MADNESS AND SET THEORY

By

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Abstract

We study the existence and possible cardinalities of Maximal Almost Disjoint (MAD) families of functions in the Baire space ω^ω that satisfy certain strong combinatorial and topological properties. We do this both in ZFC and in various models of set theory constructed using forcing. We prove in ZFC that there is a Van Douwen MAD family of size $\mathfrak{c}$. This answers a long standing question of E. van Douwen. Using ideas from this proof we show that any analytic MAD families that may exist in ω^ω must satisfy strong constraints.

We introduce a notion called the strongness of an almost disjoint family. We prove that under Martin's Axiom, for every $\kappa \leq \mathfrak{c}$, there is an almost disjoint family with strongness equal to κ.

We study the indestructibility properties of strongly MAD families. We prove that *all* strongly MAD families stay strongly MAD after forcing with any member of a certain large class of posets that do not make the ground model reals into a meager set. This class includes Cohen, Sacks and Miller forcings. We show that countable support iterations of such posets also preserve the strong MADness of strongly MAD families. En route to this result, we show that a countable support iteration of proper forcings of limit length does not make the ground model reals into a meager set if no inital segment of it does. We give a partial answer to Steprans' question of whether strongly MAD families always exist by showing that all strongly MAD families have size $\aleph_1 < \mathfrak{c}$ in the Cohen Model.

We prove a conjecture of Brendle by showing that if $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_{\mathfrak{c}}$, then very MAD families do not exist, showing thereby that they do not exist in the Random, Laver or Blass-Shelah models. We also show that strongly MAD families exist under $\mathfrak{b} = \mathfrak{c}$, proving that they exist in the Laver Model.

Jointly with Kunen, we prove in Chapter 4 that it is consistent to have no Gregory Trees while having $\mathfrak{c} > \aleph_2$.

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Contents

Chapter 1

Introduction

1.1 General Background

Two countably infinite objects a and b are said to be almost disjoint or a.d. if $a \cap b$ is finite. This notion has been considered for several different kinds of countably infinite objects, including infinite subsets of some fixed countable set X, functions from ω to ω and permutations of ω. The latter two are examples of countably infinite objects because, following standard convention in set theory, we identify functions with their graphs, turning a function f from ω to ω into the countably infinite subset $\{\langle n, f(n)\rangle : n \in \omega\} \subset \omega \times \omega$. We say that a family $\mathscr{A}$ of countably infinite objects of the same kind is almost disjoint or a.d. if its members are pairwise a.d. Such an a.d. family $\mathscr{A}$ is said to be Maximal Almost Disjoint, or MAD, if it is not properly contained within a larger a.d. family of infinite objects of the same kind. In the set theory literature, a MAD family is also required to be an infinite family. In this thesis we will take the somewhat unusual step of relaxing this requirement because it turns out that finite MAD families, when they happen to exist, are useful for our work, and their use makes certain proofs go more smoothly. This non standard terminology will matter only in Section 2.1, Section 2.2 and Section 3.7

Our work in this thesis focuses on MAD families of functions of ω to ω. We also make use of MAD families of infinite subsets of certain countable sets, and we point out some connections and differences between these two notions.

In this section we will illustrate some of the basic ideas involved using a.d. families of infinite subsets of ω as an example. Examples of such a.d. families include $\{\,\{n : n \text{ is even}\}, \{n : n \text{ is odd}\}\,\}$ and $\{\{\, p^k : k \in \omega\} : p \text{ is a prime}\}$. Observe that the first of these is maximal in the sense that it cannot be extended to a larger a.d. family of infinite subsets of ω. However, as noted above, one is usually interested only in infinite examples that have this property. Note that any infinite a.d. family, such as the second example above, can be extended to a MAD family using Zorn's Lemma.

Now, any family of pairwise *disjoint* subsets of ω must be countable. But it is a remarkable combinatorial fact that there are uncountable a.d. collections of infinite subsets of ω. Here is a quick proof. Let $\mathscr{A} = \{a_n : n \in \omega\}$ be a countable a.d. family of infinite subsets of ω. We define a sequence of natural numbers $k_0 < k_1 < \cdots$ by recursion as follows. k_0 is the least element of a_0. Given k_n, we define k_{n+1} to be the least element of $a_{n+1} \setminus (a_0 \cup \cdots \cup a_n)$ that is greater than k_n. This definition makes sense because a_{n+1} is almost disjoint from $a_0, \ldots, a_n$, and so $a_{n+1} \setminus (a_0 \cup \cdots \cup a_n)$ is an infinite set. Now, it is easy to see that the set $a = \{k_n : n \in \omega\}$ is an infinite set almost disjoint from all a_n. Thus, we have shown that no countably infinite a.d. family is maximal. Therefore, *any* MAD family of infinite subsets of ω must be uncountable.

MAD families have been intensively studied in set theory (for example, see [10], [8], [32] or [25]). They have numerous applications in set theory as well as general topology. For example, the technique of almost disjoint coding has been used in forcing theory (see [24]) and MAD families are used in the construction of the Isbell-Mrówka space in topology (see [11]). Another connection with topology is the relation between almost disjoint refinements and $\mathfrak{c}$-points in the Stone-Čech compactification of ω (see [3] and [2]).

In addition to studying purely combinatorial questions about MAD families, set theorists have also addressed the question of how "concrete" or definable MAD families can be (see [26] and [25]). This question assumes significance because the axiom of choice (in the guise of Zorn's Lemma) is necessary to construct a MAD family. One way to make this question precise is to topologize the world where MAD families live. Let ω^ω denote the Cartesian power of ω by itself. Note that ω^ω is just the set of all functions from ω to ω. Give ω the discrete topology and ω^ω the product topology. It is well known that this space is homeomorphic to the irrationals (as a subspace of the reals) (see [28, pp. 1-3]). Similarly, we give the set $\{0, 1\} = 2$, the discrete topology and the power 2^ω the product topology. Thus, 2^ω is a subspace of ω^ω and it is well known that it is homeomorphic to the Cantor Set. By identifying subsets of ω with their characteristic functions, we can identify MAD families with certain subsets of 2^ω. Now, we can phrase the question "Is there a concrete MAD family?" as the question "Can an infinite MAD family of infinite subsets of ω be a borel or an analytic subset of the space 2^ω?". It is known that the answer to this is no (Mathias [25]).

Many of these considerations also apply to MAD families of functions from ω to ω. For example, it is easy to see, as above, that *any* MAD family in ω^ω must be uncountable. Many other combinatorial properties of MAD families of sets extends to MAD families of functions. However, there are also significant differences. There has been some work on understanding such combinatorial differences (Zhang [32]). One significant difference concerns the question of "concreteness". One can also ask whether MAD families of functions can be "concrete". That is, is there a MAD family in ω^ω that is a borel or analytic subset of the space ω^ω? Unlike for MAD families of subsets of ω, this question remains open despite several attempts (see [20]). Indeed, it is not even known if there is a MAD family of functions which is a closed set in ω^ω.

1.2 Summary of Main Results

1.2.1 Background and Summary for Chapter 2

In chapter 1 we solve a long standing problem of Van Douwen and apply ideas from this solution to the question of whether analytic MAD families of functions exist. Van Douwen's question asks whether there exists a MAD family of functions satisfying a certain strong combinatorial property. Let us say that p is an infinite partial function if p is a function from an infinite subset of ω to ω. Is there a MAD family of functions which is also maximal with respect to infinite partial functions – that is, one such that there are no infinite partial functions almost disjoint from all the (total) functions in the family? This was the question posed by the late E. van Douwen. We call such a MAD family a ***Van Douwen MAD*** family. The question was included by A. Miller in his problem list [27, Problem 4.2], and an attempt to answer it was made by Zhang [33]. It is easy to construct Van Douwen MAD families if the Continuum Hypothesis (CH) is assumed. But the question of their existence in the absence of any such additional assumption remained open for at least 20 years, until its solution in this thesis. In Section 2.1 of Chapter 2 We prove:

Theorem 1.2.1 (see Theorem 2.1.13)**.** *There is a Van Douwen MAD family of size Continuum.*

In Section 2.2, we use the idea of trace (cf. Definition 1.3.8 and Definition 2.1.11) introduced in Section 2.1 to the question of whether analytic MAD families exist. Given an a.d. family $\mathscr{A} \subset \omega^\omega$ and a function $f \in \omega^\omega$, let us say that f ***avoids*** $\mathscr{A}$ if for any finite collection $\{h_0, \dots, h_k\} \subset \mathscr{A}$, $|f \setminus (h_0 \cup \dots \cup h_k)| = \omega$. Let us say that a MAD family $\mathscr{A} \subset \omega^\omega$ has ***trivial trace*** if for every $f \in \omega^\omega$ avoiding $\mathscr{A}$, there is an infinite partial function $p \subset f$ such that p is almost disjoint from everything in $\mathscr{A}$. We prove:

Theorem 1.2.2 (see Theorem 2.2.1)**.** *If $\mathscr{A} \subset \omega^\omega$ is an analytic MAD family, then $\mathscr{A}$ has trivial trace.*

This result shows that if analytic MAD families in ω^ω exist, then they must satisfy a strong combinatorial constraint. However, we also prove that it is consistent to have MAD families that satisfy this constraint (see Theorem 2.2.12). So Theorem 1.2.2 by itself does not rule out the existence of analytic MAD families. We use Theorem 1.2.2 to place some additional constraints on analytic MAD families in ω^ω. Given an a.d. family $\mathscr{A} \subset \omega^\omega$ consider the ideal $\mathcal{I}_0(\mathscr{A}) = \{a \in \mathcal{P}(\omega) : \exists p \in \omega^a [p \text{ is a.d. from } \mathscr{A}]\}$. Clearly, $\mathscr{A}$ is a MAD family *iff* $\omega \notin \mathcal{I}_0(\mathscr{A})$ *iff* $\mathcal{I}_0(\mathscr{A}) \neq \mathcal{P}(\omega)$. Thus if $\mathscr{A}$ is an analytic a.d. family, to show that it is not MAD, it suffices to prove that $\mathcal{I}_0(\mathscr{A}) = \mathcal{P}(\omega)$. While we have not been able to do this, we show in Section 2.2 that if $\mathscr{A} \subset \omega^\omega$ is an analytic a.d. family, then $\mathcal{I}_0(\mathscr{A})$ must be "large". In particular, we prove that $\mathcal{I}_0(\mathscr{A})$ must contain a copy of the ideal $0 \times Fin$ (see Theorem 2.2.24 and Corollary 2.2.25). This means that there is a partition $\{c_n : n \in \omega\}$ of ω into countably many infinite pieces such that any set $a \subset \omega$ that is a.d. from all the c_n is in the ideal $\mathcal{I}_0(\mathscr{A})$.

1.2.2 Background and Summary for Chapter 3

Juris Steprāns [20] introduced the notion of a strongly MAD family and asked whether they exist. This notion is a "σ version" of the notion of a MAD family of functions. Very roughly, this means that instead of requiring the family to be maximal just with respect to *elements* of ω^ω, we require it to be maximal with respect to *countable subsets* of ω^ω. "σ versions" of various types of subfamilies of $[\omega]^\omega = \{a \subset \omega : |a| = \omega\}$ have been considered in the literature. For example, Kamburelis and Węglorz [18] have studied the "σ version" of the notion of a splitting family. Recall that a set $a \in [\omega]^\omega$ splits a set $b \in [\omega]^\omega$ if both $a \cap b$ and $b \setminus a$ are infinite. Recall also that a family $\mathcal{F} \subset [\omega]^\omega$ is a splitting family if every $b \in [\omega]^\omega$ is split by some $a \in \mathcal{F}$. Now, the "σ version" of this notion, called an $\aleph_0$–splitting family, is simply a family $\mathcal{F} \subset [\omega]^\omega$ such that for every countable set $\{b_i : i \in \omega\} \subset [\omega]^\omega$, there is $a \in \mathcal{F}$ which splits all the b_i. We cannot simply lift this definition to the case of MAD families. That is, we cannot define a strongly MAD family to simply be an a.d. family $\mathscr{A} \subset \omega^\omega$ such that for every countable set of functions $\{f_i : i \in \omega\} \subset \omega^\omega$, there is $h \in \mathscr{A}$ such that $\forall i \in \omega\, [|h \cap f_i| = \omega]$. To see this, suppose $\mathscr{A} \subset \omega^\omega$ is an a.d. family with atleast 2 elements. Choose $h_0 \neq h_1 \in \mathscr{A}$ and consider the set $\{h_0, h_1\}$. It is clear that no element of $\mathscr{A}$ can intersect both h_0 and h_1 in an infinite set. Hence we must put some restriction on the countable sets of functions we are allowed to consider. It turns out that the restriction we need is that of avoiding discussed above. An a.d. family $\mathscr{A} \subset \omega^\omega$ is said to be ***strongly MAD*** if for any countable collection $\{f_i : i \in \omega\} \subset \omega^\omega$ of functions avoiding $\mathscr{A}$, there is $h \in \mathscr{A}$ such that $\forall i \in \omega\, [|h \cap f_i| = \omega]$. Steprāns [20] showed that strongly MAD families cannot be analytic and Kastermans [19] proved that strongly MAD families exist under Martin's Axiom (MA).

Soon after Steprāns introduced the notion of a strongly MAD family, Kastermans and Zhang [19] introduced a strenthening of this notion, called a very MAD family. Let $\mathscr{A}$ be an a.d. family and put $\kappa = |\mathscr{A}|$. We say that $\mathscr{A}$ is ***very MAD*** if for every cardinal $\lambda < \kappa$ and for every collection $\{f_\alpha : \alpha < \lambda\}$ of functions avoiding $\mathscr{A}$, there is $h \in \mathscr{A}$ such that $\forall \alpha < \lambda\, [|h \cap f_\alpha| = \omega]$. Obviously, every very MAD family is strongly MAD, and a strongly MAD family of size $\aleph_1$ must be very MAD. It turns out that every strongly MAD family is Van Douwen MAD. Thus we have a natural spectrum of combinatorial properties of increasing strength, starting with MADness, going through Van Douwen and strong MADness to very MADness. In Section 3.1, we introduce a notion which allows for systematic investigation of this spectrum. Given an a.d. family $\mathscr{A} \subset \omega^\omega$, we define the ***strongness of*** $\mathscr{A}$, written $\mathrm{st}(\mathscr{A})$, to be the least cardinal κ such that there is a family of functions $\{f_\alpha : \alpha < \kappa\}$ avoiding $\mathscr{A}$ so that $\forall h \in \mathscr{A} \exists \alpha < \kappa\, [|h \cap f_\alpha| < \omega]$. Thus to say that $\mathscr{A}$ is MAD is to say that $\mathrm{st}(\mathscr{A}) \geq 2$. To say that $\mathscr{A}$ is strongly MAD is to say that $\mathrm{st}(\mathscr{A}) \geq \omega_1$ and $\mathscr{A}$ is very MAD iff $\mathrm{st}(\mathscr{A}) \geq |\mathscr{A}|$. For which values of κ is there a MAD family $\mathscr{A}$ with $\mathrm{st}(\mathscr{A}) = \kappa$? We provide a partial answer:

Theorem 1.2.3 (see Theorem 3.1.7 and Corollary 3.3.9)**.** *Assume MA(σ–centered). For every $\kappa \leq \mathfrak{c}$ there is an a.d. family $\mathscr{A}$ with* $\mathrm{st}(\mathscr{A}) = \kappa$.

Sections 3.2 and 3.3 are devoted to addressing questions of Kastermans and to proving a conjecture of Brendle regarding very MAD families. Kastermans [19] pointed out that the standard construction of a strongly MAD family from MA actually yields a very MAD family. He asked if very MAD families always exist and if there is a construction which distinguishes between strongly MAD and very MAD families. Sections 3.2 and 3.3 address both issues. Regarding the first, J. Brendle conjectured that if $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_{\mathfrak{e}}$, then there are no very MAD families. Here $\operatorname{cov}(\mathcal{M})$ is the covering number of the meager ideal, and $\mathfrak{a}_{\mathfrak{e}}$ is the least size of a MAD family of functions in ω^ω. In Section 3.3 and 3.2 we prove:

Theorem 1.2.4 (see Theorem 3.3.5). *If $\mathscr{A} \subset \omega^\omega$ is very MAD, then $|\mathscr{A}| \leq \operatorname{cov}(\mathcal{M})$. Therefore, if $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_{\mathfrak{e}}$, there are no very MAD families.*

Theorem 1.2.5 (see Theorem 3.2.2). *If $\mathfrak{b} = \mathfrak{c}$, then there is a strongly MAD family.*

Together, these results answer both of Kastermans' questions. Theorem 1.2.4 shows that there are no very MAD families in the Random, Laver or Blass-Shelah models. Theorem 1.2.5 distinguishes between strongly MAD and very MAD families in the sense that the construction carried out there cannot be used to prove the existence of a very MAD family. This is because $\mathfrak{b} = \mathfrak{c}$ holds in the Laver model.

In Sections 3.4 and 3.5, we study the indestructibility properties of strongly MAD families. Let $\mathbb{P}$ be a forcing notion. We say that a MAD family $\mathscr{A}$ is $\mathbb{P}$–***indestructible*** if $\mathscr{A}$ stays maximal after forcing with $\mathbb{P}$. A strongly MAD family $\mathscr{A}$ is ***strongly*** $\mathbb{P}$–***indestructible*** if $\mathscr{A}$ stays strongly MAD after forcing with $\mathbb{P}$. Brendle and Yatabe [10] have studied $\mathbb{P}$–indestructibility of MAD families of subsets of ω for various posets $\mathbb{P}$. The focus of their work was to provide combinatorial characterizations of the property of being a $\mathbb{P}$–indestructible MAD family of sets for some well known posets $\mathbb{P}$. In our work, the focus is instead to find those posets $\mathbb{P}$ for which strongly MAD families of functions are strongly $\mathbb{P}$–indestructible. We show that *all* strongly MAD families are strongly $\mathbb{P}$–indestructible for a wide range of forcings which do not turn the ground model reals into a meager set. In particular,

Theorem 1.2.6 (see Theorem 3.4.9, Theorem 3.4.18 and Corollary 3.4.26)**.** *Let $\mathbb{P}$ be Cohen, Sacks or Miller forcing. All strongly MAD families are strongly $\mathbb{P}$–indestructible.*

In Section 3.4, we prove a more general theorem than the one stated above. We introduce a property of posets called having diagonal fusion (see Definition 3.4.13), and we introduce a strengthening of the relation of being strongly indestructible, called strongly preserving (see Definition 3.4.3). We show that any poset that has diagonal fusion strongly preserves every strongly MAD family. We then show that Sacks and Miller forcings have diagonal fusion (see Theorem 3.4.25), and hence that they strongly preserve all strongly MAD families.

An immediate consequence of $\mathbb{P}$-indestructibility is a strengthening of the result of Steprāns given above.

Corollary 1.2.7 (see Corollary 3.4.11)**.** *Strongly MAD families do not contain perfect sets.*

Note that it follows from Theorem 1.2.6 that strongly MAD families remain strongly MAD no matter how many Cohen reals are added. But this conclusion is not immediate for, say, Sacks reals because it is not enough to deal with just a single step. In Section 3.5 we prove a preservation theorem that takes care of this. We are unable to prove that the relation of being strongly indestructible is preserved by the countable support iteration of proper forcings. However, we are able to show that the relation of strongly preserving is preserved. This is one of the reasons for introducing the relation of strongly preserving in Section 3.4. As a consequence, we get the following theorem.

Theorem 1.2.8 (see Theorem 3.5.12 and Corollary 3.5.13)**.** *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. $\mathscr{A}$ is strongly indestructible for the countable support iteration of posets having diagonal fusion. In particular, $\mathscr{A}$ is strongly indestructible for the countable support iteration of Sacks and Miller forcings.*

En route to proving this, we prove in Section 3.5 the following general preservation theorem about not adding eventually different reals.

Theorem 1.2.9 (see Theorem 3.5.8)**.** *Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a countable support iteration of proper forcings. Suppose that for all $\alpha < \gamma$, $\mathbb{P}_\alpha$ does not add an eventually different real. $\mathbb{P}_\gamma$ does not add an eventually different real either.*

Partial results in this direction were obtained by Shelah, Goldstern and Judah [30] and by Shelah and Kellner [21].

A partial answer to the question of whether strongly MAD families always exist is provided in Section 3.6, where we show that it is consistent that there are no "large" strongly MAD families.

Theorem 1.2.10 (see Theorem 3.6.1)**.** *In the Cohen Model, all strongly MAD families have size $\aleph_1 < \mathfrak{c}$.*

We end Chapter 3 by pointing out some connections and differences between the notion of a strongly MAD family of functions and that of a strongly MAD family of subsets of ω, which are defined analogously, with ω^ω replaced everywhere by $[\omega]^\omega$, and with the additional requirement that the family be infinite. Kurilić [23] and Hrušák and García Ferreira [16] have pointed out that there is a close connection between strong MADness and Cohen indestructibility for the case of MAD families of subsets of ω. In particular, they show that a MAD family $\mathscr{A} \subset [\omega]^\omega$ is Cohen indestructible *iff* it is "somewhere strongly MAD". Hrušák suggested to us in conversation that a similar result might hold for MAD families of functions as well. We show that this is not the case in Section 3.7 by constructing under CH a Cohen indestructible MAD family of functions that is "nowhere Van Douwen MAD".

Theorem 1.2.11 (see Theorem 3.7.1)**.** *Assume CH. There is a Cohen indestructible MAD family $\mathscr{A} \subset \omega^\omega$ with trivial trace.*

We also point out in Section 3.7 that the existence of a strongly MAD family of functions implies the existence of a strongly MAD family of sets that is strongly Cohen indestructible.

Theorem 1.2.12 (see Lemma 3.7.2 and Lemma 3.7.4)**.** *Suppose $\mathscr{A} \subset \omega^\omega$ is a strongly MAD family. $\mathscr{A} \subset [\omega \times \omega]^\omega$ is a strongly MAD family of subsets of $[\omega \times \omega]^\omega$ that is strongly Cohen indestructible.*

1.2.3 Background and Summary for Chapter 4

A ***Cantor tree*** of sequences is a subset $\{p_s : s \in 2^{<\omega}\}$ of $2^{<\omega_1}$ such that for all $s \in 2^{<\omega}$, $p_{s^\frown 0}$ and $p_{s^\frown 1}$ are incompatible nodes in $2^{<\omega_1}$ that extend p_s. A subtree $\mathbb{P}$ of $2^{<\omega_1}$ is said to have the ***Cantor Tree Property (CTP)*** if: 1) for every $p \in \mathbb{P}$, $p^\frown 0, p^\frown 1 \in \mathbb{P}$; 2) given any Cantor tree $\{p_s : s \in 2^{<\omega}\} \subset \mathbb{P}$, there is $f \in 2^\omega$ and $q \in \mathbb{P}$ such that $\forall n \in \omega\, [q \leq p_{f \restriction n}]$. Finally, a subtree $\mathbb{P}$ of $2^{<\omega_1}$ is a ***Gregory tree*** if it has the CTP, but does not have a cofinal branch. In Chapter 4 we answer a question about Gregory trees due to Kunen. Gregory [13] showed that such trees exist under CH. This is of interest because a Gregory tree, viewed as a forcing order, is totally proper, and it kills itself. That is, forcing with a Gregory tree adds a cofinal branch through itself. Therefore, it is possible to kill any given Gregory tree without adding any reals. However, Gregory's result shows that it is impossible to iterate this forcing (using *any* supports) to kill off all Gregory trees without adding any new reals. Kunen and Hart [14] pointed some connections between Gregory trees and certain topological spaces. In particular, they showed that if there is a compact space which is hereditarily Lindelöf, is not totally disconnected, but does not contain a copy of the Cantor set, then there is a Gregory tree. Such a space is said to be ***weird***.

As mentioned above, a Gregory tree is a totally proper forcing. Therefore, PFA implies that there are no Gregory trees, and hence that there are no weird spaces. It is well known that PFA implies $\mathfrak{c} = \aleph_2$. Kunen asked if there is a model with $\mathfrak{c} > \aleph_2$ where there are no

Gregory trees. In Chapter 4 we provide an affirmative answer.

Theorem 1.2.13 (see Theorem 4.0.14)**.** *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ and such that $\forall\lambda < \kappa\,[\lambda^\omega < \kappa]$. There is a forcing extension preserving cardinals where there are no Gregory trees (and hence no weird spaces) and $\mathfrak{c} = \kappa$. Moreover, we can have Martin's Axiom in this model.*

The material in this Chapter is joint work with Kunen.

1.3 Basic Definitions and Notational Conventions

1.3.1 Basic Definitions

For easy reference, we will collect together here the definitions of some basic concepts that will occur throughout the thesis.

Definition 1.3.1. *Two functions f and g from ω to ω are said to be a.d. if $|f \cap g| < \omega$.*

Definition 1.3.2. *A family $\mathscr{A} \subset \omega^\omega$ is a.d. if $\forall f, g \in \mathscr{A}\,[f \neq g \implies |f \cap g| < \omega]$. An a.d. family $\mathscr{A} \subset \omega^\omega$ is MAD if $\forall f \in \omega^\omega \exists h \in \mathscr{A}\,[|h \cap f| = \omega]$.*

Definition 1.3.3. *p is said to be an infinite partial function if p is a function from some infinite subset of ω to ω.*

Definition 1.3.4. *An a.d. family $\mathscr{A} \subset \omega^\omega$ is called a Van Douwen MAD family if for any infinite partial function p, there is $h \in \mathscr{A}$ such that $|h \cap p| = \omega$*

Definition 1.3.5. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. We say that $X \in [\omega \times \omega]^\omega$ avoids $\mathscr{A}$ if for any finite collection $\{h_0, \dots h_k\} \subset \mathscr{A}$, $|\ X \setminus (h_0 \cup \dots \cup h_k)| = \omega$.*

Definition 1.3.6. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. We say that $\mathscr{A}$ is strongly MAD if for any countable family of functions $\{f_i : i \in \omega\} \subset \omega^\omega$ avoiding $\mathscr{A}$, there is $h \in \mathscr{A}$ such that $\forall i \in \omega\,[|h \cap f_i| = \omega]$.*

Definition 1.3.7. *Let X be a countable set. Two sets $a, b \in [X]^{\omega}$ are a.d. if $a \cap b$ is finite. A family $\mathscr{A} \subset [X]^{\omega}$ is a.d. if its members are pairwise a.d. An a.d. family $\mathscr{A} \subset [X]^{\omega}$ is MAD in $[X]^{\omega}$ if for every $b \in [X]^{\omega}$ there is some $a \in \mathscr{A}$ such that $|a \cap b| = \omega$. Note that we are allowing finite families to be MAD.*

Definition 1.3.8. *Let $\mathscr{A} \subset \omega^{\omega}$ be an a.d. family. Let $f \in \omega^{\omega}$. We define $\mathscr{A} \cap f = \{h \cap f : h \in \mathscr{A} \wedge |h \cap f| = \omega\}$. Note that this is an a.d. family on the countable set f. We define the trace of $\mathscr{A}$, written $\operatorname{tr}(\mathscr{A})$, to be the following set: $\{f \in \omega^{\omega} : \mathscr{A} \cap f$ is a MAD family in $[f]^{\omega}\}$.*

Definition 1.3.9. *Let $\mathscr{A} \subset \omega^{\omega}$ be a MAD family. We will say that $\mathscr{A}$ has trivial trace if no member of $\operatorname{tr}(\mathscr{A})$ avoids $\mathscr{A}$.*

1.3.2 Notational Conventions

Most of our set-theoretic notation is standard and follows Kunen [22] or Jech [17]. ω denotes the set $\{0, 1, 2, \dots\}$. Given a set a, $\mathcal{P}(a)$ denotes the power set of a, i.e. $\mathcal{P}(a) = \{b : b \subset a\}$. For a set X and a cardinal λ, $[X]^{\lambda} = \{Y \subset X : |\, Y \,| = \lambda\}$. Given sets a and b, a^b denotes $\{f : f \text{ is a function } \wedge \operatorname{dom}(f) = b \wedge \operatorname{ran}(f) \subset a\}$. We will abuse notation and sometimes also use a^b to denote the cardinality of this set. Which of these is meant will be clear from the context. Given two sets a and b, we will write $a \subset^* b$ to mean $a \setminus b$ is finite. $\exists^{\infty}$ abbreviates "there exists infinitely many" and $\forall^{\infty}$ abbreviates "for all but finitely many".

Given a set a, $\mathcal{I}$ is said to be an ideal on a if $\mathcal{I}$ is a subset of $\mathcal{P}(a)$ such that

1. if $b \subset a$ is finite, then $b \in \mathcal{I}$
2. if $b \in \mathcal{I}$ and $c \subset b$, then $c \in \mathcal{I}$
3. if $b \in \mathcal{I}$ and $c \in \mathcal{I}$, then $b \cup c \in \mathcal{I}$
4. $a \notin \mathcal{I}$.

Thus our ideals are always required to be proper and non–principal.

We will make use of forcing and iterated forcing throughout the thesis. We will fix here some notation and conventions concerning forcing. Given a notion of forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we will write $q \leq p$ to mean that q is a stronger condition than p. We will also follow the alphabet convention, whereby letters occurring later in the alphabet are used for stronger conditions. We will abuse notation and represent an iterated forcing construction of length γ as $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$, even though $\mathring{\mathbb{Q}}_\gamma$ does not exist; only $\mathbb{P}_\gamma$ does. A more correct, but also more cumbersome, notation would be $\langle\ \langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle, \langle \mathring{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle\ \rangle$. We will use CS to mean countable support and FS to mean finite support. If $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iterated forcing construction, we will require all the $\mathring{\mathbb{Q}}_\alpha$ to be full names for posets. This means that for each $\alpha < \gamma$, $\mathring{\mathbb{Q}}_\alpha$ is a full $\mathbb{P}_\alpha$ name and $\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha$ is a poset.

Given a poset $\mathbb{P}$, a full $\mathbb{P}$ name is a $\mathbb{P}$ name $\mathring{x}$ such that

1. $\forall \mathring{y} \in \mathrm{dom}\,(\mathring{x})\,[\ \Vdash \mathring{y} \in \mathring{x}]$

2. for every $\mathbb{P}$ name $\mathring{z}$, there is $\mathring{y} \in \mathrm{dom}\,(\mathring{x})$ so that $\forall p \in \mathbb{P}\,[\,p \Vdash \mathring{z} \in \mathring{x} \implies p \Vdash \mathring{z} = \mathring{y}]$.

Given any $\mathbb{P}$ name $\mathring{x}$ for which $\Vdash \mathring{x} \neq 0$ holds, it is an easy exercise to see that there is a full $\mathbb{P}$ name $\mathring{y}$ such that $\Vdash \mathring{x} = \mathring{y}$.

We will make use of the maximal principle in the following two forms. Firstly, given a formula $\phi(v)$ and a condition $p \in \mathbb{P}$, if $p \Vdash \exists x\, \phi(x)$, then there is a $\mathbb{P}$ name $\mathring{x}$ such that $p \Vdash \phi\,(\mathring{x})$. Secondly, if $A \subset \mathbb{P}$ is a maximal antichain in $\mathbb{P}$, and if $\{\mathring{x}_p : p \in A\}$ is a set of $\mathbb{P}$ names, then there is a $\mathbb{P}$ name $\mathring{x}$ so that $\forall p \in A\,[p \Vdash \mathring{x} = \mathring{x}_p]$. We will often use the maximal principle without saying so.

We will frequently use the following consequence of the maximal principle without mentioning it. Suppose $\phi(u)$ and $\psi(v)$ are two formulas. Suppose that $\Vdash \exists u\, \phi(u)$. Suppose $p \in \mathbb{P}$ and $\mathring{x} \in \mathbf{V}^{\mathbb{P}}$ are such that $p \Vdash \phi(\mathring{x}) \wedge \psi(\mathring{x})$. We may, in this situation, assume without loss of generality that $\Vdash \phi(\mathring{x})$ and $p \Vdash \psi(\mathring{x})$. To see this, note that $\Vdash \exists y\,[\phi(y) \wedge\ [\phi(\mathring{x}) \implies y = \mathring{x}]]$.

So by the maximal principle, there is $\mathring{y} \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash \phi(\mathring{y})$ and $\Vdash [\phi(\mathring{x}) \implies \mathring{y} = \mathring{x}]$. Since $p \Vdash \phi(\mathring{x})$, $p \Vdash \mathring{y} = \mathring{x}$, whence $p \Vdash \psi(\mathring{y})$.

We will often confuse sequences of names with names for sequences. Thus given a set of $\mathbb{P}$ names $\{\mathring{a}_n : n \in \omega\}$, we may write something like $p \Vdash \langle \mathring{a}_n : n \in \omega \rangle \in \omega^\omega$. Strictly speaking, this is nonsense because $\langle \mathring{a}_n : n \in \omega \rangle$ is not a name, but we are simply using it in place of a name that is forced by $\mathbb{1}_{\mathbb{P}}$ to denote the sequence $\langle \mathring{a}_n [G] : n \in \omega \rangle$, which is defined in $\mathbf{V}[G]$. Furthermore, if X is a set and $\mathring{f}$ is a $\mathbb{P}$ name such that $p \Vdash \mathring{f}$ is a function with domain X, we can find a sequence of $\mathbb{P}$ names $\langle \mathring{a}_x : x \in X \rangle$ such that $p \Vdash \langle \mathring{a}_x : x \in X \rangle = \mathring{f}$.

We will make use of elementary submodels of "the universe". We assume the reader is familiar with this concept. We will always take elementary submodels of some $H(\theta)$, where θ is some large regular cardinal. We will never explicitly say how large θ needs to be. Thus, we will often write $M \prec H(\theta)$, without saying anything further at all about θ. It will be understood that θ is some regular cardinal that is sufficiently large to carry out the argument at hand.

Chapter 2

There is a Van Douwen MAD Family

2.1 A Van Douwen MAD Family in ZFC

In this section we will prove in ZFC (Zermelo-Fraenkel set theory with the axiom of choice; for more details about ZFC see [22]) that there is a Van Douwen MAD family of size Continuum (see Definition 1.3.4). It is easily seen that Van Douwen MAD families exist under CH, and more generally under MA. The question of whether they always exist was raised by E. van Douwen and A. Miller. It occurs as problem 4.2 in A. Miller's problem list [27]. Zhang [33] discusses this problem and proves that Van Douwen MAD families of various sizes exist in certain forcing extensions.

The starting point for our construction is the following well known characterization of the cardinal $\mathrm{non}(\mathcal{M})$, due to Bartoszyński. The reader may consult [4] or [5] for a proof of this.

Definition 2.1.1. $\mathrm{non}(\mathcal{M})$ *is the least size of a non meager set of reals.*

Definition 2.1.2. *Let $h \in \omega^\omega$ be such that $\forall n \in \omega\, [h(n) \geq 1]$. An h-slalom is a function $S : \omega \to [\omega]^{<\omega}$ such that for all $n \in \omega$, $|S(n)| \leq h(n)$.*

Theorem 2.1.3 (Bartoszyński [4])**.** *Let κ be an infinite cardinal. The following are equivalent:*

1. *Every set of reals of size less than κ is meager.*

2. *For every family $F \subset \omega^\omega$ with $|F| < \kappa$, there is an infinite partial function g from ω to ω such that $\forall f \in F\,[\,|f \cap g| < \omega]$.*

3. *For every h and for every family of h-slaloms F with $|F| < \kappa$, there is a $g \in \omega^\omega$ such that $\forall S \in F\ \forall^\infty n \in \omega\,[\,g(n) \notin S(n)]$.*

$\dashv$

Our first task is to strengthen condition (3) above. We first show that if F is a family of h-slaloms of size less than $\mathrm{non}\,(\mathcal{M})$, then we can get a one-to-one function g, which is eventually outside all the slaloms in F (Lemma 2.1.4). We then show that we can, in fact, get a suitably "wide" slalom which is eventually disjoint from all slaloms in F (Lemma 2.1.6). Lemma 2.1.4 was independently discovered and used by Brendle, Spinas and Zhang [9].

