Elliptic Differential Equations and their Discretizations

By

João Paulo Fernandes Teixeira

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Abstract

A study of elliptic differential equations is carried out, from the point of view of interconnecting the discrete with the analytical. Approximate maximum principles and barrier postulates, acting on functions with hyperfinite domains, are introduced. The methods are specially adapted for proofs of convergence of discretizations for linear elliptic PDE's. The well-known Brouwer degree theory is extended to hyperfinite dimensional spaces, with the purpose of applying it to show convergence of discretizations in nonlinear elliptic problems.

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Contents

Chapter 1

Introduction

1.1 A Brief Description of this Work

A study of elliptic differential equations is carried out, from the point of view of interconnecting the discrete with the analytical.

This dissertation is roughly divided into two parts. The first part introduces approximate maximum principles and barrier postulates, acting on functions with hyperfinite domains, and includes chapters 2 and 3. The methods are specially adapted for proofs of convergence of discretizations for linear elliptic PDE's. The second part, which includes the remaining three chapters, extends the well-known Brouwer degree theory to hyperfinite dimensional spaces, with the purpose of applying it to show convergence of discretizations in nonlinear elliptic problems.

In chapter 2, the five-point Laplacian scheme is used to discretize the Laplace equation and its Dirichlet problem. An approximate maximum principle for the five-point Laplacian is proved and used to show that, under a strong barrier condition, the discrete solutions converge to the (unique) solution of the Dirichlet problem. This is done directly, i.e., existence and uniqueness of a classical solution of the problem comes also out of this proof. The barrier condition introduced here is actually a necessary and sufficient condition for the approximation process to converge to a classical solution. It is shown that this condition implies the usual one, as used in Perron's method. Finally, convergence of discrete solutions in domains with nonregular boundary points is discussed.

Chapter 3 defines a large class of "good" approximations for the Dirichlet problem of a general linear uniformly elliptic operator, on $\Omega \subset \mathbb{R}^n$ bounded and open. An approximate maximum principle is shown to be a consequence of a discrete one, and this result is used to show convergence of all "good" discretizations to the solution of the analytical problem, provided one exists and is in $C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$. Convergence rates are derived in the case that the solution of the analytical problem is in $C^2(\overline{\Omega}, \mathbb{R})$.

As an elementary example, consider the Dirichlet problem for the Laplace equation in the unit square, $\Omega = (0,1)^2$,

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{if } x \in \Omega, \\ u(x) &= f(x) \quad \text{if } x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $f \in C(\partial\Omega, \mathbb{R})$. If we divide the unit square using equally spaced gridlines, with the distance from each other equal to $h = \frac{1}{n}$, $n \in \mathbb{N} - \{0\}$, then we obtain the following discretization of $\overline{\Omega}$:

$$\overline{\Omega}_h = \{0, h, 2h, \ldots, (n-1)h, 1\}^2$$

The discrete analogue of the boundary and interior of Ω is given by:

$$\begin{aligned} \partial\Omega_h &= \overline{\Omega}_h \cap \partial\Omega, \\ \Omega_h &= \overline{\Omega}_h - \partial\Omega_h. \end{aligned}$$

To discretize Δ, just replace the second derivatives with the corresponding central

