

# TRUTH AND APPROXIMATE TRUTH IN METRIC SPACES

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# Abstract

We generalize the notion of approximate truth ( $\models_{AP}$ ) studied by Henson ([15]) and Fajardo and Keisler ([7]) to a language  $L_A$  that allows countable conjunctions, negation, and bounded existential quantification over infinitely many variables. We study approximation principles relating  $\models_{AP}$  and  $\models$  in metric structures. Let  $L_{PBA}$  be the smallest subset of  $L_A$  containing the atomic formulas and closed under countable and finite conjunction, finite disjunction, and existential and universal bounded quantification over countably many variables. Let  $L_P$  be the smallest subset of  $L_A$  containing the atomic formulas and closed under countable and finite conjunction, finite disjunction and universal bounded quantification. Let  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$  be the smallest subset of  $L_A$  containing the formulas  $\sigma(\vec{x}, \vec{y})$  and  $\exists \vec{x} \in \vec{K} \neg \sigma(\vec{x}, \vec{y})$  for  $\sigma \in L_P$  and closed under countable conjunction and negation. Finally, let  $L_{A+}$  be the smallest subset of  $L_A$  containing  $L_{PBA}$ ,  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$  and closed under countable conjunction, countable disjunction and existential bounded quantification over countable many variables. We prove the following weak approximation principle: For every metric structure  $E$ , for every sentence  $\phi$  that is in  $L_{A+}$ , if  $E \models \phi$  then  $E \models_{AP} \phi$ . We prove: For any structure  $E$ , for every sentence in  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$ ,  $E \models \phi$  iff  $E \models_{AP} \phi$ . We introduce the notion of complete and uniform classes of models of  $L_A$ . For those classes and the notion of  $\models_{AP}$  we prove a Model Existence Theorem. This result generalizes the usual Compactness Theorem for first order logic, and the Compactness Theorem for  $\models_{AP}$  obtained by Henson ([19]) for finitary positive

bounded formulas. We prove the following equivalence result: for every complete class of models  $\mathcal{M}$  axiomatized by a formula in  $L_{A+}$ , for every universal sentence  $\phi \in L_A$ ,  $(\forall E \in \mathcal{M}, E \models_{AP} \phi)$  iff  $(\forall E \in \mathcal{M}, E \models \phi)$ . We give examples where existential sentences  $\phi$  are omitted in complete classes of models in the following sense:  $\forall E \in \mathcal{M}, E \models_{AP} \phi$  but  $\exists E \in \mathcal{M}, E \models \neg\phi$ . We prove an omitting theorem for  $\exists$  formulas in  $L_A$ . Finally, we show two applications of those results. The first a uniform version of the celebrated Krivine's representation theorem in functional analysis. For the second application we give a generalization of the classical Omitting Types Theorem to infinitary formulas.

# Notation and Symbols

The notation  $A \subseteq B$  means that  $A$  is a subset of  $B$ .  $\mathfrak{R}$  denotes the real numbers and  $Q$  the rationals. The image of a map  $F$  is written as  $Im(F)$ . Likewise the domain of  $F$  is  $Dom(F)$ . In general we reserve the symbols  $\rho, d$  for metric functions and  $\|\cdot\|$  for norms.

The symbols  $E, B, \dots$  refer to models. Furthermore  $\epsilon, \delta$  refer to positive real numbers. We reserve the symbol  $\equiv$  for logical formulas. Given to formulas  $\phi$  and  $\psi$  in a logic  $L$ ,  $\phi \equiv \psi$  states that  $\phi$  and  $\psi$  are identical formulas.

We fix a collection of variables  $Var = \{x_i : i \in \omega_1\}$ . Given a logic  $L$ , and a formula  $\sigma \in L$ , the notation  $\sigma(\vec{x})$  means that the free variables of  $\sigma$  are among the variables in the vector  $\vec{x}$ . Occasionally, for a vector of function symbols  $\vec{f}$ , we will write  $\sigma(\vec{x})[\vec{f}]$ . This means that the function symbols appearing in  $\sigma$  are among the function symbols in  $\vec{f}$ . For any collection of formulas  $\Sigma$  in  $L$  and any connective  $\sharp$ , we denote by  $\sharp\Sigma$  the set of formulas  $\{\sharp\sigma : \sigma \in \Sigma\}$ . Likewise, for any quantifier  $\Delta$ , we denote by  $\Delta\Sigma$  the set of formulas  $\{\Delta\vec{x}\sigma(\vec{x}, \vec{y}) : \sigma(\vec{x}, \vec{y}) \in \Sigma\}$ . Finally, given a set of connectives and quantifiers  $\Omega$ , we denote by  $\Sigma^\Omega$  the smallest set of formulas in  $L$  that contains  $\Sigma$  and is closed under the connectives in  $\Omega$ .

Given any countable set  $F$  and a fixed enumeration of  $F$ ,  $F \uparrow n$  denotes the set made of the first  $n$  elements of  $F$ . We write  $\bigwedge_{t \in F \uparrow n}$  for a finite conjunction indexed by the first  $n$  elements of  $F$ . Likewise,  $\bigwedge_{t \in F}$  denotes a conjunction indexed by all the elements of  $F$ .

The symbol  $\square$  is used to mark the end of a definition, example or statement.

Likewise, ■ is used to mark the end of a proof.

Given a vector  $\vec{x} = (x_1, x_2, \dots, x_n)$ ,  $|\vec{x}|$  stands for the arity of the tuple. If the vector  $\vec{x} = (x_1, \dots, x_n, \dots)$  has countable arity then  $|\vec{x}| = \omega$ .

There are some fixed metric spaces that appear so often in this work that they deserve a fixed name. We denote by  $(\mathfrak{R}, d)$  the metric space of the reals with the usual distance based on the absolute value. Likewise,  $(\mathfrak{R}^2, max)$  denotes the real plane with the metric of the max.

For every  $p \in [1, \infty)$   $\ell_p$  is the Banach space of all the functions  $f : \omega \mapsto \mathfrak{R}$  such that  $\sum_{i=1}^{\infty} (f(i))^p$  is finite. The norm in this case is given by:

$$\|f\| = \sqrt[p]{\sum_{i=1}^{\infty} (f(i))^p}$$

By  $\ell_{\infty}$  we denote the Banach space of all functions  $f : \omega \mapsto \mathfrak{R}$  that are bounded in the sense:

$$\sup\{f(i) | i \in \omega\} < \infty$$

The norm is given by:

$$\|f\| = \sup\{f(i) | i \in \omega\}$$

The subspace of  $\ell_{\infty}$  generated by the sequences that converge to 0 is denoted by  $c_0$ .

Given two Banach spaces  $(E, \|\cdot\|), (F, |\cdot|)$ , we say that  $E, F$  are  $K$ -isomorphic iff there exists a linear onto map  $T : E \mapsto F$  such that for every  $x$  in  $E$ :

$$\frac{1}{K}|T(x)| \leq \|x\| \leq K|T(x)|$$

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# Chapter 1

## Introduction

The main purpose of this thesis is to develop model theoretic tools to study approximation principles for infinitary formulas in metric spaces.

One of the most successful and widely applied techniques in analysis is the notion of proof by approximation. To prove that a statement  $P$  is true in a metric structure  $E$  it is often showed that the approximations of  $P$  hold, and then special properties of the structure (i.e., compactness) are cited to conclude that the statement is true.

The natural semantic framework to study approximation proofs in metric structures is given by the notion of approximate truth of a formula in a metric space. In 1976 Henson in [15] introduced a logic of positive bounded formulas in Banach spaces to study the relationship between a Banach space  $E$  and its nonstandard hulls  $H(E)$ . This logic  $L_{PB}$  is based on a first order language  $L$  containing a binary function symbol  $+$ , unary predicate symbols  $P$  and  $Q$  to be interpreted as the closed unit ball and the closure of its complement and, for every rational number  $r$ , a unary function symbol  $f_r$  to be interpreted as the operation of scalar multiplication by  $r$ .  $L_{PB}$  is closed under finite conjunction, disjunction and bounded quantification of the form  $(\exists x)(P(x) \wedge \dots)$  or  $(\forall x)(P(x) \Rightarrow \dots)$ .

For any formula  $\phi$  in  $L_{PB}$  and for every natural number  $n$  it is possible to define



in a purely syntactical way a formula  $\phi_n$  in  $L_{PB}$ , called the *n-approximation* of  $\phi$ . Intuitively,  $\phi_n$  is the formula that results from metrically weakening the predicates that appear in  $\phi$  in such a way that as  $n$  tends to  $\infty$ ,  $\phi_n$  approaches  $\phi$ . From this notion of approximations of formulas follows the definition of approximate truth: A formula  $\phi \in L_{PB}$  is approximately true in a Banach space  $E$  (denoted by  $E \models_{AP} \phi$ ) iff for every integer  $n$ ,  $E \models \phi_n$ .

It is interesting to note that  $\models_{AP}$  is a concept that appears naturally in analysis. For example, “The normed space  $E$  is isometrically embedded in the normed space  $F$ ” is approximately true in  $F$  iff  $E$  is finitely represented in  $F$ . Likewise “the continuous map  $T : E \rightarrow E$  has a fixed point” is approximately true in  $E$  iff  $T$  has an almost fixed point ([1]).

The above definition of  $\models_{AP}$  is the starting point of the model theory of Banach spaces that has been developed in a series of papers by Henson, Heinrich and Iovino ([16, 17, 18, 14, 19]). The logic  $L_{PB}$  has been extended to have a language of  $n$ -ary function symbols (to be interpreted as uniformly continuous functions from  $E^n$  to  $E$ ) and  $n$ -ary real valued relation symbols. The notions of approximate formula and approximate truth are generalized naturally, and nice model theoretic theorems (compactness, Lowenheim-Skolem,...) are obtained.

Of particular interest to us is the interplay between the notions of approximate truth and truth. It is easy to see that the following weak approximation principle holds in any model  $E$  and for any formula  $\phi \in L_{PB}$ :

$$\text{If } E \models \phi \text{ then } E \models_{AP} \phi.$$

On the other hand, Henson proved the following approximation principle (Henson's Compactness Theorem ([19]) for particular collections of models (we will call those collections classes of models):

**Model Existence Principle.** *For any class of models in  $L_{PB}$ , for any  $\phi \in L_{PB}$ , if every approximation of  $\phi$  holds in some model of the class, then there is a model in the class where all the approximations of  $\phi$  and  $\phi$  itself hold.  $\square$*

Furthermore, Henson in [15] obtained the following strong approximation result for nonstandard hulls:

**Henson's Strong Approximation Principle.** *For any nonstandard hull  $E$  of a Banach space, and for any formula  $\phi$  in  $L_{PB}$ ,  $E \models_{AP} \phi \Leftrightarrow E \models \phi$   $\square$*

In 1986 Anderson in [2] introduced a logic  $L'$  for metric spaces that is very similar to the one previously defined by Henson. This logic is also closed under finite conjunction, finite disjunction and bounded quantification. The main difference between these two logics is that the sets bounding the quantifiers (we will call them the *bounding sets*) are interpreted in the metric space as being compact sets. Anderson also introduced a notion of approximate formulas for this logic and proved the following strong approximation principle:

**Anderson's Strong Approximation Principle.** *For any metric structure  $E$  and any sentence  $\phi$  in  $L'$  the following is true:*

$$E \models \phi \text{ iff } E \models_{AP} \phi$$

□

Let us say that a structure  $E$  is rich for a language  $L$  if for every sentence in  $L$ ,  $\models$  and  $\models_{AP}$  coincide. In a series of papers, Keisler in [25] and later Fajardo and Keisler in [7, 8] extended Anderson's Approximation Principle to very general families of metric spaces and bounding sets (the neometric families; see [8]). They introduced, for such family of spaces, a logic  $L$  (closed under countable conjunctions, finite disjunctions and bounded quantification) and a notion of approximate truth ( $\models_{AP}$ ) for a formula that extends the previous definition. They obtained the following approximation principle for neometric families and for formulas in this logic:

**Keisler & Fajardo's Strong Approximation Principle.** *For any sentence  $\phi$  and any neometric space  $\mathcal{N}$ ,  $\phi$  is approximately true in  $\mathcal{N}$  if and only if  $\phi$  is true in  $\mathcal{N}$ . In other words the neometric spaces are rich for  $L$ .□*

Since the models considered by Anderson and the nonstandard hulls are neometric structures, this approximation principle generalizes the previous ones obtained by Henson and Anderson.

In [25], [4] and in [7] this approximation principle is used to prove new existence results in Stochastic Differential Equations. The strategy is as follows: set the problem in a metric space that belongs to a neometric family, show that the statement "There exists a solution of the equation" is approximately true and use the fact that the neometric structures are rich to conclude that the statement is

true. The main interest of the above strategy is that  $\models_{AP}$  is weaker than  $\models$ . It follows that in rich spaces proofs are “easy”.

It is clear from the above account that the study of approximation principles in analysis has yielded useful results. First,  $\models_{AP}$  is the “natural” semantic notion to study metric structures from the model theoretic point of view (since most classical model theoretic results hold for  $\models_{AP}$  but not for  $\models$ ). Secondly,  $\models_{AP}$  is a concept that appears naturally in analysis. Finally, with  $\models_{AP}$  one can construct rich spaces, spaces where proofs are easy.

However, a main limitation on the applicability of the approximation principles studied above is that the languages studied lack expressive power. Most of the relevant properties in analysis require infinitary logical operations defined using the negation symbol. For example, reflexivity in Banach spaces requires countable disjunctions to be defined. Likewise, compactness and continuity also require countable disjunctions in their definitions. The principal obstacle to the extension of the above approximation principles to logics with negation is the difficulty in obtaining a notion of  $\models_{AP}$  that extends the previous ones to infinitary formulas with negation while preserving its most important properties.

Furthermore, the previous work on the interplay between  $\models_{AP}$  and  $\models$  has overlooked the study of any such relationship at the level of classes of models. For example, in functional analysis many interesting questions ask about the existence of **pathological** metric spaces: metric spaces where the approximation principles mentioned above do not hold. In particular, those problems have the following pattern:

(\*) if every structure in a fixed collection of models approximately

satisfies a sentence  $\phi$ , does it follow that every model of the collection also satisfies  $\phi$ ?

Examples of such relevant questions are the following:

- **The Subspace Problem.** This famous question of Banach asked if it is true that every infinite separable Banach space contains an infinite dimensional subspace isomorphic to one of the  $\ell_p$  ( $p \in [1, \infty)$ ) or  $c_0$ . It can be seen that in every infinite separable Banach space the finite dimensional approximations of the above statement are true. This question was finally solved by Tsirelson ([43]) in 1974 in the negative: There exists a infinite dimensional separable Banach space whose subspaces are not isomorphic to one of the  $\ell_p$  ( $p \in [1, \infty)$ ) nor to  $c_0$ .
- **The Fixed Point Property for Super-Reflexive Spaces.** This problem asks if for every super-reflexive Banach space  $E$ , for every bounded, closed and convex subset  $K$  of  $E$  and for every nonexpansive mapping  $T : K \rightarrow K$  (i.e.,  $\forall x, y \in K, \|T(x) - T(y)\| \leq \|x - y\|$ ) there exists  $x$  in  $K$  such that  $T(x) = x$ . It can easily be seen that in every Banach space, every map  $T$  that verifies the previous conditions has an “almost fixed point” (see [11]).
- **The Distortion Problem.** Given any infinite dimensional Banach space  $(E, \|\cdot\|)$  and any  $\lambda > 1$  we say that  $E$  is  $\lambda$ -distortable if there exists an equivalent norm  $|\cdot|$  in  $X$  such that for every infinite dimensional vector subspace  $Y$  of  $E$ ,  $\sup\{|y|/|z| : y, z \in Y, \|y\| = \|z\| = 1\} \geq \lambda$ .  $E$  is arbitrarily distortable if it is  $\lambda$ -distortable for every  $\lambda > 1$ .  $E$  is distortable if it is  $\lambda$ -distortable for some  $\lambda$ . It is known that in every infinite dimensional Banach space the

finite dimensional approximations of the above property are not true ([32]). The distortion problem for  $\ell_p$  ( if  $\ell_p$  is distortable) was recently solved in the affirmative by Odell and Schlumprecht ([36]). An open question concerning the notion of distortion is whether every uniformly convex space is arbitrarily distortable.

As the previous examples show, the study of approximation principles at the level of classes of models is important for its applications in analysis. In particular, note that every equivalence between  $\models_{AP}$  and  $\models$  at the level of classes of models for infinitary formulas could be seen as a bridge between the validity of infinite (dimensional) statements and the validity of finite (dimensional) ones. Any such relationship is of independent interest in fields like geometry of Banach Spaces ([3]). On the other hand, if  $\models_{AP}$  and  $\models$  are not equivalent for classes of models, this implies the existence of **non-rich** models in the class.

Our aim is to start a systematic study of the approximation principles in metric spaces for a very expressive logic that includes negation, countable conjunction and bounded quantification over countable many variables. We want to unify and generalize all the previous approaches to a fully infinitary logic. We isolate four main approximation principles to be studied:

- **Weak Approximation.** For every property  $P$  and every model  $E$  , if  $E \models P$  then  $E \models_{AP} P$ .
- **Rich Models:** Existence of structures  $E$  where approximate truth is equivalent to truth for every formula in the language.
- **Model Existence:** If every approximation of a property  $P$  holds in a model,

then there is a model where the formula is approximately true and is also true.

- **Class Approximation:** Every model of a collection of model  $\mathcal{M}$  approximately satisfies a property  $P$  iff every model of the class satisfies the formula.

In Chapter 2 we introduce an infinitary logic  $L_A$  for metric spaces closed under countable conjunction, negation and bounded existential quantification over countable many variables. This logic contains all the previously mentioned logics ( $L_{PB}$ ,  $L'$ , etc.) as fragments. Furthermore, any model of  $L_{PB}$  and any neometric space can be seen as a model of  $L_A$ . Likewise the classical first order structures can be seen as structures of  $L_A$  with the discrete metric.

We also define a notion of approximate truth ( $\models_{AP}$ ) for formulas in  $L_A$  based on approximations by finitary formulas. Recall that the natural way to obtain the analytic sets from the closed sets of a Polish space is using the ‘‘Souslin operation’’  $\mathcal{S}$  on countable collections of closed sets (see [24]). Roughly speaking, for every collection  $\mathcal{A} = \{A_{h,p} | h \in I \wedge p \in J\}$  of closed sets indexed by  $I \times J$ ,

$$\mathcal{S}(A) = \bigcup_{h \in I} \bigcap_{p \in J} A_{h,p}$$

In an analogous way, we obtain the approximation of a formula  $\phi \in L_A$  by applying a Souslin operation  $\mathcal{S}$  to a recursively defined collection of finitary formulas  $A = \{\phi_{h,n} | h \in I(\phi) \wedge n \in \omega\}$  so that the approximation of  $\phi$  is  $\mathcal{S}(A) = \bigvee_{h \in I(\phi)} \bigwedge_{n \in \omega} \phi_{h,n}$ . In other words, we associate to every formula in  $L_A$  a tree of finitary formulas. A formula will be approximately true in a structure iff for one of the branches of the tree, all the approximations along this branch hold in the structure. We call every  $h \in I(\phi)$  an **approximation path** of  $\phi$  or a branch of

the approximation tree. When the formula belongs to  $L_{PB}$ , we show that the approximation tree reduces to a single branch, and our definition of  $\models_{AP}$  coincides with the previous one by Henson.

In Chapter 3 we study some elementary properties of  $\models_{AP}$  in  $L_A$  and use these to obtain the basic approximation principles for formulas in  $L_A$ .

Let us introduce first the three main collections of formulas that we are going to deal with in the rest of this work. Let  $L_{PBA}$  be the smallest subcollection of  $L_A$  containing the atomic formulas and closed under infinite conjunction, finite disjunction and existential and universal bounded quantification over countably many variables. We call  $L_{PBA}$  the positive bounded formulas of  $L_A$ . Let  $L_P$  be the smallest subset of  $L_A$  containing the atomic formulas, and closed under countable conjunction, finite disjunction, universal quantification over countable many variables. Clearly  $L_P \subset L_{PBA}$ . Let  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  be the smallest subset of  $L_A$  containing:

- the formulas in  $L_P$ .
- The formulas of the form  $\exists \vec{x} \in \vec{K}(\neg \phi(\vec{x}, \vec{y}))$  for  $\phi(\vec{x}, \vec{y}) \in L_P$ .

and closed under countable conjunction and negation.

Let  $L_{A+} = ((L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}} \cup L_{PBA})^{\{\wedge, \vee, \exists\}}$  be the smallest subcollection of  $L_A$  containing the  $L_{PBA}$ , the  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  and closed under countable conjunction, countable disjunction and infinite existential quantification. Examples of formulas in  $L_{A+}$  are:

- The quantifier free formulas of  $L_A$ .



- Formulas of the form

$$\exists \vec{x} \in \vec{K} \bigwedge_{i=1}^{\infty} \forall \vec{y}_i \in \vec{D}_i \phi_i(\vec{x}, \vec{y}_i)$$

with the  $\phi_i$  in  $L_P$ .

We obtain the following weak approximation principle (Theorem 3.3.10):

**Weak Approximation principle for  $L_{A+}$**

*Fix a model  $E$  of  $L_A$ . Let  $\phi(\vec{x}) \in L_{A+}$ . Then for every  $\vec{a}$  in  $E$  with*

*$|\vec{a}| = |\vec{x}|$ , if  $E \models \phi(\vec{a})$  then  $E \models_{AP} \phi(\vec{a})$ .  $\square$*

We remark that the above theorem does not hold for arbitrary formulas in  $L_A$  (see Chapter 3).

We then look for stronger approximation principles holding for subsets of  $L_A$ . We obtain a strong approximation principle for the class  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ . Theorem 3.3.8 states that:

**Strong Approximation Principle for  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$**

*Let  $\phi(\vec{x}) \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ . Then for every model  $E$ , for every*

*vector  $\vec{b}$  in  $E$ ,  $E \models_{AP} \phi(\vec{b})$  iff  $E \models \phi(\vec{b})$ .  $\square$*

We conclude Chapter 3 by studying approximation principles in classical first order structures, i.e., structures where the metric is trivial (can only take the values of zero or 1). it is easily seen that any classical first order structure can be interpreted as a metric structure with the trivial metric. We will call first order formulas ( $L_{\omega\omega}$ ) all the formulas belonging to the smallest subset of  $L_A$  containing

the atomic formulas and closed under finite conjunction, negation and existential quantification over finitely many variables. Let then  $(L_{\omega,\omega})^{\{\wedge,\neg\}}$  be the smallest subset of  $L_A$  that contains all the first order formulas  $L_{\omega,\omega}$ , and is closed under countable conjunction and negation. Theorem 3.4.3 states the following:

### Strong Approximation Principle

*Let  $\phi(\vec{x}) \in (L_{\omega,\omega})^{\{\wedge,\neg\}}$ . Then for any classical first order model  $E$ , and for every  $\vec{a}$  in  $E$ ,  $E \models \phi(\vec{a})$  if and only if  $E \models_{AP} \phi(\vec{a})$ .  $\square$*

In Chapter 4 we study the third type of approximation principle. We begin by looking at a particular type of collections of models: The complete collections of models. Complete collections of models are collections of all models of  $L_A$  that share the same uniform continuity requirements for the function symbols of the language, and the same “uniform” bounds for the predicate symbols of the language. Given a complete collection of models  $\mathcal{W}$  a **complete class** of models in  $\mathcal{W}$  is defined as the subcollection of  $\mathcal{W}$  of all the models that satisfy a fixed sentence  $\phi$  in  $L_A$ . Let us remark that most of the usual collections of models appearing in analysis and first order logic are complete classes of models. For example the collection of normed spaces, the collection of Banach spaces, the collection of Banach spaces that are reflexive, the collection of Hilbert spaces, and any collection of first order models axiomatized by countable first order sentences are complete classes of models. In Chapter 4 we use complete classes to prove a Model Existence Theorem (Theorem 4.2.7) along the following lines:

### Model Existence Theorem

*Let  $\mathcal{M}$  a complete class of models axiomatized by a sentence  $\psi$ . Let  $\phi$  a sentence in  $L_A$ . If there exists a path  $h \in I(\phi \wedge \psi)$  such that every approximation of the formula  $\psi \wedge \phi$  along the path  $h$  has a model in  $\mathcal{M}$ , then there exists a model  $E$  in  $\mathcal{M}$  such that:  $E \models_{AP} \phi$ . Furthermore,  $\models_{AP}$  and  $\models$  coincide in  $E$ .  $\square$*

This theorem extends the classical Compactness Theorem for first order logic to infinitary formulas, and the Compactness Theorem obtained by Henson for  $L_{PB}$  ([19]). As a direct byproduct of the above theorem we obtain a result concerning the second approximation principle mentioned above:

### Rich Models for $L_A$

*Let  $\mathcal{W}$  a complete class of models axiomatized by a sentence in  $L_{A+}$ . There exists a rich structure for  $L_A$  in  $\mathcal{M}$ , that is, a structure  $E \in \mathcal{M}$  such that  $E \models \phi$  iff  $E \models_{AP} \phi$  for every  $\phi \in L_A$ .*

The second part of Chapter 4 and Chapter 5 are devoted to the fourth type of approximation principle. We begin by a weak approximation principle in classes of models. Theorem 4.4.1 states:

### Weak Approximation Principle for Complete Classes

*Let  $\mathcal{M}$  a complete class of models axiomatized by a sentence in  $L_{A+}$ . Fix  $\phi$  an arbitrary sentence in  $L_A$ . If  $\forall E \in \mathcal{M} E \models \phi$ , then  $\forall E \in \mathcal{M} E \models_{AP} \phi$ .  $\square$*

Using the Model Existence Theorem we show that the converse of the above theorem holds for universal formulas of the form  $\forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y})$  with  $\phi(\vec{x}, \vec{y}) \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  (Theorem 4.4.2):

### Strong Approximation Principle for Complete Classes

*For every complete class  $\mathcal{M}$  axiomatized by a sentence in  $L_{A+}$ , for every sentence  $\phi \equiv \forall \vec{x} \in \vec{K} \psi(\vec{x})$  with  $\psi(\vec{x}) \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ ,*

$$\forall E \in \mathcal{M}, E \models_{AP} \phi \text{ iff } \forall E \in \mathcal{M}, E \models \phi.$$

□

Thanks to the Strong Approximation Principle for Complete Classes, we can obtain very general results concerning the “uniformity” of the validity of a fixed formula in classes of models.

In particular, as a consequence of the above result we obtain a uniformity of paths theorem that can be seen as an equivalence statement between satisfaction of infinitary formulas in a complete class of models and satisfaction of finitary formulas in the same class. Theorem 4.4.3 says:

### Uniformity of Paths

*Fix a complete class of models  $\mathcal{M}$  axiomatized by a sentence in  $L_{PBA}$ .*

*Let  $\phi \in L_A(\Phi)$  a universal sentence*

$$\phi \equiv \forall \vec{x} \in \vec{K} \psi(\vec{x})$$

*with  $\psi \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ . Then the following are equivalent:*

- For every model  $E \in \mathcal{M}$ ,  $E \models \phi$ .
- There exists a path  $h \in I(\phi)$  such that for every model  $E \in \mathcal{M}$ ,  $E$  approximately satisfies  $\phi$  along the path  $h$ .

□

We also obtain a weak version of the above result for a bigger class of formulas.

Theorem 4.4.4 states the following:

### Weak Uniformity of Paths

Fix a signature  $\Phi$  and a complete class of models axiomatized by a sentence in  $L_{PBA}$ . Let  $\phi$  be a negative sentence ( $\phi = \neg\theta$ ). If for every model  $E \in \mathcal{M}$ ,  $E \models \phi$ , then:

For every path  $h \in I(\theta)$  there exists an integer  $n$  such that for every model  $E \in \mathcal{M}$ ,  $E \models \neg(\theta)_{h,n}$ . □

In other words, if a negative sentence  $\neg\theta$  holds for every model of a complete class, then for every path  $h$  of  $\theta$  there is an integer  $n$  such that the approximation  $\neg(\theta)_{h,n}$  holds for every model of the class.

The strong result on uniformity of paths suggests a way of proving that a sentence of the form  $\phi = \forall \vec{x} \in \vec{K} \theta(\vec{x})$  with  $\theta \in (L_P \cup \neg L_P)^{\{\wedge, \neg\}}$  does not hold for all models of a complete class  $\mathcal{M}$  axiomatized by a formula in  $L_{PBA}$ . It would be enough to show that for every path  $h \in I(\phi)$  there is a model  $E \in \mathcal{M}$  such that  $E$  does not approximately satisfy  $\phi$  along the path  $h$ .

The weak result on uniformity of paths can also be seen as a uniformity result. If a negative statement (for example  $\bigvee_{i=1}^{\infty} \theta_i$ ) holds for all the models of a complete

class of models, then it holds uniformly (along the same fixed path) for all models of the class. In other words, the above theorem gives a general “recipe” to obtain uniform versions of negative sentences that are known to be true in all the models of the class, just by decoding the meaning of  $\models_{AP}^h$ . Such a general result only requires the verification that the class of models is a complete class of models, and that the statement proved for every model of the class is a negative formula. We close Chapter 4 with an application of this result to Functional Analysis. We prove a uniform version of the celebrated theorem by Krivine concerning the finite representability of one of the  $\ell_p$  ( $p \in [1, \infty]$ ) or  $c_0$  in every infinite dimensional Banach space by simply decoding the meaning of “there exists a path, such that for every normed space structure, Krivine’s Theorem is approximately true along this path”.

Chapter 5 is devoted to the negative aspects of the following question:

(\*) if every structure in a fixed collection of models approximately satisfies a sentence  $\phi$ , does it follow that every model of the collection also satisfies  $\phi$ ?

