \mathrm{cf}(\langle \mathcal{U}, \supseteq \rangle)$. Let $Q = \bigcup_{n<\omega} Q_n$ where Q_n is ω_1-directed for all $n < \omega$. Since $\mathrm{cf}(|\mathcal{A}|) > \omega$, there exist $n < \omega$ and $\mathcal{B} \in [\mathcal{A}]^{|\mathcal{A}|}$ such that $f[\mathcal{B}] \subseteq Q_n$. Since $\mathcal{A}$ is $|\mathcal{A}|$-like, $\mathcal{B}$ is unbounded. Set $I = \omega \setminus \bigcap \mathcal{B}$. For each $i \in I$, choose $B_i \in \mathcal{B}$ such that $i \notin B_i$. Then $\bigcap_{i \in I} B_i = \bigcap \mathcal{B}$; hence, $\{B_i : i \in I\}$ is unbounded. But $\{f(B_i) : i \in I\}$ is a countable subset of Q_n, and therefore bounded. This contradicts our assumption that f is Tukey. $\square$

From Corollary 5.4 of [66] by Solecki and Todorčević, it follows that $\langle \mathcal{U}, \supseteq \rangle \not\leq_T \omega^\omega$ (where ω^ω is ordered by domination) for all $\mathcal{U} \in \omega^*$.

Our next theorem, a positive consistency result, is proved using Solovay's Lemma [43], which we now state in terms of $\mathfrak{p}$.

Lemma 6.3.13. *If $\mathcal{A}, \mathcal{B} \in [[\omega]^\omega]^{<\mathfrak{p}}$ and $|a \cap \bigcap \sigma| = \omega$ for all $a \in \mathcal{A}$ and $\sigma \in [\mathcal{B}]^{<\omega}$, then $\mathcal{B}$ has a pseudointersection b such that $|a \cap b| = \omega$ for all $a \in \mathcal{A}$.*

Theorem 6.3.14. *Assume $\mathfrak{p} = \mathfrak{c}$. Let $\omega \leq \operatorname{cf}(\kappa) = \kappa \leq \mathfrak{c}$. Then there exists $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\kappa}, \subseteq \rangle$.*

Proof. Given a set E, let $I(E)$ denote the set of injections from κ to E. Given $\mathcal{E} \subseteq \mathcal{P}(\omega)$, let $\Phi(\mathcal{E})$ denote the set of $\langle \rho, \Gamma \rangle \in [\mathcal{E}]^{<\omega} \times I(\mathcal{E})^{<\omega}$ satisfying $\bigcap \rho \subseteq^* \bigcup_{f \in \operatorname{ran} \Gamma} f(\gamma)$ for all $\gamma < \kappa$. Let $\langle \mathscr{S}_\alpha \rangle_{\alpha < \mathfrak{c}}$ enumerate $[[\omega]^\omega]^{<\kappa}$. Note that if $|\mathcal{E}| \geq \kappa$, then $\Phi(\mathcal{E}) = \emptyset$ implies that $\mathcal{E}$ has the SFIP and that $\langle \mathcal{E}, \supseteq^* \rangle$ is κ-like.

Let us construct a sequence $\langle U_\alpha \rangle_{\alpha < \mathfrak{c}}$ in $[\omega]^\omega$ such that we have the following for all $\alpha \leq \mathfrak{c}$, given the notation $\mathcal{U}_\beta = \{U_\gamma : \gamma < \beta\}$ for all $\beta \leq \mathfrak{c}$.

1. $\forall \beta < \alpha \ \forall \sigma, \tau \in [\mathcal{U}_\beta]^{<\omega} \ \ \bigcap \sigma \subseteq^* \bigcup \tau$ or $\bigcap \sigma \setminus \bigcup \tau \not\subseteq^* U_\beta$.

2. $\forall \beta < \alpha \ \exists \sigma \in [\mathscr{S}_\beta]^{<\omega} \ \ U_\beta \cap \bigcap \sigma =^* \emptyset$ or $\forall S \in \mathscr{S}_\beta \ \ U_\beta \subseteq^* S$.

3. $\Phi(\mathcal{U}_\alpha) = \emptyset$.