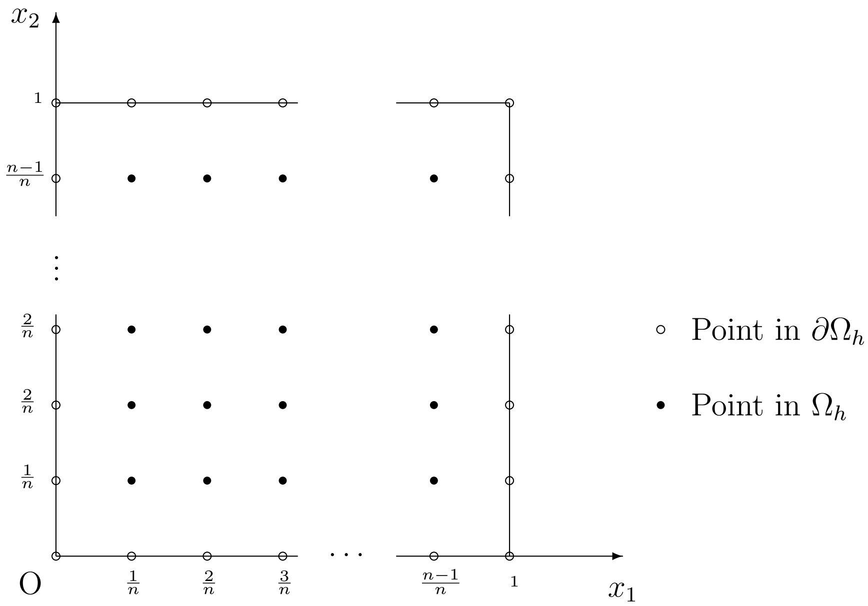

Figure 1: Discretization of $[0,1]^2$.

difference quotients, and get:

$$\Delta_h U_h(x) = \frac{1}{h^2}\Big(U_h(x_1+h,x_2) + U_h(x_1-h,x_2) + U_h(x_1,x_2+h) + U_h(x_1,x_2-h) - 4U_h(x_1,x_2)\Big).$$

This leads to the discretized problem:

$$\begin{aligned} \Delta_h U_h(x) &= 0 \quad \text{if} \ \ x \in \Omega_h, \\ U_h(x) &= f(x) \quad \text{if} \ \ x \in \partial\Omega_h, \end{aligned} \tag{1.2}$$

which is actually a family of discretized problems, indexed in $h = \frac{1}{n}$, $n \in \mathbb{N} - \{0\}$, each one leading to a unique solution, U_h[1].

Nonstandard analysis provides a very appropriate framework to deal with convergence of the U_h (as $n \to \infty$, or $h \downarrow 0$). By considering U_h, where h is a positive

[1]Problem (1.2) is a system of linear equations. It follows from the discrete maximum principle on page 12 that this has a unique solution.

infinitesimal, we are already looking at problem (1.2) ”in the limit”, i.e., in a situation where U_h is already infinitely close to the solution, u, of (1.1). Because of this, results and notions from the analytical (nondiscrete) theory — like barriers or maximum principles stronger than the usual discrete one — can be reformulated in the nonstandard setting. For a general reference in nonstandard analysis, see Stroyan [26] or Albeverio [1].

The maximum principles developed are not enough to obtain results concerning nonlinear elliptic equations. That leads us to consider degree theory as a possibility for proving convergence of discretizations.

We begin with the well-known Brouwer degree theory. A construction of this notion — this time using nonstandard analysis — is provided. It serves as a way to prepare for the work of next Chapter. This construction has some similarities with the one in Rabinowitz [23], although it gives a new formula for computing the degree in the general case.

Hyperfinite dimensional Banach spaces occur in situations where we want to study the behavior “in the limit” of some class of discrete problems. The functions U_h solving problem (1.2), for h positive and infinitesimal, are elements of such a space. By identifying infinitely close elements — with respect to some norm — we arrive at a nonstandard hull of the hyperfinite dimensional Banach space. This is somewhat similar to the standard procedure of considering the space of sequences of functions, $\left(U_{1/n}\right)_{n\in\mathbb{N}}$, identifying sequences that have the same limit with respect to some norm. The choice of the norm determines the notion of convergence being used.

Chapter 5 deals with the construction of a degree notion in nonstandard hulls of hyperfinite Banach spaces. To overcome the requirement of compactness for the map, the degree has values in the nonstandard integers, enabling the count of more than finitely many solutions.

A very important fact is that a map, φ, in the nonstandard hull is an equivalence class of many "nearby" maps, Φ, in the hyperfinite dimensional space. In our applications, each Φ represents a family $\{F_h\}$ of discrete problems indexed by h. The map φ is the nonstandard analogue of the common limit (as $h \downarrow 0$) of all $\{F_h\}$ that are near each other, in the sense of the chosen norm. So, our degree becomes quite an interesting tool, in that it may prove in one step the convergence of infinitely many "sufficiently close" discretizations.