The results in Chapter 4 imply that the simplest negative instance of question (\*) is when the formula  $\phi$  is existential. A negative instance of (\*) will be:

$$\forall E \in \mathcal{M} \ E \models_{AP} \phi \text{ but } \exists E \in \mathcal{M} \ E \models \neg\phi$$

We will say of any such occurrence that  $\phi$  is **omitted** in  $\mathcal{M}$ . We use the Model Existence Theorem to construct structures that OMIT existential sentences in the above sense. This contrasts with the results in the literature where the Model Existence Principles are used in one direction only: to obtain rich structures for some

types of infinitary formulas. Thanks to the expressive power of  $L_A$  it is possible to obtain from the Model Existence Theorem conditions for the existence of models that omit existential sentences. We give sufficient conditions on complete classes of models to have countable models that omit  $\exists$  formulas in  $L_A$  (Theorem 5.1.3). Finally, we apply the previous results to first order logic. We obtain an extension of the classical Omitting Types Theorem to infinitary formulas of the form  $\forall$  (Theorem 5.2.1). We replace the usual hypothesis of “local realization” by a statement that uses approximate truth.

## Chapter 2

### Definition of the Logic $L_A$

Our intention is to define a logic whose expressive power captures many of the concepts in Analysis. Most popular concepts refer to continuous maps between two *different* metric spaces. Examples of such maps are the metric of a metric space, the expected value of a random variable, etc.. In order to accommodate such functions we need to define a logic that accepts multisorted predicate symbols. The full description of such a language is as follows.

#### 2.1 The signature $\Phi$

We will distinguish two different types of sorts:

- Fix a collection  $S$  of metric spaces containing at least the real numbers with the usual metric  $(\mathbb{R}, d)$ . The elements of the collection  $S$  are going to be called the *fixed sorts* of the signature.
- A *true sort*. Intuitively, the true sort is going to be the metric space associated with every particular model.

For every signature to be defined, we intend the fixed sorts to be the same for all models, while the true sort will be different with every model. Every structure then will consist of an interpretation of the true sort and interpretations of the



predicate and function symbols. In this way the functions in the model will have domain the true sort and range the true sort (or one of the fixed sorts metric spaces).

Abusing the notation we will refer to the interpretation of the true sort in a model as “the model”.

A signature  $\Phi = (\mathcal{F}, \mathcal{P})$  is defined as follows:

- $\mathcal{F}$  is a collection of symbols of functions such that each element  $f \in \mathcal{F}$  has a corresponding **finite** arity  $a_f < \omega$ . The sort of  $f$  could be the true sort, or a fixed sort  $(M_f, \rho_f)$  in  $S$ . Intuitively, these symbols are going to be interpreted as maps from the model to itself, or from the model to the sort metric space  $(M_f, \rho_f)$ . We call  $\mathcal{F}$  the collection of **function symbols** of  $\Phi$ .
- $\mathcal{P}$  is the union of a collection of predicates and a collection of predicate symbols. Each element  $C \in \mathcal{P}$  has a corresponding **finite** arity  $a_C < \omega$ . The sort of  $C$  could be the true sort or a fixed sort  $(M_C, \rho_C) \in S$ . If  $C$  has a fixed sort  $(M_C, \rho_C)$  then  $C$  is an unary **predicate** that is a **closed** subset of  $M_C$ . If  $C$  has true sort then  $C$  is a predicate **symbol**. We call  $\mathcal{P}$  the collection of **predicates** of  $\Phi$ .
- There is a fixed function symbol  $\rho \in \mathcal{F}$  in every signature with arity 2 and sort  $\mathfrak{R}$ . This symbol will be interpreted as the metric of the model.

Let us see an example of a signature.

**EXAMPLE 2.1.1** *Typical signatures*

Suppose that the structures that we want to study are the normed spaces with the following basic operations:

- Sum of two vectors and multiplication by scalars.
- Norm of a vector.
- Estimates of the norms, i.e we want to refer to the relations  $|\cdot| \leq q$  on the reals for arbitrary positive rational numbers  $q$ .

A natural signature that will reflect these operations will be:

- Fixed sorts  $S = \{(\mathfrak{R}, d)\}$ .
- $\mathcal{F}$  contains the following symbols of functions:
  - the function  $(x + y)$  with arity 2 and true sort.
  - For every real number  $r$ , the function symbol  $r(x)$  with arity one and true sort space.
  - A function symbol  $\|x\|$  with arity one and sort space  $(\mathfrak{R}, d)$ .
  - The metric function  $\rho$  with arity 2 and fixed sort the real numbers.
- $\mathcal{P}$  contains for every positive rational  $q$  the compact predicate  $C_q = \{x \mid |x| \leq q\}$  of arity one and sort space  $(\mathfrak{R}, d)$ .  $\square$

### 2.1.1 Terms in $L_A$

We define terms and sort spaces of the terms by induction.

**DEFINITION 2.1.2** *Definition of terms in  $L_A$*

- Let  $\{x_i | i \in \omega_1\}$  a fixed collection of variables. The variables  $x_i$  are terms. Their arity is 1. Their sort is the true sort.
- Given a collection of terms  $\{t_i : i \leq n\}$  of true sort space, and a function symbol  $f$  of arity  $n$  and sort space  $(M_f, \rho_f)$  (or true sort space),  $f(\vec{t}) = f(t_1, \dots, t_n)$  is a term with arity given by the cardinality of the collection of variables in  $\{t_i : i \leq n\}$  and sort space  $(M_f, \rho_f)$  (or true sort space).  $\square$

In summary, terms have finite arity and they are of two different types: those which have fixed sorts and are going to be interpreted as maps from the cartesian product of the model to the sort space, and those which have true sort and are going to be interpreted as maps from cartesian product of the model to itself.

## 2.2 Formulas in $L_A$

Fix a signature  $\Phi$ . We define the formulas of  $L_A(\Phi)$  by induction. As we will see,  $L_A(\Phi)$  admits negation, finite and countable conjunctions and bounded existential quantification by  $\omega$  many variables.

As usual, for every formula  $\phi$ , the notation  $\phi(\vec{x})$  means that the free variables of  $\phi$  are among the components of the vector  $\vec{x}$ . Likewise  $\phi(\vec{x}_1, \vec{x}_2, \dots)$  means that the free variables of  $\phi$  are among the components of the vectors  $\vec{x}_1, \vec{x}_2, \dots$

We begin with the atomic formulas.

### DEFINITION 2.2.1 *Atomic formulas*

An atomic formula in  $L_A(\Phi)$  would be any expression of the form:

$$C(\vec{t})$$

where  $C \in \mathcal{P}$  and  $\vec{t}$  is a vector of terms such that the arity and sort of  $\vec{t}$  agree with the arity and sort of  $C$ .  $\square$

**DEFINITION 2.2.2** *Definition of formulas in  $L_A(\Phi)$*

- An atomic formula is a formula in  $L_A(\Phi)$ .
- If  $\phi_1, \phi_2, \dots, \phi_i, \dots$  ( $i < \omega$ ) is a collection of formulas in  $L_A(\Phi)$ , then for every  $n$ ,

$$\bigwedge_{i=1}^n \phi_i$$

is in  $L_A(\Phi)$ . Furthermore,  $\bigwedge_{i=1}^{\infty} \phi_i$  is also in  $L_A(\Phi)$ .

- If  $\phi$  is a formula in  $L_A(\Phi)$  then  $\neg\phi$  is also a formula in  $L_A(\Phi)$ .
- Consider a formula  $\phi(\vec{y}_1, \dots, \vec{y}_n, \dots, \vec{x})$  in  $L_A(\Phi)$  where every  $\vec{y}_n$  ( $n < \omega$ ) is a vector with finite arity  $a_n < \omega$ . Assume also that the set of variables in the  $\vec{y}_n$  are all disjoint. Let  $\vec{K} = (K_1, K_2, \dots)$  a corresponding vector of predicate symbols of true sort. The following formula is in  $L_A(\Phi)$ :

$$\exists(\vec{y}_1, \dots, \vec{y}_n, \dots) \in (K_1, \dots, K_n, \dots)(\phi(\vec{y}_1, \dots, \vec{y}_n, \dots, \vec{x}))$$

$\square$

We will abbreviate  $\neg\bigwedge\neg$  by  $\bigvee$  and  $\neg(\phi \wedge \neg\psi)$  by  $\phi \Rightarrow \psi$ . To avoid typing very long formulas we will also abbreviate

$$\exists(\vec{x}_1, \dots, \vec{x}_n, \dots) \in (K_1, \dots, K_n, \dots)(\phi(\vec{x}_1, \dots, \vec{x}_n, \dots, \vec{y}))$$

by  $\exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y})$ . Likewise  $\neg \exists \vec{y} \in \vec{K} \neg \phi(\vec{y}, \vec{x})$  will be abbreviated by  $\forall \vec{y} \in \vec{K} \phi(\vec{y}, \vec{x})$ . Finally, when the signature  $\Phi$  is clear from the context, we will refer to  $L_A(\Phi)$  by  $L_A$ .

Notice that the formulas in  $L_A$  admit quantification over infinitely many variables. This enhances the expressive power of  $L_A$ , as the next example shows.

**EXAMPLE 2.2.3** *Expressive power of  $L_A$*

- **The map  $T : (X, \rho) \mapsto (X, \rho)$  is continuous in the set  $K$ .** Here let us assume that we are working with a signature  $\Phi = (\mathcal{F}, \mathcal{P})$  that contains the fixed function symbol  $\rho(x, y)$  (intended to be the distance function) with arity two and fixed sort  $\mathfrak{R}$ , the function symbol  $T$  with arity one and true sort and for every integer  $m$  the closed predicate  $x \leq (1/m)$  with arity one and fixed sort  $\mathfrak{R}$ . We will assume also that the signature contains a predicate symbol  $K$  with arity one and true sort.

The desired formula is:

$$\forall x \in K \bigwedge_{n=1}^{\omega} \bigvee_{m=1}^{\omega} \forall y \in K (\rho(x, y) \leq 1/m \Rightarrow \rho(T(x), T(y)) < 1/n)$$

- **The set  $K$  is bounded and infinite dimensional in the normed space  $(X, \|\cdot\|)$ .** In this case let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature that contains a norm symbol  $\|x\|$  with arity one and fixed sort the reals, a difference symbol  $(x - y)$  with fixed sort and arity 2 and for every integer  $m$  the closed predicates  $|x| \geq (1/m)$  and  $|x| \leq m$  with arity one and fixed sort the reals. Finally, we also include in the signature a predicate  $K$  of arity one and true sort.

The desired formula is:

$$\bigvee_{M=1}^{\infty} ( \forall x \in K (0 \leq \|x\| \leq M) \wedge \bigvee_{i=1}^{\omega} (\exists \vec{x} \in \vec{K} \bigwedge_{n=1}^{\infty} \bigwedge_{m \neq n}^{\infty} 2M \geq \|x_n - x_m\| \geq 1/i) )$$

- **The set  $K$  is compact in the metric space  $(X, \rho)$ .** Assume that the signature  $\Phi = (\mathcal{F}, \mathcal{P})$  contains for every integer  $m$  the closed unary predicate  $x \leq (1/m)$  with fixed sort the reals, and a predicate symbol  $K$  with arity one and true sort.

The desired formula is:

$$\forall \vec{x} \in \vec{K} (\exists y \in K \bigwedge_{k=1}^{\omega} \bigwedge_{n=1}^{\omega} \bigvee_{m>n}^{\omega} \rho(x_m, y) \leq 1/k)$$

□

Our next step is to define the semantics for  $L_A$ .

## 2.3 Semantics for $L_A$

The definition of the structures of  $L_A$  is a natural generalization of Henson's notion of a Normed Space Structure (see[19]).

Fix a collection  $S$  of metric spaces (the fixed sort spaces).

### DEFINITION 2.3.1 *Definition of a model*

Fix a signature  $\Phi$  for  $S$ . A model  $E$  for  $\Phi$  is a collection

$$E = (X, d, F, P)$$

where:

- $(X, d)$  is a metric space.  $\rho$  is interpreted as  $d$ .
  - $F = \{f^\bullet \mid f \in \mathcal{F}\}$  with the property that:
    - $\forall f \in \mathcal{F}$  with arity  $a_f$  and sort space  $(M_f, \rho_f)$  (or true sort) the interpretation  $f^\bullet$  is a continuous function (with respect to the product topology induced by  $d$  in  $X^{a_f}$ ) from  $(X^{a_f}, d^{a_f})$  to  $(M_f, \rho_f)$  (or from  $(X^{a_f}, d^{a_f})$  to  $(X, d)$ ).
  - $P = \{C^\bullet \mid C \in \mathcal{P} \text{ and } C \text{ has true arity}\}$  with the property that:
    - $\forall C \in \mathcal{P}$  with arity  $a_C$  and true sort space, the interpretation  $C^\bullet$  is a closed set (with respect to the product topology induced by  $d$  in  $X^{a_C}$ ) in  $(X^{a_C}, d^{a_C})$ .
- 

Once the interpretations for the symbols of  $\Phi$  are defined one can define the interpretations of the terms in the model,  $t^\bullet$ , in the obvious way.

When the model and the interpretations of the elements of the language are clear from the context we will drop the symbol  $\bullet$ .

The definition of validity follows in the standard way.

**DEFINITION 2.3.2** *Definition of validity*

Fix a signature  $\Phi$  for  $S$ . Fix a model  $E = (X, d, F, P)$  of  $\Phi$ . The truth ( $\models$ ) relation in the model  $E$  is constructed by induction in formulas in the natural way from the truth definition for the atomic case:

Let  $C(\vec{t}(\vec{x}))$  be an atomic formula. Let  $\vec{a} \in X^{|\vec{x}|}$ .  $E \models C(\vec{t}(\vec{a}))$  iff it is true that  $\vec{t}^\bullet(\vec{a}) \in C^\bullet$ . □

Let us give some examples of models in  $L_A$ .

**EXAMPLE 2.3.3** *Models of  $L_A$* 

- **Classical multisorted models**

Let  $S$  be an arbitrary collection of discrete metric spaces, i.e. spaces  $(M, \rho_M)$  so that  $\text{Image}(\rho_M) = \{0, 2\}$ . Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  containing a fixed unary predicate  $K$  in  $\mathcal{P}$ . A model of  $\Phi = (\mathcal{F}, \mathcal{P})$  is a classical multisorted model if the predicate  $K$  is interpreted as the whole space  $X$  and  $\text{Im}(\rho) = \{0, 2\}$ . Any classical first order structure  $E = (X, F, P)$  can be interpreted as a classical multisorted model: Define  $d : X^2 \mapsto (\mathfrak{R}, d)$  to be the discrete metric. Under this metric every map is continuous and every set is closed.

- **The standard model of a metric space  $(X, d)$  for a signature  $\Phi$ .**

Let  $S$  be an arbitrary collection of metric spaces. Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  such that all  $C \in \mathcal{P}$  with fixed sort are compact sets. A model  $E = (X, d, F, P)$  is standard for  $\Phi$  if and only every  $C^\bullet \in P$  is a compact set in  $(X, d)$  or in  $(X^i, \tau^i)$  ( $\tau^i$  is the product topology in  $X^i$  induced by  $(X, d)$ ). The standard model is related to the standard neometric family (see [7]) and to the models studied by Anderson in [2].

- **Normed space structures (Henson).**

Let  $S = \{(\mathfrak{R}, d)\}$ . Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  such that  $\mathcal{P}$  contains all the compact subsets of the reals.  $\mathcal{F}$  contains the special functions symbols  $. + .$  of arity two and true sort, the function symbol  $\|.\|$  of arity one and sort space  $(\mathfrak{R}, d)$  and for every rational  $q$  the function symbols  $q(.)$  with arity one



and true sort. Additionally,  $\mathcal{P}$  contains the collection of unary predicates symbols (of true arity)  $\{B_r \mid r \in Q\}$ .

Furthermore, suppose that for every function symbol  $f$  in  $\mathcal{F}$  with arity  $a$ , for every rational  $q > 0$ , for every rational  $\epsilon > 0$  we select a real number  $\delta(f, q, \epsilon) > 0$ .

A model  $E = (X, d, F, P)$  of  $\Phi$  is a **Normed Space Structure** with respect to the collection of reals  $\{\delta(f, q, \epsilon) \mid f \in \mathcal{F}, q, \epsilon \in Q^+\}$  if and only if:

1.  $X$  is a normed space over the field of reals, and the metric  $d$  coincides with one induced by the norm. Furthermore,  $+$  is interpreted as the sum of two vectors,  $\|\cdot\|$  as the norm of the space, and, for every rational  $q$ ,  $q(\cdot)$  is interpreted as the multiplication by the scalar  $q$ .
2. For every rational  $r$ , the interpretation of  $B_r$  is the ball in  $X$  of radius  $r$  centered at the origin.
3. Every interpretation of a function symbol  $f$  with true sort (or fixed sort  $(M, \rho_M)$ ) and arity  $a$ , is a function  $f : X^a \mapsto X$  (or  $f : X^a \mapsto (M, \rho_M)$ ) that is uniformly continuous on every predicate symbol  $B_r$  in the following way:

$$\forall \vec{x} \left( \bigwedge_{i=1}^a B_r(x_i) \Rightarrow \forall \vec{y} \left( \bigwedge_{i=1}^a B_r(y_i) \Rightarrow (\|x_i - y_i\| < \delta(f, r, \epsilon) \Rightarrow \|f(\vec{x}) - f(\vec{y})\| \leq \epsilon) \right) \right)$$

for every  $\epsilon > 0$  (or

$$\forall \vec{x} \left( \bigwedge_{i=1}^a B_r(x_i) \Rightarrow \forall \vec{y} \left( \bigwedge_{i=1}^a B_r(y_i) \Rightarrow (\|x_i - y_i\| < \delta(f, r, \epsilon) \Rightarrow \rho_M(f(\vec{x}), f(\vec{y})) \leq \epsilon) \right) \right)$$

).

The normed space structures were introduced by Henson in [19] and extensively studied in [19, 14, 16].

- **Banach space structures.** One particular class of normed space structures is the class of Banach space structures. A normed space structure  $E = (X, d, F, P)$  is a Banach space structure if the space  $(X, \|\cdot\|)$  is a Banach space. This class has been studied by Henson & Iovino ([19]) and by Henson & Moore ([20]).
- **Hilbert space structures.** Another particular class of normed space structures is the class of Hilbert space structures. A Banach space structure  $E = (X, d, F, P)$  is a Hilbert space structure if there is a function symbol in the signature that is interpreted as the inner product. This class was also defined by Henson & Iovino in [19].
- **Nonstandard Hulls** (Fajardo & Keisler ([7]))

Fix a nonstandard universe  $(V(\Xi), V(*\Xi), *)$ . We assume that the reader is familiar with the basic concepts of nonstandard analysis (see [41] for an introduction to the subject). Let us recall some facts concerning nonstandard hulls.

Given a  $*$ -metric space  $\overline{M} = (X, \overline{\rho})$ , for every  $x$  in  $\overline{M}$  let  $[x]$  be the equivalence class of all the elements in  $\overline{M}$  infinitesimally close to  $x$ . Then for every  $c$  in  $\overline{M}$ , we define the nonstandard hull of  $\overline{M}$  with respect to  $c$  as:

$$H(\overline{M}, c) = \{[x] : x \in \overline{M} \wedge \overline{\rho}(x, c) \text{ is finite}\}$$

and the metric in  $H(\overline{M}, c)$  is given by the relation:

$$\rho([x], [y]) =^0 \overline{\rho}(x, y)$$

Let  $S$  be the class of all the nonstandard hulls  $H(\overline{M}, c)$ . Let  $\Phi = (\mathcal{F}, \mathcal{P})$  a signature for  $S$  such that any  $C \in \mathcal{P}$  with fixed sort is the standard part of an internal subset of the galaxy  $G(\overline{M}, c)$ , for some  $*$ -metric space  $\overline{M}$ .

A model  $E = (X, d, F, P)$  of  $\Phi$  is a nonstandard hull if and only if:

1.  $(X, d^I) = H(\overline{N}, a)$  for some  $*$ -metric space  $\overline{N}$ , and some  $a$  in  $\overline{N}$ .
2. If a function  $f$  belongs to  $F$  then  $f$  is uniformly liftable.
3. If  $K \in P$  then  $K$  is the standard part of an internal subset of the galaxy  $G(\overline{M}, a)$  for some  $*$ -metric space  $\overline{M}$  and some  $a$  in  $\overline{M}$ .

- **Neometric models** (Fajardo & Keisler)

We assume that the reader is familiar with the definition and notation on neometric families. A good reference for this can be found in [9]. Fix a nonstandard universe  $(V(\Xi), V(*\Xi), *)$ . Let  $S$  be the class of all metric spaces  $(M, \rho)$  such that  $M$  is a closed subspace of some nonstandard hull  $H(\overline{N}, c)$ .

Let  $\Phi = (\mathcal{F}, \mathcal{P})$  a signature for  $S$  such that any  $C \in \mathcal{P}$  with fixed sort  $(M, \rho)$  is interpreted as a closed subset in  $(M, \rho) \subseteq H(\overline{M}, c)$ . Furthermore we require  $C$  to be the standard part of the intersection of countable many internal subsets of the galaxy  $G(\overline{M}, c)$ . Any such set is said to be a neocompact set (see [9]).

A model  $E = (X, d, F, P)$  of  $\Phi$  is a neometric model if and only if:

1.  $(X, d)$  is closed subspace of a fixed nonstandard hull  $H(\overline{\mathbb{N}}, a)$ .
2. If a function  $f$  belongs to  $F$  then  $f$  is uniformly liftable.
3. If a relation  $C$  belongs to  $P$  then  $C$  is a neocompact subset of the galaxy  $G(\overline{\mathbb{N}}, a)$ .

The neometric families were introduced and studied in a series of papers by Fajardo & Keisler (see [7, 8, 9]). It is readily seen that the nonstandard hulls are particular cases of neometric models.  $\square$

As the previous example illustrates the definition of models of  $L_A$  are wide enough to encompass most of the usual structures in classical logic and in analysis.

Let us now return to the logic  $L_A$ . Our intention is to define a notion of approximate formula for formulas in  $L_A$ .

## 2.4 Approximate Formulas in $L_A$

We define a notion of approximate formula for the formulas in  $L_A$  that extends Henson's definition (see [19]) for first order positive bounded formulas to formulas in  $L_A$ . First we introduce the concept of an **approximate signature** for a fixed signature. Using this we define a first order collection  $L_{AP}$  of formulas: the approximate formulas of  $L_A$ . Finally, we associate to every formula in  $L_A$  a tree of formulas in  $L_{AP}$ .

**DEFINITION 2.4.1** *Approximate Signature of  $\Phi$*

Fix a signature  $\Phi = (\mathcal{F}, \mathcal{P})$ . The approximate signature of  $\Phi$ , denoted by  $\Phi^{app} = (\mathcal{F}, \mathcal{AP})$  is the following:

$$\mathcal{AP} = \mathcal{P} \cup \{K_n \mid K \in \mathcal{P} \text{ and } n \in \omega \text{ and } K_n \text{ has same arity and sort as } K\}.$$

Furthermore, for any closed predicate  $C \in \mathcal{P}$  with fixed sort  $(Y, \rho)$ , for any integer  $n$ ,  $C_n$  is the following closed predicate in  $(Y, \rho)$ :

$$C_n = \{y \in Y \mid \exists z \in C \rho(y, z) \leq (1/n)\}$$

□

The signature  $\Phi^{app} = (\mathcal{F}, \mathcal{AP})$  is intended to have the same function symbols as  $\Phi$ , and for every  $K \in \mathcal{P}$ , to contain the collection of metric “approximations”  $K_n$ .

Every model  $E$  of  $\Phi$  induces in a natural way a model of  $\Phi^{app}$ , the **approximate model of  $E$** .

**DEFINITION 2.4.2** *Approximate model of  $E$* 

Fix a signature  $\Phi$ . Let  $E = (X, d, F, P)$  be a model of  $\Phi$ . The approximate model of  $E$ ,  $E^{app}$  is a model of  $\Phi^{app}$  defined as follows:

$E^{app} = (X, d, F, AP)$  with

- $AP = P \cup \{K_n : K \in P \text{ and } n \in \omega\}$  and such that for every  $K \in P, \forall n$

$$K_n(\vec{b}) \text{ iff there exists } \vec{c} (E \models K(\vec{c}) \text{ and } \bigwedge_{i=1}^a d(c_i, b_i) \leq (1/n))$$

□

We will use  $\Phi^{app}$  to define in a syntactical way collections of formulas that approximate the formulas in  $L_A$ . We want these approximations to be first order formulas (i.e. to use only finite conjunctions, negation and bounded existential quantification).

**DEFINITION 2.4.3** *Definition of  $L_{AP}$* 

Fix a signature  $\Phi$ . We define the collection  $L_{AP}(\Phi)$  of **approximate formulas** of  $L_A(\Phi)$  as follows:

- Any atomic formula  $C(\vec{t}) \in L_A(\Phi^{app})$  is in  $L_{AP}(\Phi)$ .
- If  $\phi_1$  and  $\phi_2$  are formulas in  $L_{AP}(\Phi)$  then the formula  $\phi_1 \wedge \phi_2$  is also in  $L_{AP}(\Phi)$ .
- If  $\phi$  is a formula in  $L_{AP}(\Phi)$ , then the formula  $\neg\phi$  is in  $L_{AP}(\Phi)$ .

- Let  $\phi(\vec{y}, \vec{x})$  be a formula in  $L_{AP}(\Phi)$ . Let  $a < \omega$  be the finite arity of  $\vec{y}$ , and let  $K \in \mathcal{AP}$  with arity  $a$  and true sort. Then the formula

$$\exists \vec{y} \in K \phi(\vec{y}, \vec{x})$$

is in  $L_{AP}(\Phi)$ .  $\square$

Note that the logic  $L_{AP}(\Phi)$  is a finitary multisorted logic. When the signature  $\Phi$  is fixed, we will abbreviate  $L_{AP}(\Phi)$  by  $L_{AP}$ .

Our intention is to generate approximations of all the formulas in  $L_A$  by using the formulas in  $L_{AP}$  as building blocks. Inspired by the Souslin operation used in classical descriptive set theory (see for example [24]) to generate the analytic sets from the closed sets in a Polish space, we define a Souslin-type operation for the formulas in  $L_{AP}$ . We intend the approximation of a formula  $\phi$  to be the image of the Souslin-type operation applied to the collection of approximate formulas of  $\phi$  in  $L_{AP}$ .

**DEFINITION 2.4.4** *The Souslin Operation*

Fix an arbitrary signature  $\Phi$  and an arbitrary set  $I$ . Let  $Z = \{\phi_{h,n} : h \in I, n \in \omega\}$  be a collection of formulas in  $L_{AP}$  indexed by  $I \times \omega$ . We will call  $Z$  a Souslin-scheme on  $L_{AP}$ . The Souslin operation  $\mathcal{S}$  applied to  $Z$  is the following formula:

$$\mathcal{S}(Z) = \bigvee_{h \in I} \bigwedge_{n=1}^{\infty} \phi_{h,n}$$

$\square$

The final step in our process is to associate to every formula  $\phi$  in  $L_A$  a tree of approximate formulas. Formally, we will associate to every formula  $\phi$  in  $L_A$  a set

of indices  $I(\phi)$  (the branches of the tree of approximate formulas) and a Souslin scheme  $C(\phi) = \{\phi_{h,n} | h \in I(\phi), n \in \omega\}$  of finitary formulas in  $L_{AP}$ . Intuitively, for every branch  $h \in I(\phi)$ , the collection  $\{\phi_{h,n} | n \in \omega\}$  is going to “imply”  $\phi$  as  $n$  tends to  $\infty$ .

**DEFINITION 2.4.5** *Definition of approximate formulas in  $L_A$*

For any formula  $\phi(\vec{x})$  in  $L_A$  we define:

- A set of paths  $I(\phi)$ ,
- For any  $h \in I(\phi)$  and  $n \in \omega$ , a formula  $\phi_{h,n}(\vec{x}) \in L_{AP}$

The definitions is by induction in formulas as follows:

**Atomic.** For any atomic formula  $C(\vec{t})$ ,

- The set  $I(C(\vec{t})) = \{\emptyset\}$
- For every  $h$  in  $I(C(\vec{t}))$ , for every integer  $n$ ,  $(C(\vec{t}))_{h,n} \equiv C_n(\vec{t})$

**Conjunction.** For any countable collection of formulas in  $L_A$ ,  $\phi_1, \phi_2, \dots, \phi_i, \dots$ , we define:

- $I(\bigwedge_{i=1}^{\infty} \phi_i(\vec{x})) = \prod_{i=1}^{\infty} I(\phi_i)$  (the cartesian product of the  $I(\phi_i)$ ).
- For every  $h$  in  $I(\bigwedge_{i=1}^{\infty} \phi_i)$ , for every integer  $n$ ,

$$\left( \bigwedge_{i=1}^{\infty} \phi_i \right)_{h,n} \equiv \bigwedge_{i=1}^n \left( (\phi_i)_{h(i),n} \right)$$

**Negation.** For any formula  $\phi$  in  $L_A$ , we have:



- $I(\neg\phi) \subseteq (I(\phi) \times \omega)^\omega$  is the collection of all maps  $f = (f_1, f_2)$  with the following “weak” surjectivity property:

$$\forall h \in I(\phi) \exists s \in \omega, (\phi)_{h, f_2(s)} \equiv (\phi)_{f_1(s), f_2(s)}$$

- For every  $h$  in  $I(\neg\phi(\vec{x}))$ , for every integer  $n, (\neg\phi)_{h,n} \equiv \bigwedge_{i=1}^n \neg (\phi_{h_1(i), h_2(i)})$

**Existential** For every formula  $\phi((\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots), \vec{x})$ , for every vector of bounding sets  $\vec{K} = (K_1, K_2, \dots, K_i, \dots)$  of corresponding arities, we have:

- $I(\exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{x})) = I(\phi(\vec{v}, \vec{x}))$
- For every  $h$  in  $I(\exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{x}))$ , for every integer  $n$ , we have that

$$(\exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{x}))_{h,n} \equiv \exists \vec{v} \in \vec{K} (\phi(\vec{v}, \vec{x}))_{h,n}$$

□

Let us introduce some notation.