Clearly, (1) and (2) will be preserved at limit stages of the construction. Let us show that (3) will also be preserved. Let $\omega \leq \operatorname{cf}(\eta) \leq \eta \leq \mathfrak{c}$ and suppose (1) and (3) hold for all $\alpha < \eta$. Seeking a contradiction, suppose $\langle \rho, \Gamma \rangle \in \Phi(\mathcal{U}_\eta)$; we may assume $\langle \rho, \Gamma \rangle$ is chosen so as to minimize $\operatorname{dom} \Gamma$. By (1), $\langle U_\alpha \rangle_{\alpha < \eta}$ is injective; let ψ be its inverse. Since $\Phi(\mathcal{U}_{\sup(\psi[\rho])}) = \emptyset$, we have $\Gamma \neq \emptyset$. By the pigeonhole principle, there exist $A \in [\kappa]^\kappa$

and $i \in \operatorname{dom}\Gamma$ such that for all $\gamma \in A$ we have $\psi(\Gamma(i)(\gamma)) = \max_{j\in\operatorname{dom}\Gamma}\psi(\Gamma(j)(\gamma))$. By symmetry, we may assume $i = \max(\operatorname{dom}\Gamma)$. Since $\Phi(\mathcal{U}_{\sup(\psi[\rho])}) = \emptyset$, we have $|A \cap \Gamma(i)^{-1}\sup(\psi[\rho])| < \kappa$; hence, we may assume $A \cap \Gamma(i)^{-1}\sup(\psi[\rho]) = \emptyset$. By the definition of $\Phi(\mathcal{U}_\eta)$, we have $\bigcap\rho \setminus \bigcup_{j<i}\Gamma(j)(\gamma) \subseteq^* \Gamma(i)(\gamma)$ for all $\gamma \in A$. Hence, by (1), we have $\bigcap\rho \subseteq^* \bigcup_{j<i}\Gamma(j)(\gamma)$ for all $\gamma \in A$. Choose $h \in I(A)$. Then $\langle\rho, \langle\Gamma(j)\circ h\rangle_{j<i}\rangle \in \Phi(\mathcal{U}_\eta)$, in contradiction with the minimality of $\operatorname{dom}\Gamma$. Thus, (3) will be preserved at limit stages.

Given $\alpha < \mathfrak{c}$ and $\langle U_\beta\rangle_{\beta<\alpha}$ satisfying (1)-(3), let us show that there always exists $U_\alpha \in [\omega]^\omega$ such that $\langle U_\beta\rangle_{\beta\leq\alpha}$ also satifies (1)-(3). Let $g \in 2^\omega$ be sufficiently Cohen generic. There are two cases to consider. First, suppose that there exists $\sigma \in [\mathscr{S}_\alpha]^{<\omega}$ such that $\Phi(\mathcal{U}_\alpha \cup \sigma) \neq \emptyset$. Then there exists $\langle\rho_2, \Gamma_2\rangle \in \Phi(\mathcal{U}_\alpha \cup \{x_2\})$ where $x_2 = \bigcap\sigma$. For each $i < 2$, set $x_i = g^{-1}\{i\} \setminus x_2$. Seeking a contradiction, suppose there exists $\langle\rho_i, \Gamma_i\rangle \in \Phi(\mathcal{U}_\alpha \cup \{x_i\})$ for each $i < 2$. We may assume $\bigcup_{i<3}\bigcup\operatorname{ran}\Gamma_i \subseteq \mathcal{U}_\alpha$. Let Λ be a concatenation of $\{\Gamma_i : i < 3\}$ and set $\tau = \mathcal{U}_\alpha \cap \bigcup_{i<3}\rho_i$. Then, for all $\gamma < \kappa$, we have

$$\bigcap\tau = \bigcup_{i<3}\left(x_i \cap \bigcap\tau\right) \subseteq \bigcup_{i<3}\bigcap\rho_i \subseteq^* \bigcup_{f\in\operatorname{ran}\Lambda} f(\gamma).$$

Hence, $\langle\tau, \Lambda\rangle \in \Phi(\mathcal{U}_\alpha)$, in contradiction with (3). Therefore, we may choose $i < 2$ such that $\Phi(\mathcal{U}_\alpha \cup \{x_i\}) = \emptyset$. Set $U_\alpha = x_i$, which is disjoint from $\bigcap\sigma$. Then (2) and (3) are clearly satisfied for stage $\alpha + 1$, and (1) is also satisfied because of Cohen genericity.