The framework refered above is tested in the final chapter, where the general boundary value problem for the scaled Newton's law of motion, $x'' = f(x', x, t)$, with f continuous, is treated. This is a prototype for the more general Dirichlet problem for a nonlinear elliptic equation of the form $Lu = f(Du, u, t)$, with L uniformly elliptic, which we intend to study in the future.

1.2 Some Philosophical Remarks

Modeling a physical system is the art of crafting an idealized structure that reproduces, as much as possible, its observable behavior. Invariably, such structures end up taking the form of a mathematical construct of some kind. If we were to accept some form of realism, which strives for (and believes in) complete reliability of models, one would be constrained to use discrete mathematics; for the observable behavior of a system, achieved with a finite amount of effort, can only be a choice between finitely many states.

Despite all this, and since Leibniz and Newton, the physical sciences have shown a clear preference towards the use of calculus, and more generally analysis. Even classical mechanics, which was hailed by some as the hallmark of realism, relies heavily on analysis. With the twentieth century acceptance of nondeterminism bringing the introduction of state spaces even more complex than $\mathbb{R}^n$, models deviated even further from realism.

Therefore, the practical philosophy of the physical sciences has been, for quite some time, that models reproduce observable behavior only in an approximate way. The notion of approximation includes the idea of potential exactness, that is, the process of continuous refinement of our observation techniques and theoretical descriptions must be made sure to give, in the limit, predictions that are indistinguishable from observations. This puts two huge questions right at the heart of the scientific method:

(1) Is the approximation process feasible? That is, can we expect that, as our finite state space gets more and more detailed, the messy "real" (and discrete) descriptions converge to some unique and clean analytical description?

(2) Is the approximation process usable? That is, can we expect the approximation to converge fast enough so that it can lead to good practical applications in real time?

Interestingly enough, these questions have been around in mathematics (more specifically, in numerical analysis and approximation theory) for quite some time, although the philosophical considerations that lead to them seem to be a perfect mirror image of the ones we have been discussing. It is the "real" (in the sense of idealism) objects of analysis who require discretizations to be of any use in the nonplatonic world. Questions (1) and (2) also occur in mathematics, with the only change in the position of the quoted word "real".

It should be stressed that question (1) entails a very important, and often difficult, subquestion. A great deal of effort must sometimes be put into finding the type of analytical model that the discrete descriptions converge to, in some sense. A complete rethinking of the physical model may be involved in this step, so this easily turns into an interdisciplinary effort. The area of elliptic equations turned out to be a good test ground, one where this subquestion did not constitute an early burden for other needed developments.

This work is an attempt to build and apply some general tools (in the area of elliptic equations), that can be used to interconnect discrete descriptions with analytical ones.

For a mathematical overview of hyperreall numbers, and philosophical discussion of their significance, see Keisler [14].

Chapter 2

Dirichlet Problem for the Laplace Equation in the Plane

This chapter introduces the nonstandard maximum principle in the simple framework of the Laplace equation, and shows its usefulness by building the classical solution of the Dirichlet problem as a limit of discrete ones. The Dirichlet problem is:

$$\begin{aligned} \Delta u(x) &= 0, \quad && x \in \Omega, \\ u(x) &= f(x), \quad && x \in \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is open and bounded, and $f \in C(\partial\Omega, \mathbb{R})$. A function satisfying $\Delta u = 0$ on some Ω is called *harmonic* on that set. The solution of problem (2.1) describes the distribution of temperature for a steady heat flow on a plate of shape Ω, when the temperature at the boundary of the plate is constrained to be given by f. Throughout this chapter, we will always assume that Ω is an open and bounded subset of $\mathbb{R}^2$.

A classical solution of (2.1) is a function $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$, satisfying (2.1). While the smoothness requirement on u is needed for the derivatives in the differential equation to make sense, the continuity of u up to the boundary is a technical condition, imposed to ensure that the solution is "connected" to the boundary values. Nevertheless, in some situations, there are "solutions" which physically make sense (i.e., come from a limit of appropriate finite difference problems), but fail to be continuous at some boundary points.