We call  $I(\phi)$  the **set of paths** of  $\phi$ . The formulas  $\phi_{h,n}$  are the **approximate formulas** of  $\phi$ , and the collection

$$C(\phi) = \{\phi_{h,n} : h \in I(\phi), n \in \omega\}$$

is the **Souslin approximation scheme** or the approximation tree of the formula  $\phi$ . Finally, the formula

$$\mathcal{S}(C(\phi)) = \bigvee_{h \in I(\phi)} \bigwedge_{n=1}^{\infty} \phi_{h,n}$$

is the **Souslin approximation** of the formula  $\phi$ .

Note that if the formula  $\phi$  in  $L_A$  is a positive formula (i.e. a formula without negation) then  $I(\phi) = \{\emptyset\}$  has only one element, i.e. there is only one path for the

approximations of  $\phi$ . For formulas that are not positive, the set of possible paths has cardinality bigger than one ( $2^\omega$ ).

Let us see some examples of approximations of formulas in  $L_A$ .

**EXAMPLE 2.4.6** *Example of Approximate Formulas*

- Let us assume that the signature  $\Phi$  contains for every integer  $i$  the predicates ( $|y| \geq 1/i$ ) of arity one and sort  $(\mathfrak{R}, d)$  and an unary predicate  $K$  of true sort.

The formula  $\phi$ :

$$\exists \vec{x} \in \vec{K} \bigwedge_{j=1}^{\infty} \bigwedge_{n=1, n \neq j}^{\infty} \rho(x_j, x_n) \geq 1/i$$

states that there exists an infinite sequence of elements in  $K$  whose mutual distance is bigger or equal than  $1/i$ .

The reader can verify that the collection of paths  $I(\phi)$  is exactly:

$$I(\phi) = \prod_{j=1}^{\infty} \prod_{(n=1, n \neq j)}^{\infty} \{\emptyset\}$$

In other words, there is only one possible path for this formula. Using the previous definition, we have for this  $h \in I(\phi)$ , for every integer  $m$ ,

$$(\phi)_{h,m} = \exists \vec{x} \in \vec{K} \bigwedge_{j=1}^m \bigwedge_{n=1, n \neq j}^m \rho(x_j, x_n) \geq 1/i - 1/m$$

i.e. the  $m$ -approximation along the only path states that there exists  $m$  elements in the set  $K$  such that their mutual distances are bigger or equal than  $1/i - 1/m$ .

- The formula:  $\neg\phi$

$$\neg \exists \vec{x} \in \vec{K} \bigwedge_{j,n=1, n \neq j}^{\infty} \rho(x_j, x_n) \geq 1/i$$

states that for every infinite sequence in  $K$  there exists two distinct elements whose distance is smaller than  $1/i$ . This is equivalent to the fact that there are only finitely many elements in  $K$  whose distance to each other is bigger than  $1/i$ .

The reader can verify that the collection of paths  $I(\neg\phi)$  is the subset of  $I(\phi) \times \omega$  consisting of all the functions  $f = (f_1, f_2) : \omega \mapsto I(\phi) \times \omega$  such that  $\forall h \in I(\phi) \exists s, \phi_{h, f_2(s)} \equiv \phi_{f_1(s), f_2(s)}$ .

In this case, since  $I(\phi)$  contains only one path it is easy to see that  $I(\neg\phi) = (\{\emptyset\} \times \omega)^\omega$ .

Using the definition for approximation, we have then that for every function

$$f = (f_1, f_2) : \omega \mapsto \{\emptyset\} \times \omega,$$

for every integer  $m$ ,

$$(\neg\phi)_{f,m} = \bigwedge_{s=1}^m \neg \exists \vec{x} \in \vec{K} \bigwedge_{j=1}^{f_2(s)} \bigwedge_{n=1, j \neq n}^{f_2(s)} \rho(x_j, x_n) \geq 1/i - 1/f_2(s)$$

This can be rewritten in the following simplified form:

for every  $s \leq m$ :

$$\forall \vec{x} \in \vec{K} \bigvee_{j=1}^{f_2(s)} \bigvee_{n=1, n \neq j}^{f_2(s)} \rho(x_j, x_n) < 1/i - 1/f_2(s)$$

i.e. the  $m$ -approximation along the path  $f$  of  $\phi$  states that for every  $s \leq m$  for every collection  $x_1, \dots, x_{f_2(s)}$  of elements in  $K$  there exist two distinct elements such that the distance is strictly smaller than  $1/i - 1/f(s)$ .  $\square$

## 2.5 Approximate Truth

The definition that we give next is a generalization of the definition of approximate truth for first order positive formulas introduced by Henson in [15].

### DEFINITION 2.5.1 *Approximate validity*

Fix a model  $E$  for a signature  $\Phi$ . Let  $E^{app}$  be the approximate model of  $E$ . Let  $\phi(\vec{x})$  be a formula in  $L_A$  for this signature. Let  $\vec{b}$  a vector of elements of  $X$ . We say that

$$E \models_{AP} \phi(\vec{b})$$

if and only if

$$\exists h \in I(\phi(\vec{x})) \forall n \in \omega, E^{app} \models \phi_{h,n}(\vec{b})$$

Equivalently,

$$E \models_{AP} \phi(\vec{b})$$

if and only if the Souslin Approximation of  $\phi, \bigvee_{h \in I(\phi)} \bigwedge_{n=1}^{\infty} \phi_{h,n}(\vec{b})$ , is true in  $E^{app}$ .  $\square$

We illustrate this definition with an example.

### EXAMPLE 2.5.2 *Isometric Inclusion*

Consider the signature  $\Phi$  described in Example 2.3.3 for the Normed Space Structures. Recall that  $B_1$  is interpreted in those structures as the unit ball of the space. By  $\vec{B}_1$  we denote the vector:  $(B_1, B_1, \dots)$ .

Let  $(Y, \|\cdot\|)$  be a separable normed space generated by the countable set of independent vectors  $\{e_1, e_2, \dots, e_n, \dots\}$  of norm 1. It is easy to see that  $(Y, \|\cdot\|)$  is

isometrically embedded into a normed space structure  $E = (X, d, F, P)$  with norm  $|\cdot|$  if and only if:

$$E \models \exists \vec{x} \in \vec{B}_1 \bigwedge_{m=1}^{\infty} \bigwedge_{\vec{a} \in Q^m} \left| \sum_{i=1}^m a_i x_i \right| = \left\| \sum_{i=1}^m a_i e_i \right\|$$

On the other hand the statement

$$E \models_{AP} \exists \vec{x} \in \vec{B}_1 \bigwedge_{m=1}^{\infty} \bigwedge_{\vec{a} \in Q^m} \left| \sum_{i=1}^m a_i x_i \right| = \left\| \sum_{i=1}^m a_i e_i \right\|$$

is equivalent to say (since  $I(\phi) = \{\emptyset\}$ ) that for every integer  $r$

$$E^{app} \models \exists \vec{x} \in \vec{B}_1 \bigwedge_{m=1}^r \bigwedge_{\vec{a} \in Q^m \uparrow r} \left| \left| \sum_{i=1}^n a_i x_i \right| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \leq 1/r$$

It follows easily that this, in turn, is equivalent to the fact that for every finite dimensional subspace  $G$  of  $Y$  and every  $\epsilon$  there is a finite dimensional subspace  $H$  in  $E$  that is  $\epsilon$ -isomorphic to  $G$ . This is exactly the definition of the concept of finite representability of a normed space  $Y$  into  $X$  (see [29]).

In summary: the approximate validity of the statement “ $Y$  is isometrically embeddable in  $E$ ” is equivalent to the statement that  $Y$  is finitely representable in  $E$ . It is well known that the two notions are not equivalent (see for example [29]). This shows that the concept of approximate truth is different from validity.  $\square$

The following example shows a formula for which  $\models$  and  $\models_{AP}$  coincide in the collection of normed spaces models.

### **EXAMPLE 2.5.3** *Infinite Dimensional Spaces*

Consider the signature  $\Phi$  described in Example 2.3.3 for the Normed Space Structures.

The next well known remark follows immediately from the Riez's Lemma (see [6]).

**REMARK 2.5.4** *Infinite Dimensional Normed Spaces*

A normed space  $(X, \|\cdot\|)$  is infinite dimensional iff there exists a sequence  $\{x_n\}_{n=1}^{\infty}$  whose elements have norm  $\leq 1$  and such that  $\forall i \neq j \in \omega \ \|x_i - x_j\| \geq 1/2$ .  
□

It follows that for every infinite dimensional normed space structure  $E$

$$E \models \exists \vec{x} \in \vec{B}_1 \bigwedge_{n=1}^{\infty} \bigwedge_{i \neq j \leq n} 1/2 \leq \|x_i - x_j\| \leq 2$$

The approximation of this formula is simple since

$$I\left(\bigwedge_{n=1}^{\infty} \bigwedge_{i \neq j \leq n} 1/2 \leq \|x_i - x_j\| \leq 2\right) = \{\emptyset\}.$$

We get

$$\begin{aligned} & (\exists \vec{x} \in \vec{B}_1 \bigwedge_{n=1}^{\infty} \bigwedge_{i \neq j \leq n} 1/2 \leq \|x_i - x_j\| \leq 2)_{\emptyset, m} \equiv \\ & \exists \vec{x} \in \vec{B}_1 \bigwedge_{n=1}^m \bigwedge_{i \neq j \leq n+m} 1/2 - 1/m \leq \|x_i - x_j\| \leq 2 + 1/m \end{aligned}$$

i.e. for every integer  $m$  there exists  $m$  vectors with norm less than one with mutual distance bigger than  $1/2 - 1/m$ . It is easy to see that this last statement implies that  $E$  is infinite dimensional.

In summary, for every normed space structure  $E$

$$E \models \exists \vec{x} \in \vec{B}_1 \bigwedge_{n=1}^{\infty} \bigwedge_{i \neq j \leq n} 1/2 \leq \|x_i - x_j\| \leq 2$$

if and only if

$$E \models_{AP} \exists \vec{x} \in \vec{B}_1 \bigwedge_{n=1}^{\infty} \bigwedge_{i \neq j \leq n} 1/2 \leq \|x_i - x_j\| \leq 2$$

□

We end this chapter with a brief discussion on the relationship between  $\models_{AP}$  and  $\models$  for formulas in  $L_A$ .

As mentioned in the introduction, the logic  $L_{PB}$  and the logic for infinitary positive formulas introduced by Fajardo & Keisler satisfy that  $\models_{AP}$  is “weaker” than  $\models$  in the following sense:

For every structure  $E$ , for every sentence  $\phi$ ,  $E \models \phi$  implies  $E \models_{AP} \phi$ .

This is not the case for arbitrary formulas in  $L_A$ . It is not hard to see that any “decent” definition of  $\models_{AP}$  (in a logic with negation) that verifies the above property would verify:

$$E \models \phi \text{ iff } E \models_{AP} \phi \quad (2.1)$$

i.e.: The notion of  $\models_{AP}$  would be identical to  $\models$ . As the previous examples show, this is not the case for our definition of  $\models_{AP}$ . Hence  $\models_{AP}$  is not in general weaker than  $\models$ . This is the price paid for the enhanced expressability of  $L_A$ . However, we will see in the next chapter that the above property still holds for large collections of formulas in  $L_A$ . On the other hand, we can weaken the above property to the following statement concerning collection of models  $\mathcal{M}$ :

$$\text{If } \forall E \in \mathcal{M} E \models \phi \text{ then } \forall E \in \mathcal{M} E \models_{AP} \phi$$

In Chapter 4 we will show that this statement holds for formulas in  $L_A$  (under some assumptions on the collection of models  $\mathcal{M}$ ).

## Chapter 3

# Elementary Properties of Approximate Truth

In this chapter we study the basic properties of the notion of approximate truth. In the first section we show that approximate truth is a sound semantical concept. In Section 3.2 we obtain a theorem linking approximate truth and convergence in metric spaces. In Section 3.3 we use the previous results to obtain basic approximation principles for some subsets of  $L_A$ . Finally in Section 3.4 we study the notion of approximate truth in classical first order logic and obtain an approximation principle for a large class of infinitary formulas.

### 3.1 Soundness of Approximate Truth

We begin by showing that the approximate formulas behave “nicely”. We also show that every set of paths  $I(\phi)$  has a countable “dense” subset.

**LEMMA 3.1.1** *Fix a collection  $S$  of metric spaces, and a signature  $\Phi$  over  $S$ .*

*Then for every formula  $\phi(\vec{x}) \in L_A$ :*

1. *For any  $\vec{b} \in X^{|\vec{x}|}$ , for any integer  $n$  and for all  $h \in I(\phi)$ ,*

$$E^{app} \models \phi_{h,n+1}(\vec{b}) \Rightarrow \phi_{h,n}(\vec{b})$$



2. For every formula  $\phi$  there exists a set  $D(\phi) \subseteq I(\phi)$  at most countable with the following property:

$D(\phi)$  satisfies that for every  $h$  in  $I(\phi)$ , for every integer  $n$  there exists a path  $g$  in  $D(\phi)$  such that:  $\phi_{h,n}(\vec{x}) \equiv \phi_{g,n}(\vec{x})$   $\square$

PROOF: 1) By induction on the complexity of the formulas and Definition 2.4.5 of approximate formulas. Left to the reader.

2) By induction on the complexity of the formulas in  $L_A$ .

• **Atomic Formulas.** Let  $C(\vec{t})$  be an atomic formula. Then we know that  $I(C(\vec{t})) = \{\emptyset\}$  and  $\forall n \in \omega \forall h \in I(C(\vec{t}))$ :

$$(C(\vec{t}))_{h,n} = C_n(\vec{t})$$

Define then  $D(C(\vec{t})) = I(C(\vec{t}))$ . It is easy to see that this set verifies the desired property.

For the connectives and quantifier steps let us assume as induction hypothesis that for formulas  $\psi$  of less complexity than the formula  $\phi$ ,  $I(\phi)$  and  $D(\psi)$  satisfy the desired properties.

• **Conjunction.** Consider the formula

$$\phi(\vec{x}) = \bigwedge_{i=1}^{\infty} \phi_i(\vec{x})$$

Recall that

$$I(\phi) = \prod_{i=1}^{\infty} I(\phi_i)$$

and that for all integers  $n$  and for all  $h \in I(\phi)$ ,

$$(\phi(\vec{x}))_{h,n} = \bigwedge_{i=1}^n (\phi_i(\vec{x}))_{h(i),n}.$$

The construction of  $D(\bigwedge_{i=1}^{\infty} \phi_i)$  is as follows:

$\forall n \in \omega$ , let

$$p_n : \prod_{i=1}^{\infty} D(\phi_i) \mapsto \prod_{i=1}^n D(\phi_i)$$

be the usual projection map. By induction hypothesis,  $\forall i \in \omega$ ,  $D(\phi_i)$  is at most countable. It is possible then to find, for every integer  $n$ , countable sets  $Q(n) \subseteq \prod_{i=1}^{\infty} D(\phi_i) \subset I(\bigwedge_{i=1}^{\infty} \phi_i)$  such that:

$$p_n(Q_n) = \prod_{i=1}^n D(\phi_i)$$

We define  $D(\bigwedge_{i=1}^{\infty} \phi) = \cup_{n=1}^{\infty} Q_n$ .

Clearly  $D(\bigwedge_{i=1}^{\infty} \phi) \subseteq I(\bigwedge_{i=1}^{\infty} \phi)$  and is at most countable. Furthermore, by induction hypothesis, for every  $h = (h_1, ..h_i, ..) \in I(\bigwedge_{i=1}^{\infty} \phi_i)$ , for every integer  $n$ , there exists  $(g_1, ..g_n) \in \prod_{i=1}^n D(\phi_i)$  such that:

$$\bigwedge_{i=1}^n (\phi_i)_{h_i,n} \equiv \bigwedge_{i=1}^n (\phi_i)_{g_i,n}$$

It follows that there exists  $g \in D(\bigwedge_{i=1}^{\infty} \phi_i)$  such that:

$$\left( \bigwedge_{i=1}^{\infty} \phi_i \right)_{h,n} \equiv \bigwedge_{i=1}^n (\phi_i)_{h_i,n} \equiv \bigwedge_{i=1}^n (\phi_i)_{g_i,n} \equiv \left( \bigwedge_{i=1}^{\infty} \phi_i \right)_{g,n}$$

This is the desired result.

- **Negation.** Consider the formula  $\phi(\vec{x}) = \neg\psi(\vec{x})$  in  $L_A$ .

Recall that  $I(\neg\psi) \subseteq (I(\psi) \times \omega)^\omega$  is the collection of all the functions  $f = (f_1, f_2)$  with the property that:

$$\forall h \in I(\psi) \exists s, \psi_{h, f_2(s)} \equiv \psi_{f_1(s), f_2(s)}$$

Recall also that  $\forall n \in \omega \forall h = (h_1, h_2) \in I(\neg\psi(\vec{x}))$

$$(\neg\psi)_{h, n} = \bigwedge_{i=1}^n \neg(\psi_{h_1(i), h_2(i)})$$

The construction of the set  $D(\neg\psi)$  is as follows:

Let  $A \subseteq I(\neg\psi)$  be the collection of all the functions  $f = (f_1, f_2) \in I(\neg\psi)$  such that  $Image(f_1) \subseteq D(\psi)$ . From the induction hypothesis it follows that  $\forall n \in \omega$  the set  $(D(\psi) \times \omega)^{\{1, \dots, n\}}$  is at most countable. For any integer let

$$proj_n : A \mapsto (D(\psi) \times \omega)^{\{1, \dots, n\}}$$

be the natural projection map. It is possible to find for every integer  $n$  a countable set  $O(n) \subseteq A \subseteq I(\neg\psi) \subset (I(\psi) \times \omega)^\omega$  such that

$$proj_n(O(n)) = proj_n(A)$$

Let then  $D(\neg\psi) = \cup_{n=1}^\infty O(n)$ .

Let us verify the desired property. First, it is clear that  $D(\neg\psi)$  is at most countable. Fix now  $h = (h_1, h_2) \in I(\neg\psi)$  and an integer  $n$ .

By induction hypothesis for every integer  $s$  there exists  $g_s \in D(\psi)$  such that

$$\psi_{h_1(s), h_2(s)} \equiv \psi_{g_s, h_2(s)}$$

It follows that there exists  $f \in A$  such that for every integer  $m$

$$(\neg\psi)_{h, m} \equiv \bigwedge_{s=1}^m \neg(\psi)_{h_1(s), h_2(s)} \equiv \bigwedge_{s=1}^m \neg(\psi)_{f_1(s), f_2(s)} \equiv (\neg\psi)_{f, m}$$

Now, by definition of  $O(n) \subseteq A$ , there exists  $g \in O(n)$  such that  $g \upharpoonright \{1, \dots, n\} = f \upharpoonright \{1, \dots, n\}$ . We obtain then

$$(\neg\psi)_{h,n} \equiv \bigwedge_{s=1}^n \neg (\psi)_{f_1(s), f_2(s)} \equiv \bigwedge_{s=1}^n \neg (\psi)_{g_1(s), g_2(s)} \equiv (\neg\psi)_{g,n}$$

but this is the desired result.

- **Existential.** Fix a formula  $\phi$  in  $L_A$  with free variables among the collection  $\{\vec{v}_i | i < \omega\} \cup \{\vec{x}\}$ . Let  $\vec{K} = (K_1, K_2, \dots)$  a corresponding vector of predicate symbols with true sort. Consider the formula:

$$\psi(\vec{x}) = \exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{x})$$

Recall:  $I(\psi) = I(\phi)$ . Also, recall that  $\forall n \in \omega \forall h \in I(\psi(\vec{x}))$ ,

$$(\psi(\vec{x}))_{h,n} = \exists \vec{v} \in \vec{K} \phi_{h,n}(\vec{v}, \vec{x})$$

Define then  $D(\psi) = D(\phi)$ .

The verification of the desired property is trivial. ■

The first welcome consequence of this lemma is the following corollary that states that the sets  $I(\phi)$  are non empty.

**COROLLARY 3.1.2** *Fix a signature  $\Phi$ . Then for every formula  $\phi \in L_A$ ,  $I(\phi)$  is nonempty. □*

PROOF: By induction on formulas. The only interesting step is the negation step. Recall that  $I(\neg\phi) \subseteq (I(\phi) \times \omega)^\omega$  is the collection of all maps  $f = (f_1, f_2)$  such that:

$$\forall h \in I(\phi) \exists s \in \omega, \phi_{h, f_2(s)} \equiv \phi_{f_1(s), f_2(s)} \quad (3.1)$$

Lemma 3.1.1 states that  $D(\phi)$  is countable. Hence the set  $B = D(\phi) \times \omega \subseteq I(\phi) \times \omega$  is countable. Let  $f : \omega \mapsto I(\phi) \times \omega$  such that  $Im(f) = B$ . It is easy to check using Lemma 3.1.1 that  $f$  verifies statement 3.1, hence  $f \in I(\neg\phi)$ . ■

The next proposition shows that the notion of approximate truth is well behaved with respect to Boolean operations.

**PROPOSITION 3.1.3**

*Fix a signature  $\Phi$  and a model  $E$  of  $\Phi$ . Let  $\phi(\vec{x}), \phi_1, \dots, \phi_i, \dots$  be formulas in  $L_A$ .*

*Then the following is true:*

1.  $E \models_{AP} \neg\phi(\vec{a})$  if and only if  $E \not\models_{AP} \phi(\vec{a})$
2.  $E \models_{AP} \bigwedge_{i=1}^{\omega} \phi_i(\vec{a})$  if and only if for every integer  $i$ ,  $E \models_{AP} \phi_i(\vec{a})$
3.  $E \models_{AP} \neg\neg\phi(\vec{a})$  if and only if  $E \models_{AP} \phi(\vec{a})$ .
4.  $E \models_{AP} \bigvee_{i=1}^{\infty} \phi_i(\vec{a})$  if and only if there exists an  $i$  such that:

$$E \models_{AP} \phi_i(\vec{a})$$

5.  $E \models_{AP} (\phi(\vec{a}) \Rightarrow \psi(\vec{a}))$  if and only if:

$$(E \models_{AP} \phi(\vec{a})) \text{ implies } (E \models_{AP} \psi(\vec{a}))$$

□

PROOF: 1)

( $\Rightarrow$ ): Using the definition of  $I(\neg\phi)$  we have the following:

$$E \models_{AP} \neg\phi(\vec{a}) \Rightarrow \exists h = (h_1, h_2) \in I(\neg\phi), E^{app} \models \bigwedge_{n=1}^{\infty} \bigwedge_{s=1}^n \neg (\phi_{h_1(s), h_2(s)}(\vec{a}))$$

Furthermore,  $h$  (see Definition 2.4.5) verifies that for every  $f \in I(\phi) \exists s \in \omega$  such that:

$$\phi_{f,h_2(s)} \equiv \phi_{h_1(s),h_2(s)}$$

The two observations above imply:

$$\forall f \in I(\phi) \exists n, E^{app} \models \neg(\phi_{f,n}(\vec{a}))$$

but this last statement, by definition of  $\models_{AP}$ , implies

$$E \not\models_{AP} \phi(\vec{a})$$

This is the desired result.

( $\Leftarrow$ ): Suppose that

$$E \not\models_{AP} \phi(\vec{a})$$

It follows by definition of  $\models_{AP}$  that

$$\forall f \in I(\phi) \exists n E^{app} \models \neg(\phi_{f,n}(\vec{a}))$$

Consider then the set

$$W = \{(f,n) \mid f \in D(\phi) \wedge n \in \omega \wedge E^{app} \models \neg\phi_{f,n}(\vec{a})\}$$

By the properties of the set  $D(\phi)$  listed in Lemma 3.1.1 we know that:

- $W$  is countable (at most).
- If  $g \in I(\phi)$  and  $n \in \omega$  then there exists  $f \in D(\phi)$  such that  $\phi_{g,n} \equiv \phi_{f,n}$ .

Let then  $H = (h_1, h_2) : \omega \mapsto I(\phi) \times \omega$  be any function such that  $Im(H) = W$ .

From the previous remarks it follows that such a function exists. Furthermore we claim:

Claim:  $H \in I(\neg\phi)$ .

Proof: From the definition of  $I(\neg\phi)$  (Definition 2.4.5) we know that it is enough to check that

$$H : (H_1, H_2) : \omega \mapsto W \subseteq D(\phi) \times \omega \subseteq I(\phi) \times \omega$$

verifies

$$\forall f \in I(\phi) \exists s \in \omega \phi_{f, H_2(s)} \equiv \phi_{H_1(s), H_2(s)}$$

Fix then  $f \in I(\phi)$ . By hypothesis there exists an integer  $n$  such that  $E^{app} \models \neg\phi(\vec{a})_{f,n}$ . By the properties of the dense set  $D(\phi)$  mentioned above we know that there exists  $h \in D(\phi)$  such that:  $\phi_{f,n} \equiv \phi_{h,n}$ .

Furthermore, since  $E^{app} \models \neg(\phi(\vec{a})_{f,n}) \equiv \neg(\phi_{h,n}(\vec{a}))$  we know that  $(h, n) \in W$ . By definition of  $H$  we know then that there exists an  $s \in \omega$  such that  $(H_1(s), H_2(s)) = (h, n)$ .

It follows then that

$$\phi_{f,n} \equiv \phi_{h,n} \equiv \phi_{H_1(s), H_2(s)}$$

but this is the desired result. This completes the proof of the claim.

Finally, from the definition of  $H \in I(\neg\phi)$  we get:

$$E^{app} \models \bigwedge_{n=1}^{\infty} \bigwedge_{s=1}^n \neg(\phi_{H_1(s), H_2(s)}(\vec{a}))$$

i.e.

$$E \models_{AP} \neg\phi(\vec{a})$$

This is the desired result.

2)  $\Rightarrow$ . Suppose that  $E \models_{AP} \bigwedge_{i=1}^{\infty} \phi_i(\vec{a})$ . Then by definition it follows that there exists  $(h_1, h_2, \dots, h_n, \dots) \in \prod_{i=1}^{\infty} I(\phi_i)$  such that for every integer  $n$ ,

$$E^{app} \models \bigwedge_{i=1}^n (\phi_i(\vec{a}))_{h_i, n}$$

We get then that for every integer  $i$ ,

$$E^{app} \models \bigwedge_{n=i}^{\infty} (\phi_i(\vec{a}))_{h_i, n}$$

Invoking now Lemma 3.1.1 we finally get that for every integer  $i$ ,

$$E^{app} \models \bigwedge_{n=1}^{\infty} (\phi_i(\vec{a}))_{h_i, n}$$

and this is equivalent to saying that for every integer  $i$ ,

$$E \models_{AP} \phi_i(\vec{a})$$

( $\Leftarrow$ ). Left to the reader.

3), 4) and 5) follow directly from the previous items, using the definitions of the abbreviations  $\bigvee$  and  $\Rightarrow$ . Left to the reader. ■

We can partially extend the previous proposition to the quantifiers in  $L_A$ .

#### PROPOSITION 3.1.4

*Fix a signature  $\Phi$  and a model  $E$  of  $\Phi$ . Let  $\phi(\vec{x}), \phi_1, \dots, \phi_i, \dots$  be formulas in  $L_A$ .*

*Then the following is true:*

1. *If there exists  $\vec{b} = (\vec{b}_1, \vec{b}_2, \dots) \in E$  such that  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i)$  and  $E \models_{AP} \phi(\vec{b}, \vec{a})$*

*then:*

$$E \models_{AP} \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}).$$



2. If  $E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a})$  then:

For every  $\vec{b}$  in  $E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i)$  then  $E \models_{AP} \phi(\vec{b}, \vec{a})$ .

□

PROOF: 2) is the dual of 1), and follows directly from 1). So let us prove 1). Suppose that there exists  $\vec{b} = (\vec{b}_1, \vec{b}_2, \dots) \in E$  such that  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i)$  and  $E \not\models_{AP} \phi(\vec{b}, \vec{a})$ . then by definition of  $\models_{AP}$ , there exists  $h \in I(\phi)$  such that for every integer  $n$ ,

$$E^{app} \models (\phi(\vec{b}, \vec{a}))_{h,n}$$

It follows that for every integer  $n$ ,

$$E^{app} \models \exists \vec{x} \in \vec{K}(\phi(\vec{x}, \vec{a}))_{h,n} \equiv (\exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}))_{h,n}$$

but this is exactly,

$$E \models_{AP} \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a})$$

This is the desired result. ■

The above propositions state that  $\models_{AP}$  is well behaved for the usual connectives and quantifiers. Thanks to it in many of the coming proofs we will not use the cumbersome definitions of approximate formulas but rather the properties of  $\models_{AP}$  proved in Proposition 3.1.3.

In the next section we describe the relationship between approximate truth and convergence in metric spaces.

## 3.2 Approximate Truth and Convergence

We introduce some notation needed in this section.

Let us abbreviate “for all except a finite number of integers  $k$ ” by  $\forall^*k$ . Likewise  $\exists^\infty k$  would stand for “there exists infinitely many integers  $k$ ”.

Let  $\psi$  a formula in  $L_A$ . Recall that  $\psi(\vec{x})$  means that the free variables of  $\psi$  are among the (distinct) elements of  $\vec{x}$ .

Let  $\vec{f} = (f_1, \dots, f_k, \dots)$  be a vector of (distinct) function symbols in  $\Phi$  with fixed sort. The arity of  $\vec{f}$  could be finite or countable. By  $\psi(\vec{x})[\vec{f}]$  we mean that all the function symbols appearing in  $\psi(\vec{x})$  are among the elements of  $\vec{f}$ .