Now suppose that $\Phi(\mathcal{U}_\alpha \cup \sigma) = \emptyset$ for all $\sigma \in [\mathscr{S}_\alpha]^{<\omega}$. For each $\rho \in [\mathcal{U}_\alpha]^{<\omega}$, $\sigma \in [\mathscr{S}_\alpha]^{<\omega}$, and $\Gamma \in I(\mathcal{U}_\alpha)^{<\omega}$, choose $\gamma_{\rho,\sigma,\Gamma} < \kappa$ such that $\bigcap(\rho\cup\sigma) \not\subseteq^* \bigcup_{i\in\operatorname{ran}\Gamma} f(\delta)$ for all $\delta \in \kappa\setminus\gamma_{\rho,\sigma,\Gamma}$. Set $\gamma_{\rho,\Gamma} = \sup\{\gamma_{\rho,\sigma,\Gamma} : \sigma \in [\mathscr{S}_\alpha]^{<\omega}\}$; set $x_{\rho,\Gamma} = \bigcap\rho \setminus \bigcup_{f\in\operatorname{ran}\Gamma} f(\gamma_{\rho,\Gamma})$. Then $x_{\rho,\Gamma} \cap \bigcap\sigma$ is infinite for all $\sigma \in [\mathscr{S}_\alpha]^{<\omega}$. By Solovay's Lemma, $\mathscr{S}_\alpha$ has a pseudointersection y such that $y \cap x_{\rho,\Gamma}$ is infinite for all $\rho \in [\mathcal{U}_\alpha]^{<\omega}$ and $\Gamma \in I(\mathcal{U}_\alpha)^{<\omega}$, for there are at most $|\mathcal{U}_\alpha|^{<\omega}$-many possible $x_{\rho,\Gamma}$. Set $U_\alpha = y \cap g^{-1}\{0\}$. Then (2) is clearly satisfied for stage

$\alpha+1$. Since $y \cap x_{\rho,\Gamma} \cap \bigcap \sigma$ is infinite, Cohen genericity implies $U_\alpha \cap x_{\rho,\Gamma}$ is infinite, for all ρ, σ, and Γ. Hence, (3) is satisfied for stage $\alpha+1$; (1) is also satisfied because of Cohen genericity. This completes our construction of $\langle U_\alpha \rangle_{\alpha<\mathfrak{c}}$.

Let $\mathcal{U}$ be the semifilter generated by $\mathcal{U}_\mathfrak{c}$. By (3), $\mathcal{U}_\mathfrak{c}$ has the SFIP and $\mathcal{U}_\mathfrak{c}$ is κ-like with respect to $\supseteq^*$. Hence, by (2), $\mathcal{U}$ is a P_κ-point in ω^*. Therefore, $f\colon \langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle [\mathfrak{c}]^{<\kappa}, \subseteq \rangle$ for any injection f of $\mathcal{U}$ into $[\mathfrak{c}]^1$. Choose $\zeta\colon [\mathfrak{c}]^{<\kappa} \to \mathcal{U}$ such that $\zeta(\sigma)$ is a pseudointersection of $\{U_\alpha : \alpha \in \sigma\}$ for all $\sigma \in [\mathfrak{c}]^{<\kappa}$. Then ζ is Tukey because $\mathcal{U}_\mathfrak{c}$ is κ-like. Thus, $\mathcal{U} \leq_T [\mathfrak{c}]^{<\kappa} \leq_T \mathcal{U}$. □

6.4 Questions

Question 6.4.1. Is it consistent that every $\mathcal{U} \in \omega^*$ satisfies $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega} \rangle$? By Proposition 6.3.9, this is equivalent to asking if it is consistent that every $\mathcal{U} \in \omega^*$ satisfies $\langle \mathcal{U}, \supseteq \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega} \rangle$.

Question 6.4.2. Does $\diamondsuit$ imply there are at least four Tukey classes represented by $\langle \mathcal{U}, \supseteq^* \rangle$ for some $\mathcal{U} \in \omega^*$? Infinitely many Tukey classes? As many as 2^{ω_1}? (It will be shown in a forthcoming paper by Dobrinen and Todorčević that CH implies there are 2^{ω_1}-many Tukey classes represented by $\langle \mathcal{U}, \supseteq \rangle$.)

Question 6.4.3. Is it consistent that there exists $\mathcal{U} \in \omega^*$ such that $\langle \mathcal{U}, \supseteq^* \rangle \equiv_T \omega_1 \times \mathfrak{c}$ and $\omega_1 < \mathfrak{p}$?

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