We begin by looking at classical solutions, showing the classical result of existence and uniqueness of solution under a barrier condition. Unlike in the usual analytical proof, this is achieved by building it as a limit of appropriate finite difference solutions. Later, we will study a more general case, where there may not be any classical solution, but the finite difference approximations still converge pointwise to a unique function, which satisfies (2.1) and is continuous at points of $\partial\Omega$ meeting the barrier condition. The fact that this "solution" is constructed as the limit of the finite difference approximations ensures that it represents the intended physical situation.

2.1 A Finite Difference Scheme

Let $h > 0$. Let $\mathbb{R}_h^2$ be a uniform grid of points in $\mathbb{R}^2$:

$$\mathbb{R}_h^2 = \Big\{(ih, jh) : i, j \in \mathbb{Z}\Big\}.$$

Let $x_{i,j} = (ih, jh)$. The $x_{i,j}$ are sometimes called gridpoints. The set of neighbors of $x_{i,j}$ is defined as the the set of points of $\mathbb{R}_h^2$ whose distance from x is exactly h. Each gridpoint has precisely four neighbors. A function $U : \mathbb{R}_h^2 \to \mathbb{R}$ is called a gridfunction. By a finite difference scheme, we mean a family of discrete problems indexed in h, and usually obtainable from the differential equation by replacing the derivatives by finite differences. For the Laplacian operator, and using central differences:

$$u_{x_1x_1}(x_1,x_2) \longmapsto \frac{1}{h^2}\Big(U(x_1+h,x_2) - 2U(x_1,x_2) + U(x_1-h,x_2)\Big),$$

$$u_{x_2x_2}(x_1,x_2) \longmapsto \frac{1}{h^2}\Big(U(x_1,x_2+h) - 2U(x_1,x_2) + U(x_1,x_2-h)\Big).$$

Therefore, we define:

$$\begin{aligned} \Delta_h U(x_1,x_2) \quad &= \quad \frac{1}{h^2}\Big(U(x_1+h,x_2) + U(x_1-h,x_2) + U(x_1,x_2+h) \\ &\qquad\qquad + U(x_1,x_2-h) - 4U(x_1,x_2)\Big). \end{aligned} \tag{2.2}$$

If $u : \overline{\Omega} \to \mathbb{R}$, we define $\Delta_h u$ in the same way. The domain of $\Delta_h u$ will be the largest subset of $\overline{\Omega}_h$ where $\Delta_h u(x)$ is well defined. The discrete counterpart of the Laplace equation, $\Delta_h U = 0$, just says that $U(x_{i,j})$ equals the average of the values of U over the four neighbors of $x_{i,j}$ [1].

We now set up a finite difference scheme and show it has a unique solution for each $h \in \mathbb{R}^+$. Let:

$$\begin{aligned} &\overline{\Omega}_h = \overline{\Omega} \cap \mathbb{R}^2_h, \\ &\partial\Omega_h = \Big\{x_{i,j} \in \overline{\Omega}_h : x_{i+1,j} \notin \overline{\Omega}_h \vee x_{i-1,j} \notin \overline{\Omega}_h \vee x_{i,j+1} \notin \overline{\Omega}_h \vee x_{i,j-1} \notin \overline{\Omega}_h\Big\}, \\ &\Omega_h = \overline{\Omega}_h - \partial\Omega_h. \end{aligned}$$

The discrete analogue of the boundary of Ω, $\partial\Omega_h$, is the set of points in $\overline{\Omega}_h$ with less than four neighbors in $\overline{\Omega}_h$.

To define the discrete Dirichlet problem corresponding to (2.1), we need to construct a function $f_h : \overline{\Omega}_h \to \mathbb{R}$, which prescribes the values of U at the points in

[1] To give a physical interpretation, consider a plate, Ω, divided into square cells of side h, and let U give the temperature at each cell. The heat that enters a cell through each one of its boundary sides is proportional to the difference of temperature between itself and the corresponding neighboring cell. So the net flow of the heat entering the (i,j)'th cell is $\Delta_h U(x_{i,j})$; therefore $\Delta_h U(x_{i,j}) = 0$, $\forall x_{i,j} \in \Omega_h$, just says that the heat flow on Ω_h is steady.