A vector  $\vec{g}$  of function symbols of fixed sort is **similar** to  $\vec{f}$  iff for every integer  $i$ ,  $f_i, g_i$  have the same sort and the same arity. In this case, given the formula  $\psi(\vec{x})[\vec{f}]$ , the formula  $\psi(\vec{y})[\vec{g}]$  denotes the formula in  $L_A$  obtained from  $\psi(\vec{x})[\vec{f}]$  by substituting the occurrences of the variable  $x_i$  by  $y_i$  and by substituting the occurrences of the function symbol  $f_i$  by the function symbol  $g_i$ .

Note that if  $\vec{g}$  is similar to  $\vec{f}$  then for every formula  $\psi \in L_A$ ,  $I(\psi(\vec{x})[\vec{f}]) = I(\psi(\vec{y})[\vec{g}])$ .

Finally, given any structure  $E$ , and any function symbols  $f_n, f$  with the same sort, we will say that  $f_n$  **uniformly converges** to  $f$  in  $E$  ( $f_n \longmapsto f$ ) iff for every  $(\vec{a}_n)$  in  $E$  converging to  $\vec{a}$ ,  $f_n(\vec{a}_n)$  converges to  $f(\vec{a})$ . In a similar way we say that the sequence of vectors of function symbols  $\{\vec{f}_n\}_{n=1}^\infty$  uniformly converges to a vector of function symbols  $\vec{f}$  similar to the  $\vec{f}_n$  iff for every integer  $i$ , the sequence of function symbols of the  $i^{\text{th}}$  element of the vectors  $\{\vec{f}_n\}_{n=1}^\infty$  converge uniformly to the  $i^{\text{th}}$  element of the vector of function symbols  $\vec{f}$ .

We are interested, for arbitrary models  $E$ , in the class of formulas for which  $\models_{AP}$  and the metric notion of convergence are well behaved in  $E$ . Let us introduce the following definition.

**DEFINITION 3.2.1** *Definition of Class of Convergent Formulas*

Fix a signature  $\Phi$  and let  $E$  a model for  $\Phi$ . The class of the **convergent formulas** for  $E$  is the collection of all the formulas  $\phi(\vec{x})[\vec{f}] \in L_A$  verifying the following property:

- If the sequence  $\{\vec{b}_k\}_{k=1}^{\infty}$  converges  $\{\vec{b}\}$ .
- If the sequence of vectors of function symbols  $\{\vec{f}_k\}_{k=1}^{\infty}$  converge uniformly in  $E$  to the vector  $\vec{f}$ .

Then the following are equivalent:

•

$$\exists h \in I(\phi) \forall n \in \omega \forall^* k, E^{app} \models (\phi(\vec{b}_k)[\vec{f}_k])_{h,n}$$

•

$$E \models \phi(\vec{b})[\vec{f}]$$

The next theorem gives the main property of the classes of convergent formulas.

**THEOREM 3.2.2** *Classes of Convergent Formulas*

*Fix a collection  $S$  of metric spaces. Fix a signature  $\Phi$  for  $S$ , a and an arbitrary model  $E = (X, d, F, P)$  of  $\Phi$ . Let  $\Delta$  the collection of all the convergent formulas for  $E$ .*

*Then  $\Delta$  is a class of formulas closed under countable (and finite) conjunction and negation.*

□

PROOF: Fix  $\Delta$  the class of all the convergent formulas in  $E$ .

**Conjunction.** Let  $\Psi(\vec{x}) = \bigwedge_{i=1}^{\infty} \phi_i(\vec{x})$ , with, for every integer  $i$ ,  $\phi_i \in \Delta$ . Let us prove that  $\Psi$  is also in  $\Delta$ . We need to prove that the statements in Definition 3.2.1 are equivalent.

- $(\Leftarrow)$ . Direct.
- $(\Rightarrow)$  Suppose that there exists an  $H \in I(\Psi(\vec{x}))$  such that for every integer  $n$  and  $\forall^* k$  we have:

$$E^{app} \models \left( \bigwedge_{i=1}^{\infty} \phi_i(\vec{b}_k)[\vec{f}_k] \right)_{H,n}$$

then  $\forall n \in \omega \forall^* k \in \omega$ :

$$E^{app} \models \bigwedge_{i=1}^n (\phi_i(\vec{b}_k)[\vec{f}_k])_{H(i),n}$$

The last statement implies that for every  $i \in \omega$ , for every integer  $n \geq i$  and  $\forall^* k \in \omega$   $E^{app} \models (\phi_i(\vec{b}_k)[\vec{f}_k])_{H(i),n}$ . Using 1) of Lemma 3.1.1, we can transform this statement into:

$$\forall i \in \omega \forall n \in \omega \forall^* k \in \omega E^{app} \models (\phi_i(\vec{b}_k)[\vec{f}_k])_{H(i),n}$$

Using the fact that the  $\phi_i$ 's are in  $\Delta$  we conclude that  $E \models \bigwedge_{i=1}^{\omega} \phi_i(\vec{b})$ .

**Negation.** Suppose that  $\Psi(\vec{x}) = \neg\phi(\vec{x})$  and that  $\phi \in \Delta$ . Once again, we need to prove that the statements in Definition 3.2.1 are equivalent.

- $(\Leftarrow)$ . Suppose that  $E \models (\neg\phi(\vec{b}))$ . Since  $\phi \in \Delta$ , this implies that for any  $h \in I(\phi(\vec{x}))$  there exists  $n \in \omega$  such that:

$$\exists^{\infty} k E^{app} \not\models (\phi(\vec{b}_k)[\vec{f}_k])_{h,n}$$

Actually, we have an stronger result:

Claim: for any  $h \in I(\phi(\vec{x}))$  there exists  $n \in \omega$  such that:

$$\forall^* k \ E^{app} \not\models (\phi(\vec{b}_k)[\vec{f}_k])_{h,n} \quad (3.2)$$

Proof: By contradiction. If this was not true, then there would exist a  $h \in I(\phi)$  and a subsequence  $k_1 < k_2 < \dots$  of integers such that

$$\forall n \forall^* i \ E \models (\phi(\vec{b}_{k_i})[\vec{f}_{k_i}])_{h,n}$$

Since  $\{\vec{f}_{k_i}\}_{i=1}^\infty$  uniformly converges to  $\vec{f}$  and  $\{\vec{b}_{k_i}\}_{i=1}^\infty$  converge to  $\vec{b}$  we can use the fact that  $\phi \in \Delta$  to obtain:

$$E \models \phi(\vec{b})[\vec{f}]$$

but this contradicts the hypothesis. ■

Define now a function  $H : \omega \mapsto (I(\phi(\vec{x})) \times \omega)$  with the property that:

$$Image(H) = \{(g, n) \in (D(\phi(\vec{x})) \times \omega) \mid \forall^* k \in \omega \ E^{app} \not\models (\phi(\vec{b}_k)[\vec{f}_k])_{g,n}\}$$

(It is possible to define such a function because the cardinality of  $D(\phi(\vec{x}))$  is at most countable by Lemma 3.1.1).

The function  $H$  belongs to  $I(\neg\phi(\vec{x}))$  because statement 3.2) implies :

$$\forall g \in I(\phi) \exists n \forall^* k \in \omega \ E^{app} \models \neg(\phi(\vec{b}_k)[\vec{f}_k])_{g,n}$$

Hence, using the fact that  $D(\phi)$  is a “dense” subset of  $I(\phi)$  we get that:

$$\forall g \in I(\phi) \exists n \exists f \in D(\phi) \forall^* k \in \omega \ E^{app} \models \neg(\phi(\vec{b}_k)[\vec{f}_k])_{g,n} \equiv \neg(\phi(\vec{b}_k)[\vec{f}_k])_{f,n}$$

and  $(f, n) \in \text{Image}(H)$ .

Finally, from the definition of  $H$  it is clear that for every integer  $n, \forall^* k \in \omega$ :

$$E^{app} \models \bigwedge_{i=1}^n \neg((\phi(\vec{b}_k)[\vec{f}_k])_{H_1(i), H_2(i)})$$

In summary: there exists an  $H \in I(\neg\phi(\vec{x}))$  such that for every integer  $n, \forall^* k \in \omega$ :

$$E^{app} \models (\bigwedge_{i=1}^n (\neg\phi(\vec{b}_k)[\vec{f}_k])_{H, n})$$

This completes the proof.

- $(\Rightarrow)$ . Direct. ■

A result similar to Theorem 3.2.2 can be obtained for the collection of **rich formulas** for a fixed model  $E$ . The rich formulas of a fixed model  $E$  is the collection of all the formulas  $\phi(\vec{x}) \in L_A$  such that for every vector  $\vec{b}$  in  $E$ :

$$E \models \phi(\vec{b}) \text{ iff } E \models_{AP} \phi(\vec{b})$$

**COROLLARY 3.2.3** *Fix a collection  $S$  of metric spaces. Fix a signature  $\Phi$  for  $S$ , and an arbitrary model  $E = (X, d, F, P)$  of  $\Phi$ . Let  $\Gamma$  the collection of rich formulas for  $E$ .*

*Then  $\Gamma$  is closed under negation and countable conjunction.*

□

PROOF: Similar to the proof of Theorem 3.2.2. Left to the reader. ■

This result extends similar results by Henson for the case of positive bounded formulas (see [19]) and of Fajardo and Keisler ([8]).

### 3.3 Approximation Principles in $L_A$

We will use the results of the previous sections to obtain approximation principles for some classes of formulas in  $L_A$ . We begin with a corollary concerning the quantifier free formulas in  $L_A$ .

**COROLLARY 3.3.1** *Fix a collection  $S$  of metric spaces. Fix a signature  $\Phi$  for  $S$ , and an arbitrary model  $E = (X, d, F, P)$  of  $\Phi$ . Let  $\Delta$  the collection of all the convergent formulas for  $E$ . The  $\Delta$  contains the quantifier free formulas of  $L_A$ .*

□

PROOF: It follows from Theorem 3.2.2 if we verify that  $\Delta$  contains the atomic formulas. But this follows by definition of model. The predicates in  $\Phi$  are closed, have finite arity and the functions in  $\Phi$  are continuous with respect to the product topology induced by the metric.■

The next example shows that the above corollary is not true for arbitrary formulas in  $L_A$ .

**EXAMPLE 3.3.2** *Corollary 3.3.1 fails in general for the existential step*

The essential parts of this example are taken from [11].

The collection  $S$  of metric spaces contains only the real numbers with the usual metric. The signature  $\Phi = (\mathcal{F}, \mathcal{P})$  on  $S$  consists of a symbol for a function  $T$  with arity one and true sort, an unary predicate symbol  $K$  with true sort, and a closed predicate  $C = \{x \in \mathfrak{R} \mid |x| \leq 0\}$  of sort  $(\mathfrak{R}, d)$ .

Consider the following model  $E = (X, d, F, P)$  of  $\Phi$ .  $X = \mathcal{C}[0,1]$  is the space of continuous real valued functions on  $[0,1]$ , and the metric  $d$  is the sup norm.

Let  $T : X \mapsto X$  be such that for every  $x \in X = \mathcal{C}[0,1]$ ,  $(T(x))(t) = t.x(t)$ . It is easy to see that this map verifies that for every  $x, y$  in  $X$ ,  $d(T(x), T(y)) \leq d(x, y)$ . Hence the map is continuous.

Let  $K = \{x \in X : 0 = x(0) \leq x(t) \leq x(1) = 1\}$ . This set is closed and bounded.

Note first that the map  $T$  does not have a fixed point in  $K$ , i.e. for every  $x$  in  $K$ ,  $T(x) \neq x$ . If this were not true there would exist an  $x$  in  $K$  such that for every  $0 \leq t \leq 1$ ,  $x(t) = t.x(t)$ : a contradiction.

In other words, we get that:

$$E \models \neg(\exists x \in K C(\rho(x, T(x))) )$$

Finally, for every  $\epsilon \geq 0$  select any function  $x \in K$  with the property that:  $\forall t \in [0, 1], |x(t)| \leq \epsilon$ . Then it is easy to see that  $d(T(x), x) \leq \epsilon$ .

We get then:

$$E \models_{AP} \exists x \in K C(\rho(T(x), x))$$

This completes the example.  $\square$

Nevertheless we can extend Corollary 3.3.1 to some types of universal and existential formulas. We begin by proving a weak version of Corollary 3.3.1.

**LEMMA 3.3.3** *Universal/Existential formulas*

*Fix a signature  $\Phi$ . Consider a quantifier free formula:  $\phi(\vec{x}, \vec{y})$ . Let  $\vec{K}$  a vector of arity corresponding to  $\vec{x}$  and made of true sort predicate symbols. Then for every model  $E$ , for every  $\vec{b}$  in  $E$ , the following holds:*

1. *If  $E \models \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$ , then  $E \models_{AP} \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$ .*



2. If  $E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$  then  $E \models \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$ .

□

PROOF: 1) Suppose that  $E \models \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$ . Then we know that there exists  $\vec{a}$  in  $E$  such that  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  and  $E \models \phi(\vec{a}, \vec{b})$ . Since  $\phi$  is a quantifier free formula, we can invoke Theorem 3.2.2 to obtain that  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  and  $E \models_{AP} \phi(\vec{a}, \vec{b})$ . We now use Proposition 3.1.4 to obtain that:

$$E \models_{AP} \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$$

and this is the desired result.

2) Suppose that

$$E \models_{AP} \forall \vec{x} (\bigwedge_{i=1}^{\infty} K_i(\vec{x}_i) \Rightarrow \phi(\vec{x}, \vec{b}))$$

Then by the Soundness Proposition for quantifiers (Proposition 3.1.4) we obtain that: for every  $\vec{a} \in E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models_{AP} \phi(\vec{a}, \vec{b})$ .

Since  $\phi$  is a quantifier free formula, we can invoke Theorem 3.2.2 to conclude that for every  $\vec{a} = (a_1, ..a_i, ..)$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models \phi(\vec{a}, \vec{b})$ . This is the desired result.■

It is clear that the converse of this lemma does not need to be true. It could happen that every collection of vectors  $\vec{a} = (a_1, ..a_i, ..)$  approaches  $\phi(\vec{x}, \vec{b})$  along different paths so that:

$$E \models \forall \vec{x} (\bigwedge_{i=1}^{\infty} K_i(\vec{x}_i) \Rightarrow \phi(\vec{x}, \vec{b}))$$

But it is not true that:

$$E \models_{AP} \forall \vec{x} (\bigwedge_{i=1}^{\infty} K_i(\vec{x}_i) \Rightarrow \phi(\vec{x}, \vec{b}))$$

However if the formulas we are dealing with have a “minimal” path, the converse of the above holds.

**DEFINITION 3.3.4** *Definition of  $L_{PBA}$*

Let  $L_{PBA}$ , the collection of all the infinitary positive bounded formulas in  $L_A$ , be the smallest subset of  $L_A$  containing the atomic formulas, and closed under infinite conjunction ( $\bigwedge$ ), finite disjunction ( $\bigvee$ ) and bounded universal and existential quantification over countable many variables ( $\exists \vec{x} \in \vec{K}, \forall \vec{x} \in \vec{K}$ ).  $\square$

The next lemma states that, as expected, every formula in  $L_{PBA}$  has a “minimal” path.

**LEMMA 3.3.5** *Fix a signature  $\Phi$ . Consider a formula  $\phi(\vec{x}) \in L_{PBA}$ . There exists  $H \in I(\phi)$  such that for every model  $E$ , for every  $\vec{a}$  in  $E$ ,*

$$\text{If } E \models \phi(\vec{a}) \text{ then } E^{app} \models \bigwedge_{n=1}^{\infty} \phi_{H,n}(\vec{a})$$

$\square$

**PROOF:** By induction on formulas. The atomic case is trivial since there is only one path.

**Conjunction.** Consider  $\bigwedge_{i=1}^{\infty} \phi_i$ . For every  $i$ , let  $H_i \in I(\phi_i)$  be as in the statement of this lemma. Define

$$H = (H_1, \dots, H_n, \dots) \in \prod_{i=1}^{\infty} I(\phi_i) \equiv I\left(\bigwedge_{i=1}^{\infty} \phi_i\right)$$

We leave to the reader to verify that this path has the desired property.

**Finite Disjunction.** Consider  $\bigvee_{i=1}^r \phi_i$ . For every  $i$ , let  $H(i) \in I(\phi_i)$  be as in the statement of this lemma.

*Claim: For every  $i < r$ , for every  $h_i \in I(\neg\phi_i)$ , there exists an integer  $n(h_i) > r$  so that for every model  $E$ , for every  $\vec{a}$  in  $E$ ,*

$$\exists i \leq r \ E \models \phi_i \Rightarrow \forall (h_1, \dots, h_r) \in \prod_{i=1}^r I(\neg\phi_i), \ E^{app} \models \neg \bigwedge_{i=1}^r (\neg\phi_i)_{h_i, n(h_i)}$$

*Proof:* For every  $i \leq r$ , every  $h \in I(\neg\phi_i)$  is by definition a function  $h = (f, g) : \omega \mapsto I(\phi_i) \times \omega$  that verifies:

$$\forall g \in I(\phi_i) \exists s \ \phi_{f_1(s), f_2(s)} \equiv \phi_{g, f_2(s)}$$

It follows then that for every  $h \in I(\neg\phi_i)$  there exists integers  $n, m$  such that

$$(\phi_i)_{H(i), m} \equiv (\phi_i)_{h(n)}$$

Define then  $n(h) = \max\{n, r\}$ .

Using the induction hypothesis we get then the following claim:

CLAIM: for every model  $E$ , for every  $\vec{a}$  in  $E$  :

$$E \models \phi_i(\vec{a}) \Rightarrow E^{app} \not\models \bigwedge_{s=1}^{n(h)} \neg(\phi_i(\vec{a}))_{h(s)}$$

*Proof:* If this was not true, then

$$E^{app} \models \bigwedge_{s=1}^{n(h)} \neg(\phi_i(\vec{a}))_{h(s)}$$

In particular then, by the definition of  $n(h)$ , there exists an integer  $m$  such that:

$$E^{app} \models \neg(\phi_i(\vec{a}))_{H(i), m}$$

but this implies, using the induction hypothesis, that  $E \not\models \phi_i(\vec{a})$  contradicting the hypothesis of the claim. this completes the proof of the claim.

In summary, for every  $i \leq r$ , for every model  $E$ , for every  $h \in I(\neg\phi_i)$ , for every  $\vec{a}$  in  $E$ ,

$$E \models \phi_i(\vec{a}) \Rightarrow \forall h \in I(\neg\phi_i), E^{app} \models \neg (\neg\phi_i(\vec{a}))_{h,n(h)}$$

But this implies easily the desired result. This completes the proof of the claim.

Using the claim we define the following. For every  $\vec{h} = (h_1, ..h_r) \in I(\bigwedge_{i=1}^r \neg\phi_i) \equiv \prod_{i=1}^r I(\neg\phi_i)$ , let  $n(\vec{h}) = \max\{n(h_i) | i \leq r\}$ .

Finally, let  $H \in I(\neg\bigwedge_{i=1}^r \neg\phi_i)$  be any function  $H = (F, G) : \omega \mapsto I(\bigwedge_{i=1}^r \neg\phi_i) \times \omega$  such that  $Image(H) = \{(\vec{h}, n(\vec{h})) | \vec{h} \in D(\bigwedge_{i=1}^r \neg\phi_i)\}$ . It is easy to verify that such a function exists (using Lemma 3.1.1).

We obtain then for every model  $E$  and every  $\vec{a}$  in  $E$ ,

$$\begin{aligned} E \models \bigvee_{i=1}^r \phi_i(\vec{a}) &\Rightarrow \\ \forall (h_1, ..h_r) \in \prod_{i=1}^r I(\neg\phi_i), E^{app} \models \neg \bigwedge_{i=1}^r (\neg\phi_i)_{h_i, n(h_i)} &\text{ (by the above claim)} \\ \Rightarrow \forall \vec{h} \in I(\bigwedge_{i=1}^r \neg\phi_i), E^{app} \models \neg (\bigwedge_{i=1}^r \neg\phi_i(\vec{a}))_{\vec{h}, n(\vec{h})} & \\ \Rightarrow E^{app} \models \bigwedge_{n=1}^{\infty} (\neg \bigwedge_{i=1}^r \neg\phi_i(\vec{a}))_{H, n} &\text{ by definition of approximate formulas.} \end{aligned}$$

This completes the proof of the disjunction step.

**Existential.** Consider a formula

$$\exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y})$$

Suppose that the desired result holds for  $\phi$ . Let  $G$  be the “minimal” path on  $I(\phi)$ .

We claim that  $G$  is the desired path for  $\exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y})$ .

Assume that there exists  $\vec{a}$  in  $E$  such that

$$E \models \exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a})$$

Then there exists  $\vec{b} \in E$  such that  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i)$  and

$$E \models \phi(\vec{b}, \vec{a})$$

By induction hypothesis,

$$E^{app} \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i) \wedge \bigwedge_{n=1}^{\infty} (\phi(\vec{b}, \vec{a}))_{G,n}$$

It follows that

$$E^{app} \models \bigwedge_{n=1}^{\infty} (\exists \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}))_{G,n}$$

and this is the desired result.

**Universal.** Consider a formula

$$\forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y})$$

Suppose that the desired result holds for  $\phi$ . Let  $G$  be the “minimal” path on  $I(\phi)$ .

Let  $h$  be any element of  $I(\neg\phi)$ . By definition,  $h = (h_1, h_2) : \omega \mapsto I(\phi) \times \omega$  with the property that:

$$\forall g \in I(\phi) \exists s (\phi)_{h_1(s), h_2(s)} \equiv (\phi)_{g, h_2(s)}$$

In particular then there exists  $n(h)$  such that:

$$(\phi)_{h_1(n(h)), h_2(n(h))} \equiv (\phi)_{G, h_2(n(h))}$$

Define now  $H \in I(\neg\exists \vec{x} \in \vec{K} \neg\phi(\vec{x}, \vec{y}))$  as any function  $H = (H_1, H_2) : \omega \mapsto I(\neg\phi) \times \omega$  such that  $Image(H) = \{(h, n(h)) | h \in D(\neg\phi)\}$ . Once again, Lemma 3.1.1 guarantees that such a function exists.

*Claim:  $H$  is the desired minimal path.*

Proof: Assume, in order to get a contradiction, that  $E \models \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a})$  but it is false that  $E^{app} \models \bigwedge_{m=1}^{\infty} (\forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}))_{H,m}$ .

It follows that for every  $\vec{b}$  in  $E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{b}_i)$  then

$$E \models \phi(\vec{b}, \vec{a}) \quad (3.3)$$

On the other hand, since  $E^{app} \not\models \bigwedge_{m=1}^{\infty} (\forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}))_{H,m}$ , we know that there exists an integer  $m$  such that:

$$E^{app} \not\models (\forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}))_{H,m} \equiv (\neg \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a}))_{H,m} \equiv \bigwedge_{s=1}^m \neg \exists \vec{x} \in \vec{K} (\neg \phi(\vec{x}, \vec{a}))_{H_1(s), H_2(s)}$$

This is the same as saying that there exists  $h = (h_1, h_2) \in I(\neg \phi)$  such that

$$E^{app} \models \exists \vec{x} \in \vec{K} (\neg \phi(\vec{x}, \vec{a}))_{h, n(h)} \equiv \exists \vec{x} \in \vec{K} \bigwedge_{s=1}^{n(h)} \neg (\phi(\vec{x}, \vec{a}))_{h_1(n(h)), h_2(n(h))}$$

so, in particular,

$$E^{app} \models \exists \vec{x} \in \vec{K} \neg (\phi(\vec{x}, \vec{a}))_{G, n(h)}$$

By induction hypothesis on  $\phi$  and the fact that  $G$  is the “minimal” path of  $\phi$  we obtain that :

$$E \models \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a})$$

But this contradicts statement (3.3). This completes the proof of the universal quantification step and of the Lemma. ■

The existence of “minimal” paths is lost in general if we consider formulas that are not infinite positive bounded.

### EXAMPLE 3.3.6

Consider a signature that contains the closed predicate  $x \geq 1$  of sort space the reals with the usual metric. Clearly  $I(\neg\rho(x, y) \geq 1)$  can be identified with  $\omega^\omega$ , and it is easy to see that there is no “minimal” path in  $I(\neg\rho(x, y) \geq 1)$ . For every  $f \in I(\neg\rho(x, y) \geq 1)$  there exists  $g \in I(\neg\rho(x, y) \geq 1)$  such that  $\forall s, g(s) > f(s)$ . We can find a metric structure  $E$  and  $(a, b)$  in  $E$  such that:

$$E^{app} \models \bigwedge_{n=1}^{\infty} (\neg\rho(a, b) \geq 1)_{g,n} \equiv \bigwedge_{n=1}^{\infty} \bigwedge_{s=1}^n \neg(\rho(a, b) \geq 1)_{g(s)} \Leftrightarrow \\ \neg(\rho(a, b) \geq 1 - \max\{1/g(s) : s \in \omega\})$$

but there exists an integer  $s$  such that:

$$E^{app} \models (\rho(a, b) \geq 1)_{f(s)} \equiv \rho(a, b) \geq 1 - 1/f(s)$$

which implies that:

$$E^{app} \not\models \bigwedge_{n=1}^{\infty} (\neg\rho(a, b) \geq 1)_{f,n} \equiv \bigwedge_{n=1}^{\infty} \bigwedge_{s=1}^n \neg(\rho(a, b) \geq 1)_{f(s)}$$

□

Using the previous lemma we extend Corollary 3.3.1 to a subclass of  $L_A$ .

**DEFINITION 3.3.7** *Definition of  $L_P$*

Let  $L_P$  be the smallest subset of  $L_{PBA}$  containing the atomic formulas and closed under finite and infinite conjunction  $\{\wedge, \bigwedge\}$ , finite disjunction  $\{\vee\}$  and universal bounded quantification over countably many variables  $\{\forall \vec{x} \in \vec{K}\}$ . In other words,  $L_P$  is the class of all the positive bounded universal formulas in  $L_{PBA}$ . □

Recall that the notation  $\neg L_P$  denotes the set of formulas:  $\{\neg\sigma : \sigma \in L_P\}$ . Likewise, the notation  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$  denotes the smallest set of formulas in  $L_A$  containing the formulas:

- $\sigma(\vec{x}, \vec{y})$
- $\exists \vec{x} \in \vec{K} \neg \sigma(\vec{x}, \vec{y})$

for  $\sigma(\vec{x}, \vec{y}) \in L_P$  and closed under  $\wedge, \neg$ . This class is big enough to contain all the quantifier free formulas in  $L_A$ , and the formulas of the form  $\bigwedge_{i=1}^{\infty} \exists \vec{x} \in \vec{K} \neg \phi_i(\vec{x})$ ,  $\bigvee_{i=1}^{\infty} \exists \vec{x} \in \vec{K} \neg \phi_i(\vec{x})$  and  $\bigvee_{i=1}^{\infty} \forall \vec{x} \in \vec{K} \phi_i(\vec{x})$  with the  $\phi_i$  positive bounded universal formulas (i.e. in  $L_P$ ).

### THEOREM 3.3.8

*Fix a signature  $\Phi$  and an arbitrary model  $E$  for  $\Phi$ .*

1. *Consider a formula  $\phi$  in  $L_P$ . Then for every vector  $\vec{b}$  in the structure  $E$ ,*

$$E \models_{AP} \phi(\vec{b}) \text{ iff } E \models \phi(\vec{b})$$

2. *Consider a formula  $\phi$  in  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ . Then for every vector  $\vec{b}$  in the structure  $E$ ,*

$$E \models_{AP} \phi(\vec{b}) \text{ iff } E \models \phi(\vec{b})$$

□

PROOF: 1) By induction on the formulas in  $L_P$ . The only interesting step is the universal one.

**Universal.** ( $\Rightarrow$ ). Suppose that

$$E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$$



By the Soundness Proposition 3.1.4, it follows that for every  $\vec{a}$  in  $E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models_{AP} \phi(\vec{a}, \vec{b})$ . Invoking now the induction hypothesis we obtain that for every  $\vec{a}$  in  $E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models \phi(\vec{a}, \vec{b})$ . This is the desired result.

$\Leftarrow$ : Suppose that

$$E \models \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$$

Let  $H \in I(\phi)$  a minimal path as in Lemma 3.3.5 above. It follows that:

$$E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$$

This completes the proof.

2) By item 1) of this theorem we know that the desired property is true for  $L_P$ . Let us prove it for  $\exists \neg L_P$ . Fix an arbitrary formula  $\phi(\vec{x}, \vec{y}) \in L_P$ . Item 1) of this theorem implies that for every model  $E = (X, d, F, P)$ , for every  $\vec{a} \in X^{|\vec{y}|}$ ,

$$E \models \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a}) \text{ iff } E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{a})$$

But this is equivalent to the statement:

$$E \models \neg \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a}) \text{ iff } E \models_{AP} \neg \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a})$$

Using the properties of the Soundness Proposition (Proposition 3.1.3) we obtain the following equivalence:

$$E \models \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a}) \text{ iff } E \models_{AP} \exists \vec{x} \in \vec{K} \neg \phi(\vec{x}, \vec{a})$$

This is the desired result.

The final verification of the desired property for the set  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  follows by induction on the connectives  $\{\wedge, \neg\}$ . Left to the reader. ■

We close this section with a “weak” approximation principle for a big subcollection of formulas in  $L_A$ .