Lemma 2.1.4. *Let $\kappa = \mathrm{non}\,(\mathcal{M})$ and let F be a family of h-slaloms with $|F| < \kappa$. There is a one-to-one function $g \in \omega^\omega$ such that $\forall S \in F\ \forall^\infty n \in \omega\,[g(n) \notin S(n)]$.*

Proof. Our proof is similar to the argument in Bartoszyński [4]. Write $\mathrm{F} = \langle S_\xi : \xi < \lambda \rangle$, where $\lambda = |F|$. Define a new function h' and a family of h'-slaloms as follows:

$$h'(n) = \sum_{i \le n} h(i)$$
$$\forall \xi < \lambda\ S'_\xi(n) = \bigcup_{i \le n} S_\xi(i).$$

Clearly, $\langle S'_\xi : \xi < \lambda \rangle$ is a family of h'-slaloms. Now, for each $i \in \omega$, let $T_i : \omega \to [\omega]^{<\omega}$ be defined by $T_i(n) = \{i\}$. It is clear that $\langle S'_\xi : \xi < \lambda \rangle \cup \langle T_i : i \in \omega \rangle$ is a family of fewer than κ h'-slaloms. Thus by 3 of Theorem 2.1.3, we can choose $g \in \omega^\omega$ such that the following hold:

1. $\forall \xi < \lambda \forall^\infty n \in \omega\,\left[g(n) \notin S'_\xi(n)\right]$

2. $\forall i \in \omega \forall^\infty n \in \omega\,[g(n) \notin T_i(n)]$.

Property 2 implies that g takes any given value only finitely often. Thus we may choose a one-to-one infinite partial function $g' \subset g$. Let $X = \mathrm{dom}\,(g')$. By property 1 we obviously have that for any $\xi < \lambda$, $\forall^\infty n \in \omega \left[n \in X \implies g'(n) \notin S'_\xi(n)\right]$. Let $\langle x_n : n \in \omega \rangle$ be the increasing enumeration of X. For $n \in \omega$, set $g''(n) = g'(x_n)$. Since g' is one-to-one, g'' is also one-to-one. We claim that g'' is the function we are looking for. Indeed, fix $\xi < \lambda$. We know that $\exists m \in \omega \forall n \geq m \left[n \in X \implies g'(n) \notin S'_\xi(n)\right]$. We will show that $\forall n \geq m\,[g''(n) \notin S_\xi(n)]$. Suppose, for a contradiction, that $g''(n) = g'(x_n) \in S_\xi(n)$, for some $n \geq m$. Note that we have $m \leq n \leq x_n$. Thus, by the definition of S'_ξ, $S_\xi(n) \subset S'_\xi(x_n)$. Therefore, we have that $g'(x_n) \in S'_\xi(x_n)$. But this is a contradiction because $x_n \geq m$ and $x_n \in X$. $\dashv$

Convention 2.1.5. *In what follows we will only be concerned with h-slaloms for the function $h(n) = 2^n$. We will simply refer to these as slaloms, suppressing mention of h.*

Lemma 2.1.6. *Let $F = \langle S_\xi : \xi < \lambda \rangle$ be a family of slaloms with $\lambda < \mathrm{non}\,(\mathcal{M})$. There is a slalom S such that $\forall n \in \omega\,[|S(n)| = 2^n]$ and $\forall \xi < \lambda \forall^\infty n \in \omega\,[S(n) \cap S_\xi(n) = 0]$.*

Proof. For all $n \in \omega$ set $l_n = 2^n - 1$ and $I_n = [l_n, l_{n+1})$. For each $\xi < \lambda$ define S'_ξ by stipulating that $\forall k, n \in \omega \left[S'_\xi(k) = S_\xi(n) \ iff \ k \in I_n\right]$. We have that for all $k \in \omega$, $\left|S'_\xi(k)\right| \leq |S_\xi(n)| \leq 2^n$, where $k \in I_n$. But if $k \in I_n$, then $2^n \leq 2^k$ and so $\left|S'_\xi(k)\right| \leq 2^k$. Therefore, $\langle S'_\xi : \xi < \lambda \rangle$ is a family of fewer than $\mathrm{non}\,(\mathcal{M})$ many slaloms. By applying Lemma 2.1.4 we can find a one-to-one function $g \in \omega^\omega$ such that for every $\xi < \lambda$, $\forall^\infty k \in \omega \left[g(k) \notin S'_\xi(k)\right]$. Now define S by setting $S(n) = \{g(k) : k \in I_n\}$. Since g is one-to-one, $|S(n)| = |I_n| = 2^n$. Fix $\xi < \lambda$. We know that $\exists m \in \omega \forall k \geq m \left[g(k) \notin S'_\xi(k)\right]$. We claim that for any $n \geq m$, $S(n) \cap S_\xi(n) = 0$. Suppose to the contrary that for some $n \geq m$, $g(k) \in S_\xi(n)$ for some $k \in I_n$. Then since $k \in I_n$, $S'_\xi(k) = S_\xi(n)$, and so we get that $g(k) \in S'_\xi(k)$. But this is a contradiction because $m \leq n \leq l_n \leq k$. $\dashv$

Lemma 2.1.7. *Let S be a slalom such that $\forall n \in \omega\,[|S(n)| = 2^n]$. There exists an a.d. family $\mathscr{A} \subset \omega^\omega$ such that $|\mathscr{A}| = \mathfrak{c}$ and for every $f \in \mathscr{A}$, $\forall n \in \omega\,[f(n) \in S(n)]$.*

Proof. Since $|S(n)| = |2^n|$, we can assign to each $\sigma \in 2^n$ a unique number $k_\sigma \in S(n)$. Now, for each $\mu \in 2^\omega$, define $f_\mu \in \omega^\omega$ by setting $f_\mu(n) = k_{\mu\restriction n} \in S(n)$. Suppose $\mu \neq \nu \in 2^\omega$. Then there is $m \in \omega$ such that $\mu(m) \neq \nu(m)$. So for all $n > m$, $\mu \restriction n \neq \nu \restriction n$, and so $f_\mu(n) = k_{\mu\restriction n} \neq k_{\nu\restriction n} = f_\nu(n)$. Thus $\mathscr{A} = \{f_\mu : \mu \in 2^\omega\}$ is as required. $\dashv$

Definition 2.1.8. *Let $A, B \subset \omega^\omega$ be two families of functions. We will write $A \perp B$ to mean that $\forall f \in A \forall g \in B\,[\,|f \cap g| < \omega]$*

The next lemma will play an important role in our construction. The proof of this lemma uses Lemma 2.1.7 and is the reason why we set out to strengthen clause (3) of Theorem 2.1.3.

Lemma 2.1.9. *Let $\kappa = \mathrm{non}\,(\mathcal{M})$. Let $F = \langle f_\alpha : \alpha < \kappa\rangle \subset \omega^\omega$. There is a sequence $\langle \mathscr{A}_\alpha : \alpha < \kappa\rangle$ such that following hold:*

1. *$\mathscr{A}_\alpha \subset \omega^\omega$ is an a.d. family.*
2. *$|\mathscr{A}_\alpha| = \mathfrak{c}$.*
3. *for all $\beta < \alpha < \kappa$, $\mathscr{A}_\alpha \perp \mathscr{A}_\beta$*
4. *$\mathscr{A}_\alpha \perp \{f_\beta : \beta \leq \alpha\}$.*

Proof. We will construct the family $\langle \mathscr{A}_\alpha : \alpha < \kappa\rangle$ by induction. We will simultaneously build a family of slaloms $\langle S_\alpha : \alpha < \kappa\rangle$ and ensure that for all $\alpha < \kappa$, $\forall f \in \mathscr{A}_\alpha \forall n \in \omega\,[f(n) \in S_\alpha(n)]$. Fix $\alpha < \kappa$ and suppose that $\langle \mathscr{A}_\beta : \beta < \alpha\rangle$ and $\langle S_\beta : \beta < \alpha\rangle$ are already given to us. For each $\beta \leq \alpha$, define a slalom T_β by $T_\beta(n) = \{f_\beta(n)\}$. Thus, $\{S_\beta : \beta < \alpha\} \cup \{T_\beta : \beta \leq \alpha\}$ is a family of fewer than κ slaloms. So we can apply Lemma 2.1.6 to find a slalom S_α such that the following hold:

(a) $\forall n \in \omega\,[|S_\alpha(n)| = 2^n]$

(b) $\forall \beta < \alpha \forall^\infty n \in \omega\,[S_\alpha(n) \cap S_\beta(n) = 0]$

(c) $\forall \beta \leq \alpha \forall^\infty n \in \omega\,[S_\alpha(n) \cap T_\beta(n) = 0]$.

Property (a) allows us to apply Lemma 2.1.7 to S_α to find an a.d. family $\mathscr{A}_\alpha \subset \omega^\omega$ with $|\mathscr{A}_\alpha| = \mathfrak{c}$ and with the property that $\forall f \in \mathscr{A}_\alpha \forall n \in \omega\,[f(n) \in S_\alpha(n)]$. Thus $\mathscr{A}_\alpha$ satisfies requirements (1) and (2). We will check requirements (3) and (4). Fix $f \in \mathscr{A}_\alpha$ and $g \in \mathscr{A}_\beta$ for some $\beta < \alpha$. We know that there is $m \in \omega$ such that $\forall n \geq m\,[S_\alpha(n) \cap S_\beta(n) = 0]$. Since $\forall n \in \omega\,[f(n) \in S_\alpha(n) \wedge g(n) \in S_\beta(n)]$, it follows that $\forall n \geq m\,[f(n) \neq g(n)]$. To verify (4), fix $f \in \mathscr{A}_\alpha$ and some $\beta \leq \alpha$. Again we know that there is $m \in \omega$ such that $\forall n \geq m\,[S_\alpha(n) \cap \{f_\beta(n)\} = 0]$ and that $\forall n \in \omega\,[f(n) \in S_\alpha(n)]$. Therefore, it follows that $\forall n \geq m\,[f(n) \neq f_\beta(n)]$. $\dashv$

We are now ready to construct our Van Douwen MAD family. In order to ensure that our family is Van Douwen MAD we will use the notion of the trace of an a.d. family (see Definition 1.3.8). The idea is that if an a.d. family has a "sufficiently large" trace, then it must be Van Douwen MAD.

Convention 2.1.10. *By Theorem 2.1.3 there is a family $F = \langle f_\alpha : \alpha < \mathrm{non}\,(\mathcal{M})\rangle \subset \omega^\omega$ such that for every infinite partial function g there is an $\alpha < \mathrm{non}\,(\mathcal{M})$ such that $|g \cap f_\alpha| = \omega$. For the remainder of this section let us fix such a family F.*

We will remind the reader of the definition of trace of an a.d. family.

Definition 2.1.11 (see Definition 1.3.8)**.** *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. The trace of $\mathscr{A}$, written $\mathrm{tr}\,(\mathscr{A})$, is $\{f \in \omega^\omega : \mathscr{A} \cap f$ is a MAD family on $f\}$. Recall that we allow MAD families on f to be finite (see Definition 1.3.7).*

Lemma 2.1.12. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family such that $F \subset \mathrm{tr}\,(\mathscr{A})$. Then $\mathscr{A}$ is Van Douwen MAD.*

Proof. Indeed, let g be an infinite partial function. By the definition of F, there is $\alpha < \mathrm{non}(\mathcal{M})$ such that $|g \cap f_\alpha| = \omega$. Since $F \subset \mathrm{tr}(\mathscr{A})$, $\mathscr{A} \cap f_\alpha$ is a MAD family on f_α. So there is $h \in \mathscr{A}$ such that $h \cap f_\alpha$ meets $g \cap f_\alpha$ in an infinite set, whence we get that $|h \cap g| = \omega$. $\dashv$

Theorem 2.1.13. *There is a Van Douwen MAD family of size* $\mathfrak{c}$.

Proof. In view of Lemma 2.1.12, it is enough to construct an a.d. family $\mathscr{A}$ of size $\mathfrak{c}$ such that $F \subset \mathrm{tr}(\mathscr{A})$. We will use Lemma 2.1.9 to do this. Fix a sequence $\langle \mathscr{A}_\alpha : \alpha < \mathrm{non}(\mathcal{M}) \rangle$ as in Lemma 2.1.9. $\mathscr{A}$ will be constructed as the union of an increasing sequence of a.d. families. Thus, we will construct a sequence $\langle \mathscr{C}_\alpha : \alpha < \mathrm{non}(\mathcal{M}) \rangle$ such that:

1. $\mathscr{C}_\alpha \subset \omega^\omega$ is an a.d. family
2. $\forall \beta < \alpha < \mathrm{non}(\mathcal{M}) \, [\mathscr{C}_\beta \subset \mathscr{C}_\alpha]$
3. $f_\alpha \in \mathrm{tr}(\mathscr{C}_\alpha)$
4. $\forall h \in \mathscr{C}_\alpha \exists \beta \leq \alpha \exists g \in \mathscr{A}_\beta \exists X \in [\omega]^\omega \, [h = f_\beta \restriction X \cup g \restriction \omega \setminus X]$
5. $|\mathscr{C}_0| = \mathfrak{c}$.

To construct $\mathscr{C}_0$, we fix a MAD family $\{a_\xi : \xi < \mathfrak{c}\}$ on ω. Put $\mathscr{A}_0 = \{g_\xi : \xi < \mathfrak{c}\}$. For each $\xi < \mathfrak{c}$, let $h_\xi = (f_0 \restriction a_\xi) \cup (g_\xi \restriction (\omega \setminus a_\xi))$, and put $\mathscr{C}_0 = \{h_\xi : \xi < \mathfrak{c}\}$. We will check that $\mathscr{C}_0$ is a.d. Indeed, if $\xi_0 < \xi_1$, then since $a_{\xi_0} \cap a_{\xi_1}$ is finite, $|f_0 \restriction a_{\xi_0} \cap f_0 \restriction a_{\xi_1}| < \omega$. Next, since $\mathscr{A}_0 \perp \{f_0\}$, we have that both $(f_0 \restriction a_{\xi_0}) \cap (g_{\xi_1} \restriction (\omega \setminus a_{\xi_1}))$ and $(f_0 \restriction a_{\xi_1}) \cap (g_{\xi_0} \restriction (\omega \setminus a_{\xi_0}))$ are finite. Finally, since $\mathscr{A}_0$ is an a.d. family, we know that $|g_{\xi_0} \restriction (\omega \setminus a_{\xi_0}) \cap g_{\xi_1} \restriction (\omega \setminus a_{\xi_1})| < \omega$. Thus, we conclude that $|h_{\xi_0} \cap h_{\xi_1}| < \omega$. Next, it is clear from the construction that $f_0 \in \mathrm{tr}(\mathscr{C}_0)$, and that $\mathscr{C}_0$ satisfies clauses (4) and (5).

To continue the construction, suppose that we are given the sequence $\langle \mathscr{C}_\beta : \beta < \alpha \rangle$. Set $\mathscr{C} = \bigcup \mathscr{C}_\beta$ and consider $\mathscr{C} \cap f_\alpha$. This is an a.d. family on f_α. If it is a MAD family (either finite or infinite), then f_α is already in $\mathrm{tr}(\mathscr{C})$, and there is nothing more to be done. In this case, we set $\mathscr{C}_\alpha = \mathscr{C}$. So, say that $\mathscr{C} \cap f_\alpha$ is not MAD. We can extend it to a MAD family, say $\mathscr{B}$, on f_α. Consider the family $\{Y \in [\omega]^\omega : f_\alpha \restriction Y \in \mathscr{B} \setminus (\mathscr{C} \cap f_\alpha)\}$. Note that this is an a.d. family on ω. We may assume WLOG that it has size $\mathfrak{c}$. Let $\{a_\xi : \xi < \mathfrak{c}\}$ enumerate this family. Put $\mathscr{A}_\alpha = \{g_\xi : \xi < \mathfrak{c}\}$. For each $\xi < \mathfrak{c}$ set $h_\xi = (f_\alpha \restriction a_\xi) \cup (g_\xi \restriction (\omega \setminus a_\xi))$, and put $\mathscr{D} = \{h_\xi : \xi < \mathfrak{c}\}$. It is easily argued, as for $\mathscr{C}_0$, that $\mathscr{D}$ is a.d. We will check that $\mathscr{C} \perp \mathscr{D}$. Fix $h \in \mathscr{C}$ and $\xi < \mathfrak{c}$. If $h \cap f_\alpha$ is finite, then so is $h \cap f_\alpha \restriction a_\xi$. On the other hand, if $h \cap f_\alpha$ is infinite, then $h \cap f_\alpha \in \mathscr{C} \cap f_\alpha$. But then $|f_\alpha \restriction a_\xi \cap h| < \omega$ because $\mathscr{B}$ is an a.d. family. Thus in either case, $h \cap f_\alpha \restriction a_\xi$ is finite. To deal with $h \cap g_\xi \restriction (\omega \setminus a_\xi)$, by clause (4), we know that for some $\gamma \leq \beta < \alpha$, $h = (f_\gamma \restriction X) \cup (g \restriction (\omega \setminus X))$, where $X \in [\omega]^\omega$ and $g \in \mathscr{A}_\gamma$. But since $\mathscr{A}_\alpha \perp \{f_\gamma\}$, $|(f_\gamma \restriction X) \cap (g_\xi \restriction (\omega \setminus a_\xi))| < \omega$, and since $\mathscr{A}_\alpha \perp \mathscr{A}_\gamma$, we know that $|(g_\xi \restriction (\omega \setminus a_\xi)) \cap (g \restriction (\omega \setminus X))| < \omega$. Therefore, $h \cap g_\xi \restriction (\omega \setminus a_\xi)$ is also finite, and so $|h \cap h_\xi| < \omega$. Hence, we can define $\mathscr{C}_\alpha = \mathscr{C} \cup \mathscr{D}$.

Now, it is clear that $\mathscr{C}_\alpha$ satisfies clauses (1), (2) and (4). We just need to verify that $f_\alpha \in \mathrm{tr}\,(\mathscr{C}_\alpha)$. So we need to check that $\mathscr{C}_\alpha \cap f_\alpha$ is a MAD family on f_α. But clearly $\mathscr{C}_\alpha \cap f_\alpha = \mathscr{C} \cap f_\alpha \cup \mathscr{D} \cap f_\alpha$. Fix $X \in [\omega]^\omega$. Since $\mathscr{B}$ is a MAD family on f_α, there is $Y \in [\omega]^\omega$ such that $f_\alpha \restriction Y \in \mathscr{B}$ and $|f_\alpha \restriction X \cap f_\alpha \restriction Y| = \omega$. If $f_\alpha \restriction Y \in \mathscr{C} \cap f_\alpha$, then we are done. If it is not, then $Y = a_\xi$ for some $\xi < \mathfrak{c}$. It follows that $|f_\alpha \restriction X \cap h_\xi| = \omega$. But since $h_\xi \in \mathscr{D}$, we are done. $\dashv$

Definition 2.1.14. *Let $\mathfrak{a}_\mathfrak{v}$ denote the least size of a Van Douwen MAD family. By Theorem 2.1.13, this cardinal is well defined.*

Since any Van Douwen MAD family is MAD, we have $\mathfrak{a}_\mathfrak{e} \leq \mathfrak{a}_\mathfrak{v}$.

Question 2.1.15. *Is it consistent to have $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}_\mathfrak{v}$?*

2.2 Definability of MAD Families in ω^ω

Our next task is to investigate the definability of a.d. families in ω^ω. We will first prove that if $\mathscr{A}$ is an analytic MAD family in ω^ω, then $\mathscr{A}$ must satisfy some strong constraints (Theorem 2.2.1). We will then show that this is a strengthening of a result of Steprāns [20] that strongly MAD families cannot be analytic (see Definition 1.3.6). To do this, we will argue that any strongly MAD family must be Van Douwen MAD. Next, we will show that it is consistent to have MAD families in ω^ω that satisfy these strong constraints (see Theorem 2.2.12). Finally, we will argue that analytic MAD families cannot satisfy these constraints if they have some additional combinatorial properties.

Theorem 2.2.1. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family and let $X \in [\omega \times \omega]^\omega$ avoid $\mathscr{A}$ (See Definition 1.3.5). Suppose that $\mathscr{A}$ is analytic in ω^ω. There is $Y \in [X]^\omega$ such that $\forall h \in \mathscr{A}\ [\ |h \cap Y| < \omega]$.*

Proof. Let us give the space 2^X the Tychonoff product topology, with 2 having the discrete topology. Since X is a countable set, this is homeomorphic to 2^ω with the usual topology. Define a map $\Psi : \omega^\omega \to 2^X$ by stipulating that $\forall \langle n, m\rangle \in X\, [\Psi(f)\,(\langle n, m\rangle) = 1 \leftrightarrow \langle n, m\rangle \in f]$. Thus $\Psi(f)$ is the characteristic function of $X \cap f$.

This map is continious. To see this, fix finitely many members $\langle n_0, m_0\rangle, \dots, \langle n_k, m_k\rangle \in X$ and $\langle n^0, m^0\rangle, \dots, \langle n^l, m^l\rangle \in X$. A basic open subset of 2^X is of the form $U = \{\chi \in 2^X : \chi\,(\langle n_i, m_i\rangle) = 0\ \forall i \leq k \wedge \chi\left(\langle n^i, m^i\rangle\right) = 1\ \forall i \leq l\}$. Thus $\Psi^{-1}(U) = \{f \in \omega^\omega : f(n_i) \neq m_i\ \forall i \leq k \wedge f(n^i) = m^i\ \forall i \leq l\}$. It is clear that this is an open subset of ω^ω. It follows that $\Psi''\mathscr{A}$ is an analytic subset of 2^X. It is the set of characteristic functions of elements of $\{h \cap X : h \in \mathscr{A}\}$. We are only interested in the infinite elements of this set. So we will put $\mathscr{B} = \Psi''\mathscr{A} \cap \{\chi \in 2^X : \exists^\infty \langle n, m\rangle \in X\, [\chi\,(\langle n, m\rangle) = 1]\}$. It is clear that $\mathscr{B}$ is also analytic. $\mathscr{B}$ is the set of characteristic functions of elements of $\mathscr{A} \cap X = \{h \cap X : h \in \mathscr{A} \wedge |h \cap X| = \omega\}$. Now, $\mathscr{A} \cap X$ is an a.d. family on X. By a theorem of Mathias [25] we know that there are no analytic MAD families on X. Therefore, if $\mathscr{A} \cap X$ is infinite, it is not MAD on X, and we get the conclusion of the theorem. On the other hand, if $\mathscr{A} \cap X$ is finite, then since X avoids $\mathscr{A}$, $Y = X \setminus \bigcup(\mathscr{A} \cap X)$ will satisfy the conclusion of the theorem. Hence, either way, the theorem is proved. $\dashv$

Corollary 2.2.2. *Suppose $\mathscr{A} \subset \omega^\omega$ is an analytic a.d. family. Then $\mathscr{A}$ has trivial trace (see Definition 1.3.9).*

Proof. If f is a member of $\mathrm{tr}\,(\mathscr{A})$ which avoids $\mathscr{A}$, then putting $f = X$ in Theorem 2.2.1 will give a contradiction. $\dashv$

Corollary 2.2.3. *There are no analytic Van Douwen MAD families in ω^ω.*

$\dashv$

Steprāns [20] introduced the notion of a strongly MAD family (see Definition 1.3.6) and proved that they can't be analytic. We will show that this follows from Corollary 2.2.3.

Lemma 2.2.4. *Let $\mathscr{A} \subset \omega^\omega$ be strongly MAD. Let $\{g_i : i \in \omega\}$ be a collection of infinite partial functions from ω to ω such that each g_i avoids $\mathscr{A}$. There is $h \in \mathscr{A}$ such that $\forall i \in \omega\, [\, |h \cap g_i| = \omega]$. In particular, strongly MAD families are Van Douwen MAD.*

Proof. Let $h_0 \neq h_1$ be two distinct members of $\mathscr{A}$. For each $i \in \omega$, let $a_i = \mathrm{dom}\,(g_i)$ and let $b_i = \omega \setminus a_i$. For each $i \in \omega$, define $f_i^0 = g_i \cup h_0 \restriction b_i$ and $f_i^1 = g_i \cup h_1 \restriction b_i$. Since g_i avoids $\mathscr{A}$, both f_i^0 and f_i^1 avoid $\mathscr{A}$. Thus $\{f_i^j : i \in \omega \wedge j \in 2\}$ is a countable collection of total functions avoiding $\mathscr{A}$. So we may choose $h \in \mathscr{A}$ such that $\forall i \in \omega\ \forall j \in 2 \left[\, \left|h \cap f_i^j\right| = \omega\right]$. We will show that $\forall i \in \omega\, [\, |g_i \cap h| = \omega]$. If $g_i \cap h$ is finite, then since both $f_i^0 \cap h$ and $f_i^1 \cap h$ are infinite, it follows that $|h_0 \cap h| = \omega$ and that $|h_1 \cap h| = \omega$. But since $\mathscr{A}$ is an a.d. family this means that $h = h_0$ and $h = h_1$, which is a contradiction. $\dashv$

Corollary 2.2.5 (Steprāns [20])**.** *There are no analytic strongly MAD families in ω^ω.*

$\dashv$

Remark 2.2.6. *Corollary 2.2.3 is strictly stronger than Corollary 2.2.5. It is easy to modify the construction in Theorem 2.1.13 to ensure that the Van Douwen MAD family constructed there is not strongly MAD.*

It is an open problem whether there are any analytic MAD families in ω^ω. In fact, it is not even known if a MAD family in ω^ω can be closed. Since Theorem 2.2.1 puts a strong restriction on such MAD families, one might conjecture that there are no MAD families that satisfy the conclusion of Theorem 2.2.1 at all. However, we will show below that this is consistently false. We will first argue that it is sufficient to build a MAD family with trivial trace.

Lemma 2.2.7. *Let $\mathscr{A} \subset \omega^\omega$ be a MAD family with trivial trace. Suppose $X \in [\omega \times \omega]^\omega$ avoids $\mathscr{A}$. There is $Y \in [X]^\omega$ such that $\forall h \in \mathscr{A}\, [\, |h \cap Y| < \omega]$.*

Proof. Let $\mathscr{A} \cap X = \{h \cap X : h \in \mathscr{A} \wedge |h \cap X| = \omega\}$. If $\mathscr{A} \cap X$ is finite, then since X avoids $\mathscr{A}$, $Y = X \setminus \bigcup(\mathscr{A} \cap X)$ will be as required. So assume that $\mathscr{A} \cap X$ is infinite. Choose a countably infinite collection $\{h_i : i \in \omega\} \subset \mathscr{A}$ such that $|h_i \cap X| = \omega$ for each i, and put $p_i = h_i \cap X$. Thus $\{p_i : i \in \omega\}$ forms an a.d family of infinite partial functions. We may choose infinite partial functions $g_i \subset p_i$ such that $\forall i < j < \omega\, [\mathrm{dom}\,(g_i) \cap \mathrm{dom}\,(g_j) = 0]$. Now if we put $g = \bigcup g_i$, then g is an infinite partial function and $g \subset X$. Since g has infinite intersection with infinitely many things in $\mathscr{A}$, it is clear that g avoids $\mathscr{A}$. Let $a = \mathrm{dom}\,(g)$ and let $b = \omega \setminus a$. Choose $h \in \mathscr{A}$ and put $f = g \cup h \restriction b$. Obviously, f is a total function avoiding $\mathscr{A}$. So $f \notin \mathrm{tr}\,(\mathscr{A})$. Therefore, we may choose an infinite partial function $p \subset f$ such that $\forall h \in \mathscr{A}\, [|h \cap p| < \omega]$. Clearly, since $|p \cap h \restriction b| < \omega$, we have that $|p \cap g| = \omega$. Thus, $Y = p \cap g$ is as required. $\dashv$

Definition 2.2.8. *Let $\mathcal{I}$ be a proper non-principal ideal on ω. We will say that $\mathcal{I}$ is a dense ideal if $\forall a \in [\omega]^\omega \exists b \in [a]^\omega\, [b \in \mathcal{I}]$.*

Let $\mathscr{A} \subset \omega^\omega$ be a MAD family with trivial trace, and consider the ideal $\mathcal{I}_0(\mathscr{A}) = \{a \in \mathcal{P}(\omega) : \exists p \in \omega^a [p \text{ is a.d. from } \mathscr{A}]\}$. It is clear that $\mathcal{I}_0(\mathscr{A})$ must be a dense ideal on ω. So we must use such an ideal in our construction.

Lemma 2.2.9. *There is a dense ideal $\mathcal{I}$ on ω such that whenever X is a subset of $\mathcal{I}$ of size less than $\mathfrak{c}$, there is an infinite set $a \in \mathcal{I}$ such that $\forall x \in X [|a \cap x| < \omega]$.*

Proof. Let $\mathscr{A} \subset [\omega]^\omega$ be a MAD family of subsets of ω of size $\mathfrak{c}$, with $\bigcup \mathscr{A} = \omega$. Let $\mathcal{I}$ be the ideal generated by $\mathscr{A}$. It is easily checked that $\mathcal{I}$ is a dense ideal on ω. Now, suppose $X = \langle x_\alpha : \alpha < \kappa \rangle \subset \mathcal{I}$, with $\kappa < \mathfrak{c}$. As $\mathcal{I}$ is generated by $\mathscr{A}$, it is possible to find a set $\mathscr{B} \subset \mathscr{A}$, with $|\mathscr{B}| < \mathfrak{c}$, such that for every $\alpha < \kappa$, there is a finite set $\{b_0, \dots, b_k\} \subset \mathscr{B}$ so that $x_\alpha \subset b_0 \cup \dots \cup b_k$. Since $|\mathscr{A}| = \mathfrak{c}$, we may choose a set $a \in \mathscr{A}$ which is a.d. from everything in $\mathscr{B}$. Now, it is clear that a is a.d. from all the x_α. $\dashv$

Definition 2.2.10. *Let $\mathcal{I}$ be an ideal as in Lemma 2.2.9. If $\mathscr{B}$ is a family of infinite partial functions from ω to ω, we will say that $\mathscr{B}$ has domains in $\mathcal{I}$ if $\forall g \in \mathscr{B} [\mathrm{dom}(g) \in \mathcal{I}]$.*

Lemma 2.2.11. *Assume* $\mathrm{non}(\mathcal{M}) = \mathfrak{c}$. *Let $\mathcal{I}$ be an ideal as in Lemma 2.2.9. Let $\mathscr{B}$ be a family of infinite partial functions with domains in $\mathcal{I}$ and let $\mathscr{D} \subset \omega^\omega$ be a family of total functions. Suppose that both $\mathscr{B}$ and $\mathscr{D}$ have size less than $\mathfrak{c}$. Let $f \in \omega^\omega$ be a.d. from $\mathscr{D}$. There is $h \in \omega^\omega$ such that:*

1. $\forall g \in \mathscr{B} [\, |h \cap g| < \omega]$
2. $\forall h' \in \mathscr{D} [|h \cap h'| < \omega]$
3. $|h \cap f| = \omega$.

Proof. Let $X = \{\mathrm{dom}(f \cap g) : g \in \mathscr{B}\}$. Since $\mathscr{B}$ has domains in $\mathcal{I}$, it follows that $X \subset \mathcal{I}$. By assumption, $|\mathscr{B}| < \mathfrak{c}$. So by Lemma 2.2.9, we can find an infinite set $a \in \mathcal{I}$ which is a.d. from everything in X. Set $p = f \restriction a$. Since f is assumed to be a.d. from $\mathscr{D}$, p is also a.d. from $\mathscr{D}$.

Moreover, if $g \in \mathscr{B}$, then $\mathrm{dom}\,(p \cap g) \subset a \cap \mathrm{dom}\,(f \cap g)$, which is finite. Therefore, $|p \cap g| < \omega$. Now, since $\mathrm{non}\,(\mathcal{M}) = c$, there is a total function $h_0 \in \omega^\omega$ which is a.d. from $\mathscr{B} \cup \mathscr{D}$. Let $a = \mathrm{dom}\,(p)$ and $b = \omega \setminus a$. Set $h = p \cup h_0 \restriction b$. h satisfies (1) and (2) above because both p and h_0 are a.d. from $\mathscr{B} \cup \mathscr{D}$. It satisfies (3) because p is an infinite partial function contained in f. $\dashv$

Theorem 2.2.12. *Assume* $\mathrm{non}\,(\mathcal{M}) = \mathfrak{a} = \mathfrak{c}$. *There is a MAD family* $\mathscr{A} \subset \omega^\omega$ *with trivial trace* [1].

Proof. Let $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate ω^ω. Let $\mathcal{I}$ be an ideal as in Lemma 2.2.9. We will construct the MAD family $\mathscr{A}$ by induction, as the union of an increasing sequence of a.d. families. In fact, we will build two sequences $\langle \mathscr{A}_\alpha : \alpha < \mathfrak{c} \rangle$ and $\langle \mathscr{B}_\alpha : \alpha < \mathfrak{c} \rangle$ such that the following hold:

1. $\mathscr{A}_\alpha \subset \omega^\omega$ is an a.d family, with $|\mathscr{A}_\alpha| \leq |\alpha|$
2. $\mathscr{B}_\alpha$ is a family of infinite partial functions, with $|\mathscr{B}_\alpha| \leq |\alpha|$
3. $\forall \alpha < \beta < \mathfrak{c}\, [\mathscr{A}_\alpha \subset \mathscr{A}_\beta \wedge \mathscr{B}_\alpha \subset \mathscr{B}_\beta]$
4. $\mathscr{B}_\alpha$ has domains in $\mathcal{I}$
5. $\forall h \in \mathscr{A}_\alpha \forall g \in \mathscr{B}_\alpha\, [\, |h \cap g| < \omega]$
6. if f_α avoids $\bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$, then there is $g \in \mathscr{B}_\alpha$ so that $g \subset f_\alpha$
7. if f_α is a.d. from $\bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$, there is $h \in \mathscr{A}_\alpha$ so that $|h \cap f_\alpha| = \omega$.

$\mathscr{A}$ will be $\bigcup \mathscr{A}_\alpha$. Clauses (1) and (7) ensure that $\mathscr{A}$ is a MAD family in ω^ω. Clauses (5) and (6) ensure that $\mathscr{A}$ has trivial trace. It is easy to see that clause (4) is necessary because if $\mathscr{A}$ is a MAD family with trivial trace, then $\{a \in [\omega]^\omega : \exists p \in \omega^a\, [p \text{ is a.d. from } \mathscr{A}]\}$ is a proper dense ideal on ω.

[1] *An earlier version of this proof claimed to derive this theorem just from the assumption* $\mathfrak{a} = \mathfrak{c}$. *Brendle pointed out that the proof was implicitly assuming* $\mathrm{non}\,(\mathcal{M}) = \mathfrak{c}$.

Fix $\alpha < \mathfrak{c}$ and suppose that $\langle \mathscr{A}_\beta : \beta < \alpha \rangle$ and $\langle \mathscr{B}_\beta : \beta < \alpha \rangle$ are given to us. Set $\mathscr{B} = \bigcup \mathscr{B}_\beta$ and $\mathscr{D} = \bigcup \mathscr{A}_\beta$. If f_α does not avoid $\mathscr{D}$, then there is nothing to be done. In this case, we simply set $\mathscr{A}_\alpha = \mathscr{D}$ and $\mathscr{B}_\alpha = \mathscr{B}$. From now on, let us assume that f_α avoids $\mathscr{D}$. We will first define $\mathscr{B}_\alpha$. Consider $\mathscr{D} \cap f_\alpha$. This is an a.d. family on f_α. Since $|\mathscr{D}| < \mathfrak{c}$, and since, by assumption, f_α avoids $\mathscr{D}$, we can use $\mathfrak{a} = \mathfrak{c}$ to find an infinite partial function $p \subset f_\alpha$ so that $\forall h \in \mathscr{D}\,[|p \cap h| < \omega]$. Since $\mathcal{I}$ is a dense ideal, there is an infinite partial function $g_1 \subset p$ with $\mathrm{dom}\,(g_1) \in \mathcal{I}$. Now, we define $\mathscr{B}_\alpha = \mathscr{B} \cup \{g_1\}$. By our choice of p, we have that $\forall h \in \mathscr{D}\,[|h \cap g_1| < \omega]$. This completes the definition of $\mathscr{B}_\alpha$. We now define $\mathscr{A}_\alpha$. We will proceed by cases. Suppose that f_α is not a.d. from $\mathscr{D}$. In this case, we may set $\mathscr{A}_\alpha = \mathscr{D}$. Note that we have already ensured above that everything in $\mathscr{B}_\alpha$ is a.d from $\mathscr{D}$. So clause (5) will be satisfied. All the other clauses are immediate. Now, let us consider the case when f_α is a.d. from $\mathscr{D}$. $\mathscr{B}_\alpha$ is a family of infinite partial functions with domains in $\mathcal{I}$ and it's size is less than $\mathfrak{c}$. Also, $\mathscr{D}$ is a family of total functions with $|\mathscr{D}| < \mathfrak{c}$. Therefore, we can apply Lemma 2.2.11 to find a function $h \in \omega^\omega$, which is a.d. from $\mathscr{B}_\alpha \cup \mathscr{D}$, with the property that $|h \cap f_\alpha| = \omega$. Now, we can set $\mathscr{A}_\alpha = \mathscr{D} \cup \{h\}$. It is easy to see that clauses (1) – (7) are all satisfied, and so we are done. $\dashv$

Observe that if $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}$, then any MAD family $\mathscr{A} \subset \omega^\omega$ of size $\mathfrak{a}_\mathfrak{e}$ will have trivial trace because if $f \in \omega^\omega$, then $|\mathscr{A} \cap f| < \mathfrak{a}$. It is unknown if it is consistent to have $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}$. We also do not know if the construction in Theorem 2.2.12 can be carried out under $\mathfrak{a} = \mathfrak{c}$, or even just in ZFC. But we conjecture that the latter is impossible.