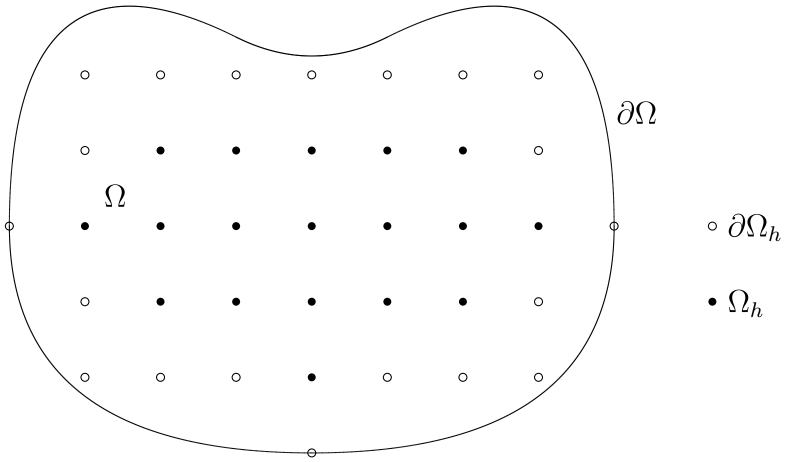

Figure 2: Ω, Ω_h, and $\partial\Omega_h$.

$\partial\Omega_h$, in a way which approximates well enough the boundary condition in (2.1). For our purpose, which is to show existence and uniqueness of solutions, the following construction will suffice.

Let $(a, b) \in \partial\Omega_h$. Let $(\tilde{a}, \tilde{b}) \in \partial\Omega$ be the intersection of the lines $x_1 = a$ and $x_2 = b$ with $\partial\Omega$ which is closest to (a, b). There may be more than one such point, but there can be only one to the North of (a, b). Similarly, there can be only one in the other three directions. So, if there are more than one, pick one of them according to a preset ordering [2]. Then, set:

$$f_h(a, b) = f(\tilde{a}, \tilde{b}).$$

[2]Say: North, East, South and West

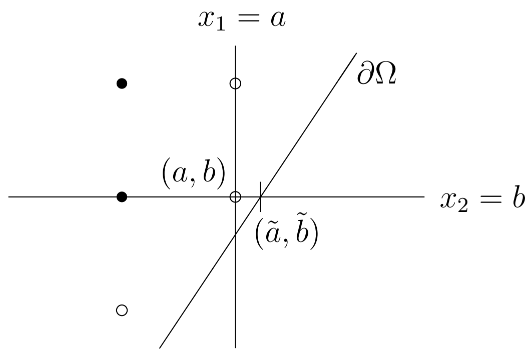

Figure 3: Construction of f_h.

The finite difference problem is:

$$\begin{aligned} \Delta_h U(x) &= 0, \quad x \in \Omega_h, \\ U(x) &= f(x), \quad x \in \partial\Omega_h. \end{aligned} \tag{2.3}$$

By construction, $\left|(\tilde{a}, \tilde{b}) - (a, b)\right| \leq h$

2.2 Approximate Maximum Principles

Maximum principles will constitute a key tool in this chapter. We begin by the discrete version. For now, and until we say the contrary, we work only with standard objects. In the end of this section, we will give a non-standard version of the maximum principle.

Theorem 2.1 (Discrete maximum principle) *Let $h > 0$. If $U : \overline{\Omega}_h \to \mathbb{R}$ satisfies*

$$\Delta_h U(x) \geq 0 \quad \forall x \in \Omega_h,$$

then

$$\max_{x\in\overline{\Omega}_h} U(x) = \max_{x\in\partial\Omega_h} U(x).$$

Proof. Let $M = \max_{x\in\partial\Omega_h} U(x)$. Assume the maximum of U occurs at some interior point, say $(x_1, x_2) \in \Omega_h$. For all elements of $\overline{\Omega}_h$ of the form $(x_1 + nh, x_2)$, $n \in \mathbb{N}$, such that all points of the form $(x_1 + ih, x_2)$, $0 \le i < n$, are in Ω_h, we show by induction on n that $U(x_1 + nh, x_2) = M$.