**DEFINITION 3.3.9** *Definition of  $L_{A+}$* 

Let  $L_{A+}$  be the smallest subset of  $L_A$  containing  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}} \cup L_{PBA}$  and closed under countable conjunction, countable disjunction and existential quantification over countably many variables. That is,

$$L_{A+} = ((L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}} \cup L_{PBA})^{\{\wedge, \vee, \exists\}}$$

.  $\square$

$L_{A+}$  contains for example all the quantifier free formulas, all the formulas in  $L_{PBA}$  and all the existential formulas  $\exists \vec{x} \in \vec{K} \phi(\vec{x})$  with  $\phi$  an infinitary quantifier free formula. It also contain formulas of the form

$$\exists \vec{x} \in \vec{K} \bigwedge_{i=1}^{\infty} \forall \vec{y} \in \vec{P}_i \phi_i(\vec{x}, \vec{y})$$

and of the form

$$\exists \vec{x} \in \vec{K} \bigvee_{i=1}^{\infty} \forall \vec{y} \in \vec{P}_i \phi_i(\vec{x}, \vec{y})$$

with the  $\phi_i$  positive bounded formulas in  $L_{PBA}$ . A typical formula that is not in  $L_{A+}$  is:

$$\forall \vec{x} \in \vec{K} \bigvee_{i=1}^{\infty} C^i(x_i)$$

where the  $C^i$ 's are unary predicates with true sort.

The following theorem says that  $\models$  implies  $\models_{AP}$  for all the formulas in  $L_{A+}$ .

**THEOREM 3.3.10** *Fix a signature  $\Phi$  and a model  $E = (X, d, F, P)$  of  $\Phi$ . Let  $\phi(\vec{x}) \in L_{A+}$ . Then for every  $\vec{a} \in X^{|\vec{x}|}$ ,*

$$\text{If } E \models \phi(\vec{a}) \text{ then } E \models_{AP} \phi(\vec{a})$$

$\square$

PROOF:By induction on formulas. The case for  $(L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  follows from Theorem 3.3.8. Likewise, the case for  $L_{PBA}$  follows from Lemma 3.3.5.

**Conjunction.** Suppose that  $E \models \bigwedge_{i=1}^{\infty} \phi_i(\vec{a})$ . Then for every  $i$ ,  $E \models \phi_i(\vec{a})$ . By the induction hypothesis for every  $i$ ,  $E \models_{AP} \phi_i(\vec{a})$ . Invoking now Proposition 3.1.3 we obtain that  $E \models_{AP} \bigwedge_{i=1}^{\infty} \phi_i(\vec{a})$ . This is the desired result.

**Disjunction.** Suppose that  $E \models \bigvee_{i=1}^{\infty} \phi_i(\vec{a})$ . Then there exists  $i$  such that  $E \models \phi_i(\vec{a})$ . By the induction hypothesis  $E \models_{AP} \phi_i(\vec{a})$ . Invoking now the Soundness Proposition (Proposition 3.1.3) we obtain that  $E \models_{AP} \bigvee_{i=1}^{\infty} \phi_i(\vec{a})$ . This is the desired result.

**Existential.** Direct. Left to the reader.■

The above Theorem can be seen as an extension of similar weak approximation results ( $\models \Rightarrow \models_{AP}$ ) proved by Keisler & Fajardo ([8]) for  $L_{PBA}$ .

## 3.4 Approximation Principles in First Order Logic

In this section we prove some approximation principles for the notion of approximate truth for first order formulas in  $L_A$  (i.e. the collection  $L_{\omega\omega}$  of formulas constructed from the atomic predicates by iterating finite conjunction, negation and bounded existential quantification) in the classical multisorted models.

Recall the definition of classical multisorted models given in Example 2.3.3:

Let  $S$  be an arbitrary collection of discrete metric spaces, i.e. spaces  $(M, \rho_M)$  so that  $\text{Image}(\rho_M) = \{0, 2\}$ . Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  containing the fixed universal predicate  $K$  in  $\mathcal{P}$ .  $E = (X, d, F, P)$  is a classical multisorted structure if  $\text{Im}(d) = \{0, 2\}$  and for every  $a$  in

$$E, E \models K(a).$$

Recall also that every first order model can be seen as a classical multisorted structure. We remark also that for classical multisorted models,  $\Phi^{app}$  is essentially the same as  $\Phi$  since the approximate predicates  $C_n$  are just identical to  $C$  (for true sort predicates or fixed sort predicates). In this section we will then omit the  $^{app}$ .

The next lemma shows that as expected, in classical multisorted models, approximate truth along a specific path is eventually constant for first order formulas in  $L_A$ .

**LEMMA 3.4.1**

*Let  $S$  be an arbitrary collection of discrete metric spaces, i.e. spaces  $(M, \rho_M)$  so that  $\text{Image}(\rho_M) = \{0, 2\}$ . Let  $\phi(x) \in L_{\omega\omega} \subset L_A$  be a first order formula. Let  $\mathcal{M}$  be the collection of all the classical multisorted models for  $\Phi$ . The following holds:*

1. *There exists a finite collection  $F$  of formulas in  $L_{AP}$  such that  $\forall h \in I(\phi), \forall n \exists \theta \in F$  such that:*

$$\forall E \in \mathcal{M} \forall \vec{a} \text{ in } E, E \models \phi_{h,n}(\vec{a}) \Leftrightarrow \theta(\vec{a})$$

2.  *$\forall h \in I(\phi) \exists m \in \omega$  such that for every classical multisorted structure  $E$  of  $\Phi$ ,*

$$\forall \vec{a} \in E \forall n \geq m, E \models \phi_{h,n}(\vec{a}) \Leftrightarrow \phi_{h,m}(\vec{a})$$

□

PROOF: 1) By induction on the collection  $L_{\omega,\omega}$  of first order formulas.

**Atomic.** Direct since the metrics involved are discrete (they take values in  $\{0, 2\}$ ) so  $\forall n, C_n$  is equivalent to  $C$  in every classical multisorted structure.

**Finite conjunction.** Direct from induction on formulas and the definition of approximate formulas for  $\bigwedge_{i=1}^p \phi_i$ .

**Negation.** Let  $\psi(\vec{x}) = \neg\phi(\vec{x})$ . By induction hypothesis there exists a finite collection  $F_1$  of formulas in  $L_{AP}$  satisfying 1) above for  $\phi$ . Let  $F_2$  the collection of all the formulas in  $L_{AP}$  of the form:

$$\bigwedge_{i=1}^r \neg\theta_i$$

where the  $\theta_i \in F_1$  and  $r \leq |F_1|$ . This set is clearly finite. Since

$$\forall H \in I(\neg\phi), \forall n, (\neg\phi)_{H,n} \equiv \bigwedge_{s=1}^n \neg (\phi)_{H_1(s), H_2(s)}$$

it is easy to see that  $F_2$  verifies the desired property for  $\neg\phi$ .

**Existential.** Direct from the definition of approximate formulas for existential and the induction hypothesis. Left to the reader.

2) Follows directly from 1) above and Lemma 3.1.1. Left to the reader. ■

The previous lemma implies the following approximation principle for first order formulas.

### THEOREM 3.4.2

*Let  $S$  be an arbitrary collection of discrete metric spaces. Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  containing the fixed universal predicate  $K$ . Let  $\phi(\vec{x}) \in L_{\omega\omega}$  be a first order formula. Then for any classical multisorted model  $E$ , and for every  $\vec{a} \in X^{|\vec{x}|}$ ,*

$$E \models \phi(\vec{a}) \text{ if and only if } E \models_{AP} \phi(\vec{a})$$

□

PROOF: By induction on formulas. The proof for the connective steps is identical to the proof of Theorem 3.2.2. The remaining case is the existential one.

$\Leftarrow$ : suppose that

$$E \models_{AP} \exists \vec{y} \in \vec{K} \phi(\vec{y}, \vec{a})$$

By definition there exists  $h \in I(\phi)$  such that for every  $n$ ,

$$E \models (\exists \vec{y} \in \vec{K} \phi(\vec{y}, \vec{a}))_{h,n} \equiv \exists \vec{y} \in \vec{K}(\phi(\vec{y}, \vec{a}))_{h,n}$$

Invoking Lemma 3.4.1 (every path of the tree of approximations of  $\phi$  is eventually “constant”) we get then that there exists a vector  $\vec{b}$  such that for every integer  $n$ :

$$E \models \bigwedge_{i=1}^a K_i(\vec{b}_i) \wedge \bigwedge_{n=1}^{\infty} \phi_{h,n}(\vec{b}, \vec{a})$$

Invoking the induction hypothesis we obtain then

$$E \models \left( \bigwedge_{i=1}^a K_i(\vec{b}_i) \wedge \phi(\vec{b}, \vec{a}) \right)$$

and this is the desired result.

$\Rightarrow$ . Direct. Left to the reader. ■

It follows that in classical structures the concept of  $\models_{AP}$  produces something new (i.e. different than the usual  $\models$ ) only for infinitary formulas.

We can extend in a natural way the previous result to a collection of infinitary formulas.  $(L_{\omega\omega})^{\{\wedge, \neg\}}$  is the smallest collection of formulas in  $L_A$  closed under countable conjunction and negation.

### THEOREM 3.4.3

*Fix  $S$  an arbitrary collection of discrete metric spaces. Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  and let  $\mathcal{M}$  be the collection of all the classical multisorted models for  $\Phi$ .*

For every  $\phi \in (L_{\omega\omega})^{\{\wedge, \neg\}}$ , for any classical multisorted model  $E$ , and for every  $\vec{a} \in X^{|\vec{a}|}$ ,

$$E \models \phi(\vec{a}) \text{ if and only if } E \models_{AP} \phi(\vec{a}) \square$$

PROOF: By induction on formulas. The case for  $L_{\omega\omega}$  was proved in Theorem 3.4.2. The induction steps are left to the reader. ■

Notice that the formulas of the form:

$$\exists \vec{x} \in \vec{K} \bigvee_{i=1}^{\infty} \phi_i(\vec{x})$$

for  $\phi_i \in L_{\omega\omega}$  are equivalent (in classical multisorted models) to:

$$\bigvee_{i=1}^{\infty} \exists \vec{x} \in \vec{K} \phi_i(\vec{x})$$

which can be seen as formulas in  $(L_{\omega\omega})^{\{\wedge, \neg\}}$ .

We close this chapter with a final weak approximation principle for classical multisorted structures. The proof of the following corollary is direct.

**COROLLARY 3.4.4** *Weak Approximation for Classical Models*

Fix  $S$  an arbitrary collection of discrete metric spaces. Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  and let  $\mathcal{M}$  be the collection of all the classical multisorted models for  $\Phi$ . For every  $\phi \in ((L_{\omega\omega})^{\{\wedge, \neg\}})^{\{\wedge, \vee, \exists\}}$ , for any classical multisorted model  $E$ , and for every  $\vec{a} \in X^{|\vec{a}|}$ ,

$$\text{if } E \models \phi(\vec{a}) \text{ then } E \models_{AP} \phi(\vec{a}) \square$$

The class  $((L_{\omega\omega})^{\{\wedge, \neg\}})^{\{\wedge, \vee, \exists\}}$  contains, for example, all the formulas of the form:

$$\exists \vec{x} \in \vec{K} \bigwedge_{i=1}^{\infty} \forall \vec{y} \in \vec{P}_i \phi_i(\vec{x}, \vec{y})$$

with the  $\phi_i$  in  $L_{\omega\omega}$ .



## Chapter 4

# Model Existence Theorem and Consequences

The aim of this chapter is to prove a Model Existence Theorem for the notion of approximate truth for formulas in  $L_A$ . In Section 4.1 we introduce the notion of a complete class of models for a signature  $\Phi$ . In Section 4.2 we prove a Model Existence Theorem for complete classes of models. Section 4.3 is devoted to some “uniformity of paths” results for complete classes of models. In Section 4.4 we use this result to study the relationship between  $\models$  and  $\models_{AP}$  for diverse collections of formulas in complete classes of models. Lastly, in Section 4.5 we give an application of the Model Existence Theorem to Functional Analysis.

Finally, since our notion of approximate truth coincides with the one given by Henson for formulas in  $L_{PB}$ , the Model Existence Theorem proved here can be seen as a generalization of the Compactness Theorem already obtained in the context of formulas in  $L_{PB}$  for Normed Space Structures (see [19]).

## 4.1 Complete Collections of Models

We want to prove a Model Existence Theorem for  $L_A$  along the following lines: In order to show that a formula  $\phi$  is consistent (i.e. there exists a model  $E$  such that  $E \models \phi$ ) it is enough to find an  $h \in I(\phi)$  such that for every integer  $n$  there exists a model  $E_n$  such that  $E_n \stackrel{app}{\models} \phi_{h,n}$ . The first step is to identify the collections of models that admit this theorem: the complete collections of models.

Intuitively, a complete collection of models is a collection of models satisfying the same uniform continuity requirements for the interpretations of the function symbols, and verifying the same uniform “bound” for the interpretation of the predicate symbols.

### DEFINITION 4.1.1 *Complete Collections of Models*

Fix a countable signature  $\Phi = (\mathcal{F}, \mathcal{P})$  over a collection  $S$  of complete metric spaces with the property that all  $C \in \mathcal{P}$  with fixed sort are **compact** sets in complete metric spaces. Fix also the following assignments:

- For every function symbol  $f \in \mathcal{F}$  with arity  $r$  and true sort,(or fixed sort  $(X, \rho)$ ), for every vector  $\vec{K} = (K_1, ..K_r)$  of true sort predicate symbols with arity 1, fix a unary true sort predicate symbol  $K^{(f, \vec{K})}$  in  $\mathcal{P}$  ( or a compact predicate  $C^{(f, \vec{K})}$  in  $(X, \rho)$ ).
- For every two predicate symbols  $Q, A$  with true sort and corresponding arities  $a, b$ , fix a unary predicate  $K^{Q, A}$  with true sort.
- For every  $f \in \mathcal{F}$  with arity  $a$  and true sort (or fixed sort  $(Y, \rho_Y)$ ), for every

vector  $\vec{K} = (K_1, ..K_a)$  of predicate symbols with arity 1 and true sort, for every rational  $\epsilon > 0$  fix a rational  $\delta^{f, \vec{K}, \epsilon} > 0$ .

The complete collection of models for the above assignments, is the collection of all the models  $E$  in  $\Phi$  that satisfy:

1. **Uniform bound for function symbols.** For every  $f \in \mathcal{F}$  with arity  $r$  and true sort (or fixed sort  $(Y, \rho_Y)$ ), for every vector  $\vec{K} = (K_1, ..K_r) \in \mathcal{P}$  of unary predicate symbols with true sort,

$$\forall \vec{x} \in \vec{K} \quad K^{(f, \vec{K})}(f(\vec{x}))$$

(or the formula

$$\forall \vec{x} \in \vec{K} \quad C^{(f, \vec{K})}(f(\vec{x}))$$

holds in  $E$  .

2.  **$\mathcal{P}$  is directed.** For every  $Q, A \in \mathcal{P}$  with arities  $a, b$  and true sort

$$\forall \vec{x}, \vec{y}(Q, A) \left( \bigwedge_{i=1}^a K^{(Q, A)}(x_i) \wedge K^{(Q, A)}(y_i) \right)$$

holds in  $E$  . Furthermore  $E$  verifies that for every  $a$  in  $E$  there exists  $r$  such that  $E \models K_r(a)$ .

3. **Uniform continuity for function symbols.** For every  $f \in \mathcal{F}$  with arity  $a$  and true sort (or fixed sort  $(Y, \rho_Y)$ ), for every vector  $\vec{K} = (K_1, ..K_a)$  of predicate symbols with arity 1 and true sort, for every rational  $\epsilon > 0$ ,

$$\forall \vec{x}, \vec{y}(\vec{K}, \vec{K}) \left( \bigwedge_{i \leq a} \rho(x_i, y_i) < \delta^{(f, \vec{K}, \epsilon)} \Rightarrow \rho(f(\vec{x}), f(\vec{y})) \leq \epsilon \right)$$

(or

$$\forall \vec{x}, \vec{y} (\vec{K}, \vec{K}) (\bigwedge_{i \leq a} \rho(x_i, y_i) < \delta^{(f, \vec{K}, \epsilon)} \Rightarrow \rho_Y(f(\vec{x}), f(\vec{y})) \leq \epsilon)$$

holds in  $E$  .

Any such collection of models is called a complete collection of models.  $\square$

We introduce some notation. If for a fixed structure  $E$  and for a sentence  $\phi \in L_A$  there exists  $h \in I(\phi)$  such that

$$E^{app} \models \bigwedge_{n=1}^{\infty} \phi_{h,n}$$

we write  $E \models_{AP}^h \phi$ .

A sub-collection  $\mathcal{M}$  of models in a complete collection  $\mathcal{W}$  is a **complete class of models** in  $\mathcal{W}$  if there exists a sentence  $\phi \in L_A$  such that:

$$E \in \mathcal{M} \text{ iff } (E \models \phi) \text{ and } E \in \mathcal{W}$$

We also say that  $\mathcal{M}$  is the complete class defined (or axiomatized) by  $\phi$ . Sometimes we will omit mentioning the complete collection  $\mathcal{W}$  and refer only to the complete class  $\mathcal{M}$ .

Recall that the abbreviation  $\forall^* k$  is to be read as saying “for all except a finite number of integers  $k$ ”.

**DEFINITION 4.1.2** *Uniform Sequence of Models*

A sequence  $\{E_n\}_{n=1}^{\infty}$  in a complete collection of models  $\mathcal{W}$  is **uniform for  $\phi$**  iff there exists  $h \in I(\phi)$  such that  $\forall n$  and  $\forall^* m$ ,

$$E_m^{app} \models (\phi_{h,n})$$

$\square$

**EXAMPLE 4.1.3** *Complete Collections of Models*

Consider the signature  $\Phi = (\mathcal{F}, \mathcal{P})$  of Example 2.3.3 for Normed Space Structures.

It is easy to see that the collection of normed space structures of  $\Phi$  is a complete class of models. The predicates in  $\mathcal{P}$  of fixed sort are by definition compact subsets of  $(\mathfrak{R}, d)$ . Furthermore  $+$ ,  $\|\cdot\|$  and  $r(\cdot)$  are uniformly continuous (and bounded) functions on the balls centered at the origin, and the balls  $B_q$  ( $q \in \mathbb{Q}^+$ ) form a directed set that covers the whole normed structure. Finally, the normed space structures are the models of  $\Phi$  that satisfy the axioms of vector space and normed space.  $\square$

**REMARK 4.1.4**

Note that in every complete class of models  $\mathcal{M}$  axiomatized by a positive bounded sentence  $\phi \in L_{PBA}$ , every sequence  $\{E_n\}_{n=1}^{\infty}$  of models in  $\mathcal{M}$  is uniform for  $\phi$ .

This is true because Lemma 3.3.5 implies that for every sentence  $\phi \in L_{PBA}$  there exists a path  $H \in I(\phi)$  such that for every model  $E$

$$\text{If } E \models \phi \text{ then } E \models_{AP}^H \phi$$

$\square$

Since the axioms of normed space can be described by a sentence  $\psi \in L_{PBA}$ , it follows from the above remark that any countable sequence of normed space structures is uniform for  $\psi$ .

**EXAMPLE 4.1.5** *Models of Multisorted Classical Logic*

Recall the definition of classical multisorted model for a fixed signature  $\Phi$ :

Let  $S$  be an arbitrary collection of discrete metric spaces, i.e. spaces  $(M, \rho_M)$  so that  $\text{Image}(\rho_M) = \{0, 2\}$ . Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  containing a fixed unary predicate  $K$  in  $\mathcal{P}$ . A model of  $\Phi = (\mathcal{F}, \mathcal{P})$  is a classical multisorted model if the predicate  $K$  is interpreted as the whole space  $X$  and  $\text{Im}(\rho) = \{0, 2\}$ .

Suppose that all the predicates with fixed sort in  $\Phi$  are compact (i.e. finite) in the corresponding fixed sort space. Suppose also that for every function symbol  $f$  with fixed sort we assign a finite set  $C^f$  in  $\mathcal{P}$  with fixed sort  $(X, \rho)$ .

Then the collection  $\mathcal{W}$  of all the classical multisorted structures  $E$  for  $\Phi$  that satisfy, for every function symbol  $f$  with fixed arity,

$$\text{Im}(f^\bullet) \subseteq C^f$$

is a complete collection of models.

The uniform continuity requirements become trivial, the predicates are directed since all the unary predicates are inside the universal predicate  $K$ , and the “uniform boundness” of the predicates is easy to see.

Furthermore, let  $\mathcal{M}$  a complete class of classical multisorted models in  $\mathcal{W}$  axiomatized by a sentence  $\phi \equiv \bigwedge_{j=1}^{\infty} \theta_j$ , with the  $\theta_j$ 's being first order sentences. Then any sequence  $\{E_n\}_{n=1}^{\infty}$  of models in  $\mathcal{M}$  has an infinite subsequence  $\{E_{n_i}\}_{i=1}^{\infty}$  (with  $n_1 < n_2 \dots$ ) that is uniform for  $\phi$ . This holds since Lemma 3.4.1 implies that for every first order formula  $\theta$  the number of non equivalent paths is finite and the approximations along every path are eventually constant. It follows from the pigeonhole principle that one can extract from any sequence of models  $\{E_n\}_{n=1}^{\infty}$  a

subsequence  $\{E_{n_i}\}_{i=1}^\infty$  such that for every integer  $j$  there exists  $h_j \in I(\theta_j)$  with:

$$\forall^* i, E_{n_i} \models_{AP}^{h_j} \theta_j$$

Using the definition of approximate formulas for the conjunction it is easy to see then that there exists an  $H \in I(\bigwedge_{j=1}^\infty \theta_j)$  such that for every integer  $n$ :

$$\forall^* i, E_{n_i}^{app} \models (\bigwedge_{j=1}^\infty \theta_j)_{H,n}$$

□

## 4.2 Model Existence Theorem for $L_A$

Our intention is to prove the a model existence theorem for complete classes of models using the ultraproduct construction.

Let us recall some notation involving ultrafilters over  $\omega$ . Let  $(M, \rho)$  be a complete metric space, and  $K$  a compact set in  $M$ . Let  $s = (s_1, s_2, \dots, s_n, \dots)$  be a sequence of elements of  $K$ . It is well known that for every ultrafilter  $U$  over  $\omega$  there exists an unique  $x \in K$  so that:

$$\forall \epsilon > 0 \exists p \in U \forall i \in p \rho(s_i, x) \leq \epsilon$$

We will denote the element  $x$  by  $\lim_{i \in U} s_i$ .

### DEFINITION 4.2.1 *Ultraproduct Construction*

Fix a countable signature  $\Phi = (\mathcal{F}, \mathcal{P})$  and a complete collection of models  $\mathcal{W}$ . Consider a sequence of models

$$\{E_i = (X^i, d^i, F^i, P^i) \mid i < \omega\}$$

of  $\mathcal{W}$ . For every  $f$  in  $\mathcal{F}$  we will denote by  $f^{\#i}$  the interpretation of this function symbol on the model  $E_i$ . Likewise for any predicate symbol  $C$  in  $\mathcal{P}$ ,  $C^{\#i}$  is the interpretation of  $C$  in  $E_i$ .

Recall that for every predicate symbol  $K \in \mathcal{P}$ , for every integer  $n$ ,  $K_n$  is the  $(1/n)$  metric deformation of  $K$ .

Define

$$X = \{g \in \prod_{i=1}^{\infty} X_i \mid \exists K \in \mathcal{P} \text{ unary predicate with true sort, } \forall n \in \omega \forall^* i, E_i^{app} \models K_n(g(i))\}$$

In other words,  $X$  is the collection of all the sequences  $g$  such that there exists a unary predicate symbol  $K$  with true sort such that:

$$\lim_{i \rightarrow \infty} d^i(g(i), K^i) = 0$$

Given an ultrafilter  $\mathcal{U}$  over  $\omega$ , we can define on  $X$  the following equivalence relation:

For arbitrary  $x, y$  in  $X$ ,  $x \sim_{\mathcal{U}} y$  if and only if for every  $\epsilon > 0$  there exists  $p \in \mathcal{U}$  such that  $\forall i \in p \ d^i(x(i), y(i)) \leq \epsilon$ . It is easy to verify that this is truly an equivalence relation. We will denote by  $[x]$  the equivalence class of  $x$ .

The set  $(X/\sim_{\mathcal{U}})$  is endowed with a metric  $D$  in the natural way:

For every pair  $[x], [y]$ , find two representatives  $x, y$  such that there exist  $K^1, K^2$  unary predicates with true sort satisfying for every integer  $i$ :

$$E_i \models K^1(x(i)) \wedge K^2(y(i))$$

By the property of directed sets of the complete collections we know that there exists a unary predicate  $K$  such that for every integer  $i$

$$E_i \models K(x(i)) \wedge K(y(i))$$



Invoking again the properties of the complete collections, we get that there exists a compact set  $C$  in  $\mathfrak{R}$  such that for every  $i$

$$E_i \models \forall(z, v) \in (K, K) C(d^i(z, v))$$

It makes sense then to define

$$D([x], [y]) = \lim_{i \in U} d^i(x(i), y(i))$$

It follows from the definition of a complete collections of models that the metric function is well defined.

Using the set  $(X/\sim_U)$  we define the interpretations of  $\Phi$ :

- For every function symbol  $f$  in  $\mathcal{F}$  with arity  $a$  and true sort we define  $f : (X/\sim_U)^a \mapsto (X/\sim_U)$  by:

$$\forall[\vec{x}] \in (X/\sim_U)^a, f([\vec{x}]) = [(f^{\#1}(\vec{x}(1)), \dots, f^{\#n}(\vec{x}(n)), \dots)]$$

The properties of the complete collections of models guarantee that the image of  $f$  is a subset of  $(X/\sim_U)$  and that  $f$  is well defined.

- For every function symbol  $f$  in  $\mathcal{F}$  with arity  $a$  and sort space  $(Y, \rho)$  we define  $f : (X/\sim_U)^a \mapsto (Y, \rho)$  by:

$$\forall[\vec{x}] \in (X/\sim_U)^a, f([\vec{x}]) = \lim_{i \in U} f^{\#i}(\vec{x}(i)).$$

The properties of the complete collection of models once again guarantee that the images of  $f$  are well defined.

- For any predicate symbol  $C$  in  $\mathcal{P}$  with arity  $a$  and true sort we define the interpretation of  $C$  in  $(X/\sim_U)^a$  as follows:

$[\vec{x}] \in C$  if and only if for every integer  $n$ ,  $\exists p \in \mathcal{U}$  such that for every  $i \in p$ ,

$$E_i^{app} \models (C(\vec{x}(i)))_{\emptyset, n}$$

Such structure is denoted by  $\prod_{\mathcal{U}} E_i$ .  $\square$

The following remark follows directly from the previous construction.

**REMARK 4.2.2** *The ultraproduct construction yields a model in  $\mathcal{W}$*

Fix  $\Phi$  a signature and  $\mathcal{W}$  a complete collection of models of  $\Phi$ . Let

$$\{E_i = (X^i, d^i, F^i, P^i) \mid i < \omega\}$$

be a sequence of models of  $\mathcal{W}$ . Let  $\mathcal{U}$  be an ultrafilter over  $\omega$ . Then the structure  $\prod_{\mathcal{U}} E_i$  is a model of  $\Phi$  in  $\mathcal{W}$ .  $\square$

PROOF: Let us verify first that  $\prod_{\mathcal{U}} E_i$  is a model of  $\Phi$ .

Clearly  $((X/\sim_{\mathcal{U}}), D)$  is a metric space. Using the property of uniform continuity of the complete collection of models  $\mathcal{W}$  we obtain that the functions  $f$  defined on  $(X/\sim_{\mathcal{U}})^a$  are continuous (in the product topology or in the fixed sort spaces). It is easy to verify that the interpretation of the predicates  $C$  with arity  $a$  and true sort are closed subsets of  $(X/\sim_{\mathcal{U}})^a$  (in the product topology).

It remains to show that  $\prod_{\mathcal{U}} E_i$  is in the complete collection  $\mathcal{W}$ , that is:  $\prod_{\mathcal{U}} E_i$  satisfy the same uniform bounds for the function symbols in  $\Phi$ , the same inclusion relationship for the true sort predicates and the same uniform continuity property for the function symbols that the models in  $\mathcal{W}$ . Left to the reader.  $\blacksquare$

This ultraproduct construction is a generalization of the ultraproduct construction for Banach spaces (see for example [1]).

We remark that this ultraproduct construction is in general not well behaved for formulas in  $L_{AP}$ .

### EXAMPLE 4.2.3

Let  $C$  be an unary predicate with true sort in  $\Phi$ . Fix  $\mathcal{W}$  a complete collection of models. It could happen that there exists a model  $E = (X, d, F, P) \in \mathcal{W}$ , a rational  $r$  and a sequence  $g : \omega \mapsto E$  satisfying the following:

For every integer  $n, g(n) \notin C_r^\bullet$

but  $\lim_{n \rightarrow \infty} g(n) \in C_r^\bullet$ .

Clearly, for every nonprincipal ultrafilter  $\mathcal{U}$ :

$$g \in (\prod_{\mathcal{U}} E) \text{ and } (\prod_{\mathcal{U}} E) \models C_r(g)$$

but for every integer  $i$ ,

$$E^{app} \models \neg C_r(g(i))$$

□

However, the next best property holds for this ultraproduct construction.