Conjecture 2.2.13. *It is consistent with ZFC that every MAD family in ω^ω has a non trivial trace.*

Theorem 2.2.12 implies that it is consistent to have a MAD family with trivial trace. However, it may still be the case that analytic MAD families cannot have trivial trace. We investigate this possibility next. We show that analytic MAD families satisfying certain extra combinatorial properties cannot have trivial trace, and hence, cannot exist. We make use of a partition theorem proved by Taylor [6] and extended by Blass [7].

Definition 2.2.14. *Let X be a countably infinite set. A non principal ultrafilter $\mathcal{U}$ on X is a P–point if for every decreasing sequence $a_0 \supset a_1 \supset \cdots$ of sets in $\mathcal{U}$, there is $a \in \mathcal{U}$ such that $\forall i \in \omega\, [a \subset^* a_i]$.*

Theorem 2.2.15 (Taylor, see [7] Theorem 4)**.** *Let $\mathcal{U}$ be a P-point on ω and let $\mathscr{X} \subset [\omega]^\omega$ be an analytic set. There is a set $E \in \mathcal{U}$ and a function $f \in \omega^\omega$ such that $\mathscr{X}$ contains all or none of the infinite subsets F of E that satisfy*

$$\forall i, j \in F\, [i < j \implies f(i) < j]\,. \tag{$*$}$$

⊣

Convention 2.2.16. *We will apply Theorem 2.2.15 to an ultrafilter $\mathcal{U}$ on $\omega \times \omega$ and an $\mathscr{X} \subset [\omega \times \omega]^\omega$. In order to make sense of the condition $(*)$ in Theorem 2.2.15, we must have a well ordering of $\omega \times \omega$ in type ω. Let us arbitrarily choose such an ordering $\prec$.*

Lemma 2.2.17. *Let $\mathscr{A} \subset \omega^\omega$ be an analytic a.d. family. Let $E \in [\omega \times \omega]^\omega$ be a set such that $\exists^\infty h \in \mathscr{A}\, [|h \cap E| = \omega]$, let $\mathscr{X} = \{F \in [\omega \times \omega]^\omega : \exists h \in \mathscr{A}\, [|h \cap F| = \omega]\}$ and let $f \in (\omega \times \omega)^\omega$. Then there are infinite sets F_0 and F_1 in $[E]^\omega$ such that $F_0 \in \mathscr{X}$, $F_1 \notin \mathscr{X}$ and*

$$\forall \langle i_0, j_0\rangle, \langle i_1, j_1\rangle \in F_0\, [\langle i_0, j_0\rangle \prec \langle i_1, j_1\rangle \implies f(\langle i_0, j_0\rangle) \prec \langle i_1, j_1\rangle] \tag{$*_0$}$$

$$\forall \langle k_0, l_0\rangle, \langle k_1, l_1\rangle \in F_1\, [\langle k_0, l_0\rangle \prec \langle k_1, l_1\rangle \implies f(\langle k_0, l_0\rangle) \prec \langle k_1, l_1\rangle]\,. \tag{$*_1$}$$

Proof. Choose $h \in \mathscr{A}$ such that $|h \cap E| = \omega$. We may choose, by recursion, an infinite set $F_0 \subset h \cap E$ that satisfies $(*_0)$ above. It is clear that $|F_0 \cap h| = \omega$, and hence that $F_0 \in \mathscr{X}$. To get F_1, we will use Theorem 2.2.1. Note that E avoids $\mathscr{A}$. So there is $F \in [E]^\omega$ such that F is a.d. from $\mathscr{A}$. Once again, we may choose, by recursion, an infinite set $F_1 \subset F$ that satisfies $(*_1)$ above. It is clear that F_1 is a.d. from $\mathscr{A}$, and hence that $F_1 \notin \mathscr{X}$. ⊣

Definition 2.2.18. *Let A be a countable set and let $\mathcal{I}$ be a non-principal ideal on A. Let $\mathcal{E} = [A]^{\omega} \setminus \mathcal{I}$. We say that $\mathcal{E}$ is a P-coideal on A if whenever $E_0 \supset E_1 \supset \cdots$ is a sequence of sets in $\mathcal{E}$, there a set $E \in \mathcal{E}$ such that $\forall n \in \omega\,[E \subset^* E_n]$.*

Theorem 2.2.19. *Let $\mathscr{A} \subset \omega^{\omega}$ be an a.d. family. Let $\mathscr{X} = \{F \in [\omega \times \omega]^{\omega} : \exists h \in \mathscr{A}\,[|h \cap F| = \omega]\}$ and let $\mathcal{E}_0 = \{E \in [\omega \times \omega]^{\omega} : \exists^{\infty} h \in \mathscr{A}\,[|h \cap E| = \omega]\}$. If there is a P-coideal $\mathcal{E}$ on $\omega \times \omega$ with $\mathcal{E} \subset \mathcal{E}_0$, then $\mathscr{A}$ is not analytic.*

Proof. By definition, there is a non-principal ideal $\mathcal{I}$ such that $\mathcal{E} = [\omega \times \omega]^{\omega} \setminus \mathcal{I}$. Let $\mathbb{P}$ be the forcing notion $\mathcal{P}(\omega \times \omega)/\mathcal{I}$. Since $\mathcal{E}$ is a P-coideal, $\mathbb{P}$ is countably closed and hence does not add any reals. Moreover, $\mathbb{P}$ generically adds a P-point $\mathcal{U} \subset \mathcal{E}$. Now, suppose for a contradiction that $\mathscr{A}$ is analytic. Identifying ω^{ω} with a G_{δ} subset of $\mathcal{P}(\omega \times \omega)$ in the natural way makes $\mathscr{A}$ into an analytic subset of $\mathcal{P}(\omega \times \omega)$. This implies that $\mathscr{X}$ is analytic because it has a Σ^1_1 definition. As $\mathbb{P}$ does not add any reals, $\mathscr{X}$ is still an analytic set in $\mathbf{V}[\mathcal{U}]$ with the same definition. Now, in $\mathbf{V}[\mathcal{U}]$, we may apply Theorem 2.2.15 to find a set $E \in \mathcal{U}$ and a function $f \in (\omega \times \omega)^{\omega}$ such that $\mathscr{X}$ contains all or none of the infinite subsets F of E that satisfy

$$\forall \langle i,j\rangle, \langle k,l\rangle \in F\,[\langle i,j\rangle \prec \langle k,l\rangle \implies f(\langle i,j\rangle) \prec \langle k,l\rangle]. \tag{$*$}$$

But $\mathbb{P}$ does not add any reals. Therefore, E and f are in the ground model $\mathbf{V}$. Note that $E \in \mathcal{E} \subset \mathcal{E}_0$ because $\mathcal{U} \subset \mathcal{E}$. This allows us to apply Lemma 2.2.17 in $\mathbf{V}$ to find $F_0, F_1 \in [E]^{\omega}$ satisfying $(*_0)$ and $(*_1)$ of Lemma 2.2.17 with $F_0 \in \mathscr{X}$ and $F_1 \notin \mathscr{X}$. But now, $F_0, F_1 \in \mathbf{V}[\mathcal{U}]$ still satisfy $(*_0)$ and $(*_1)$ in $\mathbf{V}[\mathcal{U}]$, contradicting our choice of E. $\dashv$

Remark 2.2.20. *If $\mathscr{A}$ is any infinite MAD family in $[\omega]^{\omega}$ and if $\mathcal{E}_0 = \{E \in [\omega]^{\omega} : \exists^{\infty} A \in \mathscr{A}\,[|E \cap A| = \omega]\}$, then Mathias [25] showed that $\mathcal{E}_0$ is a P-coideal. It is easy to see that for a MAD family in ω^{ω}, $\mathcal{E}_0$, as defined in Theorem 2.2.19, is not necessarily a P-coideal. This is an interesting difference between the two types of MADness.*

Next, we will explore some consequences of Theorem 2.2.19 for some ideals on ω that can be naturally defined by using a MAD family of functions $\mathscr{A} \subset \omega^\omega$.

Definition 2.2.21. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. We define $\mathcal{I}_0(\mathscr{A}) = \{a \in \mathcal{P}(\omega) : \exists p \in \omega^a \forall h \in \mathscr{A}\, [|p \cap h| < \omega]\}$. Given $E \subset \omega \times \omega$, we define $E(n) = \{m \in \omega : \langle n, m \rangle \in E\}$ and* $\mathrm{dom}\,(E) = \{n \in \omega : E(n) \neq 0\}$.

Notice that $\mathscr{A}$ is a MAD family *iff* $\omega \notin \mathcal{I}_0(\mathscr{A})$ *iff* $\mathcal{I}_0(\mathscr{A}) \neq \mathcal{P}(\omega)$. Therefore, given an analytic a.d. family $\mathscr{A}$, to show that $\mathscr{A}$ is not a MAD family, it suffices to prove that $\mathcal{I}_0(\mathscr{A}) = \mathcal{P}(\omega)$. While we don't know how to do this, we will show in what follows that $\mathcal{I}_0(\mathscr{A})$ must be "large" whenever $\mathscr{A} \subset \omega^\omega$ is an analytic a.d. family. In particular, we will show that it contains a copy of the ideal $0 \times \mathrm{Fin}$.

Definition 2.2.22. $0 \times \mathrm{Fin} = \{X \subset \omega \times \omega : \forall n \in \omega\, [\, |X(n)| < \omega]\}$.

Lemma 2.2.23. *Let $\mathscr{A} \subset \omega^\omega$ be a MAD family. Let*

$$\mathcal{E} = \{E \in [\omega \times \omega]^\omega : \forall k \in \omega\, \exists a \subset \mathrm{dom}\,(E)\, [\, a \notin \mathcal{I}_0(\mathscr{A}) \wedge \forall n \in a\, |E(n)| > k]\, \}.$$

$\mathcal{I} = \mathcal{P}(\omega \times \omega) \setminus \mathcal{E}$ is an ideal on $\omega \times \omega$.

Proof. It is easy to see that $\mathcal{I}$ is closed under subsets. We will check that it is also closed under unions. Fix $E_0, E_1 \in \mathcal{I}$ and suppose, for a contradiction, that $E_0 \cup E_1 \in \mathcal{E}$. Observe that $\mathrm{dom}\,(E_0 \cup E_1) = \mathrm{dom}\,(E_0) \cup \mathrm{dom}\,(E_1)$ and that for all $n \in \omega$, $(E_0 \cup E_1)(n) = E_0(n) \cup E_1(n)$. For each $k \in \omega$ and $i \in \{0, 1\}$, define $a^i_k = \{n \in \omega : |E_i(n)| > k\}$. Note that $\mathrm{dom}\,(E_i) = a^i_0 \supset a^i_1 \supset \cdots$. Since $\mathcal{I}_0(\mathscr{A})$ is an ideal, if $a^i_k \in \mathcal{I}_0(\mathscr{A})$ for some k, then $\forall k' \geq k \left[a^i_{k'} \in \mathcal{I}_0(\mathscr{A})\right]$. Therefore, it follows from our assumption that E_0 and E_1 are both in $\mathcal{I}$ that for some $k \in \omega$, both a^0_k and a^1_k are in $\mathcal{I}_0(\mathscr{A})$. Since $E_0 \cup E_1 \in \mathcal{E}$, $\{n \in \omega : |E_0(n) \cup E_1(n)| > 2k\} \notin \mathcal{I}_0(\mathscr{A})$. Therefore, we may choose $n \notin a^0_k \cup a^1_k$ such that $|E_0(n) \cup E_1(n)| > 2k$. But since $n \notin a^0_k \cup a^1_k$, $|E_0(n)| \leq k$ and $|E_1(n)| \leq k$, a contradiction. $\dashv$

Theorem 2.2.24. *Let $\mathscr{A} \subset \omega^\omega$ be a MAD family. If $[\omega]^\omega \setminus \mathcal{I}_0(\mathscr{A})$ is a P-coideal, then $\mathscr{A}$ is not analytic.*

Proof. Let $\mathcal{E}_0$ be defined as in Theorem 2.2.19 and $\mathcal{E}$ as in Lemma 2.2.23. Let $\mathcal{I} = \mathcal{P}(\omega \times \omega) \setminus \mathcal{E}$. Lemma 2.2.23 tells us that $\mathcal{I}$ is an ideal. Moreover, if $E \in \mathcal{E}$ and $\{h_0, \dots, h_k\} \subset \omega^\omega$, then there is an infinite partial function $p \subset E$ with $\mathrm{dom}(p) \notin \mathcal{I}_0(\mathscr{A})$, which is disjoint from $h_0, \dots, h_k$. It follows that there are infinitely many $h \in \mathscr{A}$ such that $|E \cap h| = \omega$, whence $\mathcal{E} \subset \mathcal{E}_0$. Therefore, by Theorem 2.2.19, it suffices to show that $\mathcal{E}$ is a P-coideal. Fix a sequence $E_0 \supset E_1 \supset \cdots$, with $E_i \in \mathcal{E}$. For each i and k, define $a^i_k = \{n \in \omega : |E_i(n)| > k\}$. As before, we have $\mathrm{dom}(E_i) = a^i_0 \supset a^i_1 \supset \cdots$. By assumption, no a^i_k is in $\mathcal{I}_0(\mathscr{A})$. We also have $a^0_k \supset a^1_k \supset \cdots$. Thus, $\langle a^k_k : k \in \omega \rangle$ is a decreasing sequence of sets not in $\mathcal{I}_0(\mathscr{A})$. Since we are assuming that $[\omega]^\omega \setminus \mathcal{I}_0(\mathscr{A})$ is a P-coideal, there is a set $a \notin \mathcal{I}_0(\mathscr{A})$ such that $a \subset^* a^k_k$, for all k. Let us define a set $E \subset \omega \times \omega$ with $\mathrm{dom}(E) = a$ as follows. Let $\langle n_i : i \in \omega \rangle$ enumerate a. We may assume that $a \subset a^0_0$. For each $i \in \omega$, let $l_i = \max\{k \le i : n_i \in a^k_k\}$. Note that $n_i \in a^{l_i}_{l_i}$, and hence that $|E_{l_i}(n_i)| > l_i$. Therefore, we may define $E(n_i)$ to be some (arbitrary) subset of $E_{l_i}(n_i)$ of size equal to $l_i + 1$. We will check that E is as required. Since $a \subset^* a^k_k$, $\lim l_i = \infty$, and therefore, $\lim |E(n_i)| = \infty$. As, $\mathrm{dom}(E) = a \notin \mathcal{I}_0(\mathscr{A})$, this gives us $E \in \mathcal{E}$. Next, we must check that $E \subset^* E_k$ for all k. Fix k. We know that $\forall^\infty i \in \omega\, [l_i \ge k]$. Thus $\forall^\infty i \in \omega\, [E(n_i) = E_{l_i}(n_i) \subset E_k(n_i)]$. As each $E(n_i)$ is finite, we get that $E \subset^* E_k$. $\dashv$

Corollary 2.2.25. *Suppose $\mathscr{A} \subset \omega^\omega$ is an analytic MAD family. $\mathcal{I}_0(\mathscr{A})$ contains a copy of* $0 \times \mathrm{Fin}$. *This means that there is a partition $\{c_n : n \in \omega\}$ of ω into countably many infinite pieces such that for any $a \subset \omega$, if $|a \cap c_n| < \omega$ for all $n \in \omega$, then $a \in \mathcal{I}_0(\mathscr{A})$.*

Proof. By Theorem 2.2.24 we know that there is a sequence $a_0 \supset a_1 \supset \cdots$ of subsets of ω not in $\mathcal{I}_0(\mathscr{A})$ such that for any $a \subset \omega$, if $a \subset^* a_n$ for all $n \in \omega$, then $a \in \mathcal{I}_0(\mathscr{A})$. We may assume without loss of generality that $a_0 = \omega$, that $\bigcap a_n = 0$ and that $a_n \setminus a_{n+1}$ is infinite. Put $c_n = a_n \setminus a_{n+1}$. By our assumptions, $\{c_n : n \in \omega\}$ is a partition of ω into infinite pieces. Now, suppose $a \subset \omega$ is a.d. from all the c_n. It is easy to see that for each $n \in \omega$, $a \setminus a_n \subset \bigcup_{m<n} (a \cap c_m)$, which is a finite set. So $\forall n \in \omega \, [a \subset^* a_n]$, whence $a \in \mathcal{I}_0(\mathscr{A})$. $\dashv$

Conjecture 2.2.26. *If $\mathscr{A} \subset \omega^\omega$ is a MAD family, then $\mathscr{A}$ is not analytic.*

Chapter 3

Strongly and Very MAD Families of Functions

3.1 The Strongness of an a.d. Family

We will introduce the notion of strongness of an a.d. family of functions. This notion allows for the systematic investigation of variations on the concept of a strongly MAD family (see Definition 1.3.6). Strongly MAD families were introduced by Steprāns [20], who showed that they cannot be analytic. Soon after, Kastermans and Zhang [19] introduced a strengthening of this notion, which they called very MAD family. On the other hand, E. van Douwen [27] asked if there is a MAD family of functions in ω^ω which is also maximal with respect to infinite partial functions. Such MAD families are called Van Douwen MAD families (see Definition 1.3.4). The author has proved that Van Douwen MAD families always exist, and that the notion of a Van Douwen MAD family is weaker than that of a strongly MAD family. Thus, we have a natural spectrum of combinatorial properties of increasing strength. The notion of strongness allows for the systematic investigation of this entire spectrum starting with MADness, going through Van Douwen and strong MADness, all the way up to very MADness.

We first begin with the definition of a very MAD family. The reader should see Definition 1.3.5 for the notion of avoiding and Definition 1.3.6 for the notion of a strongly MAD family.

Definition 3.1.1. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family, and let $\kappa = |\mathscr{A}|$. We say that $\mathscr{A}$ is very*

MAD if for all cardinals $\lambda < \kappa$ and for every family $\{f_\alpha : \alpha < \lambda\} \subset \omega^\omega$ of functions avoiding $\mathscr{A}$, there is $h \in \mathscr{A}$ such that $\forall \alpha < \lambda\, [|f_\alpha \cap h| = \omega]$.

Clearly, very MAD families are strongly MAD, which in turn, are MAD.

Definition 3.1.2. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. We define the strongness of $\mathscr{A}$, written $\mathrm{st}\,(\mathscr{A})$, to be the least cardinal κ such that there is a family $\{f_\alpha : \alpha < \kappa\} \subset \omega^\omega$ of functions avoiding $\mathscr{A}$ such that $\forall h \in \mathscr{A}\, \exists \alpha < \kappa\, [|h \cap f_\alpha| < \omega]$.*

Thus an a.d. family $\mathscr{A}$ is MAD *iff* $\mathrm{st}\,(\mathscr{A}) \geq 2$. It is strongly MAD *iff* $\mathrm{st}\,(\mathscr{A}) \geq \omega_1$, and it is very MAD *iff* $\mathrm{st}\,(\mathscr{A}) \geq |\mathscr{A}|$. The next Lemma points out a connection with the notion of a Van Douwen MAD family.

Lemma 3.1.3. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. If $\mathrm{st}\,(\mathscr{A}) \geq 3$, then $\mathscr{A}$ is Van Douwen MAD.*

Proof. Suppose, for a contradiction, that there is an infinite partial function f from ω to ω which is a.d. from $\mathscr{A}$. Let $h_0 \neq h_1$ be two distinct functions in $\mathscr{A}$. Let $a = \mathrm{dom}\,(f)$ and let $b = \omega \setminus a$. Let $g_0 = f \cup h_0 \restriction b$ and let $g_1 = f \cup h_1 \restriction b$. Since f is a.d. from $\mathscr{A}$, it avoids $\mathscr{A}$. So $\{g_0, g_1\} \subset \omega^\omega$ is a set of two functions avoiding $\mathscr{A}$. As $\mathrm{st}\,(\mathscr{A}) \geq 3$, there is $h \in \mathscr{A}$ such that $|h \cap g_0| = |h \cap g_1| = \omega$. We will argue that $|f \cap h| = \omega$, giving a contradiction. Indeed, suppose that $|f \cap h| < \omega$. Since h intersects both g_0 and g_1 in an infinite set, it follows that both $h \cap h_0$ and $h \cap h_1$ are infinite. But since $\mathscr{A}$ is an a.d. family, this implies that $h = h_0$ and $h = h_1$, contradicting our choice of h_0 and h_1. $\dashv$

This argument can be generalized to yield the following.

Lemma 3.1.4. *Let κ be an infinite cardinal. Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family with $\mathrm{st}\,(\mathscr{A}) > \kappa$. If $\{f_\alpha : \alpha < \kappa\}$ is a family of infinite partial functions avoiding $\mathscr{A}$, there is $h \in \mathscr{A}$ such that $\forall \alpha < \kappa\, [|h \cap f_\alpha| = \omega]$.*

Proof. Just as in Lemma 3.1.3, pick two distinct members h_0 and h_1 of $\mathscr{A}$. For each $\alpha < \kappa$, set $a_\alpha = \operatorname{dom}(f_\alpha)$ and $b_\alpha = \omega \setminus a_\alpha$. Put $g^0_\alpha = f_\alpha \cup h_0 \restriction b_\alpha$ and $g^1_\alpha = f_\alpha \cup h_1 \restriction b_\alpha$. Since f_α avoids $\mathscr{A}$, both g^0_α and g^1_α avoid $\mathscr{A}$. As κ is an infinite cardinal, $\{g^i_\alpha : i \in 2 \wedge \alpha < \kappa\} \subset \omega^\omega$ is a family of κ functions avoiding $\mathscr{A}$. As $\operatorname{st}(\mathscr{A}) > k$, there is $h \in \mathscr{A}$ such that $\forall \alpha < \kappa \forall i \in 2 \left[\left|h \cap g^i_\alpha\right| = \omega\right]$. Now, it is easily argued, just as in Lemma 3.1.3, that $\forall \alpha < \kappa \left[|h \cap f_\alpha| = \omega\right]$. ⊣

It is natural to ask for which sets of cardinals $X \subset \mathfrak{c} + 1$ is it consistent to have $X = \{\operatorname{st}(\mathscr{A}) : \mathscr{A} \subset \omega^\omega \text{ is an a.d. family}\}$. We will provide a partial answer by showing that under MA(σ-centered), every cardinal $\kappa \le \mathfrak{c}$ occurs as the strongness of some a.d. family $\mathscr{A}$. In what follows, we will prove this for the case when $\kappa < \mathfrak{c}$ (Theorem 3.1.7). We will defer the proof of the case $\kappa = \mathfrak{c}$ to section 3.3, where we will prove something slightly more general (Corollary 3.3.9).

Lemma 3.1.5. *Let $\lambda < \kappa$ be cardinals. Let $\mathscr{A} \subset \omega^\omega$ be a family of functions and let $\langle f_\alpha : \alpha < \kappa\rangle \subset \omega^\omega$ be an a.d. family. Let $\langle g_\alpha : \alpha < \lambda\rangle \subset \omega^\omega$ be any family of functions avoiding $\mathscr{A}$. There is an $\alpha < \kappa$ such that $\langle g_\alpha : \alpha < \lambda\rangle$ avoids $\mathscr{A} \cup \{f_\alpha\}$.*

Proof. Suppose not. Then for each $\alpha < \kappa$ there is a $\beta < \lambda$ and finite subset $\{h_0, \dots, h_k\} \subset \mathscr{A}$ such that $g_\beta \subset^* h_0 \cup \dots \cup h_k \cup f_\alpha$. As $\lambda < \kappa$, it follows that there are distinct $\alpha_0 \neq \alpha_1 < \kappa$ such that for the same $\beta < \lambda$ there are finite sets $\{h_0, \dots, h_k\} \subset \mathscr{A}$ and $\{h^0, \dots, h^l\} \subset \mathscr{A}$ so that both $g_\beta \subset^* h_0 \cup \dots \cup h_k \cup f_{\alpha_0}$ and $g_\beta \subset^* h^0 \cup \dots \cup h^l \cup f_{\alpha_1}$ hold. By assumption, g_β avoids $\mathscr{A}$. Therefore, $p = g_\beta \setminus (h_0 \cup \dots \cup h_k \cup h^0 \cup \dots \cup h^l)$ is an infinite partial function. But now, it follows that $p \subset^* f_{\alpha_0}$ and $p \subset^* f_{\alpha_1}$, which is a contradiction because $\langle f_\alpha : \alpha < \kappa\rangle$ is an a.d. family. ⊣

Lemma 3.1.6. *Assume MA(σ–centered). Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family with $|\mathscr{A}| < \mathfrak{c}$. Let $\lambda < \kappa < \mathfrak{c}$ be cardinals. Let $\langle f_\alpha : \alpha < \kappa\rangle \subset \omega^\omega$ be an a.d. family of functions avoiding $\mathscr{A}$. Let $\langle g_\alpha : \alpha < \lambda\rangle \subset \omega^\omega$ be another family of functions also avoiding $\mathscr{A}$. There is $h \in \omega^\omega$ such that*

1. $\forall h' \in \mathscr{A}\,[|h \cap h'| < \omega]$

2. $\langle f_\alpha : \alpha < \kappa \rangle$ *avoids* $\mathscr{A} \cup \{h\}$

3. $\forall \beta < \lambda\,[|h \cap g_\beta| = \omega]$

4. $\exists \alpha < \kappa\,[|h \cap f_\alpha| < \omega]$.

Proof. By Lemma 3.1.5, there is $\alpha < \kappa$ such that $\langle g_\beta : \beta < \lambda \rangle$ avoids $\mathscr{B} = \mathscr{A} \cup \{f_\alpha\}$. Let us fix such an α. Let $Fn(\omega, \omega)$ denote the set of finite partial functions from ω to ω. Consider the poset $\mathbb{P} = \{\langle s, H \rangle : s \in Fn(\omega, \omega) \wedge H \in [\mathscr{B}]^{<\omega}\}$. We order $\mathbb{P}$ as follows: given $\langle s_0, H_0 \rangle$ and $\langle s_1, H_1 \rangle$ in $\mathbb{P}$, $\langle s_0, H_0 \rangle \leq \langle s_1, H_1 \rangle$ *iff* $s_0 \supset s_1 \;\wedge H_0 \supset H_1 \;\wedge \forall h \in H_1 \forall n \in \operatorname{dom}(s_0) \setminus \operatorname{dom}(s_1)\,[h(n) \neq s_0(n)]$. It is easily checked that $\mathbb{P}$ is σ-centered. If $G \subset \mathbb{P}$ is a filter on $\mathbb{P}$, then $h = \bigcup\{s : \exists H\,[\langle s, H \rangle \in G]\}$ is a function, which is a.d. from $\bigcup\{H : \exists s\,[\langle s, H \rangle \in G]\}$. To see that we can get a function h that satisfies the necessary requirements, we will check that certain sets are dense.

(0) To ensure $h \in \omega^\omega$. For each $n \in \omega$, set $D_n = \{\langle s, H \rangle \in \mathbb{P} : n \in \operatorname{dom}(s)\}$. We will check that D_n is dense. Fix $\langle s_0, H_0 \rangle \in \mathbb{P}$. If $n \in \operatorname{dom}(s_0)$, there is nothing to be done. Otherwise, consider $\{h'(n) : h' \in H_0\}$. This is a finite subset of ω. So we may choose $k \in \omega \setminus \{h'(n) : h' \in H_0\}$. Now, $\langle s_0 \cup \{\langle n, k \rangle\}, H_0 \rangle$ is an extension of $\langle s_0, H_0 \rangle$ in D_n.

(1) To ensure h satisfies requirements (1) and (4). It is enough to ensure that $\forall h' \in \mathscr{B}\, \exists \langle s, H \rangle \in G\,[h' \in H]$. But it is obvious that for each $h' \in \mathscr{B}$, $D_{h'} = \{\langle s, H \rangle \in \mathbb{P} : h' \in H\}$ is dense.

(2) To ensure h satisfies requirement (2). Let F be a finite subset of $\mathscr{A}$. Let $\gamma < \kappa$. Since f_γ avoids $\mathscr{A}$, $X_\gamma^F = \{n \in \omega : \forall h' \in F\,[f_\gamma(n) \neq h'(n)]\}$ is an infinite subset of ω. For each n, consider $D\,(F, \gamma, n) = \big\{\langle s, H \rangle \in \mathbb{P} : \exists m > n\,\big[m \in X_\gamma^F \wedge m \in \operatorname{dom}(s) \wedge f_\gamma(m) \neq s(m)\big]\big\}$. If G hits $D\,(F, \gamma, n)$ for all $n \in \omega$, then f_γ avoids $F \cup \{h\}$ because there will be infinitely

many $m \in X^F_\gamma$ so that $h(m) \neq f_\gamma(m)$. To see that $D(F, \gamma, n)$ is dense, fix $\langle s_0, H_0 \rangle \in \mathbb{P}$. Since X^F_γ is an infinite set, there is $m \in X^F_\gamma$, which is greater than n and outside $\operatorname{dom}(s_0)$. Now, we can choose $k \notin \{h'(m) : h' \in H_0\} \cup \{f_\gamma(m)\}$. It is clear that $\langle s_0 \cup \{\langle m, k \rangle\}, H_0 \rangle$ is as required.

(3) To ensure h satisfies requirement (3). Let $\beta < \lambda$. It is enough to make G intersect $D^\beta_n = \{\langle s, H \rangle \in \mathbb{P} : \exists m > n\, [m \in \operatorname{dom}(s) \wedge s(m) = g_\beta(m)]\}$ for all $n \in \omega$. To see that this set is dense, fix $\langle s_0, H_0 \rangle \in \mathbb{P}$. We know, by our choice of α, that g_β avoids $\mathscr{B}$. So there are infinitely many $m \in \omega$ such that $\forall h' \in H_0\, [h'(m) \neq g_\beta(m)]$. So we can choose such an m greater than n and outside of $\operatorname{dom}(s_0)$. By our choice of m, $\langle s_0 \cup \{\langle m, g_\beta(m) \rangle\}, H_0 \rangle$ extends $\langle s_0, H_0 \rangle$ and is as required.

Since $\lambda < \kappa < \mathfrak{c}$ and $|\mathscr{A}| < \mathfrak{c}$ and since MA(σ–centered) is assumed, we can find a filter G that intersects all the sets in $\{D_n : n \in \omega\} \cup \{D_{h'} : h' \in \mathscr{B}\} \cup \{D^\beta_n : \beta < \lambda \wedge n \in \omega\} \cup \{D(F, \gamma, n) : F \in [\mathscr{A}]^{<\omega} \wedge \gamma < \kappa \wedge n \in \omega\}$. Now, h, defined as above from G, will have the required properties. ⊣

Theorem 3.1.7. *Assume MA(σ–centered). Let $\kappa < \mathfrak{c}$ be a cardinal. There is an a.d. family $\mathscr{A} \subset \omega^\omega$ with $\operatorname{st}(\mathscr{A}) = \kappa$.*

Proof. Fix an a.d. family $\langle f_\alpha : \alpha < \kappa \rangle \subset \omega^\omega$ of size κ. We will construct an a.d. family $\mathscr{A} \subset \omega^\omega$ with $\operatorname{st}(\mathscr{A}) \geq \kappa$, while at the same time ensuring that $\langle f_\alpha : \alpha < \kappa \rangle$ avoids $\mathscr{A}$, and yet nothing in $\mathscr{A}$ has infinite intersection with all the f_α. Thus $\langle f_\alpha : \alpha < \kappa \rangle$ will witness that $\operatorname{st}(\mathscr{A}) = \kappa$.

$\mathscr{A}$ will be the union of an increasing sequence of a.d. families. Since MA(σ–centered) is assumed, $\mathfrak{c}^{<\kappa} = \mathfrak{c}$. So we can let $\langle \mathscr{G}_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate all subsets of ω^ω of size less than κ. We will construct a sequence $\langle \mathscr{A}_\alpha : \alpha < \mathfrak{c} \rangle$ so that:

1. $\mathscr{A}_\alpha \subset \omega^\omega$ is an a.d. family of size $\leq |\alpha|$

2. $\forall \alpha < \beta < \mathfrak{c}\, [\mathscr{A}_\alpha \subset \mathscr{A}_\beta]$

3. If $\mathscr{G}_\alpha$ avoids $\bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$, then $\exists h \in \mathscr{A}_\alpha \forall g \in \mathscr{G}_\alpha \, [|h \cap g| = \omega]$

4. $\langle f_\alpha : \alpha < \kappa \rangle$ avoids $\mathscr{A}_\alpha$

5. $\forall h \in \mathscr{A}_\alpha \exists \beta < \kappa \, [|h \cap f_\beta| < \omega]$.

Assume that $\langle \mathscr{A}_\beta : \beta < \alpha \rangle$ is already given to us. Let $\mathscr{B} = \bigcup \mathscr{A}_\beta$. If $\mathscr{G}_\alpha$ does not avoid $\mathscr{B}$, there is nothing to be done. In this case, we simply set $\mathscr{A}_\alpha = \mathscr{B}$. Let us assume from now on that $\mathscr{G}_\alpha$ avoids $\mathscr{B}$. Notice that by clause (4), $\langle f_\alpha : \alpha < \kappa \rangle$ avoids $\mathscr{B}$ as well. By clause (1), $\mathscr{B}$ is an a.d. family with $|\mathscr{B}| < \mathfrak{c}$. Let $\lambda = |\mathscr{G}_\alpha|$. Note that we have $\lambda < \kappa < \mathfrak{c}$. Now, we can apply lemma 3.1.6 with $\mathscr{B}$ as $\mathscr{A}$ and $\mathscr{G}_\alpha$ as $\langle g_\alpha : \alpha < \lambda \rangle$ to find $h \in \omega^\omega$ so that

(a) h is a.d. from $\mathscr{B}$

(b) $\langle f_\alpha : \alpha < \kappa \rangle$ avoids $\mathscr{B} \cup h$

(c) $\forall g \in \mathscr{G}_\alpha \, [|h \cap g| = \omega]$

(d) $\exists \alpha < \kappa \, [|h \cap f_\alpha| < \omega]$.

Now, we can define $\mathscr{A}_\alpha = \mathscr{B} \cup \{h\}$. It is clear that $\mathscr{A}_\alpha$ is what is required. $\dashv$

The original motivation for the above result came from the following considerations. Under CH, all MAD families have size $\aleph_1$. Hence any strongly MAD family is automatically very MAD. Given some such consequence of CH, it is natural to ask whether this consequence also obtains under MA. So we originally wanted to know if under MA, all strongly MAD families are also very MAD. The above result shows that this fails badly.

Corollary 3.1.8. *Assume $MA + \neg\, CH$. There is a strongly MAD family that is not very MAD.*

$\dashv$

We end this section with a conjecture. We do not know for which cardinals κ is there is an a.d. family $\mathscr{A} \subset \omega^\omega$ with $\mathrm{st}(\mathscr{A}) = \kappa$ just on the basis of ZFC alone. In view of Lemma 3.1.3, the following conjecture is a natural generalization of our result that Van Douwen MAD families exist on the basis of ZFC alone.

Conjecture 3.1.9. *For every $n \in \omega$, there is an a.d. family $\mathscr{A} \subset \omega^\omega$ with $\mathrm{st}(\mathscr{A}) \geq n$.*

3.2 A Strongly MAD Family From $\mathfrak{b} = \mathfrak{c}$.

In this section we will construct a strongly MAD family from $\mathfrak{b} = \mathfrak{c}$. Kastermans [19] pointed out that the standard construction of a strongly MAD family from MA(σ–centered) actually yields a very MAD family. He asked if there is a different construction that distinguishes between strongly and very MAD families. This section is intended to address his question. The construction of a strongly MAD family given here cannot be used to build a very MAD family. This is because $\mathfrak{b} = \mathfrak{c}$ holds in the Laver model, where, as we will see in section 3.3, there are no very MAD families. The question of whether strongly MAD families exist on the basis of ZFC alone remains open.