For $n = 0$, the result is given by hypothesis. Now, suppose $U(x_1+nh, x_2) = M$. If $(x_1 + nh, x_2) \in \partial\Omega_h$, we are done. If not, that is, if $(x_1 + nh, x_2) \in \Omega_h$, then, using the induction hypothesis:

$$\begin{aligned}
0 &\le h^2\Delta_h U(x_1 + nh, x_2) \\
&= U(x_1 + (n-1)h, x_2) + U(x_1 + (n+1)h, x_2) + U(x_1 + nh, x_2 + h) \\
&\quad +U(x_1 + nh, x_2 - h) - 4U(x_1 + nh, x_2) \\
&= U(x_1 + (n-1)h, x_2) + U(x_1 + (n+1)h, x_2) + U(x_1 + nh, x_2 + h) \\
&\quad +U(x_1 + nh, x_2 - h) - 4M \\
&\le M + U(x_1 + (n+1)h, x_2) + M + M - 4M \\
&= U(x_1 + (n+1)h, x_2) - M.
\end{aligned}$$

Therefore, $U(x_1 + (n+1)h, x_2) = M$.

Now, for the least $n \in \mathbb{N}$ such that $(x_1 + nh, x_2) \in \partial\Omega_h$ (which exists because Ω is bounded), $U(x_1 + (n+1)h, x_2) = M$. Hence, we have just shown that, if the maximum of U occurs in Ω_h, then it must occur also in $\partial\Omega_h$.

Corollary 2.2 (Discrete minimum principle) *Let* $h > 0$. *If* $U : \overline{\Omega}_h \to \mathbb{R}$

satisfies

$$\Delta_h U(x) \leq 0 \quad \forall x \in \Omega_h,$$

then $\min\limits_{x \in \overline{\Omega}_h} U(x) = \min\limits_{x \in \partial\Omega_h} U(x)$.

Proof. Apply the discrete maximum principle to the function $V = -U$.

Corollary 2.3 *Let* $h > 0$*. If* $U : \overline{\Omega}_h \to \mathbb{R}$ *satisfies*

$$\Delta_h U(x) = 0 \quad \forall x \in \Omega_h,$$

then $\max\limits_{x \in \overline{\Omega}_h} U(x) = \max\limits_{x \in \partial\Omega_h} U(x)$*, and* $\min\limits_{x \in \overline{\Omega}_h} U(x) = \min\limits_{x \in \partial\Omega_h} U(x)$.

Corollary 2.4 *Let* $h > 0$*. If* $U : \overline{\Omega}_h \to \mathbb{R}$ *satisfies*

$$\begin{aligned} \Delta_h U(x) &\geq 0 \quad && \forall x \in \Omega_h, \\ U(x) &\leq a && \forall x \in \partial\Omega_h, \end{aligned}$$

for some $a \in \mathbb{R}$*, then* $U(x) \leq a \quad \forall x \in \overline{\Omega}_h$.

Proof. By the discrete maximum principle, $\max\limits_{x \in \overline{\Omega}_h} U(x) = \max\limits_{x \in \partial\Omega_h} U(x) \leq a$.

The discrete maximum principle can be used in comparison function arguments. These work as follows. Let $U : \overline{\Omega}_h \to \mathbb{R}$ be a solution of (2.3), and suppose that, for some wisely chosen $V : \overline{\Omega}_h \to \mathbb{R}$,

$$x \in \Omega_h \Rightarrow \Delta_h(U(x) - V(x)) = -\Delta_h V(x) \geq 0,$$

i.e., $\Delta_h V(x) \leq 0$, and

$$x \in \partial\Omega_h \Rightarrow U(x) - V(x) = f_h(x) - V(x) \leq 0,$$

i.e, $V(x) \geq f_h(x)$. Then, for *all* $x \in \overline{\Omega}_h$, $U(x) - V(x) \leq 0$, so $U(x) \leq V(x)$.