We introduce the following notation. Given any model  $E$ , any  $\vec{a} = (\vec{a}_1, \dots, \vec{a}_i, \dots)$  in  $E$  and any corresponding vector of true sort predicate symbols  $\vec{K} = (\vec{K}_1, \dots, \vec{K}_i, \dots)$ , by  $\vec{K}(\vec{a})$  we understand:

$$\bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$$

**LEMMA 4.2.4** *Property of the Ultraproduct*

Fix  $\Phi$  a signature and  $\mathcal{W}$  a complete collection of models of  $\Phi$ . Let

$$\{E_i = (X^i, d^i, F^i, P^i) \mid i < \omega\}$$

be a sequence of models in  $\mathcal{W}$ . Fix  $\mathcal{U}$  a nonprincipal ultrafilter over  $\omega$ . For any formula  $\phi(\vec{x}) \in L_A$ , for any sequence  $\{\vec{F}_k\}_{k=1}^\infty$  of vectors whose elements are in  $(X/\sim_{\mathcal{U}})$ , the following are equivalent:

- $\exists h \in I(\phi) \forall n \forall^* k \prod_{\mathcal{U}} E_i^{app} \models \phi_{h,n}(\vec{F}_k)$
- $\exists h \in I(\phi)$  such that  $\forall n \forall^* k \exists p \in \mathcal{U}$  satisfying:

$$\forall i \in p, E_i^{app} \models \phi_{h,n}(\vec{F}_k(i))$$

□

**PROOF:** By induction on formulas.

**Atomic Formulas.** Direct from the definition of the interpretation of the predicates in the model  $\prod_{\mathcal{U}} E_i$ , and the fact that the predicates with fixed sort are closed and the interpretation of the function symbols are continuous functions.

**Conjunction.** Both directions are direct.

**Negation.**  $\Rightarrow$ . Suppose that  $\exists g \in I(\neg\phi) \forall n \forall^* k \prod_{\mathcal{U}} E_i^{app} \models (\neg\phi(\vec{F}_k))_{g,n}$ . By the definition of the the negation step for approximate formulas we get that for every  $h$  in  $I(\phi(\vec{x}))$  there exists an integer  $n$  such that  $\forall^* k \prod_{\mathcal{U}} E_i^{app} \models \neg\phi_{h,n}(\vec{F}_k)$ .

Using the induction hypothesis on the formula  $\phi$ , and the fact that  $\mathcal{U}$  is an ultrafilter, we get that for all  $h$  in  $I(\phi)$   $\exists n \forall^* k$  there exists a  $p$  in  $\mathcal{U}$  satisfying:

$$\forall i \in p E_i^{app} \models \neg\phi_{h,n}(\vec{F}_k(i))$$

Define then the set

$$W = \{(h, n) : h \in D(\phi) \wedge n \in \omega \wedge \forall^* k \exists p \in \mathcal{U} \forall i \in p, E_i^{app} \models \neg(\phi_{h,n}(\vec{F}_k(i)))\}$$

Using the properties of the dense countable set  $D(\phi)$  (Lemma 3.1.1) it is easy to obtain a  $g = (g_1, g_2) : \omega \mapsto D(\phi) \times \omega$  such that  $Im(g) = W$ . Using the fact that  $D(\phi)$  is dense in  $I(\phi)$  it is a standard procedure to verify that this function is in  $I(\neg\phi(\vec{x}))$ .

Furthermore, by definition of  $W$ ,  $\forall n \forall^* k$  there exists a  $p$  in  $\mathcal{U}$  such that:

$$E_i^{app} \models (\neg\phi(\vec{F}_k(i)))_{g,n} \equiv \bigwedge_{s=1}^n \neg(\phi(\vec{F}_k(i))_{g_1(s),g_2(s)})$$

This is the desired result.

$\Leftarrow$ . Similar to the previous proof. Left to the reader.

**Existential.** Here we use the full power of the induction hypothesis on arbitrary sequences  $\{F_k\}_{k=1}^\infty$ .

$\Rightarrow$ . Suppose that  $\exists h \in I(\phi) \forall n \forall^* k$

$$\prod_{\mathcal{U}} E_i^{app} \models \exists(\vec{x} \in \vec{K}(\phi(\vec{F}_k, \vec{x})))_{h,n}$$

By the definition of approximate formulas, the above statement implies that  $\exists h \in I(\phi) \forall n \forall^* k$

$$\prod_{\mathcal{U}} E_i^{app} \models \exists \vec{x} \in \vec{K}(\phi(\vec{F}_k, \vec{x}))_{h,n}$$

It follows that for every integer  $k$  there exists a vector

$$\vec{G}_k$$

such that  $\forall n \forall^* k$

$$\prod_{\mathcal{U}} E_i^{app} \models \vec{K}(\vec{G}_k) \wedge \phi_{h,n}(\vec{F}_k, \vec{G}_k)$$

Using the induction hypothesis, we obtain then that there exists  $h \in I(\phi)$  such that  $\forall n \forall^* k$  there exists  $p \in \mathcal{U}$  with :

$$\forall i \in p E_i^{app} \models \vec{K}_{\emptyset, n}(\vec{G}_k(i)) \wedge \phi_{h, n}(\vec{F}_k(i), \vec{G}_k(i))$$

It is possible now to find for every  $k$  a vector of functions  $\vec{Q}_k$  such that each of its components is in the same equivalence class (for  $\sim$ ) as the corresponding component in  $\vec{G}_k$ , and such that  $\forall n, \forall^* k$

$$\forall i \in p E_i^{app} \models \vec{K}(\vec{Q}_k(i)) \wedge \phi_{h, n}(\vec{F}_k(i), \vec{Q}_k(i))$$

But this implies that  $\forall n, \forall^* k$ ,

$$\forall i \in p E_i^{app} \models \exists \vec{x} \in \vec{K} \phi_{h, n}(\vec{F}_k(i), \vec{x})$$

This is the desired result.

$\Leftarrow$ : Similar to the previous proof. Left to the reader.

This completes the proof of the lemma. ■

**REMARK 4.2.5** *Property of the Ultraproduct for  $L_{PBA}$*

For any  $\phi \in L_{PBA}$ , Lemma 3.3.5 shows that there exists a “minimal” path  $H$  such that for every model  $E$  for every  $\vec{a}$  in  $E$ ,

$$\text{If } E \models \phi(\vec{a}) \text{ then } E \models_{AP}^H \phi(\vec{a})$$

It can be shown by induction on formulas in  $L_{PBA}$  that the ultraproduct construction verifies the following stronger property for  $L_{PBA}$ :

*Fix  $\Phi$  a signature and  $\mathcal{W}$  a complete collection of models of  $\Phi$ . Let*

$$\{E_i = (X^i, d^i, F^i, P^i)\}_{i=1}^\omega$$

a sequence in  $\mathcal{W}$ . Fix  $\mathcal{U}$  a nonprincipal ultrafilter over  $\omega$ . For any formula  $\phi(\vec{x}) \in L_{PBA}$  let  $H$  the “minimal path” for  $\phi$ .

For any sequence  $\{\vec{F}_k\}_{k=1}^\infty$  of vectors whose elements are in  $(X/\sim_{\mathcal{U}})$ , the following are equivalent:

- $\forall n \forall^* k \prod_{\mathcal{U}} E_i^{app} \models \phi_{H,n}(\vec{F}_k)$
- $\forall n \forall^* k \exists p \in \mathcal{U}$  such that:

$$\forall i \in p E_i^{app} \models \phi_{H,n}(\vec{F}_k(i))$$

□

We use the previous theorem to show that the ultraproduct is a rich model, i.e. a model where  $\models_{AP}$  and  $\models$  coincide for all formulas in  $L_A$ .

**THEOREM 4.2.6** *The Ultraproduct is Rich for  $L_A$*

Fix  $\Phi$  a signature and a complete collection of models  $\mathcal{W}$ . Let  $\{E_i = (X^i, d^i, F^i, P^i)\}_{i=1}^\infty$  a sequence of models in  $\mathcal{W}$ . Fix  $\mathcal{U}$  a countably incomplete ultrafilter over  $\omega$ . Then for every formula  $\phi(\vec{x})$  in  $L_A$  and for every  $\vec{g} \in (X/\sim_{\mathcal{U}})^a$  ( $a \leq \omega$ ) the following are equivalent:

- $\prod_{\mathcal{U}} E_i \models_{AP} \phi(\vec{g})$
- $\prod_{\mathcal{U}} E_i \models \phi(\vec{g})$

□

PROOF: Recall the definition of the collection of all the rich formulas for  $\prod_{\mathcal{U}} E_i$  (above Corollary 3.2.3).

Let  $\Gamma$  the collection of all rich formulas for  $\prod_{\mathcal{U}} E_i$ . By Corollary 3.2.3,  $\Gamma$  is closed under countable conjunction and negation in every model. To prove that  $\Gamma = L_A$  it is enough then to prove that  $\Gamma$  contains the atomic formulas, and is closed under existential quantification.

However it is easy to verify that the atomic formulas are rich for every model. We are left then with the verification of the existential closure.

For the existential case, there is only one interesting direction. Suppose that  $\prod_{\mathcal{U}} E_i \models_{AP} \exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{g})$  with  $\phi \in \Gamma$ .

From Lemma 4.2.4 it follows that there exists  $h \in I(\phi)$  such that for every integer  $n \exists p \in \mathcal{U}$  such that:

$$\forall i \in p \ E_i^{app} \models \exists \vec{v} \in \vec{K} \phi_{h,n}(\vec{v}, \vec{g}(i))$$

Since the ultrafilter is countably incomplete, it is easy to construct by diagonalization a sequence  $\vec{f}$  of vectors of functions (elements of  $\prod_{\mathcal{U}} E_i$ ) corresponding to  $\vec{v}$  and such that  $\forall n \in \omega \exists p \in \mathcal{U} \forall i \in p$ :

$$E_i^{app} \models \vec{K}(\vec{f}(i)) \wedge \phi_{h,n}(\vec{f}(i), \vec{g}(i))$$

We again invoke Lemma 4.2.4 to obtain:

$$\prod_{\mathcal{U}} E_i \models_{AP} \vec{K}(\vec{f}) \wedge \phi(\vec{f}, \vec{g})$$

Since  $\phi \in \Gamma$  we get

$$\prod_{\mathcal{U}} E_i \models \vec{K}(\vec{f}) \wedge \phi(\vec{f}, \vec{g})$$

and this implies:

$$\prod_{\mathcal{U}} E_i \models \exists \vec{v} \in \vec{K} \phi(\vec{v}, \vec{g})$$



This completes the proof. ■

The main consequence of the previous lemmas is the following Model Existence Theorem. We recall the definition of a uniform sequence  $\{E_n\}_{n=1}^{\infty}$  in a complete class  $\mathcal{M}$ :

A sequence  $\{E_n\}_{n=1}^{\infty}$  in a complete class  $\mathcal{M}$  defined by  $\psi$  is **uniform for**  $\phi$  iff there exists  $h \in I(\phi)$  such that for every integer  $n$ :

$$\forall^* m, E_m^{app} \models (\phi_{h,n})$$

**THEOREM 4.2.7** *Model Existence Theorem for  $L_A$*

*Fix a signature  $\Phi$  and a fixed family of sort spaces  $S$ . Let  $\mathcal{W}$  be a complete collection of models for  $\Phi$ . Let  $\mathcal{M}$  a complete class in  $\mathcal{W}$  defined by a sentence  $\psi \in L_A$ . Let  $\phi$  a sentence in  $L_A$ . Suppose that there exists a sequence of models  $\{E_n\}_{n=1}^{\infty}$  uniform for  $\psi$  and  $\phi$ , i.e. there exists paths  $h \in I(\phi)$  and  $g \in I(\psi)$  such that for every integer  $n$ ,  $\forall^* m$ ,*

$$E_m^{app} \models (\phi_{h,n} \wedge \psi_{g,n})$$

*Then there exists a **rich** model  $E$  in  $\mathcal{M}$  with the property that:*

$$E \models \phi$$

□

PROOF: Select a countable incomplete ultrafilter over the integers and apply Lemma 4.2.4 to the sequence of models  $\{E_n\}_{n=1}^{\infty}$ . We obtain that

$$\prod_{\mathcal{U}} E_i \models_{AP} \phi \wedge \psi$$

Applying now Theorem 4.2.6 to  $\prod_{\mathcal{U}} E_i$  we get:

$$\prod_{\mathcal{U}} E_i \models \phi \wedge \psi$$

Finally, it is easy to see that  $\prod_{\mathcal{U}} E_i$  is in  $\mathcal{M}$  (using Remark 4.2.2). This completes the proof. ■

Let us remark that the existence of a uniform sequence of models  $\{E_n\}_{n=1}^{\infty}$  in a complete collection  $\mathcal{M}$  and satisfying for every  $n$

$$E_n^{app} \models_{AP} \phi_{h,n}$$

only guarantees that

$$\prod_{\mathcal{U}} E_i \models_{AP} \phi$$

It is not necessarily true that  $\prod_{\mathcal{U}} E_i$  approximately satisfy  $\phi$  along  $h$ .

In the rest of this section we obtain some direct consequences of this Model Existence Theorem. The first corollary concerns complete classes of models axiomatized by positive bounded formulas. In this case we obtain a direct extension of Henson's Compactness Theorem for  $L_{PB}$  ([19]):

**THEOREM 4.2.8** *Extension of Henson's Compactness Theorem*

Fix a signature  $\Phi$  and a fixed family of sort spaces  $S$ . Let  $\mathcal{W}$  be the complete collection of normed space structures. Let  $\mathcal{M}$  be a complete class of models in  $\mathcal{W}$  defined by a positive bounded sentence  $\psi \in L_{PBA}$ . Fix  $\phi$  a sentence in  $L_A$  and let  $h \in I(\phi)$ . Suppose that for every integer  $n$  there exists a model  $E_n$  in  $\mathcal{M}$  such that:

$$E_n^{app} \models \phi_{h,n}$$

Then there exists a **rich** model  $E$  in  $\mathcal{M}$  with the property that:

$$E \models \phi$$

□

Proof: It just follows from the Model Existence Theorem and from Remark 4.1.4 that states that any sequence of models in a complete class of models axiomatized by a positive bounded sentence  $\psi \in L_{PBA}$  is uniform for  $\psi$ . ■

In the same vein we can obtain a compactness theorem for classical multi-sorted structures. It extends the usual first order compactness result to infinitary formulas.

Note that for any first order language  $L$ , the collection of first order models in  $L$  can be seen as a complete collection of models. Fix  $S = \{(\mathfrak{R}, d)\}$  and let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature such that  $\mathcal{F}$  is the collection of function symbols in  $L$  (with true sort) and all the predicate symbols of  $L$  are true sort predicates in  $\mathcal{P}$ . Suppose also that  $\Phi$  contains an unary predicate  $K$  with true sort, and that the (compact) predicate  $\{0, 1\}$  with fixed sort the reals with the usual metric is in  $\mathcal{P}$ .

It is easy to verify that the collection  $\mathcal{W}$  of all the classical multisorted structures  $E$  for  $\Phi$  such that the predicate  $K$  is interpreted as the whole universe is a complete collection of models. Clearly this collection  $\mathcal{W}$  can also be seen as the collection of all first order models for  $L$ .

The following theorem follows from the above remark.

**THEOREM 4.2.9** *Extension of the First Order Compactness Theorem*

*Fix  $L$  a first order language, and let  $\mathcal{M}$  the class of all the models of the language  $L$  that satisfy a sentence of the form  $\bigwedge_{j=1}^{\infty} \theta_j$  with the  $\theta_j$  first order formulas. Fix  $\phi$  a sentence in  $L_A$  and let  $h \in I(\phi)$ . Suppose that for every integer  $n$  there exists a model  $E_n$  in  $\mathcal{M}$  such that:*

$$E_n \models \phi_{h,n}$$

*Then there exists a rich model  $E$  in  $\text{calM}$  with the property that:*

$$E \models \phi$$

□

**PROOF:** Consider, for the language  $L$ , the corresponding complete collection of classical multisorted models  $\mathcal{W}$ .

The desired result then follows directly from the Model Existence Theorem and Remark 4.1.5 that states that every sequence  $\{E_n\}_{n=1}^{\infty}$  in a complete class of classical multisorted models axiomatized by a sentence of the form  $\bigwedge_{j=1}^{\infty} \theta_j$  contains an infinite subsequence that is uniform for  $\bigwedge_{j=1}^{\infty} \theta_j$ . ■

At this point it is instructive to discuss, for the case of classical multisorted

structures, the relationship between the notion of rich model and the notion of  $\omega_1$ -saturated model.

Note first that rich implies  $\omega_1$ -saturation for classical first order structures. To see this consider a rich first order structure  $E$ . Fix then a countable collection of first order formulas  $\{\sigma_n(\vec{x})\}_{n=1}^\infty$  with  $|\vec{x}|$  finite. Suppose also that for every integer  $n$ ,

$$E \models \exists \vec{x} \bigwedge_{i=1}^n \sigma_i(\vec{x})$$

From Corollary 3.4.4 it follows that for every integer  $n$ ,

$$E \models_{AP} \exists \vec{x} \bigwedge_{i=1}^n \sigma_i(\vec{x})$$

Now, since the formulas  $\sigma_i(\vec{x})$  are first order formulas, we know from Lemma 3.4.1 that the number of non equivalent paths for those formulas is finite. It follows from the pigeonhole principle that for every integer  $i$  there exists a path  $h_i \in I(\sigma_i)$  such that for every integer  $n$ :

$$E \models \exists \vec{x} \bigwedge_{i=1}^n (\sigma_i(\vec{x})_{h_i, n})$$

It follows from the definition of approximate formulas for countable conjunctions that there exists an  $H \in I(\bigwedge_{i=1}^\infty \sigma_i(\vec{x}))$  such that for every integer  $n$ :

$$E \models (\exists \vec{x} \bigwedge_{i=1}^\infty \sigma_i(\vec{x}))_{H, n}$$

Invoking now the richness of the model  $E$  we finally obtain:

$$E \models \exists \vec{x} \bigwedge_{i=1}^\infty \sigma_i(\vec{x})$$

This completes the verification of the desired implication.

In the next section we look at some uniformity principles that can be deduced from the Model Existence Theorem.

### 4.3 Uniformity in Complete Classes of Models

Suppose that we know that a sentence  $\phi$  is approximately true in every model of a complete class  $\mathcal{M}$  defined by a sentence  $\psi$ . It is possible then to claim that every sequence  $\{E_n\}_{n=1}^{\infty}$  in  $\mathcal{M}$  uniform for  $\psi$  is uniform for  $\phi$ ?

We begin by defining the notion of simple sentence in a complete collection of models.

**DEFINITION 4.3.1** *Simple Sentences*

Fix a complete collection of models  $\mathcal{W}$  for a signature  $\Phi$ . A sentence  $\phi \in L_A$  is **simple** for  $\mathcal{W}$  if and only if, for every model  $E$  in  $\mathcal{W}$ ,

$$E \models \phi \text{ implies that } E \models_{AP} \phi$$

□

**EXAMPLE 4.3.2**

Theorem 3.3.10 shows that any sentence  $\phi \in L_{A+}$  is simple for every complete collection of models.

In the class of all the classical multisorted models, Corollary 3.4.4 implies that every sentence in  $((L_{\omega\omega})^{\{\wedge, \neg\}})^{\{\wedge, \vee, \exists\}}$  is simple in this collection of models. □

We also introduce some notation. For a complete class of models  $\mathcal{M}$  and a sentence  $\phi \in L_A$  we say that  $\mathcal{M} \models_{AP} \phi$  iff for every model  $E$  of  $\mathcal{M}$ ,  $E \models_{AP} \phi$ .

We are ready now for the following result that settles the question at the beginning of this section for negative formulas.

**PROPOSITION 4.3.3** *Uniformity of Paths of Formulas  $\neg\phi$* 

*Fix signature  $\Phi$  and a complete collection of models  $\mathcal{W}$ . Let  $\mathcal{M} \subseteq \mathcal{W}$  be defined by a sentence  $\psi \in L_A$  simple for  $\mathcal{W}$ . Let  $\phi$  be a sentence in  $L_A$ . The following are equivalent:*

- $\mathcal{M} \models_{AP} \neg\phi$ .

- For every sequence  $\{E_n\}_{n=1}^\infty$  in  $\mathcal{M}$  uniform for  $\psi$  there exists  $h \in I(\neg\phi)$  such that

$$\forall n \ E_n^{app} \models_{AP}^h \neg\phi$$

□

PROOF: ( $\Rightarrow$ ). Suppose that for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \neg\phi$ .

We claim that for every sequence  $\{E_n\}_{n=1}^\infty$  uniform for  $\psi$ , for every  $h \in I(\phi)$   $\exists k \in \omega$  such that:

$$\forall n, E_n^{app} \models \neg(\phi_{h,k})$$

If this were not true, then there would exist  $h \in I(\phi)$  and, for every integer  $k$ , a model  $E_{n_k}$  such that  $E_{n_k}^{app} \models \phi_{h,k}$ . Here we consider two cases:

- The collection  $\{n_k | k \in \omega\}$  is bounded. It follows then (using the properties of  $\models_{AP}$  listed in Lemma 3.1.1) that there exists an integer  $n$  such that:

$$E_n^{app} \models \bigwedge_{k=1}^\infty (\phi_{h,k})$$

but this contradicts the assumption that  $\mathcal{M} \models_{AP} \neg\phi$ .

- The collection  $\{n_k | k \in \omega\}$  is unbounded. It follows from the Model Existence Theorem (Theorem 4.2.7) that there will exist a model  $B \in \mathcal{M}$  such that  $B \models_{AP} \phi$ , contradicting the assumption.

This completes the proof of the claim.

Using the previous claim, define:

$$W = \{(h, k) | h \in D(\phi) \wedge n \in \omega \wedge \exists k \forall n \ E_n^{app} \models \neg(\phi_{h,k})\}$$



Using again the properties of the dense set  $D(\phi)$  listed in Lemma 3.1.1 we get that any function  $H = (H_1, H_2) : \omega \mapsto D(\phi) \times \omega$  with  $Im(H) = W$  is in  $I(\neg\phi)$ . Furthermore, for every integer  $m$ ,

$$\forall n, E_n^{app} \models \bigwedge_{s=1}^m \neg(\phi_{H_1(s), H_2(s)})$$

But this is exactly the statement: there exists  $H \in I(\neg\phi)$  such that for every  $m$ ,

$$\forall n E_n^{app} \models (\neg\phi)_{H,m}$$

This is the desired result.

⇐ Direct using the fact that  $\psi$  is simple for  $\mathcal{W}$ . ■

One can state the above Property 4.3.3 in a slightly different form that would be easier to apply. It suffices to decode the meaning of  $\models_{AP}^h \neg\psi$  to obtain the following:

**COROLLARY 4.3.4** *Fix a signature  $\Phi$  and a complete collection of models  $\mathcal{W}$ . Let  $\mathcal{M} \subset \mathcal{W}$  be defined by a sentence  $\psi$  in  $L_A$  simple for  $\mathcal{W}$ . Let  $\phi \in L_A$ . The following are equivalent:*

- $\mathcal{M} \models_{AP} \neg\phi$ .
- for every sequence of models  $\{E_n\}_{n=1}^\infty$  in  $\mathcal{M}$  uniform for  $\psi$ , for every  $h \in I(\phi)$  there exists an integer  $r$  such that:

$$\forall n, E_n^{app} \models \neg(\phi)_{h,r}$$

□

We now look at formulas that are not negative. Recall that  $L_{PBA}$  is the collection of all the positive bounded formulas.  $L_{PBA}$  contains the atomic formulas and is closed under countable conjunction, finite disjunction, universal and existential quantification.

The following theorem answers the uniformity question in the affirmative for sentences in the class  $(L_{PBA} \cup \exists\neg L_{PBA})^{\{\wedge, \neg\}}$ . This class contains for example all the quantifier free formulas and the formulas of the form:

$$\exists \vec{x} \in \vec{K} \neg \bigwedge_{i=1}^{\infty} \phi_i(\vec{x})$$

for  $\phi_i \in L_{PBA}$ .

**THEOREM 4.3.5** *Uniformity of Paths*

*Fix a signature  $\Phi$  and a complete collection of models  $\mathcal{W}$ . Let  $\mathcal{M} \subseteq \mathcal{W}$  be defined by a sentence  $\psi \in L_A$  simple for  $\mathcal{W}$ . Let  $\phi$  a sentence in  $(L_{PBA} \cup \exists\neg L_{PBA})^{\{\wedge, \neg\}}$ . Then the following are equivalent:*

- $\mathcal{M} \models_{AP} \phi$ .
- Every sequence  $\{E_n\}_{n=1}^{\infty}$  in  $\mathcal{M}$  uniform for  $\psi$  is uniform for  $\phi$ , i.e.: there exists  $h \in I(\phi)$  such that

$$\forall n, E_n \models_{AP} {}^h(\phi)$$

□

PROOF: The direction  $(\Leftarrow)$  is trivially true since  $\psi$  is simple for  $\mathcal{W}$ .

$(\Rightarrow)$ : By induction on formulas.

- $\phi \in L_{PBA}$ . Fix  $\phi \in L_{PBA}$ . We know by induction hypothesis that  $\mathcal{M} \models_{AP} \phi$ .

Let  $H$  the minimal path of  $I(\phi)$  as defined in Lemma 3.3.5. We claim:

CLAIM 1: For every  $E$  in  $\mathcal{M}$ ,  $E \models_{AP}^H \phi$ .

Proof: Suppose in order to obtain a contradiction that:

$$\text{there exists a model } E \text{ in } \mathcal{M} \text{ and an integer } m \text{ such that } E^{app} \models \neg(\phi_{H,m}) \quad (4.1)$$

Consider the ultrapower of  $E$  under a countably complete ultrafilter  $\mathcal{U}$ . Let us denote this ultrapower  $E^{\mathcal{U}}$ . By Remark 4.2.2 we know that  $E^{\mathcal{U}}$  is in  $\mathcal{M}$ .

It follows then that

$$E^{\mathcal{U}} \models_{AP} \phi$$

Since the ultrapower is rich for  $L_A$  it follows that:

$$E^{\mathcal{U}} \text{ verify } \phi$$

Since  $\phi \in L_{PBA}$  we can invoke Lemma 3.3.5 to obtain:

$$E^{\mathcal{U}} \models_{AP}^H \phi$$

Now we can invoke Remark prolpba concerning the property of the ultra-products for formulas in  $L_{PBA}$  to obtain:

$$\forall n, E^{app} \models \phi_{H,n}$$

but this contradicts statement 4.1. This completes the proof of the claim.

From the above claim the direction  $\Rightarrow$  follows easily.

- $\phi \in \exists\neg L_{PBA}$ . Suppose that for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \exists \vec{x} \in \vec{K} \neg \theta$  with  $\theta$  in  $L_{PBA}$ . Let  $\{E_n\}_{n=1}^\infty$  a sequence of models in  $\mathcal{M}$  uniform for  $\psi$  and let  $H \in I(\theta)$  the “minimal” path for  $\theta$ .

CLAIM 2). There exists an integer  $m$  such that:

$$\forall n, E_n^{app} \models \exists \vec{x} \in \vec{K} \neg (\theta)_{H,m}$$

Proof: If this were not true, then there would exist a subsequence  $\{E_{n_i}\}_{i=1}^\infty$  with for every integer  $i$ ,

$$E_{n_i}^{app} \models \forall \vec{x} \in \vec{K} (\theta)_{H,i}$$

We consider two cases:

- The sequence  $\{n_i | i \in \omega\}$  is bounded. In this case there exists an integer  $n$  such that:

$$E_n^{app} \models \bigwedge_{i=1}^\infty \forall \vec{x} \in \vec{K} \theta_{H,i}$$

It follows from Remark 4.2.5 that for every countably incomplete ultrafilter over  $\omega$ ,

$$(E_n^{\mathcal{U}})^{app} \models \forall \vec{x} \in \vec{K} \theta$$

but this contradicts the hypothesis that  $\mathcal{M} \models \exists \vec{x} \in \vec{K} \neg \theta$ .

- The sequence  $\{n_i | i \in \omega\}$  is unbounded. In this case there exists a subsequence  $n_1 < n_2 < \dots$  such that for every  $i$ ,

$$E_{n_i}^{app} \models \forall \vec{x} \in \vec{K} (\theta)_{H,i}$$

It follows from Remark 4.2.5 concerning the “minimal” paths  $H$  in an ultrapower, that in any ultrapower  $\prod_{\mathcal{U}} E_i$  of the sequence  $\{E_{n_i}\}_{i=1}^{\infty}$  it is true that:

$$\prod_{\mathcal{U}} E_i \models_{AP} \neg \exists \vec{x} \in \vec{K} \neg \theta(\vec{x})$$

and  $\prod_{\mathcal{U}} E_i \models \psi$ . But this contradicts the hypothesis on  $\mathcal{M}$ . This completes the proof of the claim.

Fix now  $m$  as in the above claim.

CLAIM 3): For every  $h \in I(\theta)$  there exists an integer  $r$  such that:

$$\forall n, E_n^{app} \models \forall \vec{x} \in \vec{K} (\neg(\theta(\vec{x}))_{H,m} \Rightarrow \neg(\theta(\vec{x}))_{h,r})$$

PROOF: Suppose that this statement was not true. Then we could again construct a sequence  $\{E_{n_i}\}_{i=1}^{\infty}$  such that for every integer  $i$ ,

$$E_{n_i}^{app} \models \exists \vec{x} \in \vec{K} (\neg(\theta(\vec{x}))_{H,m} \wedge \theta_{h,n}(\vec{x}))$$

Exactly as in the proof of the previous Claim, we can construct from this assumption a rich model  $E$  in  $\mathcal{M}$  such that

$$\forall n, E^{app} \models \exists \vec{x} \in \vec{K} (\neg(\theta(\vec{x}))_{H,m} \wedge (\theta(\vec{x}))_{h,r})$$

It follows then, by Remark 4.2.5, that

$$E \models \exists \vec{x} \in \vec{K} (\theta(\vec{x}) \wedge \neg \theta(\vec{x}))$$

But this is a contradiction. This completes the proof of the claim.

Finally, let  $W = \{(h, r) : h \in D(\theta) \wedge n \in \omega \text{ and } (h, n) \text{ as in the previous claim}\}$ .