Hrušák [15], Kurilić [23] and Brendle and Yatabe [10] construct a Cohen indestructible MAD family of sets from $\mathfrak{b} = \mathfrak{c}$. Our construction was inspired by theirs, although our presentation is different. We will inductively construct a strongly MAD family in $\mathfrak{c}$ steps. At each step we will deal with a given countable family of functions. We will deal with this given collection by first forming a (ω, κ) gap consisting of infinite partial functions. We will then use $\mathfrak{b} = \mathfrak{c}$ to separate this gap by an infinite partial function.

Lemma 3.2.1. *Assume $\mathfrak{b} = \mathfrak{c}$. Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family with $|\mathscr{A}| < \mathfrak{c}$. Suppose $\{g_n : n \in \omega\} \subset \omega^\omega$ is a collection of functions avoiding $\mathscr{A}$. There is $h \in \omega^\omega$ so that*

1. *$\forall h' \in \mathscr{A}\, [|h \cap h'| < \omega]$*

2. $\forall n \in \omega\,[|h \cap g_n| = \omega]$.

Proof. Firstly, observe that $\mathfrak{b} = \mathfrak{c}$ implies both $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{a}_\mathfrak{e} = \mathfrak{c}$. Now, for $n \in \omega$ consider $\mathscr{A} \cap g_n = \{h \cap g_n : h \in \mathscr{A} \wedge |h \cap g_n| = \omega\}$. This is an a.d. family on g_n. Since g_n avoids $\mathscr{A}$ and since $|\mathscr{A}| < \mathfrak{c}$, it cannot be a MAD family (either finite or infinite) on g_n. So we may find an infinite partial function $p_n \subset g_n$ which is a.d. from everything in $\mathscr{A}$. By refining their domains if necessary, we may assume that $\forall n < m < \omega\,[\operatorname{dom}(p_n) \cap \operatorname{dom}(p_m) = 0]$.

Now, $(\{p_n : n \in \omega\}, \mathscr{A})$ is the gap we would like to separate using an infinite partial function. We will use the assumption $\mathfrak{b} = \mathfrak{c}$ to do this. Let $\lambda = |\mathscr{A}|$ and put $\mathscr{A} = \{h_\alpha : \alpha < \lambda\}$. Remember that $\lambda < \mathfrak{c}$. For each $\alpha < \lambda$, define a function $F_\alpha \in \omega^\omega$ as follows. For each $n \in \omega$, $\{k \in \operatorname{dom}(p_n) : p_n(k) = h_\alpha(k)\}$ is finite. So, we can define $F_\alpha(n) = \max\{k \in \operatorname{dom}(p_n) : p_n(k) = h_\alpha(k)\}$. Since we are assuming $\mathfrak{b} = \mathfrak{c}$, the family $\{F_\alpha : \alpha < \lambda\}$ is bounded. Choose a function $F \in \omega^\omega$ so that $\forall \alpha < \lambda\,[F_\alpha <^* F]$. Define $p = \bigcup (p_n \setminus (p_n \upharpoonright F(n)))$. Clearly, p is an infinite partial function, and for all $n \in \omega$, $|p \cap g_n| = \omega$. We will check that $\forall \alpha < \lambda\,[|p \cap h_\alpha| < \omega]$. Fix $\alpha < \lambda$. Suppose $k \in \operatorname{dom}(p)$ and $p(k) = h_\alpha(k)$. By our choice of the p_n, it follows that there is a unique n such that $k \in \operatorname{dom}(p_n)$. Thus, we have that $p_n(k) = h_\alpha(k)$, and so, $k \leq F_\alpha(n)$. But since $k \in \operatorname{dom}(p)$, it follows that $k \geq F(n)$, whence $F(n) \leq F_\alpha(n)$. Thus $k \in \bigcup\{F_\alpha(n) + 1 : F(n) \leq F_\alpha(n)\}$, which is a finite set. So we conclude that $p \cap h_\alpha$ is finite.

Now, we are almost done. We just need to extend p into a total function. We will use $\mathfrak{a}_\mathfrak{e} = \mathfrak{c}$ to do this. Let $X = \operatorname{dom}(p)$ and $Y = \omega \setminus X$. $\mathscr{A}$ is an a.d. family in ω^ω with $|\mathscr{A}| < \mathfrak{c}$. So it is not maximal. Let $h_0 \in \omega^\omega$ be a.d. from $\mathscr{A}$. Clearly, $h = p \cup h_0 \upharpoonright Y$ is as needed. $\dashv$

Theorem 3.2.2. *Assume $\mathfrak{b} = \mathfrak{c}$. There is a strongly MAD family of size $\mathfrak{c}$.*

Proof. We will build the strongly MAD family, $\mathscr{A}$, in $\mathfrak{c}$ steps. Since $\mathfrak{c}^\omega = \mathfrak{c}$, we can let $\{\mathscr{G}_\alpha : \alpha < \mathfrak{c}\}$ enumerate all the countable subsets of ω^ω. We will build $\mathscr{A}$ as the union of an increasing sequence of a.d. families. We will build a sequence $\langle \mathscr{A}_\alpha : \alpha < \mathfrak{c}\rangle$ such that

1. $\mathscr{A}_\alpha \subset \omega^\omega$ is an a.d. family with $|\mathscr{A}| \leq |\alpha|$

2. if $\alpha < \beta < \mathfrak{c}$, then $\mathscr{A}_\alpha \subset \mathscr{A}_\beta$

3. if $\mathscr{G}_\alpha$ avoids $\bigcup \{\mathscr{A}_\beta : \beta < \alpha\}$, then $\exists h \in \mathscr{A}_\alpha \forall g \in \mathscr{G}_\alpha \left[|h \cap g| = \omega\right]$.

Assume that the sequence $\langle \mathscr{A}_\beta : \beta < \alpha \rangle$ has already been built. Set $\mathscr{B} = \bigcup \mathscr{A}_\beta$. $\mathscr{B} \subset \omega^\omega$ is an a.d. family with $|\mathscr{B}| < \mathfrak{c}$. If $\mathscr{G}_\alpha$ does not avoid $\mathscr{B}$, then we can simply set $\mathscr{A}_\alpha = \mathscr{B}$. So we assume that $\mathscr{G}_\alpha$ avoids $\mathscr{B}$. Now, we may apply Lemma 3.2.1 with $\mathscr{B}$ as $\mathscr{A}$ and $\mathscr{G}_\alpha$ as $\{g_n : n \in \omega\}$ to find $h \in \omega^\omega$ such that h is a.d. from $\mathscr{B}$ and $\forall g \in \mathscr{G}_\alpha \left[|h \cap g| = \omega\right]$. Now, it is clear that $\mathscr{A}_\alpha = \mathscr{B} \cup \{h\}$ is as required.

We remark that even though we have not explicitly tried to ensure that $|\mathscr{A}| = \mathfrak{c}$, it is true because $\mathfrak{b} = \mathfrak{c}$ implies $\mathfrak{a}_\mathfrak{e} = \mathfrak{c}$. ⊣

Corollary 3.2.3. *There are strongly MAD families in the Laver and Hechler models.*

⊣

As mentioned above, it is unknown if strongly MAD families always exist. We conjecture below that this is not the case. We will prove a partial result in this direction in Section 3.6, where we will show that it is consistent to have no "large" strongly MAD families (Theorem 3.6.1).

Conjecture 3.2.4. *It is consistent to have no strongly MAD families.*

3.3 Brendle's Conjecture: Consistency of no Very MAD Families

In this section we will show that if $\mathrm{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{e}$, then there are no very MAD families. This was conjectured by Brendle in an email to Kastermans. Kastermans showed that very MAD families exist under MA and asked if their existence can be proved in ZFC. Our result implies that there are no very MAD families in the Laver, Random or Blass–Shelah models. For the case of the Laver and Random models, this was already known to Brendle. Brendle also pointed out in the same email that his conjecture would imply that there are no very MAD families in a typical Template model. Our proof will use the following characterization of $\mathrm{cov}(\mathcal{M})$.

Theorem 3.3.1 (see [4] or [5])**.** *The following are equivalent for a cardinal $\kappa \geq \omega$:*

1. *The reals cannot be covered by κ meager sets.*

2. *If $\{f_\alpha : \alpha < \kappa\} \subset \omega^\omega$ is a collection of κ functions, there is $h \in \omega^\omega$ such that $\forall \alpha < \kappa\, [|f_\alpha \cap h| = \omega]$.*

$\dashv$

Now, this characterization implies that there is a family $\{f_\alpha : \alpha < \mathrm{cov}(\mathcal{M})\} \subset \omega^\omega$ such that there is no $h \in \omega^\omega$ for which $|h \cap f_\alpha| = \omega$ holds for all $\alpha < \mathrm{cov}(\mathcal{M})$. Therefore, if this $\{f_\alpha : \alpha < \mathrm{cov}(\mathcal{M})\}$ avoids an a.d. family $\mathscr{A} \subset \omega^\omega$, and if $\mathrm{cov}(\mathcal{M}) < |\mathscr{A}|$, then $\mathscr{A}$ cannot be a very MAD family. However, given an arbitrary very MAD family $\mathscr{A}$, there is no reason to expect the family $\{f_\alpha : \alpha < \mathrm{cov}(\mathcal{M})\}$ to avoid it. We will deal with this by showing that in the above Theorem one can replace functions with objects that are a bit " fatter", namely slaloms. This will provide a new characterization of the cardinal $\mathrm{cov}(\mathcal{M})$. Their "fatness" will ensure that the slaloms avoid any a.d. family.

Definition 3.3.2. *A function $S : \omega \to [\omega]^{<\omega}$ is called a slalom if $\forall n \in \omega\, [|S(n)| \leq 2^n]$. We say that S is a wide slalom if $\forall n \in \omega\, [|S(n)| = 2^n]$.*

Theorem 3.3.3. *Let κ be an infinite cardinal. The following are equivalent.*

1. *The reals cannot be covered by κ meager sets.*

2. *If $\{S_\alpha : \alpha < \kappa\}$ is a collection of κ wide slaloms, there is $h \in \omega^\omega$ such that $\forall \alpha < \kappa\, \exists^\infty n \in \omega\, [h(n) \in S_\alpha(n)]$.*

Proof. $\neg(2) \implies \neg(1)$. Fix a family of wide slaloms $\{S_\alpha : \alpha < \kappa\}$ for which the consequent of (2) fails. For each $\alpha < \kappa$ set $E_\alpha = \{h \in \omega^\omega : \forall^\infty n \in \omega\, [h(n) \notin S_\alpha(n)]\}$. It is clear that each E_α is meager. Also by assumption, we have that $\omega^\omega = \bigcup E_\alpha$. Thus (1) is false.

(2) $\implies$ (1). Assume (2). We will show that clause (2) of Theorem 3.3.1 holds. Fix a family $\{f_\alpha : \alpha < \kappa\} \subset \omega^\omega$. Since we may code functions from ω to $[\omega \times \omega]^{<\omega}$ by slaloms, our assumption entails the following:

$$\begin{aligned}&\text{For every family } \{H_\alpha : \alpha < \kappa\} \text{ of functions from } \omega \text{ to } [\omega \times \omega]^{<\omega}\\ &\text{such that } \forall n \in \omega\, [\, |H_\alpha(n)| = 2^n]\,, \text{ there is } g \in (\omega \times \omega)^\omega \text{ so that}\\ &\forall \alpha < \kappa\, \exists^\infty n \in \omega\, [g(n) \in H_\alpha(n)]\,.\end{aligned} \tag{$*$}$$

Now, for each $n \in \omega$ set $l_n = 2^n - 1$ and $I_n = [l_n, l_{n+1})$. Thus $\langle I_n : n \in \omega\rangle$ is an interval partition of ω with $|I_n| = 2^n$. Let us define a family $\{H_\alpha : \alpha < \kappa\}$ of functions from ω to $[\omega \times \omega]^{<\omega}$ by stipulating that for all $n \in \omega$, $H_\alpha(n) = f_\alpha \restriction I_n$. Since $|I_n| = 2^n$, $\forall n \in \omega\, [\, |H_\alpha(n)| = 2^n]$. Therefore, by $(*)$ above, there is a $g \in (\omega \times \omega)^\omega$ so that

$$\forall \alpha < \kappa\, \exists^\infty n \in \omega\, [g(n) \in H_\alpha(n)]\,. \tag{$**$}$$

We may assume WLOG that $\forall n \in \omega\, [g(n) \in I_n \times \omega]$ because we can modify g to make this true without affecting $(**)$ above. Now, set $p = g''\omega$. It is clear that given our assumption about g, p is an infinite partial function from ω to ω. Now, let h be a function in ω^ω which extends p (arbitrarily). We will check that h is the function we are looking for.

Indeed, fix $\alpha < \kappa$. We must show that $|h \cap f_\alpha| = \omega$. We will prove that $|p \cap f_\alpha| = \omega$. For $n \in \omega$, let us use $\langle i_n, j_n\rangle$ to denote $g(n)$. Note that by our assumption on g, $\forall n \in \omega\, [i_n \in I_n]$. Also observe that by the definition of p, $\mathrm{dom}\,(p) = \{i_n : n \in \omega\}$ and $\forall n \in \omega\, [p(i_n) = j_n]$. By $(**)$ above, the set $X = \{n \in \omega : \langle i_n, j_n\rangle \in H_\alpha(n)\}$ is infinite. By the definition of H_α, it follows that $\forall n \in X\, [f_\alpha(i_n) = j_n = p(i_n)]$. Since the I_n are disjoint, $\{i_n : n \in X\}$ is infinite, and so $|f_\alpha \cap p| = \omega$. ⊣

Lemma 3.3.4. *Let $\mathscr{A} \subset \omega^\omega$ be a Van Douwen MAD family. Let $\lambda < \mathrm{st}(\mathscr{A})$ be a cardinal and let $\{S_\alpha : \alpha < \lambda\}$ be a family of wide slaloms. There is $h \in \mathscr{A}$ such that $\forall \alpha < \lambda\, \exists^\infty n \in \omega\, [h(n) \in S_\alpha(n)]$.*

Proof. For each $\alpha < \lambda$, let $X_\alpha = \bigcup(\{n\} \times S_\alpha(n))$. Observe that for any function $f \in \omega^\omega$, $\exists^\infty n \in \omega\, [f(n) \in S_\alpha(n)] \iff |X_\alpha \cap f| = \omega$. Hence, it suffices to produce $h \in \mathscr{A}$ such that $\forall \alpha < \lambda\, [|h \cap X_\alpha| = \omega]$. For each $\alpha < \lambda$, we will produce a total function $f^\alpha \subset X_\alpha$ avoiding $\mathscr{A}$. We will first argue that each X_α has infinite intersection with infinitely many members of $\mathscr{A}$. Indeed, given any finite collection of functions $\{f_0, \dots, f_n\} \subset \omega^\omega$, there is an infinite partial function $p \subset X_\alpha$ which is a.d. from $f_0, \dots, f_n$. This is because S_α is a wide slalom. But we are assuming that $\mathscr{A}$ is Van Douwen MAD; so there are no infinite partial functions a.d. from $\mathscr{A}$. It follows that for each $\alpha < \lambda$ there is an infinite collection $\{h_i^\alpha : i \in \omega\} \subset \mathscr{A}$ such that $\forall i \in \omega\, [|h_i^\alpha \cap X_\alpha| = \omega]$. For each $i \in \omega$, set $p_i^\alpha = h_i^\alpha \cap X_\alpha$. p_i^α is an infinite partial function contained in X_α. It is possible to choose a collection of infinite partial functions $\{g_i^\alpha : i \in \omega\}$ such that $\forall i \in \omega\, [g_i^\alpha \subset p_i^\alpha]$ and $\forall\ i\ <\ j\ <\ \omega \left[\mathrm{dom}\,(g_i^\alpha) \cap \mathrm{dom}\left(g_j^\alpha\right) = 0\right]$. Now, we can find a function $f^\alpha \in \omega^\omega$ with $\bigcup_{i \in \omega} g_i^\alpha \subset f^\alpha \subset X_\alpha$. Since f^α has infinite intersection with infinitely many things in $\mathscr{A}$, it avoids $\mathscr{A}$. Now, $\lambda < \mathrm{st}(\mathscr{A})$. So there is $h \in \mathscr{A}$ such that $\forall\ \alpha\ <\ \lambda\, [|h \cap f^\alpha| = \omega]$. This h is the function we are looking for. $\dashv$

Theorem 3.3.5 (Brendle's Conjecture)**.** *If $\mathscr{A}$ is a very MAD family, then $|\mathscr{A}| \leq \mathrm{cov}(\mathcal{M})$. In particular, if $\mathrm{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{e}$, then there are no very MAD families.*

Proof. Suppose, for a contradiction, that $\mathrm{cov}(\mathcal{M}) < |\mathscr{A}|$. By Theorem 3.3.3, there is a family $\{S_\alpha : \alpha < \mathrm{cov}(\mathcal{M})\}$ of wide slaloms such that for every $h \in \omega^\omega$ there is $\alpha < \mathrm{cov}(\mathcal{M})$ such that $\forall^\infty n \in \omega\, [h(n) \notin S_\alpha(n)]$. But now, since very MAD families are Van Douwen MAD, and since $\mathrm{cov}(\mathcal{M}) < |\mathscr{A}| \leq \mathrm{st}(\mathscr{A})$, we can apply Lemma 3.3.4 to get a function $h \in \mathscr{A}$ that contradicts this. $\dashv$

Corollary 3.3.6. *There are no very MAD families in the Laver, Random or Blass-Shelah models.*

Proof. It is well-known (see [5]) that each of these forcings, as well as their respective iterations, do not add Cohen reals. Thus in all of these models $\mathrm{cov}(\mathcal{M}) = \aleph_1$. On the other hand, each of these forcings makes the ground model meager. Hence in all three of these models $\mathrm{non}(\mathcal{M})$, and hence $\mathfrak{a}_\mathfrak{e}$, is $\aleph_2$. $\dashv$

Remark 3.3.7. *Let $\mathfrak{a}_\mathfrak{v}$ be the least size of a Van Douwen MAD family. Since very MAD families are Van Douwen MAD, Theorem 3.3.5 implies that there are no very MAD families as long as $\mathrm{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{v}$. It is concievable that $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}_\mathfrak{v}$ is consistent, but no models of this are known.*

In Section 3.1 we promised to give a proof of Theorem 3.1.7 for the case when $\kappa = \mathfrak{c}$ in this section. We will end this section by fulfilling this promise.

Corollary 3.3.8. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family.* $\mathrm{st}(\mathscr{A}) \leq \mathrm{cov}(\mathcal{M})$.

Proof. Suppose for a contradiction that $\mathrm{st}(\mathscr{A}) > \mathrm{cov}(\mathcal{M})$. By Lemma 3.1.3, $\mathscr{A}$ is Van Douwen MAD. But now, we can argue just as in Theorem 3.3.5 to get a contradiction using Lemma 3.3.4. $\dashv$

Corollary 3.3.9. *Assume MA(σ–centered). There is an a.d. family $\mathscr{A} \subset \omega^\omega$ with* $\mathrm{st}(\mathscr{A}) = \mathfrak{c}$.

Proof. Kastermans [19] showed that there is a very MAD family under MA(σ–centered). Let $\mathscr{A}$ be a very MAD family. Clearly, $\mathfrak{c} = |\mathscr{A}| \leq \mathrm{st}(\mathscr{A})$. On the other hand, by Corollary 3.3.8, $\mathrm{st}(\mathscr{A}) \leq \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, whence $\mathrm{st}(\mathscr{A}) = \mathfrak{c}$. $\dashv$

3.4 Indestructibility Properties of Strongly MAD Families

In this section we will study the effect of forcing on strongly MAD families. In particular, we will be interested in showing that certain posets preserve strongly MAD families.

Definition 3.4.1. *Let $\mathbb{P}$ be a notion of forcing and let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. We say that $\mathscr{A}$ is $\mathbb{P}$-indestructible if $\Vdash_{\mathbb{P}} \mathscr{A}$ is MAD. We say that $\mathscr{A}$ is strongly $\mathbb{P}$-indestructible if $\Vdash_{\mathbb{P}} \mathscr{A}$ is strongly MAD.*

Brendle and Yatabe [10] have studied $\mathbb{P}$-indestructibility of MAD families of subsets of ω for various posets $\mathbb{P}$. The focus of their work was to provide combinatorial characterizations of the property of being a $\mathbb{P}$-indestructible MAD family of sets for some well known posets $\mathbb{P}$. Here our focus is instead to find those posets $\mathbb{P}$ for which strongly MAD families of functions are strongly $\mathbb{P}$-indestructible.

If $\mathbb{P}$ is a poset which turns the ground model reals into a meager set, then it is clear that no MAD family $\mathscr{A} \subset \omega^\omega$ can be $\mathbb{P}$-indestructible. This is because any such poset adds an element of ω^ω which is eventually different from all the elements of ω^ω in the ground model (see [5]). In this section, we will show that a strong converse to this observation holds for a certain class of posets. That is, we will show that if $\mathscr{A} \subset \omega^\omega$ is strongly MAD, then $\mathscr{A}$ is strongly $\mathbb{P}$-indestructible for a wide range of posets $\mathbb{P}$ that do not make the ground model meager.

We will assume familiarity with the basic theory of proper forcing. The reader may consult Abraham [1], Goldstern [12] or Shelah [31] for an introduction.

We are ultimately interested not only in treating indestructibility for single step forcing extensions, but also for countable support iterations. We are unable to show that every strongly MAD $\mathscr{A}$ is strongly indestructible for any countable support iteration of posets for which $\mathscr{A}$ is strongly indestructible. However, we are able to prove the preservation of a slightly stronger property, which we introduce next.

Definition 3.4.2. *Let $\mathscr{A} \subset \omega^\omega$ be an a.d. family. Let $M \prec H(\theta)$ be countable with $\mathscr{A} \in M$. We say that $h \in \mathscr{A}$ covers M with respect to $\mathscr{A}$ if whenever $f \in M$ is an infinite partial function avoiding $\mathscr{A}$, $|h \cap f| = \omega$.*

Definition 3.4.3. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family and let $\mathbb{P}$ be a poset. Let $M \prec H(\theta)$ be countable with $\mathscr{A}, \mathbb{P} \in M$. We will say that $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ holds if whenever p is a condition in $\mathbb{P} \cap M$ and $h \in \mathscr{A}$ covers M with respect to $\mathscr{A}$, there is $q \leq p$ which is $(M, \mathbb{P})$ generic such that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. We say that $\mathbb{P}$ strongly preserves $\mathscr{A}$ if for every $M \prec H(\theta)$, M countable, with $\mathscr{A}, \mathbb{P} \in M$, $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ holds.*

Remark 3.4.4. *Notice that if $\mathscr{A} \subset \omega^\omega$ is a strongly MAD family and if $\mathbb{P}$ is a poset which strongly preserves $\mathscr{A}$, then $\mathbb{P}$ is proper.*

Our definition of strongly preserving requires $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ to hold for *all* elementary submodels containing $\mathscr{A}$ and $\mathbb{P}$. But as is usual in the theory of proper forcing, it is sufficient if this is true for a club of such elementary submodels. We prove this next, and in what follows, we will use this fact without further comment.

Lemma 3.4.5. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family and let $\mathbb{P}$ be a poset. If $\{M \prec H(\theta) : |M| = \omega \wedge \mathscr{A}, \mathbb{P} \in M \wedge \mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ holds$\}$ contains a club in $[H(\theta)]^\omega$, then $\mathbb{P}$ strongly preserves $\mathscr{A}$.*

Proof. Arguments of this sort are standard in the theory of properness; so we merely outline the steps. We must find a set $X \in H(\theta)$ that "captures" all the information necessary for deciding the truth of $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$, for any M with $\mathscr{A}, \mathbb{P} \in M$. We define two sets as follows. Let $\mathcal{F}_{\mathscr{A}} = \{f : f$ is an infinite partial function avoiding $\mathscr{A}\}$ and let $A_{\mathbb{P}} = \{\mathring{x} \in \mathbf{V}^{\mathbb{P}} : \mathring{x}$ is a nice $\mathbb{P}$ name for a subset of $\omega \times \omega\}$. Put $X = \mathbb{P} \cup \mathcal{P}(\mathbb{P}) \cup A_{\mathbb{P}} \cup \mathscr{A} \cup \mathcal{F}_{\mathscr{A}}$. We will argue that this X does the job. Notice that if $\mathcal{M} \prec H(\theta)$ is countable with $\mathscr{A}, \mathbb{P} \in M$, then X, and hence $[X]^\omega$ are elements of M. Now, given $a \in [X]^\omega$ and $q \in \mathbb{P}$, say that q is $(a, \mathbb{P})$

generic if whenever $D \subset \mathbb{P}$ is a dense open set in a, $q \Vdash a \cap D \cap \mathring{G} \neq 0$. Similarly, say that $h \in \mathscr{A}$ covers a with respect to $\mathscr{A}$ if $|h \cap f| = \omega$ for all $f \in a \cap \mathcal{F}_{\mathscr{A}}$. Finally, say that $a \in [X]^{\omega}$ is good if whenever $p \in \mathbb{P} \cap a$ and $h \in \mathscr{A}$ covers a with respect to $\mathscr{A}$, there is $q \leq p$ which is $(a, \mathbb{P})$ generic such that $q \Vdash h$ covers $a[\mathring{G}]$ with respect to $\mathscr{A}$. It is easy to see that if $M \prec H(\theta)$ is countable with $\mathscr{A}, \mathbb{P} \in M$ and if $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ holds, then $M \cap X$ is good. Thus our assumption implies that $C = \{a \in [X]^{\omega} : a \text{ is good}\}$ contains a club in $[X]^{\omega}$. Now, fix a countable $M \prec H(\theta)$ with $\mathscr{A}, \mathbb{P} \in M$. We must show that $\mathbf{R}(\mathscr{A}, \mathbb{P}, M)$ holds. Notice that $C \in M$ and since C contains a club, $X \cap M \in C$. Therefore, $X \cap M$ is good. Now, fix $p \in \mathbb{P} \cap M$ and let $h \in \mathscr{A}$ cover M with respect to $\mathscr{A}$. Obviously, $p \in \mathbb{P} \cap X \cap M$ and h covers $X \cap M$ with respect to $\mathscr{A}$. So, we can find $q \leq p$ which is $(X \cap M, \mathbb{P})$ generic such that $q \Vdash h$ covers $(X \cap M)[\mathring{G}]$ with respect to $\mathscr{A}$. It is easily seen that q is in fact $(M, \mathbb{P})$ generic. We will argue that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. Indeed, let G be a $(\mathbf{V}, \mathbb{P})$ generic filter with $q \in G$ and suppose $f \in M[G]$ is an infinite partial function avoiding $\mathscr{A}$. By elementarity of M, there is $\mathring{x} \in A_{\mathbb{P}} \cap M$ such that $\mathring{x}[G] = f$. But then, $\mathring{x} \in X \cap M$, and so $f \in (X \cap M)[G]$. Therefore, $|h \cap f| = \omega$, and we are done. $\dashv$

Lemma 3.4.6. *Let $\mathscr{A} \subset \omega^{\omega}$ be a strongly MAD family and let $\mathbb{P}$ be a poset that strongly preserves $\mathscr{A}$. Then $\mathscr{A}$ is strongly $\mathbb{P}$-indestructible.*

Proof. Firstly, note that if $\mathscr{A}$ is strongly MAD and M is a countable elementary submodel, then, by Lemma 3.1.4, there is $h \in \mathscr{A}$ which covers M with respect to $\mathscr{A}$. Suppose for a contradiction that $\mathscr{A}$ is not strongly $\mathbb{P}$-indestructible. Fix $M \prec H(\theta)$ with $|M| = \omega$ and $\mathbb{P}, \mathscr{A} \in M$. Now, by our assumption, we can find a set of $\mathbb{P}$-names $\left\{\mathring{f}_i : i \in \omega\right\} \in M$ and $p \in \mathbb{P} \cap M$ such that

1. $\forall i \in \omega \left[p \Vdash \mathring{f}_i \in \omega^{\omega} \wedge \mathring{f}_i \text{ avoids } \mathscr{A}\right]$

2. $\forall h \in \mathscr{A} \left[p \Vdash \exists i \in \omega \left[\left|h \cap \mathring{f}_i\right| < \omega\right]\right]$.

Now, fix $h \in \mathscr{A}$ which covers M with respect to $\mathscr{A}$. By assumption, we can choose $q \leq p$ such that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. By elementarity, $\left\{\mathring{f}_i : i \in \omega\right\} \subset M$. So, for each $i \in \omega$, $q \Vdash \mathring{f}_i \in \omega^\omega \cap M[\mathring{G}] \wedge \mathring{f}_i$ avoids $\mathscr{A}$. But then, for each $i \in \omega$, $q \Vdash \left|h \cap \mathring{f}_i\right| = \omega$, contradicting (2) above. $\dashv$

Our aim in the rest of this section will be to show that a large class of posets not turning the ground model reals into a meager set strongly preserve all strongly MAD families. As a warm up, we will first show that the Cohen poset strongly preserves all strongly MAD families. The following lemma will play an important role in all our proofs of strong preservation. It allows us to transfer the property of avoiding from an infinite partial function in some generic extension to certain infinite partial functions in the ground model.

Lemma 3.4.7. *Let $\mathscr{A} \subset \omega^\omega$ be an infinite a.d. family and let $\mathbb{P}$ be any poset. Suppose $\mathring{f}$ is a $\mathbb{P}$ name such that $\Vdash \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. There is a countable set of $\mathbb{P}$ names $\left\{\mathring{f}_i : i \in \omega\right\}$ such that*

1. *$\Vdash \mathring{f}_i \subset \mathring{f}$ is an infinite partial function*
2. *$\Vdash \forall g \subset \omega \times \omega \left[\text{if } \forall i \in \omega \left[\left|g \cap \mathring{f}_i\right| = \omega\right], \text{ then } g \text{ avoids } \mathscr{A}\right]$.*

Proof. Let G be any $(\mathbf{V}, \mathbb{P})$ generic filter. We will work inside $\mathbf{V}[G]$. By assumption, $\mathring{f}[G]$ is an infinite partial function avoiding $\mathscr{A}$. We will find a countable set $\{f_i : i \in \omega\}$ of infinite partial sub–functions of $\mathring{f}[G]$ such that any $g \subset \omega \times \omega$ having infinite intersection with all the f_i avoids $\mathscr{A}$. Consider $\mathscr{A} \cap \mathring{f}[G] = \{h \cap \mathring{f}[G] : h \in \mathscr{A} \wedge \left|h \cap \mathring{f}[G]\right| = \omega\}$. This is an a.d. family on $\mathring{f}[G]$. The proof will break into two cases depending on whether $\mathscr{A} \cap \mathring{f}[G]$ is finite or infinite.

First, consider the case when $\mathscr{A} \cap \mathring{f}[G]$ is finite. Since $\mathring{f}[G]$ avoids $\mathscr{A}$, we can find an infinite partial function $f_0 \subset \mathring{f}[G]$ that is a.d. from $\mathscr{A}$. Now, for each $i \in \omega$, we can simply set f_i equal to f_0. We check that this will do. Indeed, suppose $g \subset \omega \times \omega$ has infinite intersection with f_0.

If g did not avoid $\mathscr{A}$, then since f_0 is a.d. from $\mathscr{A}$, f_0 would also be a.d. from g. Therefore, g avoids $\mathscr{A}$.

Next, suppose that $\mathscr{A} \cap \mathring{f}[G]$ is infinite. Choose an infinite set $\{h_i : i \in \omega\} \subset \mathscr{A}$ such that $\forall i \in \omega \left[\left|h_i \cap \mathring{f}[G]\right| = \omega\right]$. Now, for each $i \in \omega$ set f_i equal to $h_i \cap \mathring{f}[G]$. Thus f_i is an infinite partial sub–function of $\mathring{f}[G]$. Now, suppose $g \subset \omega \times \omega$ has infinite intersection with all the f_i. Clearly then, $|g \cap h_i| = \omega$, for all $i \in \omega$. Since g has infinite intersection with infinitely many members of $\mathscr{A}$, it avoids $\mathscr{A}$.

Now, back in the ground model $\mathbf{V}$, since G was an arbitrary $(\mathbf{V}, \mathbb{P})$ generic filter, we can use the maximal principle to find a countable set of names $\left\{\mathring{f}_i : i \in \omega\right\}$ which are forced to have the same properties as $\{f_i : i \in \omega\}$ defined above. $\dashv$

Lemma 3.4.8. *Let $\mathbb{P} = Fn(\omega, 2)$. Let $\mathring{f}$ be a name and suppose that $\Vdash \mathring{f}$ is an infinite partial function. Suppose that $\left\{\mathring{f}_i : i \in \omega\right\}$ is a set of names so that for each $i \in \omega$, $\Vdash \mathring{f}_i \subset \mathring{f}$ is an infinite partial function. Let $p \in \mathbb{P}$. Then there is an infinite partial function g such that*

1. $\forall i \in \omega \left[p \Vdash \left|g \cap \mathring{f}_i\right| = \omega\right]$
2. $\forall n \in \operatorname{dom}(g)\, \exists q \leq p \left[q \Vdash n \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(n) = g(n)\right]$.

Proof. Let $\{q_j : j \in \omega\}$ enumerate $\{q \in \mathbb{P} : q \leq p\}$. We will build g by induction as the union of an increasing sequence of finite partial functions g_j. We will build a sequence $\langle g_j : -1 \leq j < \omega\rangle$ such that for each $j \geq 0$

(a) $g_{-1} = 0$ and $g_{j-1} \subset g_j$ is a finite partial function

(b) $\exists q \leq q_j \forall i \leq j \exists k^i_j > j \exists m^i_j \in \omega \left[q \Vdash k^i_j \in \operatorname{dom}(\mathring{f}_i) \wedge \mathring{f}_i(k^i_j) = m^i_j\right]$

(c) $g_j = g_{j-1} \cup \left\{\langle k^i_j, m^i_j\rangle : i \leq j\right\}$.

We will first argue that $g = \bigcup g_j$ will satisfy requirements (1) and (2) above. To see that (1) holds, suppose for a contradiction that for some $i \in \omega$, there is a $p_0 \leq p$ and $k \in \omega$ such that

$p_0 \Vdash \forall n > k \left[n \in \mathrm{dom}\,(\mathring{f}_i) \cap \mathrm{dom}\,(g) \implies \mathring{f}_i(n) \neq g(n)\right]$. There are infinitely many conditions below p_0. So it is possible to find $j \geq k, i$ such that $q_j \leq p_0$. But then by clause (b) and (c), there is a $q \leq q_j$ and numbers $k^i_j > j \geq k$ and $m^i_j \in \omega$ such that $k^i_j \in \mathrm{dom}\,(g)$, $g(k^i_j) = m^i_j$, and $q \Vdash k^i_j \in \mathrm{dom}\,(\mathring{f}_i) \wedge \mathring{f}_i(k^i_j) = m^i_j$, which is a contradiction.

Next, to see that (2) holds, suppose that $n \in \mathrm{dom}\,(g)$. By clause (c) above, $n = k^i_j$ for some $i \leq j$, and $g(n) = m^i_j$. But then by clause (b), there is a $q \leq q_j \leq p$ such that $q \Vdash n \in \mathrm{dom}\,(\mathring{f}_i) \wedge \mathring{f}_i(n) = m^i_j = g(n)$. Since $\mathring{f}_i$ is forced to be a sub–function of $\mathring{f}$, we have $q \Vdash n \in \mathrm{dom}\,(\mathring{f}) \wedge \mathring{f}(n) = g(n)$, which is as required.