Many variations of this argument can be used to get bounds for a solution, U, of (2.3), without knowing its form, or even if it exists. This is why these sorts of estimates are called a priori bounds.

We will now apply the discrete maximum principle to show that the discrete problem (2.3) has a unique solution. First note that (2.3) is no more than a system of linear equations. In fact, the set of grid functions with domain $\overline{\Omega}_h$, with the usual (pointwise) sum and scalar product, is a vector space of finite dimension, and its dimension equals the number of elements of $\overline{\Omega}_h$. So, (2.3) consists exactly of $\left|\overline{\Omega}_h\right|$ equations in $\left|\overline{\Omega}_h\right|$ unknowns.

Example 2.5 *Let $\Omega = (0,1)^2$, and $h = 1/n$ (see Fig. 1 on page 3). Then $\left|\overline{\Omega}_h\right| = (n+1)^2$, $|\partial\Omega_h| = 4n$, and $|\Omega_h| = (n-1)^2$. The dimension of $\mathbb{R}^{\overline{\Omega}_h}$ is $(n+1)^2$.*

Lemma 2.6 *Let $h > 0$. Let $f : \partial\Omega \to \mathbb{R}$. Then the problem (2.3) has a unique solution $U : \overline{\Omega}_h \to \mathbb{R}$.*

Proof. Write $\overline{\Omega}_h = \{x_{i,j} : (i,j) \in I\}$, where $I = \left\{(i,j) \in \mathbb{Z}^2 : x_{i,j} \in \overline{\Omega}_h\right\}$. Well-order I (e.g., lexicographically), and set $v_k = u(x_{i,j})$, where (i,j) is the k'th element of I. We know $1 \leq k \leq \left|\overline{\Omega}_h\right|$. Then, equations (2.3) set up a system of linear equations:

$$Av = b \tag{2.4}$$

The vector b comes from the right-hand side of (2.3), so its entries are either 0 or $f_h(x)$, for some $x \in \partial\Omega_h$. A is a $\left|\overline{\Omega}_h\right| \times \left|\overline{\Omega}_h\right|$ matrix. To show that (2.4) has a unique solution, it is enough to show that the linear map,

$$\mathbb{R}^{\overline{\Omega}_h} \ni v \longmapsto Av \in \mathbb{R}^{\overline{\Omega}_h}$$

is injective.

Suppose $Av = 0$. From (2.3), this means that $f_h(x) = 0$, for all $x \in \partial\Omega_h$. By the discrete maximum principle, $v = 0$.

We now work in a superstructure $\langle V(\mathbb{R}), {}^*V(\mathbb{R}), *\rangle$. We will omit the stars on all standard functions of one or several variables and usual binary relations. Each finite [3] $x \in {}^*\mathbb{R}$ can be uniquely decomposed as $x = r + \epsilon$, where $r \in \mathbb{R}$ and ϵ is an infinitesimal; r is called the standard part of x, and denoted by $\mathrm{st}\, x$. If $x, y \in {}^*\mathbb{R}$ are such that $x - y$ is infinitesimal, then we say that x is infinitesimally close to y, and write $x \approx y$. Similarly, if $x, y \in {}^*\mathbb{R}^n$:

$$x \approx y \quad \text{iff} \quad |x - y| \approx 0 \quad \text{iff} \quad x_i \approx y_i, \quad \text{for each } \ i = 1, \dots, n.$$

If $x \in {}^*\mathbb{R}^n$ is finite then let:

$${}^\circ x = {}^\circ(x_1, \dots, x_n) = (\,\mathrm{st}\, x_1, \dots,\, \mathrm{st}\, x_n)\ {}^4.$$

For other functions $F : A \subset {}^*\mathbb{R}^n \to {}^*\mathbb{R}^m$, define also ${}^\circ F$ by:

$${}^\circ F({}^\circ x) = {}^\circ(F(x)) \quad \forall x \in A.$$

For sets $A \in \mathbb{R}^n$, let:

$${}^\circ A = \Big\{ {}^\circ x : \text{“}x \text{ is finite” and } x \in A \Big\}.$$

Each “circle” map as introduced above is sometimes called a standard part map.