It is easy to see that  $W$  is countable. Let  $G : \omega \mapsto I(\theta) \times \omega$  such that

$Image(G) = W$ . Using the property of the dense set  $D(\theta)$  (Lemma 3.1.1) we have that  $G \in I(\neg\theta) = I(\exists \vec{x} \in \vec{K} \neg\theta(\vec{x}))$ . Using the last claim we can also verify that for every integer  $r$ ,

$$\forall^* n, E_n^{app} \models (\exists \vec{x} \in \vec{K} \neg\theta(\vec{x}))_{G,r} \equiv \exists \vec{x} \in \vec{K} \bigwedge_{s=1}^r \neg(\theta(\vec{x}))_{G_1(s), G_2(s)}$$

But this is the desired result.

We are now ready for the induction steps.

**Conjunction:** Assume that for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \bigwedge_{i=1}^{\infty} \phi_i$ . It follows from Lemma 3.1.3 that for every integer  $i$ , for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \phi_i$ . By induction hypothesis we conclude that for every sequence  $\{E_n\}_{n=1}^{\infty}$  in  $\mathcal{M}$  uniform for  $\psi$ , for every  $i$  there exists  $h_i \in I(\phi_i)$  such that

$$\forall n, E_n \models_{AP}^{h_i} \phi_i$$

It follows then by definition of approximate formulas that there exists  $H \in I(\bigwedge_{i=1}^{\infty} \phi_i)$  such that

$$\forall n, E_n \models_{AP}^H \left( \bigwedge_{i=1}^{\infty} \phi_i \right)$$

This is the desired result.

**Negation.** Direct from Proposition 4.3.3. ■

The rest of the section is devoted to applications of the above results on uniformity of paths.

We begin with an application that concerns the complete classes of models  $\mathcal{M}$  defined by positive bounded sentences ( $L_{PBA}$ ). We prove a strong uniformity principle for negative formulas or formulas in  $(L_{PBA} \cup \exists \neg L_{PBA})^{\{\wedge, \neg\}}$ .

Recall that sentences in  $L_{PBA}$  are simple for any complete collection of models since they are in the collection  $L_{A+}$ . Furthermore, Remark 4.1.4 implies, for any complete class  $\mathcal{M}$  defined by a  $\psi \in L_{PBA}$ , that any sequence  $\{E_n\}_{n=1}^{\infty}$  in  $\mathcal{M}$  is uniform for  $\psi$ . Hence we get for these classes the following corollary:

**COROLLARY 4.3.6** *Strong Uniformity Principle*

*Fix a signature  $\Phi$  and a complete class of models  $\mathcal{M}$  defined by a sentence  $\psi$  in  $L_{PBA}$ . Let  $\phi \in L_A$  be a sentence of the form  $\phi \equiv \neg\theta$  or a sentence in  $(L_{PBA} \cup \exists\neg L_{PBA})^{\{\wedge, \neg\}}$ . Then the following are equivalent:*

- $\mathcal{M} \models_{AP} \phi$ .
- There exists  $h \in I(\phi)$  such that for every model  $E \in \mathcal{M}$ ,

$$E \models_{AP} {}^h\phi$$

□

PROOF:  $\Leftarrow$ : Direct.

$\Rightarrow$ : By contradiction. Suppose that  $\forall h \in I(\phi)$  there exists an integer  $n$  and a model  $E \in \mathcal{M}$  such that  $E^{app} \models \neg(\phi_{h,n})$ . Let

$$W = \{(h, n) \mid h \in D(\phi) \wedge n \in \omega \wedge \exists E \in \mathcal{M}, E^{app} \models \neg(\phi_{h,n})\}$$

Note first that  $W$  is countable. Furthermore, by the properties of the dense set  $D(\phi)$  listed in Lemma 3.1.1 we know that for every  $f \in I(\phi)$  there exists an  $(h, n) \in W$

and a model  $E \in \mathcal{M}$  such that:

$$E^{app} \models \neg(\phi_{f,n}) \equiv \neg(\phi_{h,n})$$

Define now, for every pair  $(h, n) \in W$  a model  $E_{h,n}$  in  $\mathcal{M}$  such that:

$$E_{h,n}^{app} \models \neg(\phi_{h,n})$$

Clearly the collection  $\{E_{h,n} \mid (h, n) \in W\}$  is countable. Consider then a sequence of models in  $\mathcal{M}$ ,  $\{E_n\}_{n=1}^\infty$  such that for every  $(h, n) \in W$  there exists an integer  $i$  such that:

$$E_i = E_{h,n}$$

Since  $\mathcal{M}$  is axiomatized by a sentence  $\psi \in L_{PBA}$  that is simple, we know that every sequence of models in  $\mathcal{M}$  is uniform for  $\psi$ . Invoking then Theorem 4.3.3 (if  $\phi \equiv \neg\theta$ ) or Theorem 4.3.5 (if  $\phi \in (L_{PBA} \cup \exists\neg L_{PBA})^{\{\wedge, \neg\}}$ ) we know that the sequence  $\{E_n\}_{n=1}^\infty$  is uniform for  $\phi$ . This implies that there exists an  $h \in I(\phi)$  such that  $\forall n, E_n \models_{AP}^h \phi$ . However, by the definition of the set  $W$  we know that there exists a pair  $(g, m) \in W$  such that  $\phi_{h,m} \equiv \phi_{g,m}$  and such that for some  $n \in \omega$ ,

$$E_n^{app} \models \neg(\phi_{g,m}) \equiv \neg(\phi_{h,m})$$

But this contradicts the fact that  $\forall n E_n \models_{AP}^h \phi$ . This completes the proof of the corollary. ■

The above corollary is a “uniformity of paths ” property with respect to the  $\models_{AP}$ . If a sentence  $\phi$  that is negative or belongs to  $(L_{PBA} \cup \exists\neg L_{PBA})^{\{\wedge, \neg\}}$  is approximately true in every model of a class defined by a positive bounded sentence, then every model of the class verifies  $\phi$  along the same path.

The next application is a weak uniformity principle for formulas in  $L_A$ .

First we introduce some notation. Fix a formula  $\psi$  in  $L_A$ . By a covering  $C$  of  $\psi$  we understand a set

$$C \subseteq I(\psi) \times \omega$$



with the additional property that for every  $h \in I(\psi)$  there exists  $(h_s, n_s) \in C$  such that:

$$(\psi)_{h,n_s} \equiv (\psi)_{h_s,n_s}$$

It is easy to verify that for every  $H \in I(\neg\psi)$ ,  $Image(H)$  is a covering of  $\psi$ . From Lemma 3.1.1 we know also that for every covering  $C$  of  $\psi$  there exists a countable covering  $W$  of  $\psi$  such that for every  $(h, n) \in C$  there exists  $(g, n)$  in  $W$  such that:

$$(\psi)_{h,n} \equiv (\psi)_{g,n}$$

We are ready for the following theorem.

**THEOREM 4.3.7** *Weak Uniformity Principle*

*Fix a signature  $\Phi$  and a complete class of models  $\mathcal{M}$  defined by a sentence  $\psi \in LPBA$ . Let  $\phi$  a sentence in  $L_A$ . The following are equivalent:*

- $\mathcal{M} \models_{AP} \phi$ .
- *For every countable covering  $C = \{(h, n) : h \in I(\phi) \wedge n \in \omega\}$  of  $\phi$ , there exists a finite  $F \subseteq C$  such that:*

$$\forall E \in \mathcal{M}, E^{app} \models \bigvee_{(h,n) \in F} (\phi)_{h,n}$$

□

**PROOF:** It follows from the Soundness Proposition (Proposition 3.1.3) that  $\forall E \in \mathcal{M}, E \models_{AP} \phi$  if and only if  $\forall E \in \mathcal{M}, E \models_{AP} \neg\neg\phi$ .

Invoking now Corollary 4.3.4 for the formula  $\neg(\neg\phi)$  and using the fact that every sequence of models  $\{E_n\}_{n=1}^\infty$  in a class of models axiomatized by a positive bounded formula is uniform for this formula, we get that the following two statements are equivalent:

- for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \phi$ .
- for every sequence of models  $\{E_n\}_{n=1}^\infty$ , for every  $g \in I(\neg\phi)$  there exists an integer  $r$  such that:

$$\forall n, E_n^{app} \models \neg(\neg\phi)_{g,r} \equiv \neg \bigwedge_{s=1}^r \neg(\phi)_{g(s)}$$

Finally, using the fact that for every  $g \in I(\neg\phi)$ ,  $Image(g)$  is a countable covering of  $\phi$ , we obtain the following equivalence:

- for every  $E \in \mathcal{M}$ ,  $E \models_{AP} \phi$ .
- For every countable covering  $C \subseteq \{(h,n) : h \in I(\phi) \wedge n \in \omega\}$  of  $\phi$  there exists a finite  $F \subseteq C$  such that:

$$\forall E \in \mathcal{M}, E^{app} \models \bigvee_{(h,n) \in F} (\phi)_{h,n}$$

But this is the desired result. ■

In the next section we use the Model Existence Theorem to get some elementary results for approximation principles in complete classes of models.

## 4.4 Approximation Principles in Complete Classes

As mentioned in Chapter 1 we are interested in the following question:

Given a complete class of models  $\mathcal{M}$ , for which kind of formulas  $\phi \in L_A$  does the following hold:

$$\mathcal{M} \models \phi \text{ iff } \mathcal{M} \models_{AP} \phi \quad (4.2)$$

We begin with the following theorem that gives a “weak” answer to the above question.

Recall that  $L_{PBA}$  is the smallest subset of  $L_A$  containing the atomic formulas and closed under countable conjunction, finite disjunction and bounded existential and universal quantification over countable many variables. Likewise,  $L_P$  is the smallest subset of  $L_{PBA}$  containing the atomic formulas and closed under countable conjunction, finite disjunction and bounded universal quantification over countable many variables. Finally,  $L_{A+} = ((L_P \cup \exists \neg L_P) \{\wedge, \neg\} \cup L_{PBA}) \{\wedge, \forall, \exists\}$ .

**THEOREM 4.4.1** *Let  $\Phi$  be a signature and let  $\mathcal{M}$  be a complete class of models in  $\Phi$  axiomatized by a sentence in  $L_{A+}$ . Fix  $\phi$  an arbitrary sentence in  $L_A$ . If  $\mathcal{M} \models \phi$ , then  $\mathcal{M} \models_{AP} \phi$ .  $\square$*

**PROOF:** Suppose that the statement is not true. Then there exists a model  $E \in \mathcal{M}$  such that:

$$E \not\models_{AP} \phi$$

It follows from the soundness properties of approximate truth (Proposition 3.1.3) that

$$E \models_{AP} \neg \phi$$

Furthermore, if  $\psi$  defines  $\mathcal{M}$ , then  $\psi \in L_{A+}$ . Theorem 3.3.10 imply that for this formula  $\psi$ ,  $E \models_{AP} \psi$ .

Clearly then  $\{E_n\}_{n=1}^\infty$ , with  $E_n = E$  for every  $n$ , is a sequence in  $\mathcal{M}$  uniform for  $\neg\phi \wedge \psi$ . Hence we can apply the Model Existence Theorem 4.2.7 to obtain that there exists a model  $B \in \mathcal{M}$  such that

$$B \models \neg\phi$$

But this contradicts the hypothesis. ■

The converse of this remark is true for a particular class of formulas. Recall that  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$  is the smallest set of formulas in  $L_A$  containing:

- $L_P$ .
- The formulas of the form:

$$\exists \vec{x} \in \vec{K} \neg\phi(\vec{x}, \vec{y}) \text{ for } \phi(\vec{x}, \vec{y}) \in L_P,$$

and closed under countable conjunction and negation. The next theorem shows that the converse of the previous result holds for the class  $\forall((L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}})$ .

### **THEOREM 4.4.2**

*Let  $\Phi$  be a signature, and  $\mathcal{M}$  a complete class of models. The following holds:*

*Suppose that  $\mathcal{M}$  is axiomatized by a sentence in  $L_{A+}$ . Let  $\phi(\vec{x}_1, \dots, \vec{x}_j, \dots)$  be a formula in  $(L_P \cup \exists\neg L_P)^{\{\wedge, \neg\}}$ , and let  $K_1, \dots, K_j, \dots$  be a corresponding collection of elements of  $\mathcal{P}$  with true sort. Then*

$$\mathcal{M} \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x})$$

*if and only if*

$$\mathcal{M} \models \forall \vec{x} \in \vec{K} \phi(\vec{x})$$

□

PROOF:  $\Leftarrow$ : Directly from Theorem 4.4.1.

$\Rightarrow$ : Suppose that

$$E \models_{AP} \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{b})$$

Then by the Soundness Proposition (Proposition 3.1.3) we obtain that: for every  $\vec{a} \in E$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models_{AP} \phi(\vec{a}, \vec{b})$ .

Since  $\phi \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$ , we can invoke Theorem 3.3.8 to conclude that for every  $\vec{a} = (a_1, \dots, a_i, \dots)$ , if  $E \models \bigwedge_{i=1}^{\infty} K_i(\vec{a}_i)$  then  $E \models \phi(\vec{a}, \vec{b})$ . This is the desired result. ■

In summary, the simplest interesting case where question 4.2 could have a negative answer is when  $\phi \equiv \exists \vec{x} \in \vec{K} \psi(\vec{x})$  and  $\psi$  is a quantifier free positive formula.

Let us close this section with an application of the previous results to complete classes of models defined by positive bounded sentences. We prove an equivalence principle between satisfaction of an infinitary formula in a complete class  $\mathcal{M}$  and satisfaction of a fixed path of finitary approximations.

**THEOREM 4.4.3** *Strong Equivalence Principle*

*Fix a signature  $\Phi$  and a complete class of models  $\mathcal{M}$  axiomatized by a sentence in  $L_{PBA}$ . Let  $\phi \in L_A$  be an universal sentence ( $\phi \equiv \forall \vec{x} \in K \theta(\vec{x})$  with  $\theta \in ((L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}})$ ). Then the following are equivalent:*

- $\mathcal{M} \models \phi$ .
- There exists  $h \in I(\phi)$  such that for every model  $E \in \mathcal{M}$ ,  $E \models_{AP} {}^h \phi$ .

□

PROOF: Follows directly from Theorem 4.4.2 and Corollary 4.3.6.■

A weak version of the previous result holds for negative formulas.

**THEOREM 4.4.4** *Weak Uniformity of paths*

*Fix a signature  $\Phi$  and a complete class of models axiomatized by a sentence in  $L_{PBA}$ . Let  $\phi$  be a negative sentence ( $\phi = \neg\theta$ ). If  $\mathcal{M} \models \phi$ , then:*

*For every  $h \in I(\theta)$  there exists an integer  $n$  such that for every model  $E \in \mathcal{M}$ ,  $E^{app} \models \neg(\theta)_{h,n}$ .□*

PROOF: Assume that  $\forall E \in \mathcal{M}, E \models \phi$ . Since  $\mathcal{M}$  is axiomatized by a formula in  $L_{PBA} \subset L_{A+}$  we can invoke Theorem 4.4.1 to obtain that:

$$\forall E \in \mathcal{M}, E \models_{AP} \phi \equiv \neg\theta$$

Now we use Corollary 4.3.4 to obtain the desired result.■

The above theorems can be used to obtain uniformity statements in complete classes of models. An universal sentence defined as above holds for all the models of a complete class axiomatized by a sentence in  $L_{PBA}$  iff the “uniform” version of the property also holds in all the models of  $\mathcal{M}$ . We will give an example of the applicability of this theorem in functional analysis in the next section.

Furthermore, this theorem suggests a way of proving that a sentence  $\phi$  of the form  $\forall \vec{x} \in \vec{K} \psi(\vec{x})$  or of the form  $\bigvee_{i=1}^{\infty} \psi_i$  does not hold in all the models of a complete class of models axiomatized by countably many positive bounded formulas. It is enough to show that for every path  $h \in I(\phi)$  there is a model  $E \in \mathcal{M}$  such that  $E \not\models_{AP} {}^h\phi$ .

## 4.5 Application to Functional Analysis

We begin by giving a brief summary of some fundamental results in Functional Analysis.

We need first some notation. Let  $N$  a normed space,  $\{x_n\}_{n=1}^{\infty}$  a collection of vectors in  $N$ , and  $u = \sum_{i \in E_1} a_i x_i$  and  $v = \sum_{i \in E_2} a_i x_i$  where  $E_1, E_2$  are subsets of the integers. We say that  $u < v$  iff

$$\forall n \in E_1 \forall m \in E_2 \ n < m \text{ (i.e. } E_1 < E_2)$$

Likewise we say that  $n < u$  iff  $\forall m \in E_1 \ m > n$ . Finally, we say that  $u$  has rational coefficients with respect to  $\{x_n\}_{n=1}^{\infty}$  iff  $\forall i \in E_1 \ a_i \in Q$ .

Here are some basic definitions concerning Banach spaces. Most of the statements in this section are taken from the classical book of Lindenstrauss & Tzafriri ([29]).

### DEFINITION 4.5.1 *Schauder Basis*

A sequence  $(x_i)_{i=1}^r$  (or  $(x_i)_{i=1}^{\infty}$ ) in a Banach space  $(E, \|\cdot\|)$  is a basic sequence iff there exists a real  $K \geq 1$  such that for all integers  $n, m$  (with  $n + m \leq r$  in the case of a finite sequence  $(x_i)_{i=1}^r$ ), for every  $\vec{a} \in R^{n+m}$ :

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n+m} a_i x_i \right\|$$

Clearly the span of every countable basic sequence is an infinite dimensional Banach space. The minimal value of the constant  $K$  is called the basis constant of the sequence  $(x_i)_{i=1}^r$  (or of the sequence  $(x_i)_{i=1}^{\infty}$ ).  $\square$

**DEFINITION 4.5.2** *K-Finite Representability*

A Banach space  $(Y, |\cdot|)$  is  $K$ -finitely representable in a Banach space  $(X, |\cdot|)$  if and only if for every finite dimensional subspace  $E$  of  $Y$  there exists a linear isomorphism  $T : E \mapsto X$  with  $\|T\|\|T^{-1}\| \leq K$ .  $Y$  is said to be finitely representable in  $X$  if it is  $K$ -finitely representable in  $X$  for all  $K > 1$ .

A sequence  $\{y_n\}_{n=1}^{\infty}$  in a Banach space  $(Y, \|\cdot\|)$  is  $K$ -block finitely representable in a sequence  $(x_n)_{n=1}^{\infty}$  in a Banach space  $(X, |\cdot|)$  if for each integer  $N$  there exist, in the span of  $(x_n)_{n=1}^{\infty}$ , vectors  $u_1 < u_2 < \dots < u_N$  such that for every choice of scalars  $\{\alpha_i\}_{i=1}^N$ :

$$\left\| \sum_{i=1}^N \alpha_i y_i \right\| \leq \left| \sum_{n=1}^N \alpha_n u_n \right| \leq K \left\| \sum_{i=1}^N \alpha_i y_i \right\|$$

(For the previous definition it is enough to require the scalars  $\{\alpha_n\}$  to be rational).

The sequence  $u_1 < u_2 < \dots < u_N$  is called a block sequence of  $(x_n)_{n=1}^{\infty}$ .

A sequence  $\{y_n\}$  is block finitely representable in a sequence  $\{x_n\}_{n=1}^{\infty}$  if and only if  $\{y_n\}_{n=1}^{\infty}$  is  $K$ -block finitely representable in  $\{x_n\}_{n=1}^{\infty}$  for every  $K > 1$ .  $\square$

It is easy to see that finite representability is an approximate property (see Example 2.5.2). Henson & Moore studied this property for the positive bounded logic  $L_{PB}$  (see [15, 20]).

Krivine ([27]) obtained the following deep theorem concerning the finite representability of the  $\ell_p$  and  $c_0$  spaces in the infinite dimensional Banach spaces.

**THEOREM 4.5.3** *Krivine's Theorem*

*Let  $E$  a Banach space. Let  $\{x_n\}_{n=1}^{\infty} \subseteq E$  be a sequence whose span is infinite dimensional. Then the usual basis of one of the spaces  $\ell_p$  for  $p \in [1, \infty)$  or  $c_0$  is block finitely representable in  $\{x_n\}_{n=1}^{\infty}$ .*



□

The proof of this theorem can be found in [31].

Lemberg ([28]) and Rosenthal ([39]) obtained a stronger and uniform version of the above theorem. They proved:

**THEOREM 4.5.4** *Uniform version of Krivine's Theorem*

*Fix arbitrary  $K \geq 1$ ,  $n \in \omega$  and  $\epsilon > 0$ . There exists an  $m(K, n, \epsilon)$  such that if  $(x_i)_{i=1}^m$  is a finite basic sequence in some Banach space with basis constant  $K$ , then there exists  $1 \leq p \leq \infty$  and a block sequence  $(y_i)_{i=1}^n$  of  $(x_i)_{i=1}^m$  so that  $(y_i)_{i=1}^n$  is  $(1 + \epsilon)$  isomorphic to the unit vector basis of  $\ell_p^n$ . □*

Let us see how we can get the above result directly from Krivine's Theorem using the Weak Uniformity Principle (Theorem 4.4.4) developed in the previous section.

First notice that Krivine's Theorem clearly implies the following weak statement:

*For every basic sequence  $(x_i)_{i=1}^\infty$  in a normed space  $(N, \|\cdot\|)$ , for every  $\epsilon > 0$ , for every integer  $n$  there exists  $\ell_p^n$  ( $p \in Q \cap [1, \infty]$ ) such that the usual basis of the  $\ell_p^n$  is  $(1 + \epsilon)$ -block finitely representable in  $(x_i)_{i=1}^\infty$ .*

Consider then the class  $\mathcal{M}$  of normed space structures. Here we need more notation. For every real number  $p \in \{0\} \cup [1, \infty)$  and every integer  $n$  let:

- For every  $\vec{a} \in Q^n$ , if  $p \in [1, \infty)$ ,  $f(\vec{a}, n, p) = \sqrt[p]{\sum_{i=1}^n |a_i|^p}$ .
- if  $p \in \{0\}$ ,  $f(\vec{a}, n, 0) = \max\{|a_i| \mid i \leq n\}$ .

- For every  $\epsilon > 0$ , let  $\theta[n, p, \epsilon](\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i)$  be the formula that states that the usual basis of the space  $\ell_p^n$  is  $(1 + \epsilon)$ -equivalent to the vectors

$$\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i$$

This can be written as the following positive formula in  $L_A$ :

$$\bigwedge_{\vec{c} \in Q^n} f(\vec{c}, n, p) \leq \left\| \sum_{j=1}^n c_j \left( \sum_{i=q_j+1}^{q_{j+1}} b_i x_i \right) \right\| \leq (1 + \epsilon) f(\vec{c}, n, p)$$

Using the finite dimensionality of the  $\ell_p^n$  it can be easily seen that in the complete class of normed structures, for every  $n \in \omega, p \in Q \cap [1, \infty], \epsilon > 0$  there exists an integer  $w$  such that the  $w$  approximation of the formula  $\theta[n, p, \epsilon]$  implies the formula  $\theta[n, p, 2\epsilon]$ . In other words, there exists an integer  $w$  such that:

$$\mathcal{M} \models (\theta[n, p, \epsilon](\vec{y}))_w \Rightarrow \theta[n, p, 2\epsilon](\vec{y}) \quad (4.3)$$

- For every  $K \geq 1$ , let  $\sigma[K](\vec{x})$  be the positive formula in  $L_A$  that states that the sequence  $(x_i)_{i=1}^\infty$  is a basic sequence with basic constant  $K$ :

$$\bigwedge_{n, m \in \omega} \bigwedge_{\vec{a} \in Q^{n+m}} \left\| \sum_{i=1}^n a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n+m} a_i x_i \right\|$$

We are now ready to write the weak version of Krivine's Theorem in  $L_A$ . Let us call  $Q^\sharp$  the set:  $\{0\} \cup (Q \cap [1, \infty))$ . Likewise for a fixed integer  $n$  let  $\vec{q}$  denote a vector  $(q_1, q_2, \dots, q_{n+1})$  of integers such that  $q_1 < q_2 < \dots < q_{n+1}$  and let  $V_n \subset \omega^{n+1}$  be the collection of all such vectors.

Fix now an enumeration of  $Q^\sharp$  and an enumeration of  $V_n$ . For every integer  $r$ ,  $Q^\sharp \uparrow r$  denotes the restriction of  $Q^\sharp$  to the first  $r$  elements of its (fixed) enumeration.

Likewise,  $V_n \upharpoonright r$  denotes the restriction of  $V_n$  to the first  $r$  elements of its (fixed) enumeration.

The weak version of Krivine's Theorem just says that for every  $K \geq 1$ , for every  $\epsilon \geq 0$ , for every  $n \in \omega$ ,  $\forall E \in \mathcal{M}$ ,

$$E \models \neg \exists \vec{x} \in \vec{B}_1(\sigma[K](\vec{x}) \wedge$$

$$\bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg \theta[n, p, \epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right)$$

Since  $\mathcal{M}$  is a complete class of models axiomatized by a sentence in  $LPBA$  (Example 4.1.3) we can invoke the Weak Uniformity of Paths (Theorem 4.4.4).

We get:

for every  $K \geq 1$ , for every  $\epsilon \geq 0$ , for every  $n \in \omega$ , for every path

$$g \in I(\sigma[K] \wedge \bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg \theta[n, p, \epsilon])$$

there exists an integer  $r$  such that for every normed space structure  $E$ ,

$$E^{app} \models \neg (\exists \vec{x} \in \vec{B}_1(\sigma[K](\vec{x}) \wedge$$

$$\bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg \theta[n, p, \epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right))_{g,r} \quad (4.4)$$

Select then, for any  $\epsilon > 0$ , for every integer  $n$  and every  $K \geq 1$ , a function

$$h : Q^\# \times V_n \times Q^{q_{n+1}} \mapsto \omega$$

such that  $h(p, (q_1, q_2, \dots, q_{n+1}), \vec{b}) \geq w$ , where the  $w$  is defined above in statement 4.3 for the fixed values of  $n, p, \epsilon$ .

It is easy to see that

$$(\emptyset, h) \in I(\sigma[K] \wedge \bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg\theta[n, p, \epsilon])$$

By the statement 4.4 above, there exists an integer  $r$  such that for every normed space structure  $E$ ,

$$\begin{aligned} E^{app} \models & \neg(\exists \vec{x} \in \vec{B}_1 \sigma[K](\vec{x}) \wedge \\ & \bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg\theta[n, p, \epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right) )_{h,r} \equiv \\ & \neg \exists \vec{x} \in \vec{B}_1 (\sigma[K](\vec{x}))_r \wedge \\ & \bigwedge_{p \in Q^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \bigwedge_{s=1}^r \neg (\theta[n, p, \epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right)_{h(p, (q_1, \dots, q_{n+1}), \vec{b})}) \equiv \\ & \forall \vec{x} \in \vec{B}_1 (\vec{x}) ((\sigma[K](\vec{x}))_r \Rightarrow \\ & \bigvee_{p \in Q^\#} \bigvee_{\vec{q} \in V_n} \bigvee_{\vec{b} \in Q^{q_{n+1}}} \bigvee_{s=1}^r (\theta[n, p, \epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right)_{h(p, (q_1, \dots, q_{n+1}), \vec{b})})) \end{aligned}$$

and this implies, using property 4.3 and the definition of  $h$ ,

$$\begin{aligned} & \forall \vec{x} \in \vec{B}_1 ((\sigma[K](\vec{x}))_r \Rightarrow \\ & \bigvee_{p \in Q^\#} \bigvee_{\vec{q} \in V_n} \bigvee_{\vec{b} \in Q^{q_{n+1}}} (\theta[n, p, 2\epsilon] \left( \sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3}, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right)) \end{aligned}$$

This last statement can be written as follows:

*For every  $K \geq 1$ , for every  $\epsilon > 0$ , for every integer  $n$ , there exists a finite collection  $I = \{p_1, \dots, p_r\} \in Q \cap [1, \infty) \cup \{0\}$ , an integer  $m$ , and a finite collection of vectors of terms*

$$V = \{\vec{u}(\vec{x}) = (u_1, \dots, u_n) : u_1(\vec{x}) < u_2(\vec{x}) < u_n(\vec{x}) \text{ are terms with variables among } (x_1, \dots, x_m)\}$$

*such that for every normed space  $(E, \|\cdot\|)$  and every basic sequence*

*$(x_i)_{i=1}^m$  with basic constant  $K$  there exists a  $\vec{u}(\vec{x}) \in V$  and a  $p \in I$  such*

*that the block sequence of vectors  $u_1 < u_2 < \dots u_n$  with respect to  $(x_i)_{i=1}^n$  is  $1 + \epsilon$ -equivalent to the usual basis of  $\ell_p^n$  (or  $c_0^n$  if  $p = 0$ ).  $\square$*

This is the desired result.

# Chapter 5

## Omitting Formulas in $L_A$

The aim of this chapter is to develop tools to construct countable models that omit infinitary formulas in  $L_A$  in this sense:

Given a complete class of models  $\mathcal{M}$  and a sentence  $\phi$  such that  $\forall E \in \mathcal{M}, E \models_{AP} \phi$ , there exists a countable model  $B \in \mathcal{M}$  with  $B \models \neg\phi$ .

We already know from Chapter 4 that the above can not happen if  $\phi = \forall \vec{x} \in \vec{K} \psi(\vec{x})$  with  $\psi \in (L_P \cup \exists \neg L_P)^{\{\wedge, \neg\}}$  for a complete class of models  $\mathcal{M}$  axiomatized by a sentence in  $L_{A+}$  (Theorem 4.4.2).

It follows that the simplest case where one can omit  $\phi$  in complete classes of models would be when  $\phi$  is a  $\exists$  formula in  $L_A$ . In Section 5.1 we prove an omitting theorem (Theorem 5.1.3) for  $\exists$  formulas in complete classes of models. Section 5.2 gives an application of this theorem to classical infinitary multisorted logic. We show that Theorem 5.1.3 generalizes the Omitting Types Theorem in classical logic.