Now, let us build the sequence $\langle g_j : -1 \leq j < \omega\rangle$. At stage j, suppose that g_{j-1} is given to us. As all the $\mathring{f}_i$ are forced to be infinite partial functions, we can successively extend q_j $j+1$ times to find a condition $q \leq q_j$ and numbers $\max\,(\mathrm{dom}\,(g_{j-1}) \cup \{j\}) < k^0_j < \cdots < k^j_j$ and $m^0_j, \ldots, m^j_j \in \omega$ such that $\forall i \leq j \left[q \Vdash k^i_j \in \mathrm{dom}\,(\mathring{f}_i) \wedge \mathring{f}_i(k^i_j) = m^i_j\right]$. Since the k^i_j are different for different values of i, we can set $g_j = g_{j-1} \cup \left\{\langle k^0_j, m^0_j\rangle, \ldots, \langle k^j_j, m^j_j\rangle\right\}$. It is clear that g_j satisfies conditions (a)–(c). $\dashv$

Theorem 3.4.9. *Let $\mathbb{P} = Fn(\omega, 2)$. If $\mathscr{A} \subset \omega^\omega$ is a strongly MAD family, then $\mathbb{P}$ strongly preserves $\mathscr{A}$.*

Proof. Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. Fix a countable elementary submodel $M \prec H(\theta)$ with $\mathbb{P}, \mathscr{A} \in M$. Choose $h \in \mathscr{A}$ which covers M with respect to $\mathscr{A}$, and let $p \in \mathbb{P} \cap M$ be any condition. It is well known that p is always $(M, \mathbb{P})$ generic. We will argue that $p \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$.

Suppose for a contradiction that there is $q \leq p$, $\mathring{f} \in M \cap \mathbf{V}^{\mathbb{P}}$ and $n \in \omega$ so that

$(*)$ $\Vdash \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$

$(**)$ $q \Vdash \forall m > n \left[m \in \mathrm{dom}\,(\mathring{f}) \implies \mathring{f}(m) \neq h(m)\right]$.

Since $\mathbb{P}$ is countable, $\mathbb{P} \subset M$. Therefore, $q \in M$. Now, we can apply Lemma 3.4.7 to $\mathring{f}$ to find a countable set of names $\left\{\mathring{f}_i : i \in \omega\right\} \in M$ that satisfy clauses (1) and (2) of Lemma 3.4.7. Clause (1) of Lemma 3.4.7 implies that $\mathring{f}$ and $\left\{\mathring{f}_i : i \in \omega\right\}$ satisfy the hypothesis of Lemma 3.4.8. Thus we can apply Lemma 3.4.8 with the condition q in place of p to find an infinite partial function $g \in M$ which satisfies clauses (1) and (2) of Lemma 3.4.8 (with respect to q). Now, clause (2) of Lemma 3.4.7 and clause (1) of Lemma 3.4.8 together imply that g avoids $\mathscr{A}$. As h covers M with respect to $\mathscr{A}$, $|h \cap g| = \omega$. Choose $n < m \in \operatorname{dom}(g)$ such that $g(m) = h(m)$. But by clause (2) of Lemma 3.4.8, there is $r \leq q$ such that $r \Vdash m \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(m) = g(m) = h(m)$, which contradicts $(**)$ above. ⊣

Corollary 3.4.10. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. Let $\mathbb{P} = Fn(\omega, 2)$. $\mathscr{A}$ is $\mathbb{P}$-indestructible. In fact, $\mathscr{A}$ is strongly $\mathbb{P}$-indestructible.*

⊣

An immediate consequence of the Cohen indestructibility of strongly MAD families is a strengthening of a result of Steprāns [20] which says that strongly MAD families cannot be analytic.

Corollary 3.4.11. *If $\mathscr{A} \subset \omega^\omega$ is strongly MAD, then $\mathscr{A}$ does not contain perfect sets.*

Proof. Suppose for a contradiction that $T \subset \omega^{<\omega}$ is a perfect tree such that $[T] \subset \mathscr{A}$. Let $\mathbb{P} = Fn(\omega, 2)$ be Cohen forcing and G be a $(\mathbf{V}, \mathbb{P})$ generic filter. Since T is perfect, $[T]$ has a new branch in $\mathbf{V}[G]$. That is, there is a $b \in [T] \cap \mathbf{V}[G]$, which is not a member of $\mathbf{V}$. We will argue that b is a.d. from $\mathscr{A}$, contradicting the Cohen indestructibility of $\mathscr{A}$. First of all, notice that in $\mathbf{V}$, the following statement is true: any two distinct branches through T are a.d. This statement is Π^1_1 and hence absolute. So it is still true in $\mathbf{V}[G]$ that any two distinct branches through T are a.d. In particular, b is a.d. from $[T] \cap \mathbf{V}$. Next, suppose that $f \in \mathscr{A} \setminus ([T] \cap \mathbf{V})$. Notice that in $\mathbf{V}$, the following statement holds: f is a.d. from every branch through T. This is again Π^1_1, and hence absolute. Thus in $\mathbf{V}[G]$ every branch through T is a.d. from f. In particular, b is a.d. from f, and we are done. $\dashv$

We now proceed to demonstrate that a certain class of forcings, including such well–known ones as the Miller and Sacks forcings, strongly preserve strongly MAD families. We adopt a general framework and show that all forcings that satisfy certain conditions (Definition 3.4.13) have this property. We will then show that Miller and Sacks forcings do satisfy these conditions. These conditions might seem technical, but they are a natural abstraction of the properties of the above mentioned forcings. This will be clear from the proof of Theorem 3.4.25

Definition 3.4.12. *Let $(\mathbb{P}, \leq)$ be a poset. We will say that $(\mathbb{P}, \leq)$ has fusion if there is a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings on $\mathbb{P}$ such that*

1. *$\forall p, q \in \mathbb{P}\, [q \leq_n p \implies q \leq_{n-1} p]$, with $\leq_{-1}$ being $\leq$*
2. *if $\langle p_n : n \in \omega \rangle$ is a sequence with $p_{n+1} \leq_n p_n$, then $\exists q \in \mathbb{P} \forall n \in \omega\, [\, q \leq_n p_n]$.*

Definition 3.4.13. *Let $\mathbb{P}$ be a poset. We will say that $\mathbb{P}$ has diagonal fusion if there exist a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings on $\mathbb{P}$, a strictly increasing sequence of natural numbers $\langle i_n : n \in \omega \rangle$ with $i_0 = 0$, and for each $p \in \mathbb{P}$ a sequence $I_p = \langle p_i : i \in \omega \rangle \in \mathbb{P}^\omega$ such that the following hold*

1. *$\mathbb{P}$ has fusion with respect to $\langle \leq_n : n \in \omega \rangle$*
2. *$\forall i \in \omega\, [p_i \leq p]$*
3. *if $q \leq p$, then $\exists^\infty i \in \omega\, [q \not\perp p_i]$*
4. *if $q \leq_n p$, then $\forall i < i_n\, [q_i \leq p_i]$, where $I_q = \langle q_i : i \in \omega \rangle$*
5. *if $\langle r_i : i_n \leq i < i_{n+1} \rangle$ is a sequence so that $\forall i \in [i_n, i_{n+1})\, [r_i \leq p_i]$, then $\exists q \leq_n p\; \forall i \in [i_n, i_{n+1})\, [q_i \leq r_i]$, where $I_q = \langle q_i : i \in \omega \rangle$.*

Our terminology is motivated by analogy with Miller forcing, where the notion of diagonal fusion across a Miller tree occurs. It will be shown in Theorem 3.4.25 that conditions (1)–(5) are abstractions of what goes on in the case of diagonal fusion through a Miller tree. In the case of Miller forcing, I_T, as a set, is just the collection of all subtrees of T that correspond to the successors of split nodes of the tree T. Condition (5) corresponds to amalgamating extensions of these into the Miller tree T.

Remark 3.4.14. *We point out here that all our proofs will go through under the following slight weakening of condition (5) above:*

$$\text{for any } i \in [i_n, i_{n+1}), \text{ if } r_i \leq p_i, \text{ then there is a } q \leq_n p \text{ such that } q_i \leq r_i \tag{5'}$$
$$\text{and } \forall i_n \leq i' < i \, [q_{i'} \leq p_{i'}] \, .$$

Intuitively, (5) seems stronger than (5') because (5) allows us to "amalgamate" the r_i into q simultaneously, whereas with (5'), we must do this successively, one i at a time. We do not use (5') in our proofs because it makes the notation more cumbersome, and it doesn't introduce any new ideas into the proofs. We will leave it to the reader to verify that (5') is indeed enough for the proofs in this section.

Lemma 3.4.15. *If $\mathbb{P}$ has diagonal fusion, then $\mathbb{P}$ is Axiom A.*

Proof. By assumption, there is a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings on $\mathbb{P}$ witnessing that $\mathbb{P}$ has fusion. We will check that this same sequence also witnesses that $\mathbb{P}$ is Axiom A. To this end, suppose that $\mathring{x}$ is a $\mathbb{P}$ name such that $\Vdash \mathring{x} \in \mathbf{V}$. Let p be any condition and let $n \in \omega$. We build a sequence $\langle p^m : m \in \omega \rangle$ with $p^0 = p$ such that $p^{m+1} \leq_{n+m} p^m$ as follows. Suppose that at stage $m+1$ we are given p^m. Let $I_{p^m} = \langle p_i^m : i \in \omega \rangle$. For each $i \in [i_{n+m}, i_{n+m+1})$ choose $r_i^m \leq p_i^m$ and x_i^m such that $r_i^m \Vdash \mathring{x} = x_i^m$. Now, we can find a condition $p^{m+1} \leq_{n+m} p^m$ such that $\forall i \in [i_{n+m}, i_{n+m+1}) \left[p_i^{m+1} \leq r_i^m \right]$, where $I_{p^{m+1}} = \langle p_i^{m+1} : i \in \omega \rangle$. Let $X = \{ x_i^m : m \in \omega \wedge i \in [i_{n+m}, i_{n+m+1}) \}$ and let $q \in \mathbb{P}$ be such that $\forall m \in \omega \, [q \leq_{n+m} p^m]$. X is clearly a countable set. We will argue that $q \Vdash \mathring{x} \in X$. Put $I_q = \langle q_i : i \in \omega \rangle$. Let $r \leq q$. Find m and $i \in [i_{n+m}, i_{n+m+1})$ such that $r \not\perp q_i$. Since $q \leq_{n+m+1} p^{m+1}$, we have $q_i \leq p_i^{m+1} \leq r_i^m$. But then we can choose $s \in \mathbb{P}$ extending both r and r_i^m, whence $s \Vdash \mathring{x} = x_i^m \in X$. $\dashv$

We now show that if $\mathbb{P}$ has diagonal fusion, then $\mathbb{P}$ strongly preserves strongly MAD families. The steps are analogous to the steps for Cohen forcing.

Lemma 3.4.16. *Let $\mathbb{P}$ be a poset with diagonal fusion. Let $\mathring{f}$ be a $\mathbb{P}$ name and suppose that $\Vdash \mathring{f}$ is an infinite partial function. Let $\left\{\mathring{f}_l : l \in \omega\right\}$ be a set of $\mathbb{P}$ names such that $\forall l \in \omega \left[\Vdash \mathring{f}_l \subset \mathring{f} \text{ is an infinite partial function}\right]$. Let $p \in \mathbb{P}$ and let $n \in \omega$. There is $q \leq_n p$ and an infinite partial function g such that*

1. $\forall l \in \omega \left[q \Vdash \left|g \cap \mathring{f}_l\right| = \omega\right]$
2. $\forall k \in \mathrm{dom}\,(g) \exists i \left[q_i \Vdash k \in \mathrm{dom}\,(\mathring{f}) \wedge \mathring{f}(k) = g(k)\right]$, *where $I_q = \langle q_i : i \in \omega \rangle$.*

Proof. We build g by induction as the union of an increasing sequence of finite partial functions g_j. In fact, we will build two sequences $\langle g_j : -1 \leq j < \omega \rangle$ and $\langle p^j : j \in \omega \rangle \subset \mathbb{P}$ such that for each $j \geq 0$

(a) $p^0 = p$ and $p^{j+1} \leq_{n+j} p^j$

(b) $g_{-1} = 0$ and $g_{j-1} \subset g_j$ is a finite partial function

(c) for each $i \in [i_{n+j}, i_{n+j+1})$ and for each $l \leq j$ there is a $k(i,j,l) > j$ and a $m(i,j,l) \in \omega$ so that $\left[p_i^{j+1} \Vdash k(i,j,l) \in \mathrm{dom}\,(\mathring{f}_l) \wedge \mathring{f}_l(k(i,j,l)) = m(i,j,l)\right]$, where $I_{p^{j+1}} = \langle p_i^{j+1} : i \in \omega \rangle$

(d) $g_j = g_{j-1} \cup \{\langle k(i,j,l), m(i,j,l) \rangle : i \in [i_{n+j}, i_{n+j+1}) \wedge l \leq j\}$.

Let $q \in \mathbb{P}$ be such that $q \leq_{n+j} p^j$ and let $g = \bigcup g_j$. We will first argue that q and g satisfy clauses (1) and (2). Put $I_q = \langle q_i : i \in \omega \rangle$. To verify (1), fix $l \in \omega$. Let $r \leq q$ and $k \in \omega$ be given. We know $\exists^\infty i \in \omega\, [r \not\perp q_i]$. Choose $j \geq k, l$ and $i \in [i_{n+j}, i_{n+j+1})$ such that $r \not\leq q_i$. As $l \leq j$, there is a $k(i,j,l) > j \geq k$ and a $m(i,j,l) \in \omega$ such that $k(i,j,l) \in \operatorname{dom}(g)$, $g(k(i,j,l)) = m(i,j,l)$, and $p_i^{j+1} \Vdash k(i,j,l) \in \operatorname{dom}(\mathring{f}_l) \wedge \mathring{f}_l(k(i,j,l)) = m(i,j,l)$. But $q \leq_{n+j+1} p^{j+1}$. Therefore, $q_i \leq p_i^{j+1}$. Hence, we can find $s \in \mathbb{P}$ extending both r and p_i^{j+1}, whence $s \Vdash k(i,j,l) \in \operatorname{dom}(\mathring{f}_l) \wedge \mathring{f}_l(k(i,j,l)) = m(i,j,l) = g(k(i,j,l))$. As $k(i,j,l) > k$, this verifies (1).

Next to verify (2), suppose that $k \in \operatorname{dom}(g)$. By clause (d), $k = k(i,j,l)$ and $g(k) = m(i,j,l)$ for some $l \leq j$ and $i \in [i_{n+j}, i_{n+j+1})$. We can conclude from clause (c) that $p_i^{j+1} \Vdash k \in \operatorname{dom}(\mathring{f}_l) \wedge \mathring{f}_l(k) = m(i,j,l) = g(k)$. But since $\mathring{f}_l$ is forced to be a sub–function of $\mathring{f}$, $p_i^{j+1} \Vdash k \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(k) = g(k)$. As $q \leq_{n+j+1} p^{j+1}$, $q_i \leq p_i^{j+1}$. Therefore, we get that $q_i \Vdash k \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(k) = g(k)$, which is as required.

Now, we describe the construction of $\langle g_j : -1 \leq j < \omega \rangle$ and $\langle p^j : j \in \omega \rangle$. We set $g_{-1} = 0$ and $p^0 = p$. At stage $j \geq 0$ suppose we are given g_{j-1} and p^j. Put $I_{p^j} = \langle p_i^j : i \in \omega \rangle$. g_{j-1} being a finite partial function, we can set $k = \max(\operatorname{dom}(g_{j-1}))$. As each $\mathring{f}_l$ is forced to be an infinite partial function, we can find a sequence of conditions $\langle r_i^j : i \in [i_{n+j}, i_{n+j+1}) \rangle$ and two sequences of numbers $\langle k(i,j,l) : i \in [i_{n+j}, i_{n+j+1}) \wedge l \leq j \rangle$ and $\langle m(i,j,l) : i \in [i_{n+j}, i_{n+j+1}) \wedge l \leq j \rangle$ satisfying

(i) $r_i^j \leq p_i^j$ and $r_i^j \Vdash k(i,j,l) \in \operatorname{dom}(\mathring{f}_l) \wedge \mathring{f}_l(k(i,j,l)) = m(i,j,l)$

(ii) $k(i,j,l) > \max(\{k,j\})$ and $k(i,j,l) < k(i',j,l')$ whenever $(i,l) < (i',l')$ lexicographically.

By clause (5) of Definition 3.4.13, we can find $p^{j+1} \leq_{n+j} p^j$ such that for each $i \in [i_{n+j}, i_{n+j+1})$, $p_i^{j+1} \leq r_i^j$. Since the $k(i,j,l)$ are distinct for distinct pairs (i,l), we can set $g_j = g_{j-1} \cup \{\langle k(i,j,l), m(i,j,l) \rangle : i \in [i_{n+j}, i_{n+j+l}) \wedge l \leq j\}$. It is clear that g_j and p^{j+1} are as required. $\dashv$

Lemma 3.4.17. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family and let $\mathbb{P}$ be a poset with diagonal fusion. Let $M \prec H(\theta)$ be countable with $\mathscr{A}, \mathbb{P} \in M$. Suppose $\mathring{f} \in M$ is a $\mathbb{P}$ name such that $\Vdash \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. Suppose $h \in \mathscr{A}$ covers M with respect* *to $\mathscr{A}$. Let $p \in M$ be a condition and let $j \in \omega$. There is a $r \leq p$ such that $r \in M$ and $\exists k > j \left[r \Vdash k \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(k) = h(k) \right]$.*

Proof. We can apply Lemma 3.4.7 to $\mathring{f}$ to find a set of names $\left\{ \mathring{f}_l : l \in \omega \right\} \in M$ that satisfy clauses (1) and (2) of Lemma 3.4.7. Notice that the hypotheses of Lemma 3.4.16 are satisfied by $\mathbb{P}, \mathring{f}, \left\{ \mathring{f}_l : l \in \omega \right\}$, and p. So we can find $q \leq p$ with $q \in M$ and an infinite partial function $g \in M$ which satisfy clauses (1) and (2) of Lemma 3.4.16. Put $I_q = \langle q_i : i \in \omega \rangle$. Notice that since $q \in M$, $I_q \in M$, and by elementarity, $I_q \subset M$. Now, observe that clause (2) of Lemma 3.4.7 and clause (1) of Lemma 3.4.16 together imply that g avoids $\mathscr{A}$. Therefore, $|h \cap g| = \omega$. Choose $k > j$ such that $k \in \operatorname{dom}(g)$ and $h(k) = g(k)$. By clause (2) of Lemma 3.4.16, there is $i \in \omega$ such that $q_i \Vdash k \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(k) = g(k) = h(k)$. As $q_i \in M$ and as $q_i \leq q$, we can set $r = q_i$. Clearly, r and k are as required. $\dashv$

Theorem 3.4.18. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. Let $\mathbb{P}$ be a poset with diagonal fusion. Then $\mathbb{P}$ strongly preserves $\mathscr{A}$.*

Proof. Fix a countable elementary submodel $M \prec H(\theta)$ with $\mathscr{A}, \mathbb{P} \in M$. Let $h \in \mathscr{A}$ cover M with respect to $\mathscr{A}$ and let $p \in M$ be a condition. We must find a $q \leq p$ which is $(M, \mathbb{P})$ generic such that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. We will use Lemma 3.4.15 to ensure that q is $(M, \mathbb{P})$ generic and use Lemma 3.4.17 to ensure that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. Let $\langle \mathring{\alpha}_n : n \in \omega \rangle$ enumerate all $\mathring{\alpha} \in M \cap \mathbf{V}^{\mathbb{P}}$ such that $\Vdash \mathring{\alpha}$ is an ordinal. Let $\langle \mathring{f}_j : j \in \omega \rangle$ enumerate all $\mathring{f} \in M \cap \mathbf{V}^{\mathbb{P}}$ such that $\Vdash \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. We will build a sequence $\langle p^n : n \in \omega \rangle \subset \mathbb{P} \cap M$ such that the following hold

(a) $p^0 = p$ and $p^{n+1} \leq_n p^n$

(b) $p^{n+1} \Vdash \mathring{\alpha}_n \in M$

(c) for each $i \in [i_n, i_{n+1})$ and for each $j \leq n$ there is a number $k(i, n, j) > n$ such that $\left[p_i^{n+1} \Vdash k(i,n,j) \in \operatorname{dom}(\mathring{f}_j) \wedge \mathring{f}_j(k(i,n,j)) = h(k(i,n,j)) \right]$.

Let $q \in \mathbb{P}$ be a condition so that $q \leq_n p^n$ for all $n \in \omega$. We will first argue that q is as required. Indeed, it is clear from (b) above that q is $(M, \mathbb{P})$ generic. We will argue that $q \Vdash h$ covers $M[\mathring{G}]$ with respect to $\mathscr{A}$. Let G be a $(\mathbf{V}, \mathbb{P})$ generic filter with $q \in G$ and let $f \in M[G]$ be an infinite partial function avoiding $\mathscr{A}$. By elementarity of M, there is $\mathring{f} \in M$ with $\mathring{f}[G] = f$ such that $\Vdash \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. Therefore, $\mathring{f} = \mathring{f}_j$ for some $j \in \omega$. It is enough to show that $q \Vdash \left| h \cap \mathring{f}_j \right| = \omega$. Fix $r \leq q$ and $k \in \omega$. Put $I_q = \{q_i : i \in \omega\}$. We know that $\exists^\infty i \in \omega \, [r \not\perp q_i]$. So we can choose $n \geq \max\{j, k\}$ and $i \in [i_n, i_{n+1})$ such that $r \not\perp q_i$. Since $j \leq n$, by (c) above, there is $k(i,n,j) > n$ such that $p_i^{n+1} \Vdash k(i,n,j) \in \operatorname{dom}(\mathring{f}_j) \wedge \mathring{f}_j(k(i,n,j)) = h(k(i,n,j))$. But since $q \leq_{n+1} p^{n+1}$, $q_i \leq p_i^{n+1}$. So we may choose $s \leq r$ with the property that $s \Vdash k(i,n,j) \in \operatorname{dom}(\mathring{f}_j) \wedge \mathring{f}_j(k(i,n,j)) = h(k(i,n,j))$. This is as required because $k(i,n,j) > k$.

We now describe how to construct $\langle p^n : n \in \omega \rangle$. Set $p^0 = p$ and suppose that at stage n, $p^n \in M$ is given to us. We first apply Lemma 3.4.15 to p^n and $\mathring{\alpha}_n$ within M to find $\tilde{p}^n \leq_n p^n$ with $\tilde{p}^n \in M$ such that $\tilde{p}^n \Vdash \mathring{\alpha}_n \in M$. Put $I_{\tilde{p}^n} = \langle \tilde{p}_i^n : i \in \omega \rangle$. Note that $I_{\tilde{p}^n} \subset M$. Fix any $i \in [i_n, i_{n+1})$. Notice that $\tilde{p}_i^n \in M$. As $\langle \mathring{f}_j : j \leq n \rangle \subset M$, we can apply Lemma 3.4.17 to $\tilde{p}_i^n$ $n+1$ times to find $r_i^n \leq \tilde{p}_i^n$ with $r_i^n \in M$ and numbers $k(i,n,0), \ldots, k(i,n,n) \in \omega$, all of them greater than n, such that $\forall j \leq n \left[r_i^n \Vdash k(i,n,j) \in \operatorname{dom}(\mathring{f}_j) \wedge \mathring{f}_j(k(i,n,j)) = h(k(i,n,j)) \right]$. Now, $\langle r_i^n : i \in [i_n, i_{n+1}) \rangle$ is a finite sequence of things in M. Therefore $\langle r_i^n : i \in [i_n, i_{n+1}) \rangle \in M$. Hence, we can apply (5) of Definition 3.4.13 to $\tilde{p}^n$ to find $p^{n+1} \leq_n \tilde{p}^n$ with $p^{n+1} \in M$ such that $\forall i \in [i_n, i_{n+1}) \left[p_i^{n+1} \leq r_i^n \right]$. It is clear that p^{n+1} is as needed. $\dashv$

We now prove that Miller and Sacks forcings have diagonal fusion. We check the details only for Miller forcing, as it is the more difficult case. The proof for Sacks forcing is very similar, but easier.

Definition 3.4.19. *Let T be a sub–tree of either $\omega^{<\omega}$ or $2^{<\omega}$. If $s \in T$, we will write* $\operatorname{succ}_T(s)$ *to denote $\{s^\frown\langle n \rangle : s^\frown\langle n \rangle \in T\}$. If $T \subset 2^{<\omega}$, then* $\operatorname{split}(T) = \{s \in T : |\operatorname{succ}_T(s)| = 2\}$*, while if*

$T \subset \omega^{<\omega}$, *then* $\operatorname{split}(T) = \{s \in T : |\operatorname{succ}_T(s)| = \omega\}$. *In both cases,* $\operatorname{split}^+(T) = \bigcup\{\operatorname{succ}_T(s) : s \in \operatorname{split}(T)\}$. *If* $n \in \omega$, $\operatorname{split}_n(T) = \{s \in \operatorname{split}(T) : |\{t \subsetneq s : t \in \operatorname{split}(T)\}| = n\}$. $\operatorname{split}^+_n(T)$ *will denote* $\bigcup\{\operatorname{succ}_T(s) : s \in \operatorname{split}_n(T)\}$.

Definition 3.4.20. *Let T be a sub–tree of $2^{<\omega}$. We will say that T is perfect if* $\forall s \in T \exists t \in \operatorname{split}(T)\,[s \subset t]$. *If T is a sub–tree of $\omega^{<\omega}$, we will say that T is superperfect if for each $s \in T$ either* $|\operatorname{succ}_T(s)| = 1$ *or* $|\operatorname{succ}_T(s)| = \omega$ *and if in addition to this* $\forall s \in T \exists t \in \operatorname{split}(T)\,[s \subset t]$.

Definition 3.4.21. $\mathbb{M}$ *will denote Miller forcing,* $\mathbb{M} = \{T \subset \omega^{<\omega} : T$ *is superperfect*$\}$*, ordered by inclusion.* $\mathbb{S}$ *will denote Sacks forcing,* $\mathbb{S} = \{T \subset 2^{<\omega} : T$ *is perfect*$\}$*, ordered by inclusion.*

Several distinct notions of fusion can be defined on $\mathbb{M}$. The strongest such notion requires a $\leq_n$ extension to preserve *all* n^{th} split nodes. However, this is too strong for proving that $\mathbb{M}$ strongly preserves strongly MAD families because in order to ensure that our extensions stay within the elementary submodel M, we need to be able to get away with preserving only finitely many nodes at a time. So we will use the weaker notion of fusion which is sometimes known as diagonal fusion across the Miller tree T (hence the terminology of Definition 3.4.13).

Definition 3.4.22. *Let $T \in \mathbb{M}$. As T is superperfect, there is a natural bijection from $\omega^{<\omega}$ onto* $\operatorname{split}(T)$. *If $s \in \omega^{<\omega}$, we will let $T(s)$ denote the split node of T corresponding to s under this bijection. If $i \in \omega$, then $T(s,i)$ will denote the i^{th} element of* $\operatorname{succ}_T(T(s))$ *under the natural ordering on* $\operatorname{succ}_T(T(s))$. *Finally, if $s \in T$, we will write T_s to denote* $\{t \in T : s \subset t \vee t \subset s\}$.

Given $T \in \mathbb{M}$, I_T as a set, will just be $\{T_t : t \in \operatorname{split}^+(T)\}$. But to ensure that the conditions of Definition 3.4.13 are satisfied we must enumerate this set in a very particular way.

Definition 3.4.23. *We define a sequence of finite subsets of $\omega^{<\omega}$ as follows.* $\Sigma_0 = \{\langle\rangle\}$. *Given* Σ_n, $\Sigma_{n+1} = \{s^\frown\langle i\rangle : s \in \Sigma_n \wedge i \leq n\} \cup \{\langle\rangle\}$. *Notice that* $\Sigma_{n+1} \supset \Sigma_n$ *and that* $\bigcup \Sigma_n = \omega^{<\omega}$.

Definition 3.4.24. *Let $T^1 \leq T^0 \in \mathbb{M}$. For any $n \in \omega$, we will say $T^1 \leq_n T^0$ if $\forall s \in \Sigma_n \left[T^1(s) = T^0(s)\right]$.*

Theorem 3.4.25. $\mathbb{M}$ *has diagonal fusion.*

Proof. We will show that $\langle \leq_n : n \in \omega \rangle$ as defined above witnesses that $\mathbb{M}$ has diagonal fusion. Indeed, it is clear that $\mathbb{M}$ has fusion with respect to $\langle \leq_n : n \in \omega \rangle$. We will check that the other conditions hold. Fix $T \in \mathbb{M}$. We will define I_T as follows. Set $i_n = |\Sigma_n| - 1$. Now, let $e : \omega \to (\omega^{<\omega} \setminus \{\langle\rangle\})$ be a one–to–one onto enumeration such that $e''[0, i_n) = \Sigma_n \setminus \{\langle\rangle\}$. If $i \in [i_n, i_{n+1})$, then $e(i) \in \Sigma_{n+1} \setminus \Sigma_n$. So there is a unique $s \in \Sigma_n$ and a unique $j \leq n$ such that $e(i) = s^\frown\langle j \rangle$. Let $t_i = T(s, j)$. We will set $T_i = T_{t_i}$. Now, it is clear that $I_T = \langle T_i : i \in \omega \rangle = \{T_t : t \in \mathrm{split}^+(T)\}$. Therefore, $T_i \leq T$, and if $T' \leq T$, then $\exists^\infty i \in \omega \, [T' \not\perp T_i]$.

Now suppose that $T^1 \leq_n T^0$ and let $i < i_n$. We must argue that $T^1_i \leq T^0_i$. Indeed, if $n = 0$, there is nothing to be proved. So suppose that $n > 0$. As $e(i) \in \Sigma_n \setminus \{\langle\rangle\}$, we can find (unique) $s \in \Sigma_{n-1}$ and $j \leq n-1$ so that $e(i) = s^\frown\langle j \rangle$. Notice that $T^1_i = T^1_{t^1_i}$, where $t^1_i = T^1(s, j)$ and that $T^0_i = T^0_{t^0_i}$, where $t^0_i = T^0(s, j)$. Since $T^1 \leq_n T^0$, we know that $T^1(s^\frown\langle j \rangle) = T^0(s^\frown\langle j \rangle)$. It follows from this that $t^1_i = t^0_i$. But since $T^1 \subset T^0$, it is easy to see that $T^1_{t^1_i} = T^1_{t^0_i} \subset T^0_{t^0_i}$, whence $T^1_i \leq T^0_i$.

Now it only remains to verify clause (5) of Definition 3.4.13. To this end, fix $n \in \omega$ and $T^0 \in \mathbb{M}$. Let $\langle T'_i : i \in [i_n, i_{n+1}) \rangle$ be a sequence such that $\forall i \in [i_n, i_{n+1}) \left[T'_i \leq T^0_i\right]$. We wish to amalgamate the T'_i into T_0. It is clear that any two distinct $s \neq t \in \Sigma_{n+1} \setminus \Sigma_n$ are incomparable nodes in $\omega^{<\omega}$. Therefore, if $s = \tilde{s}^\frown\langle j \rangle$ and if $t = \tilde{t}^\frown\langle k \rangle$, then $T^0(\tilde{s}, j)$ and $T^0(\tilde{t}, k)$ are incomparable nodes in the tree T^0. Thus it follows that if $i \neq i'$ are distinct elements in $[i_n, i_{n+1})$, then t^0_i and $t^0_{i'}$ are incomparable nodes in T^0. But now, we can get $T^1 \leq_n T^0$ simply by replacing $T^0_{t^0_i}$ in T^0 with T'_i for each $i \in [i_n, i_{n+1})$. Now, T^1 is as required, and this finishes the proof. $\dashv$

Corollary 3.4.26. *If $\mathscr{A} \subset \omega^\omega$ is a strongly MAD family, then $\mathbb{M}$ and $\mathbb{S}$ strongly preserve $\mathscr{A}$.*

$\dashv$

We end this section with a conjecture regarding preservation of strongly MAD families. Our notion of strongly preserving a strongly MAD family (Definition 3.4.3) is very similar to the following notion which has been considered in the literature in connection with the problem of preserving non meager sets (see [5], [12] or [30]).

Definition 3.4.27. *Let $\mathbb{P}$ be a poset. We will say that $\mathbb{P}$ preserves $\sqsubseteq_{\mathbf{C}}$ if the following holds. For every $M \prec H(\theta)$, M countable, with $\mathbb{P} \in M$, whenever $p \in \mathbb{P} \cap M$ and x is a Cohen real over M, there is $q \leq p$ which is $(M, \mathbb{P})$ generic such that $q \Vdash x$ is a Cohen real over $M[\mathring{G}]$.*

The proof of Theorem 3.4.18 can be easily modified to show that if $\mathbb{P}$ has diagonal fusion, then $\mathbb{P}$ preserves $\sqsubseteq_{\mathbf{C}}$. We are not aware of any posets that preserve $\sqsubseteq_{\mathbf{C}}$ but do not strongly preserve strongly MAD families. We make the following conjecture.

Conjecture 3.4.28. *Let $\mathscr{A} \subset \omega^{\omega}$ be a strongly MAD family and let $\mathbb{P}$ be a poset that preserves $\sqsubseteq_{\mathbf{C}}$. $\mathscr{A}$ is strongly $\mathbb{P}$–indestructible. Moreover, $\mathbb{P}$ strongly preserves $\mathscr{A}$.*

3.5 Some Preservation Theorems for Countable Support Iterations

Our main goal in this section is to prove that the property of strongly preserving a strongly MAD family is preserved by the countable support iteration of proper posets. By the results of the last section, this will imply that the countable support iteration of Sacks and Miller forcings strongly preserve strongly MAD families.

We assume that the reader is familiar with the basic theory of iterated forcing, including some preservation theorems, such as the preservation of properness. The reader may consult [1], [12], [30] or [31] for a good introduction. Our presentation will generally follow that of Abraham [1].

En route to proving our main theorem, we will show the following. Suppose γ is a limit ordinal and that $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a countable support iteration of proper posets. If for all $\alpha < \gamma$, $\mathbb{P}_\alpha$ does not add an eventually different real, then $\mathbb{P}_\gamma$ does not add an eventually different real either. To the best of our knowledge, this result (Theorem 3.5.8) is new. Shelah, Judah and Goldstern (see [30]) have shown that the countable support iteration of posets which preserve $\sqsubseteq_{\mathbf{C}}$ itself preserves $\sqsubseteq_{\mathbf{C}}$ (Definition 3.4.27). The property of preserving $\sqsubseteq_{\mathbf{C}}$ appears to be slightly stronger than that of not adding eventually different reals. On the other hand, our result puts a condition on initial segments of the iteration, and not on the iterands. A partial result towards our theorem was obtained by Shelah and Kellner [21], who proved it for the case of Suslin forcings as well as some nep forcings.

Before giving the proofs of our results, we collect together some basic facts about countable support iterations that we will use.

Lemma 3.5.1 (See the proof of Lemma 2.8 in [1])**.** *Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a CS iteration. Let $M \prec H(\theta)$ be countable and suppose that $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle \in M$. Put $\gamma' = \sup(\gamma \cap M)$ and let $\langle \gamma_n : n \in \omega \rangle \subset \gamma \cap M$ be an increasing sequence that is cofinal in γ'. Suppose that $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$ are two sequences such that the following hold:*

1. $q_n \in \mathbb{P}_{\gamma_n}$ *and* $q_{n+1} \restriction \gamma_n = q_n$
2. $\mathring{p}_n \in \mathbf{V}^{\mathbb{P}_{\gamma_n}}$ *and* $q_n \Vdash_{\gamma_n} \mathring{p}_n \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_n \restriction \gamma_n \in \mathring{G}_{\gamma_n}$
3. $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \leq \mathring{p}_n$.

If $q = (\bigcup q_n)^\frown \mathbb{1} \in \mathbb{P}_\gamma$, then $\forall n \in \omega \left[q \Vdash_\gamma \mathring{p}_n \in \mathring{G}_\gamma \right]$.

$\dashv$

Lemma 3.5.2 (See Lemma 2.8 of [1])**.** *Let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha \text{ is proper} \right]$. Let $M \prec H(\theta)$ be countable and suppose that $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle \in M$.*

Let $\gamma_0 \in \gamma \cap M$ and suppose that $q_0 \in \mathbb{P}_{\gamma_0}$ is a $(M, \mathbb{P}_{\gamma_0})$ generic condition. Suppose $\mathring{p}_0 \in \mathbf{V}^{\mathbb{P}_{\gamma_0}}$ and $q_0 \Vdash_{\gamma_0} \mathring{p}_0 \in \mathbb{P}_\gamma \cap M \wedge \mathring{p}_0 \restriction \gamma_0 \in \mathring{G}_{\gamma_0}$. There is a $(M, \mathbb{P}_\gamma)$ generic condition $q \in \mathbb{P}_\gamma$ such that $q \restriction \gamma_0 = q_0$ and $q \Vdash_\gamma \mathring{p}_0 \in \mathring{G}_\gamma$.