### 5.1 Omitting $\exists$ Formulas in $L_A$

Our intention is to prove a omitting theorem using the notion of approximate formulas in complete classes of models.

We begin with a definition concerning formulas in  $L_A$ .

**DEFINITION 5.1.1** *Limited and Strong Sentences*

Fix a complete collection of models  $\mathcal{W}$  for a signature  $\Phi$ . A sentence  $\psi$  is limited for  $\mathcal{W}$  if and only if:

- For every model  $E \in \mathcal{W}$ ,  $E \models \psi$  iff  $E \models_{AP} \psi$ .
- Every infinite sequence  $(E_n)_{n=1}^{\infty}$  of models of  $\mathcal{W}$  that approximately satisfy  $\psi$  contains an infinite subsequence  $(E_{n_i})_{i=1}^{\infty}$  that is uniform for  $\psi$ .

A formula  $\psi(\vec{x})$  in  $L_A$  is strong for  $\mathcal{W}$  iff for every model  $E \in \mathcal{W}$ , for every  $\vec{a}$  in  $E$ ,

$$\text{if } E \models_{AP} \psi(\vec{a}) \text{ then } E \models \psi(\vec{a})$$

□

Next we give examples of limited sentences and strong formulas.

**EXAMPLE 5.1.2**

Consider the complete collection of all classical multisorted models for a language  $\Phi = (\mathcal{F}, \mathcal{P})$  where all the fixed sort predicates are compact and for every fixed sort function symbol there exists a  $C \in \mathcal{P}$  such that in every model of  $\mathcal{W}$ ,  $Im(f^\bullet) \subseteq C$ . Then Lemma 3.4.1 and Lemma 3.4.2 imply that every first order sentence is limited in this class of models (since for those models and formulas,  $\models$  and  $\models_{AP}$  coincide and the number of non equivalent paths of any first order formula in this class is finite). It follows then that any sentence of the form  $\bigwedge_{i=1}^{\infty} \psi_i$

with the  $\psi_i$  first order sentences is also limited in the complete collection of all the classical multisorted models. For those classes of models, it is easy to verify that the formulas of the form  $\bigwedge_{i=1}^{\infty} \psi_i$  are also strong formulas for this collection of models.

Consider the complete collection of all the normed space structures. The sentence  $\psi$  that states that the normed space is infinite dimensional,

$$\psi \equiv \exists \vec{x} \in \vec{B}_1 \left( \bigwedge_{n=1}^{\infty} \bigwedge_{m=1, m \neq n}^{\infty} \|x_n - x_m\| \geq 1/2 \right)$$

is positive bounded, and we know (see Example 2.5.3) that for every normed space structure  $E$ ,

$$E \models \psi \text{ iff } E \models_{AP} \psi$$

It follows then that  $\psi$  is limited for the complete collection of normed space structures.  $\square$

Fix  $\Phi$  a countable signature, and let  $C = \{c_1, \dots, c_k, \dots\}$  be a collection of constant symbols with true sort and not appearing in  $\Phi$ . Let  $\Phi(C)$  be the signature that extends  $\Phi$  by adding  $C$ . Let  $T$  be the collection of all variable free terms in  $L_A(\Phi(C))$ .

Given any complete collection of models  $\mathcal{W}$  for  $\Phi$  and a countable collection of unary predicate symbols in  $\Phi$ ,  $\{P_i : i \in \omega\}$ , we can define a complete collection of models  $\mathcal{W}(\{P_i : i \in \omega\})$  for  $\Phi(C)$  in the following way:

A model  $E$  in  $\Phi(C)$  belongs to  $\mathcal{W}(\{P_i : i \in \omega\})$  if and only if

- the restriction of  $E$  to  $\Phi$  belongs to  $\mathcal{W}$ .
- For every integer  $i$ ,  $E$  satisfies  $P_i(c_i)$ .



It is easy to verify that such a collection is a complete class of models in  $\phi(C)$ .

The last step before the omitting theorem is to define some notation. Given any universal formula  $\forall(\vec{y}, \vec{x}) \in (\vec{K}, \vec{D})\theta(\vec{y}, \vec{x})$  in  $L_A$  with  $|\vec{y}| = r < \infty$ , and given any arbitrary:

- path  $h \in I(\theta)$ .
- A vector of terms  $\vec{t} = (t_1, t_2, \dots) \in T^r$ .
- A function  $f : \omega \mapsto \omega$ .

by  $\theta^{[\vec{t}, h, n, f]}$  we mean the following formula in  $L_{AP}$ :

$$\bigwedge_{i=1}^r (K_i(\vec{t}_i))_{f(n)} \Rightarrow \forall \vec{x} \in \vec{D}_{f(n)}(\theta(\vec{t}, \vec{x})_{h,n})$$

Intuitively speaking, for every  $f : \omega \mapsto \omega$ , the formulas  $\theta^{[\vec{t}, h, n, f]}$  are approximations to the formula  $\vec{K}(\vec{t}) \Rightarrow \forall \vec{x} \in \vec{D}\theta(\vec{t}, \vec{x})$ .

Recall the definition of a simple sentence for a complete class of models: A sentence  $\phi$  is going to be simple for  $\mathcal{M}$  if for every model structa in  $\mathcal{M}$ ,  $E \models \phi$  implies  $E \models_{AP} \phi$ .

The following theorem gives a tool to decide when a complete class of models omits an  $\exists$  sentence in  $L_A$ .

**THEOREM 5.1.3** *Omitting  $\exists$  Formulas in  $L_A$*

*Fix  $\mathcal{W}$  a complete collection of models for  $\Phi$ . Let  $\mathcal{M} \subseteq \mathcal{W}$  a complete class of models defined by a sentence  $\psi$  limited in  $\mathcal{W}$ . Fix a collection  $\{P_i : i \in \omega\}$  of unary predicate symbols in  $\Phi$  with true sort. Consider the  $\forall$  formula*

$$\forall(\vec{y}, \vec{x}) \in (\vec{K}, \vec{D})\theta(\vec{y}, \vec{x})$$

in  $L_A$  with  $\theta$  simple for  $\mathcal{W}$  and  $|\vec{y}| = r < \infty$ .

Suppose that the following property is true:

- For every finite collection  $F \subset T^r$ , for all functions  $\{f_{\vec{t}} \in \omega^\omega : \vec{t} \in F\}$ , for every function  $H : F \mapsto I(\theta)$  and for every sentence  $\Delta \in L_{AP}(C)$ , if  $\forall n$

$$\Delta \wedge \bigwedge_{\vec{t} \in F} \theta^{[\vec{t}, H(\vec{t}), n, f_{\vec{t}}]}$$

is consistent in  $\mathcal{M}(\{P_i : i \in \omega\})$ ,

then  $\forall \vec{q} \in T^r$ , there exists  $h \in I(\theta)$  and a function  $g : \omega \mapsto \omega$  such that  $\forall n$ ,

$$\Delta \wedge \bigwedge_{\vec{t} \in F} \theta^{[\vec{t}, H(\vec{t}), n, f_{\vec{t}}]} \wedge \theta^{[\vec{q}, h, n, g]}$$

is consistent in  $\mathcal{M}(\{P_i : i \in \omega\})$ .

Then there exists a countable model  $E \in \mathcal{M}$  such that

$$E \models \forall (\vec{y}, \vec{x}) \in (\vec{K}, \vec{D}) \theta(\vec{y}, \vec{x})$$

□

PROOF: List all the elements of  $T^r$ :

$$\vec{t}_1, \dots, \vec{t}_j, \dots$$

List also all the sentences in  $L_{AP}(\Phi(C))$ :

$$\sigma_1, \dots, \sigma_j, \dots$$

We will construct by induction :

- a collection  $\Gamma_n$  of finite sentences in  $L_{AP}(\Phi(C))$  consistent with  $\mathcal{M}(\{P_i : i \in \omega\})$ .

- A function  $H : \omega \mapsto I(\theta)$ .
- A collection of functions  $\{f_i \in \omega^\omega : i \in \omega\}$ .

with the following properties:

- $\Gamma_n \subseteq \Gamma_{n+1}$  and  $\Gamma_n$  is consistent with  $\mathcal{M}(\{P_i : i \in \omega\})$ .
- $\sigma_n$  or  $\neg\sigma_n$  is in  $\Gamma_n$ .
- if  $\sigma_n \equiv \exists \vec{y} \in \vec{D}\phi(\vec{x})$  is in  $\Gamma_n$  then there exists a vector of constants  $\vec{c}$  in  $C$  such that  $\vec{D}(\vec{c}) \wedge \phi(\vec{c})$  is in  $\Gamma_n$ .
- For every  $i \leq n$ , for every integer  $m$ ,  $\{\theta^{[\vec{t}_i, H(i), m, f_i]}\} \cup \Gamma_n$  is consistent in  $\mathcal{M}(\{P_i : i \in \omega\})$ .

We leave to the reader the verification that such a collection of  $\Gamma_n$ , and the map  $H : \omega \mapsto I(\theta)$  verifying the above properties can be constructed. It is enough to use the hypothesis of the theorem.

Note that for every integer  $n$ , for every integer  $m$ ,

$$\theta^{[\vec{t}_n, H(n), m, f_n]} \equiv \bigwedge_{i=1}^r (K_i(\vec{t}_i))_{\emptyset, f_n(m)} \Rightarrow \forall \vec{x} \in \vec{D}_{f_n(m)}(\theta(\vec{t}_n, \vec{x})_{H(n), m}) \in \bigcup_{j=1}^{\infty} \Gamma_j$$

Consider now the sequence  $(E_n)_{n=1}^{\infty}$  of models in  $\mathcal{M}(\{P_i : i \in \omega\})$  such that for every integer  $n$ ,  $E_n^{app} \models \Gamma_n$ .

Since  $\psi$  is a limited formula for  $\mathcal{M}$ , it is easy to see that it is also a limited formula for  $\mathcal{M}(\{P_i : i \in \omega\})$ . Hence there is a subsequence of models  $(E_{n_i})_{i=1}^{\infty}$  that is uniform for  $\psi$ . Consider now the ultraproduct generated by this family over a non principal ultrafilter  $\mathcal{U}$ . Call this ultraproduct  $\prod_{\mathcal{U}} E_{n_i}$ . We know that

$\prod_{\mathcal{U}} E_{n_i}$  is in  $\mathcal{M}(\{P_i : i \in \omega\})$ . It follows from the Property of the Ultraproduct (Lemma 4.2.4) and the properties of the  $\Gamma_{n_i}$  that:

- For every  $(\vec{q}, \vec{z}) \in T^r \times T^\infty$  such that  $\prod_{\mathcal{U}} E_{n_i} \models \vec{K}(\vec{q}) \wedge \vec{D}(\vec{z})$ ,

$$\prod_{\mathcal{U}} E_{n_i} \models_{AP} \theta(\vec{q}, \vec{z})$$

Proof: Fix  $\vec{q} \in T^r$  such that  $\prod_{\mathcal{U}} E_{n_i} \models \vec{K}(\vec{q})$ . Then for every integer  $n$  there exists a  $p \in \mathcal{U}$  such that:

$$\forall i \in p, E_{n_i}^{app} \models \bigwedge_{i=1}^r (K_i(t_i))_n$$

Let  $\vec{q} = \vec{t}_k$ . Since for every  $m$ ,  $\theta[\vec{t}_k, H^{(k), m, f_k}]$  in  $\bigcup_{j=1}^\infty \Gamma_j$ , it follows from the above statement, that for every integer  $n$  there exists  $p \in \mathcal{U}$  such that:

$$\forall i \in p, E_{n_i}^{app} \models \forall \vec{x} \in \vec{D}_{\emptyset, f_k(n)}(\theta(\vec{q}, \vec{x})_{H^{(k), n}}) \quad (5.1)$$

Consider now any vector of elements  $\vec{z}$  in  $T^\infty$ . Suppose that  $\prod_{\mathcal{U}} E_{n_i} \models \bigwedge_{j=1}^\infty D_j(z_j)$ .

Then by the property of the ultraproduct (Lemma 4.2.4) we get that for every integer  $m$  there exists a  $p \in \mathcal{U}$  such that:

$$\forall i \in p, E_{n_i}^{app} \models \bigwedge_{j=1}^m (D_j(z_j))_{\emptyset, m}$$

From this remark and statement 5.1 we obtain then that for every integer  $n$  there exists a  $p \in \mathcal{U}$  such that:

$$\forall i \in \mathcal{U}, E_{n_i}^{app} \models (\theta(\vec{q}, \vec{z}))_{H^{(k), n}}$$

Invoking the property of the ultraproduct (Lemma 4.2.4) again we finally get:

$$\prod_{\mathcal{U}} E_{n_i} \models_{AP} \theta(\vec{q}, \vec{z})$$

This completes the proof of the statement.

- $\prod_{\mathcal{U}} E_i \models_{AP} \psi$  since  $\psi$  is limited.
- The interpretations of the constants  $c_i$  in  $\prod_{\mathcal{U}} E_{ni}$  define a submodel of  $\prod_{\mathcal{U}} E_i$  (because the  $(\Gamma_{ni})_{i=1}^{\infty}$  decide every sentence in  $L_{AP}(\Phi(C))$ , and have witnesses for every existential formula in  $(\Gamma_{ni})_{i=1}^{\infty}$ ). Let us call this submodel  $B$ . Since  $B \subseteq \prod_{\mathcal{U}} E_{ni} \in \mathcal{W}(\{P_i : i \in \omega\})$ , it follows that  $B \in \mathcal{W}(\{P_i : i \in \omega\})$ . Furthermore, we leave to the reader to verify by induction on formulas in  $L_A(\Phi(C))$  that:

$$\forall \text{ formula } \phi(\vec{x}) \in L_A(\Phi(C)), \text{ for every } \vec{a} \text{ in } B, \prod_{\mathcal{U}} E_{ni} \models_{AP} \phi \text{ iff } B \models_{AP} \phi$$

From the previous remarks it follows then that:

- For every  $(\vec{t}, \vec{q}) \in T^r \times T^\infty$  such that  $B \models \vec{K}(\vec{t}) \wedge \vec{D}(\vec{q})$ ,

$$B \models_{AP} \text{theta}(\vec{t}, \vec{q})$$

- $B \models_{AP} \psi$ .

Invoking now the fact that  $\psi$  is limited for  $\mathcal{M}(\{P_i : i \in \omega\})$  and that  $\theta$  is strong for  $\mathcal{W}$ , we finally get:

- For every  $\vec{q} \in T^\infty$  such that  $B \models \vec{K}(\vec{q})$ ,

$$B \models \forall \vec{x} \in \vec{D} \theta(\vec{q}, \vec{x})$$

- The restriction of  $B$  to  $\Phi$  is in  $\mathcal{M}$ .

But this is the desired result. ■

## 5.2 Applications to Classical Logic

We show that the previous theorem yields an extension of the Omitting Types Theorem in classical logic.

In this section we fix  $S$  a collection of discrete metric spaces (i.e.  $(M, \rho_M)$  so that  $\text{Image}(\rho_M) = \{0, 2\}$ ). We also fix  $\Phi = (\mathcal{F}, \mathcal{P})$  a signature for  $S$  containing a fixed universal predicate  $K$  in  $\mathcal{P}$  and such that every fixed sort predicate is compact. Recall that a classical multisorted model of  $\Phi$  is a structure  $E = (X, d, F, P)$  such that the metric  $d$  is discrete, and for every  $a \in X$ ,

$$E \models K(a)$$

Let  $\mathcal{W}$  be a collection (complete) of all classical multisorted models for  $\Phi$  with the property that for every fixed sort function symbol  $f$  there exists a compact predicate of the same sort  $\mathcal{P}$  such that for every classical multisorted model  $E \in \mathcal{W}$ ,  $\text{Im}(f^\bullet) \subset C$ . We already know that any such collection  $\mathcal{W}$  is a complete collection of models. The Omitting Theorem proved in the previous section can be stated in a simplified way for  $\mathcal{W}$ .

**THEOREM 5.2.1** *Extension of the Omitting Types Theorem*

Let  $\mathcal{M} \subseteq \mathcal{W}$  be the collection of all the models that satisfy the sentence  $\psi \equiv \bigwedge_{i=1}^{\infty} \psi_i$ , where the  $\psi_i$  are first order sentences. Consider the  $\forall$  formula

$$\forall \vec{y}, \vec{x} \theta(\vec{y}, \vec{x})$$

with  $\theta$  in  $(L_{\omega, \omega})^{\{\wedge, \neg\}}$  and  $|\vec{x}| = r < \infty$  (the arity of  $|\vec{x}|$  can be infinite).

Suppose that the following property is true:

- For every finite collection  $F \subset T^r$ , for every function  $H : F \mapsto I(\theta)$  and for every sentence  $\Delta \in L_{AP}(C)$ , if  $\forall n$

$$\Delta \wedge \bigwedge_{\vec{t} \in F} \forall \vec{x} (\theta(\vec{t}, \vec{x}))_{H(\vec{t}, n)}$$

is consistent in  $\mathcal{M}$ ,

then  $\forall \vec{q} \in T^r$ , there exists  $h \in I(\theta)$  such that  $\forall m$

$$\Delta \wedge \left( \bigwedge_{\vec{t} \in F} \forall \vec{x} (\theta(\vec{t}, \vec{x}))_{H(\vec{t}, m)} \right) \wedge \forall \vec{x} ((\theta(\vec{q}, \vec{x}))_{h, m})$$

is consistent in  $\mathcal{M}$ .

Then there exists a countable model  $E \in \mathcal{M}$  such that

$$E \models \forall \vec{x} \theta(\vec{x})$$

□

PROOF: Since we are dealing with classical multisorted models, it is clear that in every such model,

$$E \models \forall \vec{x} \in \vec{K} \phi(\vec{x}, \vec{y}) \text{ iff } E \models \forall \vec{x} \phi(\vec{x}, \vec{y})$$

where  $K$  is the universal predicate for the models in  $\mathcal{W}$ . Similarly the existential quantification is equivalent to bounded existential quantification. We can safely omit the bounding predicates from the existential and universal quantifiers.

Furthermore, we saw in Chapter 3 (Corollary 3.4.4) that every sentence in  $(L_{\omega, \omega})^{\{\wedge, \neg\}}$  is simple in  $\mathcal{W}$ . Likewise, every infinite conjunction of first order formulas is limited in the complete class of classical multisorted models (see Example 5.1.2).

Since in every multisorted model, and for every predicate symbol  $K$ , for every integer  $n$ ,  $K_n \equiv K$ , we can see that:

$$\theta^{[\vec{t}, h, n, f]} \equiv \bigwedge_{i=1}^r (K(t_i))_{f(n)} \Rightarrow \forall \vec{x} \in \vec{K}_{f(n)}(\theta(\vec{t}, \vec{x})_{h, n})$$

is equivalent, for every  $n, h \in I(\theta)$ ,  $f \in \omega^\omega$  and  $\vec{t} \in T^r$ , and for the universal predicate  $K$ , to:

$$\forall \vec{x}(\theta(\vec{t}, \vec{x})_{h, n})$$

Finally, consider the collection  $\{P_i : i \in \omega\}$  of unary predicate symbols with true sort such that for every  $i$ ,  $P_i = K$ . it is clear that any interpretation of the constants  $\{c_i : i \in \omega\}$  in any classical multisorted model verifies  $K(c_i)$ .

From these remarks it is clear that this theorem follows from Theorem 5.1.3.■

let us discuss the relationship between the previous theorem and the classical Omitting Types Theorem for first order logic.

We show that Theorem 5.2.1 implies the Omitting Types Theorem for classical multisorted structures.

We begin with a lemma that states that countable disjunctions of first order formulas behave like first order formulas for its approximations.

### LEMMA 5.2.2

*Fix  $S$  an arbitrary collection of discrete metric spaces. Let  $\Phi = (\mathcal{F}, \mathcal{P})$  be a signature for  $S$  and let  $\mathcal{W}$  be a the collection of classical multisorted models for  $\Phi$ .*

*For every countable collection of first order formulas  $\phi_i$ , for every  $h \in I(\bigvee_{i=1}^{\infty} \phi_i)$   $\exists m \in \omega$  such that for every classical multisorted structure  $E \in \mathcal{W}$ ,  $\forall \vec{a}$  in  $E$*

$$\forall n \geq m \ E^{app} \models \left( \bigvee_{i=1}^{\infty} \phi_i(\vec{a}) \right)_{h, n} \Leftrightarrow \left( \bigvee_{i=1}^{\infty} \phi_i(\vec{a}) \right)_{h, m}$$



□

PROOF: For every integer  $i$ , Lemma 3.4.1 implies that there exists a finite collection  $F_i \subseteq L_{AP}$  such that  $\forall h \in I(\neg\phi_i), \forall m, \exists \theta\langle i, h, m \rangle \in F_i$  such that

$$\forall E \in \mathcal{M}, \forall \vec{a}_i \text{ in } E, E \models (\neg\phi_i)_{h,m} \Leftrightarrow \neg\theta\langle i, h, m \rangle \quad (5.2)$$

Fix then  $H \in I(\neg\bigwedge_{i=1}^{\infty} \neg\phi_i)$ . By definition

$$H = (H_1, H_2) : \omega \mapsto \left( \prod_{i=1}^{\infty} I(\phi_i) \right) \times \omega$$

with the property that:

$$\forall h \in I\left(\bigwedge_{i=1}^{\infty} \neg\phi_i\right) \exists s, \left(\bigwedge_{i=1}^{\infty} \neg\phi_i\right)_{h, f_2(s)} \equiv \left(\bigwedge_{i=1}^{\infty} \neg\phi_i\right)_{f_s, f_2(s)}$$

Using (5.2) and the definition of approximate formulas we obtain for every multi-sorted structure  $E$

$$\begin{aligned} E \models (\neg \bigwedge_{i=1}^{\infty} \neg\phi_i(\vec{a}_i))_{H,n} &\equiv \bigwedge_{s=1}^n \neg \left( \bigwedge_{i=1}^{\infty} \neg\phi_i(a_i) \right)_{H_1(s), H_2(s)} \Leftrightarrow \\ &\bigwedge_{s=1}^n \neg \bigwedge_{i=1}^{H_2(s)} (\neg\phi_i(\vec{a}_i))_{H_1(s,i), H_2(s)} \Leftrightarrow \bigwedge_{s=1}^n \neg \bigwedge_{i=1}^{H_2(s)} \neg\theta\langle i, H_1(s, i), H_2(s) \rangle(\vec{a}_i) \\ &\Leftrightarrow \bigwedge_{s=1}^n \bigvee_{i=1}^{H_2(s)} \theta\langle i, H_1(s, i), H_2(s) \rangle(\vec{a}_i) \end{aligned}$$

Fix  $k$  such that  $H_2(k)$  is the minimal value of  $H_2$ . Since  $\prod_{i=1}^k F_i$  is finite there exists an integer  $m$  satisfying:

$$\forall s \exists p \leq m \ E \models \bigvee_{i=1}^{H_2(k)} \theta\langle i, H_1(s, i), H_2(s) \rangle \equiv \bigvee_{i=1}^{H_2(k)} \theta\langle i, H_1(p, i), H_2(p) \rangle$$

and satisfying also that  $H_2(p) \leq H_2(s)$ .

For every integer  $n \geq m$  and every classical multisorted model  $E$  we obtain then:

$$\begin{aligned} E \models (\neg \bigwedge_{i=1}^{\infty} \neg \phi_i(\vec{a}_i))_{H,n} &\Leftrightarrow \bigwedge_{s=1}^n \bigvee_{i=1}^{H_2(s)} \theta\langle i, H_1(s, i), H_2(s) \rangle(\vec{a}_i) \\ &\Leftrightarrow \bigwedge_{s=1}^m \bigvee_{i=1}^{H_2(s)} \theta\langle i, H_1(s, i), H_2(s) \rangle(\vec{a}_i) \Leftrightarrow (\neg \bigwedge_{i=1}^{\infty} \neg \phi_i(\vec{a}_i))_{H,m} \end{aligned}$$

This is the desired result. ■

Let us now recall one of the classical versions of the Omitting Types Theorem in classical logic (see [26]):

### Omitting Types Theorem

Let  $L$  a first order countable language, and let  $C$  be a countable collection of constants not appearing in  $L$ . Suppose that  $\mathcal{M}$  is the collection of all models of a countable first order set of sentences  $\psi_j$  in  $L$ .

Consider the formula:  $\forall \vec{x} \bigvee_{i=1}^{\infty} \theta_i(\vec{x})$  where the  $|\vec{x}| = r < \infty$  and the  $\theta_j$ 's are first order formulas. If for every first order quantifier free and consistent sentence  $\Delta$  (with constants from  $C$ ) and for every tuple  $\vec{c}$  of constants in  $C$  with arity  $r$ , there exists a  $j$  such that

$$\Delta \wedge \theta_j(\vec{c})$$

is consistent, then there is a countable model  $E$  in  $\mathcal{M}$  such that

$$E \models \forall \vec{x} \bigvee_{j=1}^{\infty} \theta_j(\vec{x})$$

In order to see that this theorem follows from Theorem 5.2.1 it is enough to check that the hypothesis of the Omitting Types Theorem implies the hypothesis of Theorem 5.2.1.

We begin by noting that the collection of all the classical first order models of a fixed language  $L$  can be seen as the collection of all classical multisorted models for a signature  $\Phi = (\mathcal{F}, \mathcal{P})$  that contain all the function and predicate symbols of  $L$  as true sort function symbols and predicates, and such that the only fixed sort metric space is the reals with the usual metric. The only fixed sort predicate on the reals is the set  $\{0, 1\}$ . It is easy to see then that the collection of all the classical multisorted models for  $\Phi$  coincide with the collection of all the classical first order models for  $L$ . Furthermore, this collection, let us call it  $\mathcal{W}$ , is a complete collection of models.

Fix then  $\mathcal{M} \subset \mathcal{W}$  the collection of all models of the sentence  $\bigwedge_{j=1}^{\infty} \psi_j$  with the  $\psi_j$  first order sentences. As we saw before, the formula  $\bigwedge_{j=1}^{\infty} \psi_j$  is simple for the class  $\mathcal{W}$ . Note also that  $\mathcal{M}$  is a complete class of models. Consider the formula:  $\forall \vec{x} \bigvee_{i=1}^{\infty} \theta_i(\vec{x})$  where the  $\alpha_j$ 's are first order formulas and  $|\vec{x}| = r$ .

Assume that the hypothesis of the Omitting Types Theorem are true for these formulas.

Suppose also that

For every finite collection  $F \subset T^r$ , for every function  $H : F \mapsto$

$I(\bigvee_{i=1}^{\infty} \theta_i)$  and for every sentence  $\Delta \in L_{AP}(C)$ ,  $\forall n$

$$\Lambda_n \equiv \Delta \wedge \bigwedge_{\vec{t} \in F} \left( \bigvee_{i=1}^{\infty} \theta_i(\vec{t}) \right)_{H(\vec{t}), n}$$

is consistent in  $\mathcal{M}$ ,

Fix an arbitrary  $\vec{q}$  in  $T^r$ .

It follows from Lemma 5.2.2 for classical multisorted structures that there exists

an integer  $k$  such that for every  $m \geq k$ ,

$$\Lambda_m \text{ is equivalent, for every model in } \mathcal{W}, \text{ to } \Lambda_k \quad (5.3)$$

We can now apply the hypothesis of the Omitting Types Theorem to  $\Lambda_k$  to obtain that there exists a  $i \in \omega$  such that

$$\Lambda_k \wedge \theta_i(\vec{q})$$

is consistent in  $\mathcal{M}$ . This implies that there exists a model  $E$  in  $\mathcal{M}$  and interpretations of the constants of  $C$  appearing in  $\Lambda_k \wedge \theta_i(\vec{q})$  such that:

$$E^{app} \models \Lambda_k \wedge \theta_i(\vec{q})$$

Since  $\theta$  is a first order formula, it is strong for  $\mathcal{W}$ , i.e.

$$E \models_{AP} \theta_i(\vec{q})$$

but by the soundness properties of  $\models_{AP}$ , this implies that:

$$E \models_{AP} \bigvee_{i=1}^{\infty} \theta_i(\vec{q})$$

In summary, there exists  $h \in I(\bigvee_{i=1}^{\infty} \theta_i)$  such that for every integer  $n$ :

$$\Lambda_k \wedge \left( \bigvee_{i=1}^{\infty} \theta_i(\vec{q}) \right)_{h,n}$$

is consistent in  $\mathcal{M}$ . Invoking now statement 5.3 we finally obtain that for every integer  $n$ ,

$$\Lambda_n \wedge \left( \bigvee_{i=1}^{\infty} \theta_i(\vec{q}) \right)_{h,n} \equiv \Delta \wedge \bigwedge_{\vec{t} \in F} \left( \bigvee_{i=1}^{\infty} \theta_i(\vec{t}) \right)_{H(\vec{t}),n} \wedge \left( \bigvee_{i=1}^{\infty} \theta_i(\vec{q}) \right)_{h,n}$$

is consistent in  $\mathcal{M}$ . But this is the hypothesis of Theorem 5.2.1. We just verified that the hypothesis of the Omitting types Theorem imply the hypothesis of Theorem 5.2.1. This completes the proof of the desired statement.

We close this section with some remarks concerning the differences between Theorem 5.2.1 and the classical Omitting Types Theorem. Note that Theorem 5.2.1 omits formulas of the form

$$\forall \vec{x} \theta(\vec{x})$$

where  $\theta \in (L_{\omega, \omega})^{\{\wedge, \neg\}}$  and the arity of  $\vec{x}$  can be countable. The classical Omitting Types Theorem on the other hand, only works for formulas of the form:

$$\forall \vec{x} \bigvee_{i=1}^{\infty} \theta_i(\vec{x})$$

with the  $\theta_i$  first order formulas and the arity of  $\vec{x}$  finite.

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