$\dashv$

Definition 3.5.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is an onto map. We will say that π is a projection if the following hold:*

1. *π is order preserving. That is, if $q_1 \leq q_0$, then $\pi(q_1) \leq \pi(q_0)$*

2. *for every $q_0 \in \mathbb{Q}$ if $p \leq \pi(q_0)$, then $\exists q_1 \leq q_0 \, [\pi(q_1) = p]$.*

Definition 3.5.4. *Let $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be posets and let $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection. If $G \subset \mathbb{P}$ is a $(\mathbf{V}, \mathbb{P})$ generic filter, then in $\mathbf{V}[G]$ we define the poset $\mathbb{Q}/G = \{q \in \mathbb{Q} : \pi(q) \in G\}$ ordered by $\leq_{\mathbb{Q}}$. In $\mathbf{V}$, we let $\mathbb{Q}/\mathring{G}$ be a full $\mathbb{P}$ name for $\mathbb{Q}/G$.*

Lemma 3.5.5 (See section 4 of [12])**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and let $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection. There is a dense embedding $i : \mathbb{Q} \to \mathbb{P} * \mathbb{Q}/\mathring{G}$ given by $i(q) = \langle \pi(q), q \rangle$. Moreover, if $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iteration, then for each $\alpha \leq \gamma$ the map $\pi_{\gamma\alpha} : \mathbb{P}_\gamma \to \mathbb{P}_\alpha$ given by $\pi_{\gamma\alpha}(p) = p \restriction \alpha$ is a projection. Therefore, if $\alpha \leq \gamma$ and if $G_\gamma \subset \mathbb{P}_\gamma$ is a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter, then there is a $(\mathbf{V}[G_\alpha], \mathbb{P}_\gamma/G_\alpha)$ generic filter H so that in $\mathbf{V}[G_\gamma]$, $G_\gamma = G_\alpha * H$ holds. In fact, this H is equal to G_γ.*

$\dashv$

Definition 3.5.6. *Suppose $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is an iteration and let $\alpha \leq \gamma$. By Lemma 3.5.5, $\mathbb{P}_\gamma$ densely embeds into $\mathbb{P}_\alpha * \mathbb{P}_\gamma/\mathring{G}_\alpha$. Thus we may think of any $\mathbb{P}_\gamma$ name as a $\mathbb{P}_\alpha$ name for a $\mathbb{P}_\gamma/\mathring{G}_\alpha$ name. Thus, given a $\mathbb{P}_\gamma$ name $\mathring{x}$, we use $\mathring{x}[\mathring{G}_\alpha]$ to denote a canonical $\mathbb{P}_\alpha$ name for a $\mathbb{P}_\gamma/\mathring{G}_\alpha$ name representing $\mathring{x}$. If G_α is a $(\mathbf{V}, \mathbb{P}_\alpha)$ generic filter, we will write $\mathring{x}[G_\alpha]$ to denote the evaluation of $\mathring{x}[\mathring{G}_\alpha]$ by G_α. Therefore, if G_γ is a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter, then in $\mathbf{V}[G_\gamma]$,*

$\mathring{x}[G_\gamma] = \mathring{x}[G_\alpha][H]$ holds, where H is a $(\mathbf{V}[G_\alpha], \mathbb{P}_\gamma/G_\alpha)$ generic filter such that $G_\gamma = G_\alpha * H$. By Lemma 3.5.5, $H = G_\gamma$.

We are now ready to prove our main results. We begin by showing that a CS iteration of proper forcings of limit length does not add an eventually different real if no initial segment does.

Lemma 3.5.7. *Let $\mathbf{V}_0 \subset \mathbf{V}_1$ be transitive universes satisfying ZFC and suppose that for every $f \in \omega^\omega \cap \mathbf{V}_1$ there is a slalom $S \in \mathbf{V}_0$ such that $\exists^\infty n \in \omega\,[f(n) \in S(n)]$. No $f \in \omega^\omega \cap \mathbf{V}_1$ is eventually different from $\omega^\omega \cap \mathbf{V}_0$.*

Proof. Working in $\mathbf{V}_0$ partition ω into a sequence of intervals $\langle I_n : n \in \omega\rangle \in \mathbf{V}_0$ such that $\forall n \in \omega\,[|I_n| = 2^n]$. Put $X = \bigcup \omega^{I_n}$. Notice that for each $n \in \omega$, $\omega^{I_n} \cap \mathbf{V}_1 = \omega^{I_n} \cap \mathbf{V}_0$. Let $f \in \omega^\omega \cap \mathbf{V}_1$. Working in $\mathbf{V}_1$ define a function $F : \omega \to X$ by stipulating that $F(n) = f \restriction I_n \in \omega^{I_n}$. Since we can code elements of X^ω by elements of ω^ω, we can find $S \in \mathbf{V}_0$ such that

1. $S : \omega \to [X]^{<\omega}$
2. $\forall n \in \omega\,[|S(n)| \le 2^n]$
3. $\exists^\infty n \in \omega\,[F(n) \in S(n)]$.

WLOG we may assume that for all $n \in \omega$, $S(n) \subset \omega^{I_n}$ and that $|S(n)| = 2^n$ because we may modify S to make both of these things true without affecting the truth of (3) above. Put $S(n) = \{\sigma^n_0, \ldots, \sigma^n_{2^n-1}\}$ and $I_n = \{i^n_0, \ldots, i^n_{2^n-1}\}$. For each $n \in \omega$ and for each $0 \le j < 2^n$ define $g(i^n_j) = \sigma^n_j(i^n_j)$. This definition makes sense because by assumption, $\sigma^n_j \in \omega^{I_n}$, and so σ^n_j is defined at i^n_j. Clearly, $g \in \omega^\omega \cap \mathbf{V}_0$. We will argue in $\mathbf{V}_1$ that $\exists^\infty n \in \omega\,[f(n) = g(n)]$. We know that $A = \{n \in \omega : F(n) \in S(n)\}$ is infinite. For each $n \in A$, there is a $0 \le j_n < 2^n$ such that $f \restriction I_n = \sigma^n_{j_n}$. Therefore, for each $n \in A$, $g\left(i^n_{j_n}\right) = \sigma^n_{j_n}\left(i^n_{j_n}\right) = f\left(i^n_{j_n}\right)$. Since the I_n are disjoint, the set $\left\{i^n_{j_n} : n \in A\right\}$ is infinite, and we are done. $\dashv$

Theorem 3.5.8. *Let γ be a limit ordinal and let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha \text{ is proper} \right]$. Suppose that for all $\alpha < \gamma$, $\mathbb{P}_\alpha$ does not add an eventually different real. $\mathbb{P}_\gamma$ does not add an eventually different real either.*

Proof. Let $\mathring{f}$ be a $\mathbb{P}_\gamma$ name such that $\Vdash_\gamma \mathring{f} \in \omega^\omega$, and let $p_0 \in \mathbb{P}_\gamma$ be a condition. Fix a countable $M \prec H(\theta)$ with $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle, \mathring{f}, p_0 \in M$. Let $S : \omega \to [\omega]^{<\omega}$ be a slalom such that for all $f \in \omega^\omega \cap M$, $\forall^\infty n \in \omega \, [f(n) \in S(n)]$. We will find $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p_0 \in \mathring{G}_\gamma$ and $q \Vdash_\gamma \exists^\infty n \in \omega \left[\mathring{f}(n) \in S(n) \right]$. By Lemma 3.5.7, this is sufficient. Put $\gamma' = \sup(M \cap \gamma)$ and let $\langle \gamma_n : n \in \omega \rangle \subset M \cap \gamma$ be an increasing sequence that is cofinal in γ'. We will build two sequences $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$ such that the following hold

1. $q_n \in \mathbb{P}_{\gamma_n}$, q_n is $(M, \mathbb{P}_{\gamma_n})$ generic, and $q_{n+1} \restriction \gamma_n = q_n$
2. $\mathring{p}_0 = p_0$, $\mathring{p}_n \in \mathbf{V}^{\mathbb{P}_{\gamma_n}}$, and $q_n \Vdash_{\gamma_n} \mathring{p}_n \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_n \restriction \gamma_n \in \mathring{G}_{\gamma_n}$
3. $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \leq \mathring{p}_n$.
4. $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k \geq n \left[\mathring{f}[\mathring{G}_{\gamma_{n+1}}] \, (k) \in S(k) \right]$.

Before showing how to build such a sequence, we show that it is sufficient to do so. Let $q = (\bigcup q_n)^\frown \mathring{\mathbb{1}} \in \mathbb{P}_\gamma$. By Lemma 3.5.1, $\forall n \in \omega \left[q \Vdash_\gamma \mathring{p}_n \in \mathring{G}_\gamma \right]$. In particular, $q \Vdash_\gamma p_0 \in \mathring{G}_\gamma$. We will argue that $q \Vdash_\gamma \exists^\infty n \in \omega \left[\mathring{f}(n) \in S(n) \right]$. Indeed let $r \leq q$ and let $n \in \omega$. Fix a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter G_γ, with $r \in G_\gamma$. By Lemma 3.5.5, we know that G_γ is $\left(\mathbf{V}[G_{\gamma_{n+1}}], \mathbb{P}_\gamma / G_{\gamma_{n+1}} \right)$ generic and that in $\mathbf{V}[G_\gamma]$, $G_\gamma = G_{\gamma_{n+1}} * \ G_\gamma$ holds. Notice that $q \in G_\gamma$ and therefore, $q_{n+1} \in G_{\gamma_{n+1}}$. Also, since $\mathring{p}_{n+1}$ is a $\mathbb{P}_{\gamma_{n+1}}$ name, $\mathring{p}_{n+1}[G_\gamma] = \mathring{p}_{n+1}[G_{\gamma_{n+1}}]$. It follows from clauses (2) and (4) that $\mathring{p}_{n+1}[G_\gamma] \in M \cap \mathbb{P}_\gamma$, that $\mathring{p}_{n+1}[G_\gamma] \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$, and that in $\mathbf{V}[G_{\gamma_{n+1}}]$, $\mathring{p}_{n+1}[G_\gamma] \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \exists k \geq n \left[\mathring{f}[G_{\gamma_{n+1}}] \, (k) \in S(k) \right]$. However, as observed above, Lemma 3.5.1 implies that $\mathring{p}_{n+1}[G_\gamma] \in G_\gamma$. Therefore in $\mathbf{V}[G_\gamma]$, there is a $k \geq n$ such that $\mathring{f}[G_{\gamma_{n+1}}][G_\gamma](k) \in S(k)$. But $\mathring{f}[G_{\gamma_{n+1}}][G_\gamma] = \mathring{f}[G_\gamma]$. So $\mathring{f} \, [G_\gamma] \, (k) \in S(k)$. Since $r \in G_\gamma$, we may find $s \leq r$ such that $s \Vdash_\gamma \mathring{f}(k) \in S(k)$. As $k \geq n$, this finishes the proof.

We now describe how to construct $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$. $\mathring{p}_0$ is just p_0, the given condition. Since $p_0 \in M \cap \mathbb{P}_\gamma$ and since $\gamma_0 \in M$, $p_0 \restriction \gamma_0 \in M \cap \mathbb{P}_{\gamma_0}$. As $\mathbb{P}_{\gamma_0}$ is proper, we may find a $(M, \mathbb{P}_{\gamma_0})$ generic condition $q_0 \leq p_0 \restriction \gamma_0$. Because $q_0 \leq p_0 \restriction \gamma_0$, $q_0 \Vdash_{\gamma_0} p_0 \restriction \gamma_0 \in \mathring{G}_{\gamma_0}$. Now suppose that q_n and $\mathring{p}_n$ are given to us. By clause (1), q_n is $(M, \mathbb{P}_{\gamma_n})$ generic, and by clause (2), $q_n \Vdash_{\gamma_n} \mathring{p}_n \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_n \restriction \gamma_n \in \mathring{G}_{\gamma_n}$. Now this means that the hypothesis of Lemma 3.5.2 are satisfied by the iteration $\langle \mathbb{P}_\alpha, \mathring{Q}_\alpha : \alpha \leq \gamma_{n+1} \rangle$, the elementary submodel M, the ordinal γ_n, the condition q_n and by a $\mathbb{P}_{\gamma_n}$ name forced by q_n to equal $\mathring{p}_n \restriction \gamma_{n+1}$. So by Lemma 3.5.2, we can find a $(M, \mathbb{P}_{\gamma_{n+1}})$ generic condition q_{n+1} such that

(a) $q_{n+1} \restriction \gamma_n = q_n$

(b) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_n \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$.

To find $\mathring{p}_{n+1}$, we proceed as follows. Choose a $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$ filter $G_{\gamma_{n+1}}$ with $q_{n+1} \in G_{\gamma_{n+1}}$. Since $\mathring{p}_n$ is a $\mathbb{P}_{\gamma_n}$ name, $\mathring{p}_n[G_{\gamma_{n+1}}] = \mathring{p}_n[G_{\gamma_n}]$. Therefore $\mathring{p}_n[G_{\gamma_{n+1}}] \in M \cap \mathbb{P}_\gamma$ and by (b) above, $\mathring{p}_n[G_{\gamma_{n+1}}] \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$. Now, notice that $M[G_{\gamma_{n+1}}] \prec H(\theta)[G_{\gamma_{n+1}}]$, and that $H(\theta)[G_{\gamma_{n+1}}]$ is the same as $H(\theta)$ as computed within the universe $\mathbf{V}[G_{\gamma_{n+1}}]$. Observe also that both $\mathbb{P}_\gamma / G_{\gamma_{n+1}}$ and $\mathring{f}[G_{\gamma_{n+1}}]$ are elements of $M[G_{\gamma_{n+1}}]$. Thus we conclude that $\mathring{p}_n[G_{\gamma_{n+1}}] \in \mathbb{P}_\gamma / G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$. Moreover, we have that $\Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \mathring{f}[G_{\gamma_{n+1}}] \in \omega^\omega$. Thus by elementarity, we can find a sequence of conditions $\langle p^i : i \in \omega \rangle \in M[G_{\gamma_{n+1}}]$ and a function $f \in \omega^\omega \cap M[G_{\gamma_{n+1}}]$ such that the following hold:

(i) $p^0 = \mathring{p}_n[G_{\gamma_{n+1}}]$ and $\forall i \in \omega \left[p^i \in \mathbb{P}_\gamma / G_{\gamma_{n+1}} \right]$

(ii) $p^{i+1} \leq p^i$

(iii) $\forall i \in \omega \left[p^i \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \mathring{f}[G_{\gamma_{n+1}}]\,(i) = f(i) \right]$.

Recall that q_{n+1} is a $(M, \mathbb{P}_{\gamma_{n+1}})$ generic condition. Therefore, $M[G_{\gamma_{n+1}}] \cap \mathbb{P}_\gamma = M \cap \mathbb{P}_\gamma$. It follows that $\langle p^i : i \in \omega \rangle \subset M$ (even though it is not an element of M). Now, since $\mathbb{P}_{\gamma_{n+1}}$ does not add eventually different reals, we can find $g \in \omega^\omega \cap M$ such that $|f \cap g| = \omega$. But we chose S so that $\forall^\infty i \in \omega\,[g(i) \in S(i)]$. Therefore, we can find $k \geq n$ such that $f(k) \in S(k)$. Now by (i)–(iii) above, p^k has the following properties in $\mathbf{V}[G_{\gamma_{n+1}}]$

(+) $p^k \in M \cap \mathbb{P}_\gamma$ and $p^k \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$

(++) $p^k \leq \mathring{p}_n[G_{\gamma_{n+1}}]$

(+++) $p^k \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \exists k \geq n \left[\mathring{f}[G_{\gamma_{n+1}}]\,(k) \in S(k)\right]$.

Since $G_{\gamma_{n+1}}$ was an arbitrary $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$ generic filter containing q_{n+1}, we can use the maximal principle in $\mathbf{V}$ to end the proof by finding a $\mathbb{P}_{\gamma_{n+1}}$ name $\mathring{p}_{n+1}$ so that

(∗) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_{n+1} \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$

(∗∗) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \leq \mathring{p}_n$

(∗∗∗) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k \geq n \left[\mathring{f}[\mathring{G}_{\gamma_{n+1}}]\,(k) \in S(k)\right]$.

$\dashv$

We will use Theorem 3.5.8 to show that the property of strongly preserving a strongly MAD family is preserved. Our proof of this will proceed by induction. However, just as in the case of the proof of the preservation of properness, we will have to make an inductive assumption that is stronger than simply what we want to prove. We state it below for the case of a two step iteration.

Convention 3.5.9. *In the context of the next Lemma, in order to avoid unnecessary repetitions, we will adopt the convention that for any poset $\mathbb{P}$, $\mathring{G}_{\mathbb{P}}$ is the canonical $\mathbb{P}$ name for a $\mathbb{P}$ generic filter.*

Lemma 3.5.10. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. Let $\mathbb{P}$ be a poset which strongly preserves $\mathscr{A}$ and let $\mathring{\mathbb{Q}}$ be a $\mathbb{P}$ name for a poset so that $\Vdash_{\mathbb{P}} \mathring{\mathbb{Q}}$ strongly preserves $\mathscr{A}$. Suppose*

$M \prec H(\theta)$ is countable with $\mathscr{A}, \mathbb{P}, \mathring{\mathbb{Q}} \in M$. Fix $h \in \mathscr{A}$ that covers M with respect to $\mathscr{A}$. Let $p \in \mathbb{P}$ and let $\mathring{r}$ be a $\mathbb{P}$ name such that

1. *p is $(M, \mathbb{P})$ generic*

2. *$p \Vdash_{\mathbb{P}} h$ covers $M[\mathring{G}_{\mathbb{P}}]$ with respect to $\mathscr{A}$*

3. *$p \Vdash_{\mathbb{P}} \mathring{r} \in \mathbb{P} * \mathring{\mathbb{Q}} \cap M \wedge \mathring{r}(0) \in \mathring{G}_{\mathbb{P}}$.*

In this case, there is a $\mathring{q} \in \operatorname{dom}(\mathring{\mathbb{Q}})$ such that

(1^+) $\langle p, \mathring{q}\rangle$ is $(M, \mathbb{P} * \mathring{\mathbb{Q}})$ generic

(2^+) $\langle p, \mathring{q}\rangle \Vdash_{\mathbb{P}*\mathring{\mathbb{Q}}} h$ covers $M[\mathring{G}_{\mathbb{P}*\mathring{\mathbb{Q}}}]$ with respect to $\mathscr{A}$

(3^+) $\langle p, \mathring{q}\rangle \Vdash_{\mathbb{P}*\mathring{\mathbb{Q}}} \mathring{r} \in \mathring{G}_{\mathbb{P}*\mathring{\mathbb{Q}}}$.

Proof. Let $G_{\mathbb{P}}$ be a $(\mathbf{V}, \mathbb{P})$ generic filter with $p \in G_{\mathbb{P}}$. Within $\mathbf{V}[G_P]$, form $M[G_{\mathbb{P}}]$ and notice that $M[G_{\mathbb{P}}] \prec H(\theta)[G_{\mathbb{P}}]$ and that $H(\theta)[G_{\mathbb{P}}]$ is the same as $H(\theta)$ as computed within $\mathbf{V}[G_{\mathbb{P}}]$. Now, by assumption, h covers $M[G_{\mathbb{P}}]$ with respect to $\mathscr{A}$. Also, $\mathscr{A}, \mathring{\mathbb{Q}}[G_{\mathbb{P}}] \in M[G_{\mathbb{P}}]$, and $\mathring{\mathbb{Q}}[G_{\mathbb{P}}]$ strongly preserves $\mathscr{A}$. Next, by assumption, $\mathring{r}[G_{\mathbb{P}}] \in \mathbb{P} * \mathring{\mathbb{Q}} \cap M$. So there are $p' \in \mathbb{P} \cap M$ and $\mathring{q}' \in \operatorname{dom}(\mathring{\mathbb{Q}}) \cap M$ such that $\mathring{r}[G_{\mathbb{P}}](0) = p'$ and $\mathring{r}[G_{\mathbb{P}}](1) = \mathring{q}'$. Moreover, $p' \in G_{\mathbb{P}}$. It follows that $\mathring{q}'[G_{\mathbb{P}}] \in \mathring{\mathbb{Q}}[G_{\mathbb{P}}] \cap M[G_{\mathbb{P}}]$. Thus, we may find a $q \leq \mathring{q}'[G_{\mathbb{P}}]$ such that in $\mathbf{V}[G_{\mathbb{P}}]$

(a) $q \in \mathring{\mathbb{Q}}[G_{\mathbb{P}}]$ is $(M[G_{\mathbb{P}}], \mathring{\mathbb{Q}}[G_{\mathbb{P}}])$ generic

(b) $q \Vdash_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]} h$ covers $M[G_{\mathbb{P}}]\left[\mathring{G}_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}\right]$ with respect to $\mathscr{A}$.

(c) $q \leq \mathring{r}[G_{\mathbb{P}}](1)[G_{\mathbb{P}}]$.

Therefore, since $G_{\mathbb{P}}$ was an arbitrary $(\mathbf{V}, \mathbb{P})$ generic filter containing p, we may use the maximal principle in $\mathbf{V}$ to find $\mathring{q} \in \operatorname{dom}(\mathring{\mathbb{Q}})$ such that

(a') $p \Vdash_{\mathbb{P}} \mathring{q} \in \mathring{\mathbb{Q}}$ is $(M[\mathring{G}_{\mathbb{P}}], \mathring{\mathbb{Q}})$ generic

(b') $p \Vdash_{\mathbb{P}} \mathring{q} \Vdash_{\mathring{\mathbb{Q}}} h$ covers $M[\mathring{G}_{\mathbb{P}}][\mathring{G}_{\mathring{\mathbb{Q}}}]$ with respect to $\mathscr{A}$.

(c') $p \Vdash_{\mathbb{P}} \mathring{q} \leq \mathring{r}(1)[\mathring{G}_{\mathbb{P}}]$.

We argue that $\mathring{q}$ is as needed. Indeed, by (a') above and by the fact that p is $(M, \mathbb{P})$ generic, it is easily follows that $\langle p, \mathring{q}\rangle$ is $(M, \mathbb{P} * \mathring{\mathbb{Q}})$ generic.

Next, we argue that $\langle p, \mathring{q}\rangle \Vdash_{\mathbb{P}*\mathring{\mathbb{Q}}} h$ covers $M[\mathring{G}_{\mathbb{P}*\mathring{\mathbb{Q}}}]$ with respect to $\mathscr{A}$. Let $G_{\mathbb{P}*\mathring{\mathbb{Q}}}$ be a $(V, \mathbb{P} * \mathring{\mathbb{Q}})$ generic filter with $\langle p, \mathring{q}\rangle \in G_{\mathbb{P}*\mathring{\mathbb{Q}}}$. Notice that there is a $(\mathbf{V}, \mathbb{P})$ generic filter $G_{\mathbb{P}}$ and a $(\mathbf{V}[G_{\mathbb{P}}], \mathring{\mathbb{Q}}[G_{\mathbb{P}}])$ generic filter $G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$ such that in $\mathbf{V}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}]$, $G_{\mathbb{P}*\mathring{\mathbb{Q}}} = G_{\mathbb{P}} * G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$. Moreover, $p \in G_{\mathbb{P}}$ and $\mathring{q}[G_{\mathbb{P}}] \in G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$. Therefore in $\mathbf{V}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}]$, we have that h covers $M[G_{\mathbb{P}}]\left[G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}\right]$ with respect to $\mathscr{A}$. Now, let $f \in M[G_{\mathbb{P}*\mathring{\mathbb{Q}}}]$ be an infinte partial function avoiding $\mathscr{A}$. We can find a $\mathbb{P} * \mathring{\mathbb{Q}}$ name $\mathring{f} \in M$ with $\mathring{f}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}] = f$. But we can think of $\mathring{f}$ as a $\mathbb{P}$ name for a $\mathring{\mathbb{Q}}$ name. So there is a $\mathbb{P}$ name $\mathring{f}[\mathring{G}_{\mathbb{P}}] \in M$ such that $\mathring{f}[G_{\mathbb{P}}]\left[G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}\right] = \mathring{f}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}] = f$. Thus $\mathring{f}[G_{\mathbb{P}}]$ is a $\mathring{Q}[G_{\mathbb{P}}]$ name in $M[G_{\mathbb{P}}]$ and so $f \in M[G_{\mathbb{P}}]\left[G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}\right]$. Therefore, $|h \cap f| = \omega$, as needed.

Finally we must argue that $\langle p, \mathring{q}\rangle \Vdash_{\mathbb{P}*\mathring{\mathbb{Q}}} \mathring{r} \in \mathring{G}_{\mathbb{P}*\mathring{\mathbb{Q}}}$. Let $G_{\mathbb{P}*\mathring{\mathbb{Q}}}, G_{\mathbb{P}}$ and $G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$ be as in the last paragraph. Once again, notice that since $\langle p, \mathring{q}\rangle \in G_{\mathbb{P}*\mathring{\mathbb{Q}}}$, $p \in G_{\mathbb{P}}$ and $\mathring{q}[G_{\mathbb{P}}] \in G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$. Notice also that since $\mathring{r}$ is a $\mathbb{P}$ name, $\mathring{r}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}] = \mathring{r}[G_{\mathbb{P}}]$. Within $\mathbf{V}[G_{\mathbb{P}}]$, we have that $\mathring{r}[G_{\mathbb{P}}] = \langle p', \mathring{q}'\rangle \in \mathbb{P} * \mathring{\mathbb{Q}}$, where $p' \in \mathbb{P}$ and $\mathring{q}' \in \operatorname{dom}(\mathring{\mathbb{Q}})$. Also by (3) above, $p' \in G_{\mathbb{P}}$, and so, $\mathring{q}'[G_{\mathbb{P}}] \in \mathring{\mathbb{Q}}[G_{\mathbb{P}}]$. Moreover, by (c') above, $\mathring{q}[G_{\mathbb{P}}] \leq \mathring{q}'[G_{\mathbb{P}}]$. Since $\mathring{q}[G_{\mathbb{P}}] \in G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$, $\mathring{q}'[G_{\mathbb{P}}] \in G_{\mathring{\mathbb{Q}}[G_{\mathbb{P}}]}$ as well. Therefore, in $\mathbf{V}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}]$, it follows that $\langle p', \mathring{q}'\rangle = \mathring{r}[G_{\mathbb{P}*\mathring{\mathbb{Q}}}] \in G_{\mathbb{P}*\mathring{\mathbb{Q}}}$, as required. $\dashv$

We will now prove the same for iterations of arbitrary length. We will make use of Theorem 3.5.8 in conjunction with the following, which is similar to Lemma 3.4.7.

Lemma 3.5.11. *Let $\mathscr{A} \subset \omega^{\omega}$ be an infinite a.d. family and let $\mathbb{P}$ be a poset that does not add any eventually different reals. Let $\mathring{f}$ be a $\mathbb{P}$ name for which the following holds: $\Vdash$ $\mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. For each $p \in \mathbb{P}$, there is $q \leq p$ and an infinite partial function f avoiding $\mathscr{A}$ such that for each $n \in \operatorname{dom}(f)$ there exists $r \leq q$ such that*

$r \Vdash n \in \operatorname{dom}(\mathring{f}) \wedge \mathring{f}(n) = f(n)$.

Proof. We use the well known fact that a poset which does not add eventually different reals does not make $\mathbf{V} \cap \omega^{\omega}$ meager (see proof of Lemma 2.4.8 in [5]). By Lemma 3.4.7, there is a countable set of $\mathbb{P}$ names $\left\{\mathring{f}_i : i \in \omega\right\}$ such that

1. $\Vdash \mathring{f}_i \subset \mathring{f}$ is an infinite partial function

2. $\Vdash \forall g \subset \omega \times \omega \left[\text{if } \forall i \in \omega \left[\left|g \cap \mathring{f}_i\right| = \omega\right], \text{ then } g \text{ avoids } \mathscr{A}\right]$.

Fix a condition $p \in \mathbb{P}$. As $\mathbb{P}$ does not make $\mathbf{V} \cap \omega^{\omega}$ meager, there is $q \leq p$ and $h \in \omega^{\omega}$ such that for each $i \in \omega$, $q \Vdash \left|h \cap \mathring{f}_i\right| = \omega$. Put $X = \left\{n \in \omega : \exists r \leq q \left[r \Vdash n \in \operatorname{dom}(\mathring{f}) \wedge h(n) = \mathring{f}(n)\right]\right\}$ and set $f = h \restriction X$. To finish the proof, by (2) above, it is enough to check that for each $i \in \omega$, $q \Vdash \left|f \cap \mathring{f}_i\right| = \omega$. Indeed, suppose $r \leq q$ and $k \in \omega$. Since $q \Vdash \left|h \cap \mathring{f}_i\right| = \omega$, there is $s \leq r$ and $n > k$ such that $s \Vdash n \in \operatorname{dom}(\mathring{f}_i) \wedge h(n) = \mathring{f}_i(n)$. But since $\Vdash \mathring{f}_i \subset \mathring{f}$, it follows that $s \Vdash n \in \operatorname{dom}(\mathring{f}) \wedge h(n) = \mathring{f}(n)$, which means that $n \in \operatorname{dom}(f)$. $\dashv$

Theorem 3.5.12. *Let $\mathscr{A} \subset \omega^{\omega}$ be a strongly MAD family. Let $\langle \mathbb{P}_{\alpha}, \mathring{\mathbb{Q}}_{\alpha} : \alpha \leq \gamma \rangle$ be a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_{\alpha} \mathring{\mathbb{Q}}_{\alpha} \text{ strongly preserves } \mathscr{A}\right]$. Let $M \prec H(\theta)$ be countable with $\langle \mathbb{P}_{\alpha}, \mathring{\mathbb{Q}}_{\alpha} : \alpha \leq \gamma \rangle, \mathscr{A} \in M$. Let $h \in \mathscr{A}$ cover M with respect to $\mathscr{A}$. Suppose $\gamma_0 \in \gamma \cap M$ and let $q_0 \in \mathbb{P}_{\gamma_0}$ be a $(M, \mathbb{P}_{\gamma_0})$ generic condition such that $q_0 \Vdash_{\gamma_0} h$ covers $M[\mathring{G}_{\gamma_0}]$ with respect to $\mathscr{A}$. Let $\mathring{p}_0$ be a $\mathbb{P}_{\gamma_0}$ name such that $q_0 \Vdash_{\gamma_0} \mathring{p}_0 \in \mathbb{P}_{\gamma} \cap M \wedge \mathring{p}_0 \restriction \gamma_0 \in \mathring{G}_{\gamma_0}$. There is a $q \in \mathbb{P}_{\gamma}$ with $q \restriction \gamma_0 = q_0$, which is $(M, \mathbb{P}_{\gamma})$ generic, such that $q \Vdash_{\gamma} h$ covers $M[\mathring{G}_{\gamma}]$ with respect to $\mathscr{A}$, and such that $q \Vdash_{\gamma} \mathring{p}_0 \in \mathring{G}_{\gamma}$. In particular, $\mathbb{P}_{\gamma}$ strongly preserves $\mathscr{A}$.*

Proof. Before proving the main claim of the Theorem, we remark that the last sentence of the Theorem easily follows from the main claim. To see this, suppose $p_0 \in \mathbb{P}_{\gamma} \cap M$ is a condition. Now, apply the main claim of the Theorem with $\gamma_0 = 0$, the trivial condition as q_0, and $\mathring{p}_0 = p_0$.

The proof of the main claim is by induction on γ. Let us assume that the theorem holds for all $\alpha < \gamma$. The case when γ is a successor has already been dealt with in Lemma 3.5.10. So we assume that γ is a limit ordinal. We observe that it follows from our inductive hypothesis that no $\mathbb{P}_\alpha$ adds eventually different reals, for $\alpha < \gamma$. As γ is a limit ordinal, it follows from Theorem 3.5.8 that $\mathbb{P}_\gamma$ does not add an eventually different real. We will make use of this observation in what follows.

Put $\gamma' = \sup(\gamma \cap M)$ and let $\langle \gamma_n : n \in \omega \rangle \subset M \cap \gamma$ be an increasing sequence that is cofinal in γ'. Let $\langle D_n : n \in \omega \rangle$ enumerate all the dense open subsets of $\mathbb{P}_\gamma$ that are elements of M. Also, we let $\langle \mathring{f}_i : i \in \omega \rangle$ enumerate all $\mathbb{P}_\gamma$ names in M for which it is the case that $\Vdash_\gamma \mathring{f}_i$ is an infinite partial function avoiding $\mathscr{A}$. We will build two sequences $\langle q_n : n \in \omega \rangle$ and $\langle \mathring{p}_n : n \in \omega \rangle$ such that the following hold:

1. q_0 is given, $q_n \in \mathbb{P}_{\gamma_n}$, q_n is $(M, \mathbb{P}_{\gamma_n})$ generic, and $q_{n+1} \restriction \gamma_n = q_n$
2. $q_n \Vdash_{\gamma_n} h$ covers $M[\mathring{G}_{\gamma_n}]$ with respect to $\mathscr{A}$
3. $\mathring{p}_0 = p_0$, $\mathring{p}_n \in \mathbf{V}^{\mathbb{P}_{\gamma_n}}$, and $q_n \Vdash_{\gamma_n} \mathring{p}_n \in M \cap \mathbb{P}_\gamma \wedge \mathring{p}_n \restriction \gamma_n \in \mathring{G}_{\gamma_n}$
4. $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \in D_n \wedge \mathring{p}_{n+1} \leq \mathring{p}_n$.
5. $\forall i \leq n \left[q_{n+1} \Vdash_{\gamma_{n+1}} \Phi_i\right]$, where Φ_i is this formula in $\mathbb{P}_{\gamma_{n+1}}$ forcing language:
   $\mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k_i^n \geq n \left[k_i^n \in \operatorname{dom}\left(\mathring{f}_i[\mathring{G}_{\gamma_{n+1}}]\right) \wedge \mathring{f}_i[\mathring{G}_{\gamma_{n+1}}]\,(k_i^n) = h(k_i^n)\right]$.

Before describing how to construct such sequences, we will argue that it is enough to do so. Put $q = (\bigcup q_n)^\frown \mathring{\mathbb{1}} \in \mathbb{P}_\gamma$. By lemma 3.5.1, $\forall n \in \omega \left[q \Vdash_\gamma \mathring{p}_n \in \mathring{G}_\gamma\right]$. We will first argue that q is $(M, \mathbb{P}_\gamma)$ generic. It suffices to show that for each $n \in \omega$, $q \Vdash_\gamma D_n \cap M \cap \mathring{G}_\gamma \neq 0$. But by clauses (3) and (4) and by Lemma 3.5.1, it is clear that $q \Vdash_\gamma \mathring{p}_{n+1} \in D_n \cap M \cap \mathring{G}_\gamma$. That $q \Vdash_\gamma h$ covers $M[\mathring{G}_\gamma]$ with respect to $\mathscr{A}$ will be verified next. We will first argue that it is sufficient to show that $\forall i \in \omega \left[q \Vdash_\gamma \left|h \cap \mathring{f}_i\right| = \omega\right]$. Assume this and let G_γ be a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter with $q \in G_\gamma$. Let $f \in M[G_\gamma]$ be an infinite partial function avoiding $\mathscr{A}$. There is a $\mathbb{P}_\gamma$ name $\mathring{f} \in M$ such that $\mathring{f}[G_\gamma] = f$. But by elementarity of M, we can find such a $\mathring{f}$ with the additional property that $\Vdash_\gamma \mathring{f}$ is an infinite partial function avoiding $\mathscr{A}$. Thus, $f = \mathring{f}_i[G_\gamma]$, for some $i \in \omega$, and so $|h \cap f| = \omega$. We will now check that $\forall i \in \omega \left[q \Vdash_\gamma \left|h \cap \mathring{f}_i\right| = \omega\right]$. Fix $i \in \omega$. Let $r \leq q$ and let $m \in \omega$. Choose a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter with $r \in G_\gamma$. Choose $n \in \omega$ with $m, i \leq n$. By Lemma 3.5.5, we know that G_γ is $\left(\mathbf{V}[G_{\gamma_{n+1}}], \mathbb{P}_\gamma / G_{\gamma_{n+1}}\right)$ generic and that in $\mathbf{V}[G_\gamma]$, $G_\gamma = G_{\gamma_{n+1}} * G_\gamma$ holds. Notice that $q \in G_\gamma$ and therefore, $q_{n+1} \in G_{\gamma_{n+1}}$. Also, since $\mathring{p}_{n+1}$ is a $\mathbb{P}_{\gamma_{n+1}}$ name, $\mathring{p}_{n+1}[G_\gamma] = \mathring{p}_{n+1}[G_{\gamma_{n+1}}]$. It follows from clauses (3) and (5) that $\mathring{p}_{n+1}[G_\gamma] \in M \cap \mathbb{P}_\gamma$, that $\mathring{p}_{n+1}[G_\gamma] \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$, and that in $\mathbf{V}[G_{\gamma_{n+1}}]$, $\mathring{p}_{n+1}[G_\gamma] \Vdash_{\mathbb{P}_\gamma / G_{\gamma_{n+1}}} \exists k_i^n \geq n \left[k_i^n \in \operatorname{dom}\left(\mathring{f}_i[G_{\gamma_{n+1}}]\right) \wedge \mathring{f}_i[G_{\gamma_{n+1}}](k_i^n) = h(k_i^n)\right]$. On the other hand, we know from Lemma 3.5.1 that $\mathring{p}_{n+1}[G_\gamma] \in G_\gamma$. Therefore, in $\mathbf{V}[G_\gamma]$, we are able to find a $k_i^n \geq n \geq m$ such that $k_i^n \in \operatorname{dom}\left(\mathring{f}_i[G_{\gamma_{n+1}}][G_\gamma]\right)$ and $\mathring{f}_i[G_{\gamma_{n+1}}][G_\gamma]\ (k_i^n) = h(k_i^n)$. However, $\mathring{f}_i[G_{\gamma_{n+1}}][G_\gamma] = \mathring{f}_i[G_\gamma]$. As $r \in G_\gamma$, there is $s \leq r$ so that $s \Vdash_\gamma k_i^n \in \operatorname{dom}(\mathring{f}_i) \wedge \mathring{f}_i(k_i^n) = h(k_i^n)$. As $k_i^n \geq m$, this is as needed.

Next we describe how to construct $\langle q_n : n \in \omega\rangle$ and $\langle \mathring{p}_n : n \in \omega\rangle$. q_0 and $\mathring{p}_0$ are both given to us. Now assume that q_n and $\mathring{p}_n$ are given. We can apply the inductive hypothesis to the iteration $\langle \mathbb{P}_\alpha, \mathring{Q}_\alpha : \alpha \leq \gamma_{n+1}\rangle$, the elementary submodel M, the ordinal γ_n, the condition q_n and a $\mathbb{P}_{\gamma_n}$ name forced by q_n to equal $\mathring{p}_n \restriction \gamma_{n+1}$ to find a $(M, \mathbb{P}_{\gamma_{n+1}})$ generic condition q_{n+1} such that

(a) $q_{n+1} \restriction \gamma_n = q_n$

(b) $q_{n+1} \Vdash_{\gamma_{n+1}} h$ covers $M[\mathring{G}_{\gamma_{n+1}}]$ with respect to $\mathscr{A}$

(c) $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_n \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$.

To find $\mathring{p}_{n+1}$ we proceed as follows. Let $G_{\gamma_{n+1}}$ be a $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$ generic filter with $q_{n+1} \in G_{\gamma_{n+1}}$. We begin with some general observations. Note that $\mathring{p}_n[G_{\gamma_{n+1}}] \in \mathbb{P}_\gamma \cap M$. Also, $\mathring{p}_n[G_{\gamma_{n+1}}] \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$. Thus we conclude that $\mathring{p}_n[G_{\gamma_{n+1}}]$ is a condition in $\mathbb{P}_\gamma / G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$. Moreover, $\mathbb{P}_\gamma / G_{\gamma_{n+1}} \in M[G_{\gamma_{n+1}}]$. Now, we will describe how to take care of the dense open set D_n. We make use of the fact that if $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection, and if $D \subset \mathbb{Q}$ is dense and if $G \subset \mathbb{P}$ is a $(\mathbf{V}, \mathbb{P})$ generic filter, then in $\mathbf{V}[G]$, $D/G = D \cap \mathbb{Q}/G$ is dense in $\mathbb{Q}/G$. Applying this to $\mathbb{P}_\gamma$, $\mathbb{P}_{\gamma_{n+1}}$ and D_n, we conclude that $D_n \cap \mathbb{P}_\gamma/G_{\gamma_{n+1}}$ is dense in $\mathbb{P}_\gamma/G_{\gamma_{n+1}}$. Since $D_n, \mathring{p}_n[G_{\gamma_{n+1}}] \in M[G_{\gamma_{n+1}}]$, we can find a $p^0 \in D_n \cap \mathbb{P}_\gamma/G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$ such that $p^0 \leq \mathring{p}_n[G_{\gamma_{n+1}}]$. We note here that since D_n is open, any further extension of p^0 will stay within D_n. Next, we describe how to deal with $\mathring{f}_0[G_{\gamma_{n+1}}]$. First of all, since in the ground model $\mathbf{V}$, $\Vdash_\gamma \mathring{f}_0$ is an infinite partial function avoiding $\mathscr{A}$, we have that in $\mathbf{V}[G_{\gamma_{n+1}}]$, $\Vdash_{\mathbb{P}_\gamma/G_{\gamma_{n+1}}} \mathring{f}_0[G_{\gamma_{n+1}}]$ is an infinite partial function avoiding $\mathscr{A}$. Moreover, we observed above that $\mathbb{P}_\gamma$ does not add eventually different reals. As $\mathbb{P}_\gamma$ is forcing equivalent to $\mathbb{P}_{\gamma_{n+1}} * \mathbb{P}_\gamma/\mathring{G}_{\gamma_{n+1}}$, it follows that $\mathbb{P}_\gamma/G_{\gamma_{n+1}}$ does not add eventually different reals over $\mathbf{V}[G_{\gamma_{n+1}}]$. As $\mathring{f}_0[G_{\gamma_{n+1}}] \in M[G_{\gamma_{n+1}}]$, we can apply Lemma 3.5.11 to $\mathbb{P}_\gamma/G_{\gamma_{n+1}}$ to find a $\tilde{p}^0 \in \mathbb{P}_\gamma/G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$ with $\tilde{p}^0 \leq p^0$ and an infinite partial function $f \in M[G_{\gamma_{n+1}}]$ as in the Lemma which avoids $\mathscr{A}$. But by (b) above, h covers $M[G_{\gamma_{n+1}}]$ with respect to $\mathscr{A}$. Therefore, $|h \cap f| = \omega$. Choose $k_0^n \geq n$ such that $k_0^n \in \operatorname{dom}(f)$ and $h(n) = f(n)$. By the Lemma, there is a $p^1 \leq \tilde{p}^0$ in $\mathbb{P}_\gamma/G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$ so that $p^1 \Vdash_{\mathbb{P}_\gamma/G_{\gamma_{n+1}}} k_0^n \in \operatorname{dom}\left(\mathring{f}_0[G_{\gamma_{n+1}}]\right) \wedge \mathring{f}_0[G_{\gamma_{n+1}}](k_0^n) = f(k_0^n) = h(k_0^n)$. Repeating this argument another n times we get $p^{n+1} \in \mathbb{P}_\gamma/G_{\gamma_{n+1}} \cap M[G_{\gamma_{n+1}}]$ with $p^{n+1} \leq \mathring{p}_n[G_{\gamma_{n+1}}]$ as well as numbers $k_i^n \geq n$, for each $i \leq n$ so that for each such i, we have $p^{n+1} \Vdash_{\mathbb{P}_\gamma/G_{\gamma_{n+1}}} k_i^n \in \operatorname{dom}\left(\mathring{f}_i[G_{\gamma_{n+1}}]\right) \wedge \mathring{f}_i[G_{\gamma_{n+1}}](k_i^n) = h(k_i^n)$. Now, we note that since q_{n+1} is a $(M, \mathbb{P}_{\gamma_{n+1}})$ condition and $q_{n+1} \in G_{\gamma_{n+1}}$, $M[\mathbb{P}_{\gamma_{n+1}}] \cap \mathbb{P}_\gamma = M \cap \mathbb{P}_\gamma$. Therefore, p^{n+1} is in fact in M. Thus we have found a condition p^{n+1} with the following properties:

(i) $p^{n+1} \in \mathbb{P}_\gamma \cap M$ and $p^{n+1} \restriction \gamma_{n+1} \in G_{\gamma_{n+1}}$.

(ii) $p^{n+1} \leq \mathring{p}_n[G_{\gamma_{n+1}}]$ and $p^{n+1} \in D_n$

(iii) $\forall i \leq n \left[p^{n+1} \Vdash_{\mathbb{P}_\gamma/G_{\gamma_{n+1}}} \Phi_i\right]$, where Φ_i is this formula
$\exists k_i^n \geq n \left[k_i^n \in \operatorname{dom}\left(\mathring{f}_i[G_{\gamma_{n+1}}]\right) \wedge \mathring{f}_i[G_{\gamma_{n+1}}](k_i^n) = h(k_i^n)\right]$.

Since $G_{\gamma_{n+1}}$ was an arbitrary $(\mathbf{V}, \mathbb{P}_{\gamma_{n+1}})$ generic filter containing q_{n+1} we can use the maximal principle in $\mathbf{V}$ to find a $\mathbb{P}_{\gamma_{n+1}}$ name $\mathring{p}_{n+1}$ so that

(i') $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \in \mathbb{P}_\gamma \cap M \wedge \mathring{p}_{n+1} \restriction \gamma_{n+1} \in \mathring{G}_{\gamma_{n+1}}$.

(ii') $q_{n+1} \Vdash_{\gamma_{n+1}} \mathring{p}_{n+1} \leq \mathring{p}_n \wedge \mathring{p}_{n+1} \in D_n$

(iii') $\forall i \leq n \left[q_{n+1} \Vdash_{\gamma_{n+1}} \Phi_i\right]$, where Φ_i is this formula

$$\mathring{p}_{n+1} \Vdash_{\mathbb{P}_\gamma / \mathring{G}_{\gamma_{n+1}}} \exists k_i^n \geq n \left[k_i^n \in \operatorname{dom}\left(\mathring{f}_i[\mathring{G}_{\gamma_{n+1}}]\right) \wedge \mathring{f}_i[\mathring{G}_{\gamma_{n+1}}](k_i^n) = h(k_i^n)\right].$$

$\dashv$

Corollary 3.5.13. *Let $\mathscr{A} \subset \omega^\omega$ be a strongly MAD family. If $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a CS iteration such that $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha \text{ has diagonal fusion}\right]$, then $\mathbb{P}_\gamma$ strongly preserves $\mathscr{A}$. In particular is $\forall \alpha < \gamma \left[\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha = \mathbb{M} \wedge \mathring{\mathbb{Q}}_\alpha = \mathbb{S}\right]$, then $\mathbb{P}_\gamma$ strongly preserves $\mathscr{A}$.*

$\dashv$

3.6 It is Consistent That There are no Strongly MAD Families of Size $\mathfrak{c}$

In this section we show that it is consistent that there are no strongly MAD families of size continuum. In fact, we will show that this holds in the Cohen model: the model one gets by adding $\aleph_2$ Cohen reals to a ground model satisfying CH. The proof is similar to the well known result, due to Kunen, that in the Cohen model there are no well ordered chains of length ω_2 in $\mathcal{P}(\omega)/\operatorname{Fin}$. Given a name for a strongly MAD family of size $\aleph_2$, we use an elementary submodel to restrict the name to an initial segment of the iteration, and then we get a contradiction using the Cohen-indestructability of strongly MAD families.

Theorem 3.6.1. *There are no strongly MAD families of size continuum in the Cohen model.*

Proof. Let $\langle \mathring{h}_\alpha : \alpha < \omega_2 \rangle$ be a collection of $Fn(\omega_2, 2)$ names such that $\Vdash_{\omega_2} \langle \mathring{h}_\alpha : \alpha < \omega_2 \rangle$ is a strongly MAD family in ω^ω. We may assume that each $\mathring{h}_\alpha$ is a nice name for a subset of $\omega \times \omega$. Fix M so that $M \prec H(\theta)$ with $\langle \mathring{h}_\alpha : \alpha < \omega_2 \rangle \in M$, $M^\omega \subset M$ and $|M| = \omega_1$. We can do this because CH is assumed to be true in the ground model. Let $\delta = M \cap \omega_2$. Note that for each $\alpha < \delta$, $\mathring{h}_\alpha \in M$. Thus, $\mathring{h}_\alpha$ is a $Fn(\delta, 2)$ name, for each $\alpha < \delta$. We will now prove:

Lemma 3.6.2. *$\Vdash_\delta \langle \mathring{h}_\alpha : \alpha < \delta \rangle$ is a strongly MAD family in ω^ω.*

Proof. It is clear that for any $\alpha < \beta < \delta$, $\Vdash_\delta \left| \mathring{h}_\alpha \cap \mathring{h}_\beta \right| < \omega$. Now, fix $p \in Fn(\delta, 2)$ and let $\langle \mathring{f}_i : i \in \omega \rangle$ be a collection of $Fn(\delta, 2)$ names for reals such that:

$$p \Vdash_\delta \langle \mathring{f}_i : i \in \omega \rangle \text{ is a countable family avoiding } \langle \mathring{h}_\alpha : \alpha < \delta \rangle. \tag{$*_1$}$$

Again, we may assume that each $\mathring{f}_i$ is a nice name for a subset of $\omega \times \omega$. We will find a $q \in Fn(\delta, 2)$ extending p, and $\alpha \in \delta$ such that $\forall i \in \omega \left[q \Vdash_\delta \left| \mathring{f}_i \cap \mathring{h}_\alpha \right| = \omega \right]$. Clearly, this is sufficient to prove the Lemma. We will prove this via a series of claims. We first claim:

Claim 3.6.3. *Let $X \in M$ be a countable set. Any nice $Fn(\delta, 2)$ name for a subset of X is an element of M.*

Proof. Any nice $Fn(\delta, 2)$ name for a subset of X may be identified with a set of the form $\{A_x : x \in X\}$, where each A_x is an antichain in $Fn(\delta, 2)$. Since $Fn(\delta, 2)$ is CCC, each A_x is a countable subset of M. Since $M^\omega \subset M$, it follows that $A_x \in M$, for each $x \in X$. Now, since X is countable, we conclude that $\{A_x : x \in X\} \in M$. $\dashv$

Claim 3.6.4. *$p \Vdash_{\omega_2} \langle \mathring{f}_i : i \in \omega \rangle$ is a countable family of reals avoiding $\langle \mathring{h}_\alpha : \alpha < \omega_2 \rangle$.*

Proof. Suppose not. Then the following statement is true:

$$\exists q \in Fn(\omega_2, 2) \exists i \in \omega \exists \alpha_0, \ldots, \alpha_n \in \omega_2 \tag{$*_2$}$$
$$\left[q \leq p \wedge q \Vdash_{\omega_2} \mathring{f}_i \subset^* \mathring{h}_{\alpha_0} \cup \cdots \cup \mathring{h}_{\alpha_n} \right].$$

From Claim 3.6.3, we gather that for each $i \in \omega$, $\mathring{f}_i \in M$, and hence that $\langle \mathring{f}_i : i \in \omega \rangle \in M$. Thus, all of the parameters occurring in $(*_2)$ are elements of M, whence we conclude that $(*_2)$ is true in M. Therefore, we get $q \in Fn(\delta, 2)$, $i \in \omega$ and $\alpha_0, \dots, \alpha_n \in \delta$ so that $q \leq p$ and $q \Vdash_{\omega_2} \mathring{f}_i \subset^* \mathring{h}_{\alpha_0} \cup \dots \cup \mathring{h}_{\alpha_n}$. But since $q \in Fn(\delta, 2)$ and since $\mathring{f}_i$ and $\mathring{h}_{\alpha_0}, \dots, \mathring{h}_{\alpha_n}$ are all $Fn(\delta, 2)$ names, we get that $q \Vdash_\delta \mathring{f}_i \subset^* \mathring{h}_{\alpha_0} \cup \dots \cup \mathring{h}_{\alpha_n}$. But this contradicts $(*_1)$ above. $\dashv$

Now to prove the Lemma, by claim 3.6.4, the following is true:

$$\exists q \in Fn(\omega_2, 2) \exists \alpha \in \omega_2 \left[q \leq p \wedge \forall i \in \omega \left[q \Vdash_{\omega_2} \left| \mathring{f}_i \cap \mathring{h}_\alpha \right| = \omega \right] \right] \qquad (*_3)$$

Again, all the parameters occurring in $(*_3)$ are in M. So $(*_3)$ is true in M, and so we get $q \in Fn(\delta, 2)$ and $\alpha \in \delta$ such that $q \leq p$ and $\forall i \in \omega \left[q \Vdash_{\omega_2} \left| \mathring{f}_i \cap \mathring{h}_\alpha \right| = \omega \right]$. But again, since $q \in Fn(\delta, 2)$ and since $\mathring{f}_i$ and $\mathring{h}_\alpha$ are $Fn(\delta, 2)$ names, we get that $\forall i \in \omega \left[q \Vdash_\delta \left| \mathring{f}_i \cap \mathring{h}_\alpha \right| = \omega \right]$, which is exactly as required. $\dashv$

Now we can get a contradiction using the Cohen indestructibility of strongly MAD families. Let G be a $(\mathbf{V}, Fn(\omega_2, 2))$ generic filter, and let G_δ denote the restriction of G to $Fn(\delta, 2)$. By Lemma 3.6.2, $\mathscr{A} = \{\mathring{h}_\alpha [G_\delta] : \alpha \in \delta\}$ is a strongly MAD family in $\mathbf{V}[G_\delta]$. In $\mathbf{V}[G]$, $\mathring{h}_\delta [G]$ is almost disjoint from $\mathscr{A}$. However, every real in $\mathbf{V}[G]$ is in an extension of the form $\mathbf{V}[G_\delta][H]$, where H is $(\mathbf{V}[G_\delta], Fn(\omega, 2))$ generic. So for some H that is $(\mathbf{V}[G_\delta], Fn(\omega, 2))$ generic, $\mathbf{V}[G_\delta][H]$ thinks that $\mathscr{A}$ is not maximal. But this contradicts Corollary 3.4.10 and we are done. $\dashv$

Remark 3.6.5. *Our argument actually shows that if any number of Cohen reals are added to a model satisfying CH, then there are no strongly MAD families of size greater than $\aleph_1$ in the resulting model.*

Remark 3.6.6. *We can use the fact that strongly MAD families are iterated Sacks and iterated Miller indestructible to prove a similar result for the Sacks and Miller models.*

3.7 Miscellaneous Results

We gather together here some assorted results that do not belong in any of the previous sections. Our first result grew out of a conversation we had in a bar with Michael Hrušák. It is possible to consider the notion of a strongly MAD family for families of subsets of ω too. The definition of this concept is identical to our Definition 1.3.6, but with ω^ω replaced everywhere by $[\omega]^\omega$, and with the additional requirement that the family be infinite. It is shown in Hrušák and García Ferreira [16] and Kurilić [23] that a MAD family of subsets of ω is Cohen–indestructible *iff* it is "somewhere" strongly MAD. This led Hrušák to suggest that a similar result is true for MAD families of functions as well. We will show below that this is not the case. Indeed, we will show that assuming CH, we can construct a Cohen–indestructible MAD family of functions that is "nowhere" Van Douwen MAD (and hence "nowhere" Strongly MAD). This shows that Cohen–indestructibility is somewhat different for MAD families of functions.

Theorem 3.7.1. *Assume CH. There is a Cohen–indestructible MAD family $\mathscr{A} \subset \omega^\omega$ with trivial trace (see Definition 1.3.9).*

Proof. To ensure that our family is Cohen–indestructible, we will do a construction similar to the one in Kunen [22]. Let $\mathbb{P} = Fn(\omega, 2)$. Since we are assuming CH, there are only ω_1 nice $\mathbb{P}$ names for elements of ω^ω. Let $\langle\langle p_\alpha, \mathring{f}_\alpha\rangle : \alpha < \omega_1\rangle$ enumerate all pairs $\langle p, \mathring{f}\rangle$ such that $p \in \mathbb{P}$ and $\mathring{f}$ is a nice $\mathbb{P}$ name for an element of ω^ω. Let $\langle g_\alpha : \alpha < \omega_1\rangle$ enumerate ω^ω. An ideal $\mathcal{I}$ of subset of ω is said to be dense if $\forall a \in [\omega]^\omega \exists b \in [a]^\omega\, [\, b \in \mathcal{I}]$. Fix a proper, non–principal dense ideal on ω. Notice that for any such ideal $\mathcal{I}$, if $\{a_i : i \in \omega\} \subset \mathcal{I}$ is a countable collection of infinite sets, then there is an infinite set $b \in \mathcal{I}$ such that $\forall i \in \omega\, [\, |b \cap a_i| < \omega]$. Now, we will build two sequences $\langle \mathscr{A}_\alpha : \alpha < \omega_1\rangle$ and $\langle \mathscr{B}_\alpha : \alpha < \omega_1\rangle$ such that the following hold:

1. $\mathscr{A}_\alpha \subset \omega^\omega$ is a countable a.d. family

2. $\mathscr{B}_\alpha$ is a countable set of infinite partial functions

3. $\forall f \in \mathscr{B}_\alpha\, [\mathrm{dom}\,(f) \in \mathcal{I}]$

4. $\forall \alpha < \beta < \omega_1\, [\mathscr{A}_\alpha \subset \mathscr{A}_\beta \wedge \mathscr{B}_\alpha \subset \mathscr{B}_\beta]$

5. $\forall h \in \mathscr{A}_\alpha \forall f \in \mathscr{B}_\alpha\, [|h \cap f| < \omega]$

6. if g_α avoids $\bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$, then there is $f \subset g_\alpha$ so that $f \in \mathscr{B}_\alpha$

7. if $p_\alpha \Vdash \mathring{f}_\alpha$ is a.d. from $\bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$, then $\exists h \in \mathscr{A}_\alpha \left[p_\alpha \Vdash \left|h \cap \mathring{f}_\alpha\right| = \omega\right]$.

Our MAD family $\mathscr{A}$ will be $\bigcup \mathscr{A}_\alpha$. It is clear from clauses (5) and (6) that $\mathscr{A}$ has trivial trace, while it is easy to see that clause (7) implies that $\mathscr{A}$ is Cohen–indestructible.

Assume that $\langle \mathscr{A}_\beta : \beta < \alpha \rangle$ and $\langle \mathscr{B}_\beta : \beta < \alpha \rangle$ have already been constructed. Set $\mathscr{C} = \bigcup \mathscr{A}_\beta$ and $\mathscr{B} = \bigcup \mathscr{B}_\beta$. $\mathscr{C} \subset \omega^\omega$ is a countable a.d. family and $\mathscr{B}$ is a countable family of infinite partial functions. Moreover, $\forall f \in \mathscr{B} \forall h \in \mathscr{C}\, [|h \cap f| < \omega]$. We will first define $\mathscr{B}_\alpha$, taking care of clause (6). Consider g_α. If g_α does not avoid $\mathscr{C}$, there is nothing to be done, and we simply set $\mathscr{B}_\alpha = \mathscr{B}$. Now, let us assume that g_α avoids $\mathscr{C}$. Since $\mathscr{C}$ is countable, this assumption implies that $\mathscr{C} \cap g_\alpha$ is neither a finite nor an infinite MAD family on g_α. So there is an infinite partial function $p \subset g_\alpha$ which is a.d. from $\mathscr{C}$. Since $\mathcal{I}$ is a dense ideal, there is an infinite partial functions $f \subset p$ with $\mathrm{dom}\,(f) \in \mathcal{I}$. As p is a.d. from $\mathscr{C}$, f is also a.d. from $\mathscr{C}$, and therefore, we can set $\mathscr{B}_\alpha = \mathscr{B} \cup \{f\}$.

Next, we define $\mathscr{A}_\alpha$. Once again, if $p_\alpha \nVdash \mathring{f}_\alpha$ is a.d. from $\mathscr{C}$, there is nothing to be done, and we set $\mathscr{A}_\alpha = \mathscr{C}$. Now, assume that $p_\alpha \Vdash \mathring{f}_\alpha$ is a.d. from $\mathscr{C}$. Put $\mathscr{B}_\alpha = \{f_i : i \in \omega\}$ and $\mathscr{C} = \{h_i : i \in \omega\}$. For each $i \in \omega$, put $a_i = \mathrm{dom}\,(f_i)$. Thus $\{a_i : i \in \omega\}$ is a countable collection of infinite sets in $\mathcal{I}$. By our observation above, there is an infinite set $b \in \mathcal{I}$ such that $\forall i \in \omega\, [\, |b \cap a_i| < \omega]$. We will define an infinite partial function h^0 with $\mathrm{dom}\,(h^0) \subset b$ such that $p_\alpha \Vdash \left|h^0 \cap \mathring{f}_\alpha\right| = \omega$. Observe that for any $i \in \omega$, $h^0 \cap f_i$ will be finite. To get h^0 we proceed as follows. Let $\{q_i : i \in \omega\}$ enumerate $\{q \in \mathbb{P} : q \leq p_\alpha\}$. We will build h^0 as the union of an increasing sequence of finite partial functions. We will build a sequence $\langle h^0_i : -1 \le i < \omega \rangle$ such that for each $i \ge 0$

(a) $h^0_{-1} = 0$ and h^0_i is a finite partial function with $\mathrm{dom}\,(h^0_i) \subset b$

(b) $h^0_{i-1} \subset h^0_i$ and $\forall j \le i \left[h^0_i \cap h_j \subset h^0_{i-1} \cap h_j \right]$

(c) $\exists k_i \ge i\; \exists r \le q_i \left[k_i \in \mathrm{dom}\,(h^0_i) \wedge r \Vdash h^0_i(k_i) = \mathring{f}_\alpha(k_i) \right]$.

Put $h^0 = \bigcup h^0_i$. It is clear from clause (b) that h^0 is a.d. from h_i for all $i \in \omega$. Also, we have from clause (a) that $\mathrm{dom}\,(h^0) \subset b$. We will argue that $p_\alpha \Vdash \left| h^0 \cap \mathring{f}_\alpha \right| = \omega$. Let $q \le p_\alpha$ and let $n \in \omega$. There are infinitely many conditions below q. Hence we can find $i > n$ such that $q_i \le q$. But now by clause (c) there is a $k_i \ge i > n$ and $r \le q_i \le q$ such that $k_i \in \mathrm{dom}\,(h^0_i) \subset \mathrm{dom}\,(h^0)$ and $r \Vdash \mathring{f}_\alpha(k_i) = h^0_i(k_i) = h^0(k_i)$.

We will now describe how to construct $\langle h^0_i : -1 \le i < \omega \rangle$. h^0_{-1} is 0. At stage $i \ge 0$, assume that h^0_{i-1} is given to us. We wish to define h^0_i so that clause (c) is satisfied. But we need to be sure that we introduce no new agreements between h^0_i and any of the members of $\{h_0, \dots, h_i\}$. We know that $q_i \Vdash \mathring{f}_\alpha$ is a.d. from $\{h_0, \dots, h_i\}$. Hence, there is $\tilde{r} \le q_i$ and $l \in \omega$ such that $\tilde{r} \Vdash \forall k > l \left[\mathring{f}_\alpha(k) \notin \{h_0(k), \dots, h_i(k)\} \right]$. Put $m = \max\,(\mathrm{dom}\,(h^0_{i-1}))$. Since b is an infinite set there is a $k_i \in b$ with $k_i > \max\{m, l, i\}$. Now, since $\mathring{f}_\alpha$ is a name for an element of ω^ω, we can find $r \le \tilde{r}$ and $n \in \omega$ such that $r \Vdash \mathring{f}_\alpha(k_i) = n$. Notice that our choice of $\tilde{r}$ entails that $n \notin \{h_0(k_i), \dots, h_i(k_i)\}$. Since $k_i > m$, we can define $h^0_i = h^0_{i-1} \cup \{\langle k_i, n \rangle\}$. As $k_i \in b$, this is as required.

We are almost done. We just need to extend h^0 to a total function. Since both $\mathscr{B}_\alpha$ and $\mathscr{C}$ are countable, there is a total function $h' \in \omega^\omega$ such that $\forall i \in \omega\, [\, |h' \cap f_i| < \omega \wedge |h' \cap h_i| < \omega]$. Put $X = \mathrm{dom}\,(h^0)$ and $Y = \omega \setminus X$. Put $h^1 = h' \restriction Y$ and set $h = h^0 \cup h^1$. It is clear that h is a.d. from both $\mathscr{B}_\alpha$ and $\mathscr{C}$. So we may set $\mathscr{A}_\alpha = \mathscr{C} \cup \{h\}$, and this ends the proof. $\dashv$

Despite certain differences, there are close connections between the notion of a strongly

MAD family of functions and the notion of a strongly MAD family of sets. In particular, the existence of the former implies the existence of the latter.

Lemma 3.7.2. *If there is a strongly MAD family in ω^ω, then there is a strongly MAD family in $[\omega]^\omega$.*

Proof. Let $\mathscr{A} \subset \omega^\omega$ be strongly MAD. For each $n \in \omega$, let C_n be the n^{th} vertical column of $\omega \times \omega$. That is, $C_n = \{\langle n, m \rangle : m \in \omega\}$. It is clear that each C_n is a.d. from $\mathscr{A}$. Thus $\mathscr{A} \cup \{C_n : n \in \omega\}$ is an infinite a.d. family in $[\omega \times \omega]^\omega$. We will argue that it is strongly MAD in $[\omega \times \omega]^\omega$. Let $\{A_n : n \in \omega\} \subset [\omega \times \omega]^\omega$ be a countable family avoiding $\mathscr{A} \cup \{C_n : n \in \omega\}$. We will find infinite partial functions $f_n \subset A_n$ avoiding $\mathscr{A}$. The argument is similar to the proof of Lemma 3.3.4. We will first argue that A_n has infinite intersection with infinitely many members of $\mathscr{A}$. Suppose this if false. Fix $\{h_0, \ldots, h_k\} \subset \mathscr{A}$ such that for any $h \in \mathscr{A}$, if $|h \cap A_n| = \omega$, then $h = h_i$ for some $0 \leq i \leq k$. Put $B = A_n \setminus h_0 \cup \cdots \cup h_k$. Our assumption implies that B is a.d. from $\mathscr{A}$. Therefore, since strongly MAD families are Van Douwen MAD, it follows that there is no infinite partial function $p \subset B$. Thus for all but finitely many $n \in \omega$, $C_n \cap B = 0$. But then there is $n \in \omega$ such that $B \subset C_0 \cup \cdots \cup C_n$, whence $A_n \subset h_0 \cup \cdots \cup h_k \cup C_0 \cup \cdots \cup C_n$, contradicting our assumption that A_n avoids $\mathscr{A} \cup \{C_n : n \in \omega\}$. Hence we can fix an infinite set $\{h_i : i \in \omega\} \subset \mathscr{A}$ such that $\forall i \in \omega\, [|h_i \cap A_n| = \omega]$. Now, put $p_i = h_i \cap A_n$. This is an infinite partial function. It is possible to choose infinite partial functions $g_i \subset p_i$ such that $\forall\, i < j < \omega\, [\operatorname{dom}(g_i) \cap \operatorname{dom}(g_j) = 0]$. Put $f_n = \bigcup g_i$. This is an infinite partial function and clearly $f_n \subset A_n$. Moreover, f_n has infinite intersection with infinitely many members of $\mathscr{A}$. So f_n avoids $\mathscr{A}$. Thus $\{f_n : n \in \omega\}$ is a countable family of infinite partial functions avoiding $\mathscr{A}$. So by Lemma 3.1.4 we can find $h \in \mathscr{A}$ such that $\forall n \in \omega\, [\, |h \cap f_n| = \omega]$. But since $f_n \subset A_n$, we get that $\forall n \in \omega\, [|h \cap A_n| = \omega]$. $\dashv$

We do not know if the converse is true:

Question 3.7.3. *Suppose that there is a strongly MAD family in $[\omega]^\omega$. Is there a strongly MAD family in ω^ω?*

Lemma 3.7.2 also yields a connection between the indestructability properties of strongly MAD families in ω^ω and those of strongly MAD families in $[\omega]^\omega$.

Lemma 3.7.4. *Let $\mathbb{P}$ be any poset. Suppose that any strongly MAD family in $[\omega]^\omega$ is strongly $\mathbb{P}$-indestructible (see Definition 3.4.1). Let $\mathscr{A} \subset \omega^\omega$ be strongly MAD. $\mathscr{A}$ is strongly $\mathbb{P}$-indestructible.*

Proof. As in Lemma 3.7.2, let C_n be the n^{th} vertical column of $\omega \times \omega$. We know from Lemma 3.7.2 that $\mathscr{A} \cup \{C_n : n \in \omega\}$ is a strongly MAD family in $[\omega \times \omega]^\omega$. Now, let G be a $(\mathbf{V}, \mathbb{P})$ generic filter. By assumption, in $\mathbf{V}[G]$, $\mathscr{A} \cup \{C_n : n \in \omega\}$ remains a strongly MAD family in $[\omega \times \omega]^\omega$. In $\mathbf{V}[G]$, let $\{f_i : i \in \omega\} \subset \omega^\omega$ be a countable family avoiding $\mathscr{A}$. As each f_i is a.d. from each C_n, it follows that $\{f_i : i \in \omega\} \subset [\omega \times \omega]^\omega$ still avoids $\mathscr{A} \cup \{C_n : n \in \omega\}$. But then there must be $h \in \mathscr{A}$ such that $\forall i \in \omega\, [|h \cap f_i| = \omega]$. $\dashv$

Chapter 4

Consistency of no Gregory Trees With Large Continuum

Kunen and Hart [14] introduced the notion of a weird topological space in connection with the question of which compact spaces satisfy the Complex version of the Stone–Weierstrass theorem. In this context, they wanted to know if there is a compact Hausdorff space which is hereditarily Lindelöf (HL), is not totally disconnected, but does not contain a copy of the Cantor set. They showed that this question is independent of the axioms of ZFC. Their proof involved looking at a certain kind of subtree of $2^{<\omega_1}$ that was implicit in the work of Gregory [13], which they called a Gregory tree. The present chapter addresses some questions about Gregory trees that were left open in Kunen and Hart [14]

Definition 4.0.5 (Definition 1.2 of [14])**.** *A space X is weird if X is compact and not scattered, and there is no $P \subset X$ such that P is perfect and totally disconnected.*

Definition 4.0.6. *A Cantor tree of sequences (a CT) is a subset $\{f_\sigma : \sigma \in 2^{<\omega}\}$ of $2^{<\omega_1}$ such that for all $\sigma \in 2^{<\omega}$, $f_{\sigma^\frown 0}$ and $f_{\sigma^\frown 1}$ are incompatible nodes in $2^{<\omega_1}$ that extend f_σ. A subtree $\mathbb{T}$ of $2^{<\omega_1}$ is said to have the Cantor Tree Property (CTP) if: 1) for every $f \in \mathbb{T}$, $f^\frown 0, f^\frown 1 \in \mathbb{T}$; 2) given any Cantor tree $\{f_\sigma : \sigma \in 2^{<\omega}\} \subset \mathbb{T}$, there is $x \in 2^\omega$ and $g \in \mathbb{T}$ such that $\forall n \in \omega\, [f_{x \restriction n} \subset g]$. Finally, a subtree $\mathbb{T}$ of $2^{<\omega_1}$ is a Gregory tree if it has the CTP, but does not have a cofinal branch.*

Gregory trees occur in Gregory [13], where it is proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies that there is a Gregory tree. This result is of interest because a Gregory tree, when viewed as a forcing notion, is totally proper, and moreover, it kills itself. That is, given a Gregory tree $\mathbb{T}$, we can view $\mathbb{T}$ as a forcing notion with the following ordering: $\forall f, g \in \mathbb{T}\,[g \leq f \textit{ iff } g \supset f]$. It is easy to see that $\mathbb{T}$ is a totally proper poset, that is, it is proper and does not add any reals (see Lemma 5.5 of [14]). Moreover, forcing with $\mathbb{T}$ adds a cofinal branch through $\mathbb{T}$. Thus it is possible to kill any given Gregory tree without adding reals. But Gregory's result shows that it is not possible to iterate this without adding reals.

The connection between Gregory trees and weird spaces is given by the following result of Kunen and Hart [14].

Theorem 4.0.7 (see Lemma 5.7 of [14]). *Assume that X is compact, HL, and not totally disconnected, and assume that X has no subspace homeomorphic to the Cantor set. Then there is a Gregory tree.* $\dashv$

As Gregory trees are totally proper, PFA implies that there are no Gregory trees (and hence no weird spaces). It is well known that PFA implies that $\mathfrak{c} = \aleph_2$. So a natural question is:

Question 4.0.8. *Is it consistent to have no Gregory trees and $\mathfrak{c}$ arbitrarily large?*

Firstly, we remark that even though $2^{\aleph_0} < 2^{\aleph_1}$ implies that there is a Gregory tree, these two statements are not equivalent. To see this start with a model where $2^{\aleph_0} = \aleph_2 < \aleph_3 = 2^{\aleph_1}$. In this model there is a Gregory tree. Now, force with $Fn(\omega_2, 2, \omega_1) = \{p : p \text{ is a partial function from } \omega_2 \text{ to } 2 \wedge |p| \leq \omega_1\}$. This poset is closed under decreasing sequences of length ω_1. So it add no new functions from either ω or ω_1 into the ground model. So any Gregory tree remains a Gregory tree in the generic extension. Moreover, $2^{\aleph_0} = \aleph_2$ holds, and $2^{\aleph_1}$ gets collapsed to $\aleph_2$.

In this chapter we will show that the answer to Question 4.0.8 is affirmative. Given any

$\kappa > \omega$ satisfying $\kappa^{<\kappa} = \kappa$ and $\forall \lambda < \kappa\, [\lambda^\omega < \kappa]$, there is a c.c.c. forcing extension where there are no Gregory trees and $\mathfrak{c} = \kappa$ and Martin's Axiom (MA) holds.

It is well–known that we cannot construct a model where $\mathfrak{c}$ is large using countable support iterations. If we are to use finite support, then to ensure ω_1 is preserved, our iterands must be c.c.c. posets. Hence, given a Gregory tree $\mathbb{T}$, we would like to find a c.c.c. poset that adds a cofinal branch through $\mathbb{T}$. The most natural candidate is a Suslin subtree of the Gregory tree $\mathbb{T}$. It is shown in Kunen and Hart [14] that if we assume $\diamondsuit$, then every subtree of $2^{<\omega_1}$ with the CTP contains a Suslin subtree (see Lemma 5.8 of [14]). Now, if we want to do an iteration of length ω_2, this is enough because $\diamondsuit$ is preserved by posets of size atmost $\aleph_1$, and so, at any stage $\alpha < \omega_2$, $\diamondsuit$ will be true (assuming it holds in the ground model), allowing us to force with a Suslin subtree of some Gregory tree we want to kill at that stage. But if we want to make $\mathfrak{c} > \aleph_2$, then we must do a longer iteration, and then, $\diamondsuit$ will fail at stages $\alpha \geq \omega_2$. In general, it will not be the case that at *every* stage $\alpha \geq \omega_2$, every Gregory tree contains a Suslin subtree. The main lemma of this chapter will be that this still holds at *certain* stages: those where $\mathrm{cf}\,(\alpha) = \omega_1$. We will then show, by an elementary submodel argument, that any name for a Gregory tree in the final model reflects to a Gregory tree on a club of ω_1 limits of κ. Therefore, we can use $\diamondsuit$ on the ω_1 limits of κ to kill off all Gregory trees.

We proceed to the proof of the main lemma (see Theorem 4.0.12). It is well known that one can construct a Suslin tree after adding $\aleph_1$ Cohen reals (see, for example, Theorem 3.1 and Theorem 6.1 of [29]). Our argument is similar to this.

Lemma 4.0.9. *Suppose γ is a limit ordinal with $\mathrm{cf}\,(\gamma) \geq \omega_1$. Let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ be a FS iteration of c.c.c. forcings. Suppose $|X| \leq \omega$ and $\mathring{a} \in \mathbf{V}^{\mathbb{P}_\gamma}$ such that $\Vdash_\gamma \mathring{a} \subset X$. There is a sequence $\langle \mathring{a}_\alpha : \alpha < \gamma \rangle$ such that:*

1. *$\mathring{a}_\alpha$ is a nice $\mathbb{P}_\alpha$ name for a subset of X*

2. *$\exists\, \alpha_0 < \gamma \forall\, \alpha_0 \leq \alpha < \gamma\, [\, \Vdash_\gamma \mathring{a}_\alpha = \mathring{a}]$.*

Proof. Without loss of generality, $\mathring{a}$ is a nice $\mathbb{P}_\gamma$ name for a subset of X. Thus we may identify $\mathring{a}$ with $\langle A_x : x \in X \rangle$, where each $A_x \subset \mathbb{P}_\gamma$ is an antichain in $\mathbb{P}_\gamma$. For each $\alpha < \gamma$ and $x \in X$, put $A_x^\alpha = \{p \in A_x : \operatorname{suppt}(p) \subset \alpha\}$. Define $B_x^\alpha = \{p \restriction \alpha : p \in A_x^\alpha\}$. It is easy to see that B_x^α is an antichain in $\mathbb{P}_\alpha$. Thus, for each $\alpha < \gamma$, the set $\{B_x^\alpha : x \in X\}$ may be identified with a nice $\mathbb{P}_\alpha$ name for a subset of X, say $\mathring{a}_\alpha$. We will argue that $\langle \mathring{a}_\alpha : \alpha < \gamma \rangle$ has property (2) above.

Fix G_γ, a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter. In $\mathbf{V}[G_\gamma]$, each $\mathring{a}_\alpha[G_\gamma]$ is a subset of X. So is $\mathring{a}[G_\gamma]$. Now, for each $x \in \mathring{a}[G_\gamma]$, choose $p_x \in G_\gamma \cap A_x$. As the cofinality of γ is unchanged and $|X| \leq \omega$, we may choose $\alpha_0 < \gamma$ such that $\forall x \in X\, [\operatorname{suppt}(p_x) \subset \alpha_0]$. Now, take $\alpha_0 \leq \alpha < \gamma$. We will argue that $\mathring{a}_\alpha[G_\gamma] = \mathring{a}[G_\gamma]$. Firstly, suppose $x \in \mathring{a}_\alpha[G_\gamma] = \mathring{a}_\alpha[G_\alpha]$. We can find $p \in G_\gamma$ such that $p \restriction \alpha \in G_\alpha \cap B_x^\alpha$. Now, it is easy to see that $(p \restriction \alpha)^\frown \mathring{\mathbb{1}} \in G_\gamma \cap A_x$, whence $x \in \mathring{a}[G_\gamma]$. Next, suppose that $x \in \mathring{a}[G_\gamma]$. Then $p_x \in G_\gamma \cap A_x$. But $\operatorname{suppt}(p_x) \subset \alpha_0 \subset \alpha$. So $p_x \in A_x^\alpha$. Therefore, $p_x \restriction \alpha \in B_x^\alpha \cap G_\alpha$, whence $x \in \mathring{a}_\alpha[G_\alpha] = \mathring{a}_\alpha[G_\gamma]$.

Now, back in the ground model $\mathbf{V}$, we can use the maximal principal to find $\mathring{\alpha}_0 \in \mathbf{V}^{\mathbb{P}_\gamma}$ such that $\Vdash_\gamma \mathring{\alpha}_0 < \gamma \wedge \forall \alpha \geq \mathring{\alpha}_0\, [\mathring{a}_\alpha = \mathring{a}]$. But $\mathbb{P}_\gamma$ is a c.c.c. poset and $\operatorname{cf}(\gamma) \geq \omega_1$. So there is $\alpha_0 < \gamma$ such that $\Vdash_\gamma \mathring{\alpha}_0 < \alpha_0$. If $\alpha_0 \leq \alpha < \gamma$, $\Vdash_\gamma \mathring{a}_\alpha = \mathring{a}$. $\dashv$

Lemma 4.0.10. *Suppose γ is a limit ordinal with $\operatorname{cf}(\gamma) = \omega_1$. Let $\langle \gamma_\alpha : \alpha < \omega_1 \rangle \subset \gamma$ be an increasing sequence of limit ordinals cofinal in γ. Suppose $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a FS iteration of c.c.c. forcings. For each $\delta, \alpha < \omega_1$, let $\mathring{2}_\alpha^{<\delta}$ and $\mathring{\mathcal{F}}_\alpha^\delta$ be $\mathbb{P}_{\gamma_\alpha}$ names such that $\Vdash_{\gamma_\alpha} \mathring{2}_\alpha^{<\delta} = 2^{<\delta} \wedge \mathring{\mathcal{F}}_\alpha^\delta = \left[2^{<\delta}\right]^{\leq\omega}$. Let G_γ be a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter. Working within $\mathbf{V}[G_\gamma]$, define for each $\delta, \alpha < \omega_1$, $2_\alpha^{<\delta} = \mathring{2}_\alpha^{<\delta}[G_\gamma]$ and $\mathcal{F}_\alpha^\delta = \mathring{\mathcal{F}}_\alpha^\delta[G_\gamma]$. Still within $\mathbf{V}[G_\gamma]$, suppose $T \subset 2^{<\omega_1}$ such that $\forall \delta < \omega_1 \left[\left|T \cap 2^{<\delta}\right| \leq \omega\right]$. There is a club $C \subset \omega_1$ such that for each $\delta \in C$, $T \cap 2^{<\delta} \in \mathcal{F}_\delta^\delta$, and hence $T \cap 2^{<\delta} \subset 2_\delta^{<\delta}$.*

Proof. First note that in $\mathbf{V}[G_\gamma]$, $|T| \leq \omega_1$. So put $T = \{f_i^\alpha : \alpha < \omega_1 \wedge i \in \omega\}$, where $\{f_i^\alpha : i \in \omega\} = T \cap 2^\alpha$. In the ground model $\mathbf{V}$, choose $\mathbb{P}_\gamma$ names $\{\mathring{a}(\alpha, i) : \alpha < \omega_1 \wedge i \in \omega\}$ with $\Vdash_\gamma \mathring{a}(\alpha, i) \in 2^\alpha$ such that $\mathring{a}(\alpha, i)[G_\gamma] = f_i^\alpha$. Now, by Lemma 4.0.9 we can find a sequence

$\langle \mathring{a}(\alpha, i, \beta) : \alpha < \omega_1 \wedge i \in \omega \wedge \beta < \gamma \rangle$ so that

1. $\mathring{a}(\alpha, i, \beta)$ is a nice $\mathbb{P}_\beta$ name for an element of 2^α
2. $\forall \alpha < \omega_1 \forall i \in \omega \exists \beta_i^\alpha < \gamma \forall\ \beta_i^\alpha \leq \beta < \gamma\, [\, \Vdash_\gamma \mathring{a}(\alpha, i, \beta) = \mathring{a}(\alpha, i)]$.

Now, we return to $\mathbf{V}[G_\gamma]$. Let $M \prec H(\theta)$ be a countable elementary submodel containing the relevant objects. Put $\delta = M \cap \omega_1$. To prove the Lemma, it suffices to show that $T \cap 2^{<\delta} \in \mathcal{F}_\delta^\delta$. First of all, note that $T \cap 2^{<\delta} = \{f_i^\alpha : i \in \omega \wedge \alpha < \delta\} = \{\mathring{a}(\alpha, i)[G_\gamma] : \alpha < \delta \wedge i \in \omega\}$. This follows from our stipulation above that for all $\alpha < \omega_1$, $\{f_i^\alpha : i \in \omega\} = T \cap 2^\alpha$, and that $\mathring{a}(\alpha, i)[G_\gamma] = f_i^\alpha$. It follows from this observation that $T \cap 2^{<\delta} \subset M$. Now, from (2) above we know that for each $\alpha < \delta$ and $i \in \omega$, there is $\beta_i^\alpha < \gamma$ such that $\forall\ \beta_i^\alpha \leq \beta < \gamma\, [\mathring{a}(\alpha, i, \beta)[G_\gamma] = f_i^\alpha]$. Since all the relevant parameters are in M, we conclude that $\beta_i^\alpha \in M \cap \gamma$, for each such α and i. Now, for each such β_i^α, there is a $\eta < \omega_1$ such that $\beta_i^\alpha < \gamma_\eta$. Again, since $\beta_i^\alpha \in M$, $\eta < \delta$, and so $\beta_i^\alpha < \gamma_\eta < \gamma_\delta$. Thus we conclude that for each $\alpha < \delta$ and $i \in \omega$, $f_i^\alpha = \mathring{a}(\alpha, i, \gamma_\delta)[G_\gamma]$. Therefore, $T \cap 2^{<\delta} = \{\mathring{a}(\alpha, i, \gamma_\delta)[G_\gamma] : \alpha < \delta \wedge i \in \omega\}$. But each $\mathring{a}(\alpha, i, \gamma_\delta)$ is a $\mathbb{P}_{\gamma_\delta}$ name. So $\mathring{a}(\alpha, i, \gamma_\delta)[G_\gamma] = \mathring{a}(\alpha, i, \gamma_\delta)[G_{\gamma_\delta}]$. Therefore, $T \cap 2^{<\delta} = \{\mathring{a}(\alpha, i, \gamma_\delta)[G_{\gamma_\delta}] : \alpha < \delta \wedge i \in \omega\} \in \mathbf{V}[G_{\gamma_\delta}]$, and so, $T \cap 2^{<\delta} \in \mathcal{F}_\delta^\delta$. $\dashv$

Lemma 4.0.11. *Assume that the hypotheses of the previous lemma (Lemma 4.0.10) are satisfied. Fix a limit ordinal $\delta < \omega_1$. Work within $\mathbf{V}[G_\gamma]$. Fix a countable subtree $T_0 \subset 2_\delta^{<\delta}$ such that $T_0 \in \mathcal{F}_\delta^\delta$. Assume that*

1. *for all $f \in T_0$, $f^\frown 0 \in T_0$ and $f^\frown 1 \in T_0$*
2. *for all $f \in T_0$, if $\mathrm{ht}\,(f) = \alpha$, then $\forall\ \alpha < \beta < \delta\ \exists g \in T_0\, [\mathrm{ht}\,(g) = \beta \wedge f \subset g]$.*

Fix $x \in T_0$. There is a Cantor tree of sequences $\{f_\sigma : \sigma \in 2^{<\omega}\} \subset T_0$ so that

(a) $f_0 = x$

(b) *for every* $\alpha < \delta$ *there is* $m \in \omega$ *so that* $\forall \sigma \in 2^m \left[\operatorname{ht}(f_\sigma) > \alpha\right]$

(c) *if* $A \subset T_0$ *is a maximal antichain in* T_0 *with* $A \in \mathcal{F}_\delta^\delta$*, then* $\exists m \in \omega \forall \sigma \in 2^m \exists f \in A\left[f \subset f_\sigma\right]$.

Proof. We will use the well known fact that any finite support iteration of non–trivial forcings of length ξ adds a Cohen real whenever $\operatorname{cf}(\xi) = \omega$. Moreover, the Cohen poset is forcing equivalent to any countable non atomic poset. Therefore, if within $\mathbf{V}[G_{\gamma_\delta}]$, $\mathbb{P}$ is a countable poset, then there is $H \in \mathbf{V}[G_\gamma]$ which is a $(\mathbf{V}[G_{\gamma_\delta}], \mathbb{P})$ generic filter.

Now, notice that $T_0 \in \mathbf{V}[G_{\gamma_\delta}]$ and that in $\mathbf{V}[G_{\gamma_\delta}]$, T_0 is a subtree of $2^{<\delta}$ having properties (1) and (2) above. We will force with the poset of "partial Cantor tree sequences starting at x". More formally, working within $\mathbf{V}[G_{\gamma_\delta}]$, we say that p is a partial Cantor tree of sequences starting at x if there is $n \in \omega$ such that

(i) $p : 2^{\leq n} \to T_0$

(ii) $p(0) = x$

(iii) $\forall \sigma \in 2^{<n}\left[\, p(\sigma^\frown 0)\right.$ and $p(\sigma^\frown 1)$ are incompatible extensions of $\left.p(\sigma)\right]$.

We then define $\mathbb{P}_x = \{p : p$ is a partial Cantor tree of sequences starting at $x\}$, and we order $\mathbb{P}_x$ by stipulating that $q \leq p$ *iff* $q \supset p$. It is clear that $\mathbb{P}_x$ is a countable poset, and so there will be a $(\mathbf{V}[G_{\gamma_\delta}], \mathbb{P}_x)$ generic filter H in $\mathbf{V}[G_\gamma]$. We will check that $\bigcup H$ will yield a Cantor tree of sequences satisfying properties (a)–(c) above. Firstly, since T_0 has property (1) above, it is clear that $D_n = \left\{p \in \mathbb{P}_x : \exists m \geq n \left[\operatorname{dom}(p) = 2^{\leq m}\right]\right\}$ is dense for each n. Therefore, $\bigcup H$ will be defined on all of $2^{<\omega}$. Next, to check that $\bigcup H$ has property (b), fix $\alpha < \delta$, we will check that $D_\alpha = \{q \in \mathbb{P}_x : \exists m \in \omega\, [2^m \subset \operatorname{dom}(q) \wedge \forall \sigma \in 2^m \left[\operatorname{ht}(q(\sigma)) > \alpha\right]]\}$ is dense. Fix $p \in \mathbb{P}_x$ and put $\operatorname{dom}(p) = 2^{\leq n}$. Now choose $\alpha < \beta < \delta$ such that for each $\sigma \in 2^n$, $\operatorname{ht}(p(\sigma)) + 1 < \beta$. For each $\sigma \in 2^n$, $p(\sigma)^\frown 0$ and $p(\sigma)^\frown 1$ are in T_0. By property (2), we can find g_0^σ and g_1^σ in

T_0 with $\mathrm{ht}\,(g_0^\sigma) = \mathrm{ht}\,(g_1^\sigma) = \beta$ such that $p(\sigma)^\frown 0 \subset g_0^\sigma$ and $p(\sigma)^\frown 1 \subset g_1^\sigma$. Now, define $q \supset p$ by setting $q(\sigma^\frown 0) = g_0^\sigma$ and $q(\sigma^\frown 1) = g_1^\sigma$, for each $\sigma \in 2^n$. It is clear that q is as required. Finally, for property (c), fix $A \subset T_0$, a maximal antichain in T_0, with $A \in \mathcal{F}_\delta^\delta$. We will check that $D_A = \{q \in \mathbb{P}_x : \exists m \in \omega\,[2^m \subset \mathrm{dom}\,(q) \wedge \forall\sigma \in 2^m \exists f \in A\,[f \subset q(\sigma)]]\}$ is dense. First of all, observe that since A is a maximal antichain in T_0, for every $h \in T_0$, there is $g \in T_0$ such that $h \subset g \wedge \exists f \in A\,[f \subset g]$. Fix $p \in \mathbb{P}_x$ and put $\mathrm{dom}\,(p) = 2^{\leq n}$. For each $\sigma \in 2^n$, $p(\sigma)^\frown 0$ and $p(\sigma)^\frown 1$ are in T_0. Therefore, there exist $g_0^\sigma \in T_0$ and $g_1^\sigma \in T_0$ such that $p(\sigma)^\frown 0 \subset g_0^\sigma$ and $p(\sigma)^\frown 1 \subset g_1^\sigma$, and $\exists f_0, f_1 \in A\,[f_0 \subset g_0^\sigma \wedge f_1 \subset g_1^\sigma]$. Now, define $q \supset p$ by setting $q(\sigma^\frown 0) = g_0^\sigma$ and $q(\sigma^\frown 1) = g_1^\sigma$. It is clear that q is a required. $\dashv$

Theorem 4.0.12. *Let γ be a limit ordinal with $\mathrm{cf}\,(\gamma) = \omega_1$. Let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma\rangle$ be a FS iteration of (non–trivial) c.c.c. posets. Let G_γ be a $(\mathbf{V}, \mathbb{P}_\gamma)$ generic filter. In $\mathbf{V}[G_\gamma]$, let $T \subset 2^{<\omega_1}$ be a subtree having CTP. There is a Suslin subtree $T_0 \subset T$.*

Proof. We will work within $\mathbf{V}\,[G_\gamma]$. We will construct a Suslin subtree of T in a manner similar to the usual construction of a Suslin tree from $\diamondsuit$. But we will substitute appeals to $\diamondsuit$ by appeals to Lemma 4.0.10 and Lemma 4.0.11. We will construct the levels of T_0 by induction. For $\delta < \omega_1$, let T_0^δ denote $T_0 \cap 2^\delta$ – i.e. the δ^{th} level of T_0. Thus, we will specify T_0^δ at stage δ.

Now, fix an increasing sequence of limit ordinals $\langle \gamma_\alpha : \alpha < \omega_1\rangle \subset \gamma$ cofinal in γ. For each $\alpha, \delta < \omega_1$, define $2_\alpha^{<\delta}$ and $\mathcal{F}_\alpha^\delta$ exactly as in Lemma 4.0.10. Now, we will construct $T_0 \subset T$ so that

1. T_0^δ is countable
2. $T_0^{\delta+1} = \{f^\frown 0, f^\frown 1 : f \in T_0^\delta\}$
3. if $\xi < \delta$, then $\forall f \in T_0^\xi \exists g \in T_0^\delta\,[f \subset g]$
4. Put $T_0^{<\delta} = \bigcup_{\xi<\delta} T_0^\xi$. Suppose $T_0^{<\delta} \subset 2_\delta^{<\delta}$ with $T_0^{<\delta} \in \mathcal{F}_\delta^\delta$. If $A \subset T_0^{<\delta}$ is a maximal antichain in $T_0^{<\delta}$ with $A \in \mathcal{F}_\delta^\delta$, then $\forall g \in T_0^\delta \exists f \in A\,[f \subset g]$.

We will first check that it is sufficient to do this. As usual, because of properties (2) and (3), it is enough to check that T_0 has no uncountable antichains. Towards a contradiction, fix $A \subset T_0$, an uncountable antichain. By the usual argument (see Lemma 7.6 of [22]), there is a club $C_0 \subset \omega_1$ such that for all $\delta \in C_0$, $A \cap T_0^{<\delta}$ is a maximal antichain in $T_0^{<\delta}$. Because of property (1), we can apply Lemma 4.0.10 to both T_0 and to A to find a club $C_1 \subset \omega_1$ such that for all $\delta \in C_1$, $T_0^{<\delta} \subset 2_\delta^{<\delta}$ and both $T_0^{<\delta}$ and $A \cap T_0^{<\delta}$ are elements of $\mathcal{F}_\delta^\delta$. But now, if $\delta \in C_0 \cap C_1$, then by property (4), $A \cap T_0^{<\delta} = A$, a contradiction.

We now describe how to construct T_0^δ. If δ is a successor, then the construction is fully specified by property (2). So we assume that δ is a limit ordinal. Put $T_0^{<\delta} = \bigcup_{\xi<\delta} T_0^\xi$. If $T_0^{<\delta} \not\subset 2_\delta^{<\delta}$ or $T_0^{<\delta} \notin \mathcal{F}_\delta^\delta$, then arbitrarily choose T_0^δ satisfying properties (1) and (3). It is possible to do this while staying within T because T has CTP. Now, suppose $T_0^{<\delta} \subset 2_\delta^{<\delta}$ and $T_0^{<\delta} \in \mathcal{F}_\delta^\delta$. For each $x \in T_0^{<\delta}$, we can apply Lemma 4.0.11 to find a Cantor tree of sequences $\{f_\sigma^x : \sigma \in 2^{<\omega}\} \subset T_0^{<\delta}$ satisfying properties (a)–(c) of Lemma 4.0.11. Because T has CTP and because of property (b) of Lemma 4.0.11, there is $g_x \in T \cap 2^\delta$ and $y_x \in 2^\omega$ such that $\forall n \in \omega \left[f_{y_x \restriction n}^x \subset g_x\right]$. Put $T_0^\delta = \{g_x : x \in T_0^{<\delta}\}$. Property (a) of Lemma 4.0.11 ensures that $x \subset g_x$. Property (c) of Lemma 4.0.11 ensures the property (4) above is satisfied. $\dashv$

We are now ready to answer Question 4.0.8. Given $\kappa > \omega$ satisfying $\kappa^{<\kappa} = \kappa$ and $\forall \lambda < \kappa\, [\lambda^\omega < \kappa]$, we do an interation of length κ. We will use Theorem 4.0.12 to kill off some specific Gregory tree at stages of cofinality ω_1. To ensure there are no Gregory trees in the final model, we need to be able to guess names for Gregory trees, and for these names to reflect on a club of ω_1 limits of κ. We will show next that this reflection happens. We will use $\forall \lambda < \kappa\, [\lambda^\omega < \kappa]$ here.

Lemma 4.0.13. *Let $\kappa > \omega$ be a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and $\forall \lambda < \kappa\, [\lambda^\omega < \kappa]$. Let $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \kappa \rangle$ be a FS iteration of c.c.c. forcings, with $|\mathbb{P}_\alpha| < \kappa$, for all $\alpha < \kappa$. Say $\{\mathring{a}(\alpha, \beta) : \alpha < \omega_1 \wedge \beta < \kappa\} \subset \mathbf{V}^{\mathbb{P}_\kappa}$ with*

1. $\Vdash_\kappa \mathring{a}(\alpha, \beta) \in 2^\alpha$

2. $\Vdash_\kappa \{\mathring{a}(\alpha, \beta) : \alpha < \omega_1 \wedge \beta < \kappa\}$ *is a subtree of* $2^{<\omega_1}$ *with the CTP.*

By Lemma 4.0.9, we can fix a sequence $\langle \mathring{a}(\alpha, \beta, \gamma) : \alpha < \omega_1 \wedge \beta < \kappa \wedge \gamma < \kappa \rangle$ *such that* $\mathring{a}(\alpha, \beta, \gamma)$ *is a nice* $\mathbb{P}_\gamma$ *for an element of* 2^α *and* $\forall \alpha < \omega_1 \forall \beta < \kappa \exists \gamma^\alpha_\beta < \kappa \forall \gamma \geq \gamma^\alpha_\beta \, [\Vdash_\kappa \mathring{a}(\alpha, \beta, \gamma) = \mathring{a}(\alpha, \beta)]$. *There is a* ω_1*–club* $C \subset \kappa$ *such that for all* $\delta \in C$

(a) $\forall \alpha < \omega_1 \forall \beta < \delta \, [\Vdash_\kappa \mathring{a}(\alpha, \beta, \delta) = \mathring{a}(\alpha, \beta)]$

(b) $\Vdash_\delta \{\mathring{a}(\alpha, \beta, \delta) : \alpha < \omega_1 \wedge \beta < \delta\}$ *is a subtree of* $2^{<\omega_1}$ *with CTP.*

Proof. Let $M \prec H(\theta)$ be an elementary submodel containing the relevant objects satisfying

(i) $|M| < \kappa$, $\delta = M \cap \kappa$ is an ordinal less than κ and $\mathrm{cf}\,(\delta) = \omega_1$

(ii) $M^\omega \subset M$.

It is possible to construct M having these properties because of our assumption that $\forall \lambda < \kappa \, [\lambda^\omega < \kappa]$. It suffices to check that δ has properties (a) and (b). First of all, note that for all $\alpha < \omega_1$ and $\beta < \delta$, $\gamma^\alpha_\beta \in M$, and so $\gamma^\alpha_\beta < \delta$. It follows from this that $\Vdash_\kappa \mathring{a}(\alpha, \beta, \delta) = \mathring{a}(\alpha, \beta)$. Next, it is easy to check that $\Vdash_\delta \{\mathring{a}(\alpha, \beta, \delta) : \alpha < \omega_1 \wedge \beta < \delta\}$ is a subtree of $2^{<\omega_1}$. It is also easy to verify that $\Vdash_\delta \{\mathring{a}(\alpha, \beta, \delta) : \alpha < \omega_1 \wedge \beta < \delta\}$ satisfies part (1) of the definition of CTP (see Definition 4.0.6). We will check that this holds for part (2) of the definition.

Say $\{\mathring{b}_\sigma : \sigma \in 2^{<\omega}\} \subset \mathbf{V}^{\mathbb{P}_\delta}$ such that

$$\Vdash_\delta \{\mathring{b}_\sigma : \sigma \in 2^{<\omega}\} \text{ is a CT in } \{\mathring{a}(\alpha, \beta, \delta) : \alpha < \omega_1 \wedge \beta < \delta\}. \tag{$*_1$}$$

Since $\mathbb{P}_\delta$ is c.c.c. there is $\alpha_0 < \omega_1$ so that $\Vdash_\delta \{\mathring{b}_\sigma : \sigma \in 2^{<\omega}\} \subset 2^{<\alpha_0}$. Since $\mathrm{cf}\,(\delta) = \omega_1$, by Lemma 4.0.9, there is a sequence $\langle \mathring{b}^\beta_\sigma : \sigma \in 2^{<\omega} \wedge \beta < \delta \rangle$ such that $\mathring{b}^\beta_\sigma$ is a nice $\mathbb{P}_\beta$ name for an element of $2^{<\alpha_0}$ and $\forall \sigma \in 2^{<\omega} \exists \beta_\sigma < \delta \forall \beta \geq \beta_\sigma \left[\Vdash_\delta \mathring{b}^\beta_\sigma = \mathring{b}_\sigma \right]$. We may choose $\beta_0 < \delta$ such that $\beta_\sigma < \beta_0$ for all $\sigma \in 2^{<\omega}$. Therefore, $\forall \sigma \in 2^{<\omega} \left[\Vdash_\delta \mathring{b}^{\beta_0}_\sigma = \mathring{b}_\sigma \right]$. Thus, we may assume without loss of generality that each $\mathring{b}_\sigma$ is a nice $\mathbb{P}_{\beta_0}$ name for an element of $2^{<\alpha_0}$. Now, $|\mathbb{P}_{\beta_0}| < \kappa$ and $\mathbb{P}_{\beta_0} \in M$. It follows from this that $\mathbb{P}_{\beta_0} \subset M$. As $M^\omega \subset M$, we conclude that $\{\mathring{b}_\sigma : \sigma \in 2^{<\omega}\} \in M$.

Now, it is easy to check that

$$\Vdash_\kappa \{\mathring{b}_\sigma : \sigma \in 2^{<\omega}\} \text{ is a CT in } \{\mathring{a}(\alpha, \beta) : \alpha < \omega_1 \wedge \beta < \kappa\}. \tag{$*_2$}$$

By elementarity of M, there is $\mathring{x} \in \mathbf{V}^{\mathbb{P}_\kappa} \cap M$ such that

$$\Vdash_\kappa \mathring{x} \in 2^\omega \wedge \exists \alpha < \omega_1 \exists \beta < \kappa \forall n \in \omega \left[\mathring{b}_{\mathring{x} \restriction n} \subset \mathring{a}(\alpha, \beta) \right]. \tag{$*_3$}$$

Once again applying 4.0.9, we may assume without loss of generality that $\mathring{x}$ is a nice $\mathbb{P}_\delta$ name for an element of 2^ω. Now, since $\mathbb{P}_\kappa$ is c.c.c. , $(*_3)$ above together with the elementarity of M implies that $\Vdash_\kappa \exists \beta < \delta \exists \alpha < \omega_1 \forall n \in \omega \left[\mathring{b}_{\mathring{x} \restriction n} \subset \mathring{a}(\alpha, \beta) \right]$. But since $\Vdash_\kappa \mathring{a}(\alpha, \beta, \delta) = \mathring{a}(\alpha, \beta)$, for all $\alpha < \omega_1$ and $\beta < \delta$, and since $\mathring{x}$ is a $\mathbb{P}_\delta$ name, it is easy to check that

$$\Vdash_\delta \exists \beta < \delta \exists \alpha < \omega_1 \forall n \in \omega \left[\mathring{b}_{\mathring{x} \restriction n} \subset \mathring{a}(\alpha, \beta, \delta) \right]$$

This is as required, and we are done. $\dashv$

Theorem 4.0.14. *Let $\kappa > \omega$ be a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and $\lambda < \kappa\, [\lambda^\omega < \kappa]$. Let S be the stationary set $\{\alpha < \kappa : \mathrm{cf}\,(\alpha) = \omega_1\}$. Assume $\diamondsuit_\kappa(S)$ holds. There is a c.c.c. forcing extension where there are no Gregory trees and $\mathfrak{c} = \kappa$. Moreoever, we can have MA in this model.*

Proof. Fix a sequence $\langle A_\delta : \delta \in S \rangle$ witnessing $\diamondsuit_\kappa(S)$. We will build a FS iteration of c.c.c. forcings $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \kappa \rangle$ with $\left| \mathring{\mathbb{Q}}_\alpha \right| < \kappa$, for all $\alpha < \kappa$. This will ensure that for all $\alpha < \kappa$, $|\mathbb{P}_\alpha| < \kappa$ and that $|\mathbb{P}_\kappa| = \kappa$. Now, fix $\delta \in S$ and suppose that $\mathbb{P}_\delta$ has been constructed. Suppose A_δ codes a sequence $\{\mathring{a}(\alpha, \beta, \gamma) : \alpha < \omega_1 \wedge \beta < \delta \wedge \gamma < \delta\}$ such that

1. $\mathring{a}(\alpha, \beta, \gamma)$ is a nice $\mathbb{P}_\gamma$ name for an element of 2^α

2. $\forall\alpha < \omega_1 \forall\beta < \delta \exists\gamma_\beta^\alpha < \delta\ \forall\gamma_\beta^\alpha \leq \gamma < \delta \left[\Vdash_\delta \mathring{a}(\alpha, \beta, \gamma_\beta^\alpha) = \mathring{a}(\alpha, \beta, \gamma) \right]$

3. $\Vdash_\delta \{\mathring{a}(\alpha, \beta, \gamma_\beta^\alpha) : \alpha < \omega_1 \wedge \beta < \delta\}$ is a subtree of $2^{<\omega_1}$ with CTP

Since $\mathrm{cf}\,(\delta) = \omega_1$, we can apply Theorem 4.0.12 to get that

$$\Vdash_\delta \exists T_0 \left[T_0 \subset \{\mathring{a}(\alpha, \beta, \gamma_\beta^\alpha) : \alpha < \omega_1 \wedge \beta < \delta\} \wedge T_0 \text{ is a Suslin tree}\right] \tag{$*$}$$

Since $\mathbb{P}_\delta$ is a c.c.c. poset with $|\mathbb{P}_\delta| < \kappa$ and since $\forall\lambda < \kappa\,[\lambda^\omega < \kappa]$, we can find a *full* $\mathbb{P}_\delta$ name $\mathring{\mathbb{Q}}$ with $\left|\mathring{\mathbb{Q}}\right| < \kappa$ such that

$$\Vdash_\delta \left[\mathring{\mathbb{Q}} \subset \{\mathring{a}(\alpha, \beta, \gamma_\beta^\alpha) : \alpha < \omega_1 \wedge \beta < \delta\} \wedge \mathring{\mathbb{Q}} \text{ is a Suslin tree}\right] \tag{$**$}$$

Now, we put $\mathring{\mathbb{Q}}_\delta = \mathring{\mathbb{Q}}$. It follows from the proof of Lemma 4.0.13 that doing this is sufficient to ensure that there are no Gregory trees in the final model.

If we also want to get MA in the final model, we can do the usual bookkeeping argument. It is clear that this can be combined with the construction above. We must only be sure that $\left|\mathring{\mathbb{Q}}_\alpha\right| < \kappa$, for all $\alpha < \kappa$. But the usual argument for MA only requires us to consider, at any given stage $\alpha < \kappa$, a $\mathbb{P}_\alpha$ name for a poset $\mathring{\mathbb{Q}}$ such that $\Vdash_\alpha \left|\mathring{\mathbb{Q}}\right| < \kappa$. Therefore, since $\forall\lambda < \kappa\,[\lambda^\omega < \kappa]$, we can find a *full* $\mathbb{P}_\alpha$ name $\mathring{\mathbb{Q}}_\alpha$ with $\left|\mathring{\mathbb{Q}}_\alpha\right| < \kappa$ such that $\Vdash_\alpha \mathring{\mathbb{Q}} = \mathring{\mathbb{Q}}_\alpha$. $\dashv$

We have had to assume that $\forall\lambda < \kappa\,[\lambda^\omega < \kappa]$. The first place where this assumption matters is $\aleph_1$, and there we know from Gregory's result that it is impossible to have $\mathfrak{c} = \aleph_1$ and not have a Gregory tree. The next place where this matters is $\aleph_{\omega+1}$. Here we do not know the answer.

Question 4.0.15. *Assume MA and $\mathfrak{c} = \aleph_{\omega+1}$. Is there a Gregory tree?*

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