Whenever $h \approx 0$, the sets $\overline{\Omega}_h$, $\partial\Omega_h$, and Ω_h will be internal sets; also, f_h will be an internal function. In this chapter, we stick to the convention that objects

[3] $x \in {}^*\mathbb{R}^n$ is finite iff there exists $m \in \mathbb{N}$ such that $|x| < m$.

[4] Obviously, if $x \in {}^*\mathbb{R}$ (the $n = 1$ case) then ${}^\circ x = \mathrm{st}\, x$.

subscripted by $h \approx 0$ will be internal subsets of $\mathbb{R}^2_h$ or gridfunctions. The capital letters U, V, ... will preferably be used to designate internal gridfunctions. Since we will sometimes need to designate standard elements of some other sets, we will not omit the stars on sets [5]. By transfer, the *size of $\overline{\Omega}_h$ will be some hyperinteger $N \in {}^*\mathbb{N}$. Since Ω is open, whenever h is infinitesimal, N will be infinitely large.

Recall the construction of f_h on page 12. By construction, $\left|(\tilde{a}, \tilde{b}) - (a, b)\right| \leq h$, so if $h \approx 0$ and f is continuous, then for all $x \in \partial\Omega_h$, $^\circ f_h(x) = f(^\circ x)$.

By transfer of Lemma (2.6) and the internal definition principle, we obtain:

Lemma 2.7 *Let $h \approx 0$. Let $f : \partial\Omega \to {}^*\mathbb{R}$. Then the problem (2.3) has a unique internal solution $U : \overline{\Omega}_h \to \mathbb{R}$.*

We now turn to the approximate version of the maximum principle.

Definition 2.8 *The relations $\lesssim$ and $\gtrsim$ are defined in $^*\mathbb{R}^2$ by:*

$$a \lesssim b \Leftrightarrow a < b \vee a \approx b,$$
$$a \gtrsim b \Leftrightarrow b \lesssim a.$$

Theorem 2.9 (Approximate Maximum Principle) *Let $h > 0$, $h \approx 0$. Let $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ be internal and $a \in \mathbb{R}$. Suppose:*

$$\Delta_h U(x) \gtrsim 0, \quad \forall x \in \Omega_h;$$
$$U(x) \lesssim a, \qquad \forall x \in \partial\Omega_h.$$

Then, $U(x) \lesssim a \; \forall x \in \overline{\Omega}_h$.

[5] For example, $\mathbb{R}^+$ is the set of *standard* positive reals, while $^*\mathbb{R}^+$ designates the positive hyperreals.

Proof. We use a comparison function argument. Fix some $\bar{x} = (x_1, x_2) \in \Omega_h$. For each $c \in \mathbb{R}^+$, define:

$$V(x) = U(x) + w(x) \qquad \forall x \in \Omega_h,$$

where $w(x) = c\,|x - \bar{x}|^2$. Let $R \in \mathbb{R}^+$ be such that $B_R(\bar{x}) \supset \overline{\Omega}$. Computing the laplacian of the (standard) function w yields:

$$\Delta w(x_1, x_2) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(c(x_1 - \bar{x}_1)^2 + c(x_2 - \bar{x}_2)^2\right) = 4c.$$

From our hypothesis on U, and the differentiability of w:

$$\Delta_h V(x) = \Delta_h U(x) + \Delta_h w(x) \gtrsim 0 + 4c > 0.$$

Hence, $\Delta_h V(x) > 0$. On the other hand, if $x \in \partial\Omega_h$, then:

$$V(x) = U(x) + c\,|x - \bar{x}|^2 \leq U(x) + cR^2 \lesssim a + cR^2 \leq a + 2cR^2$$

By the transfer of the discrete maximum principle, $V(x) \leq a + 2cR^2$, for all $x \in \overline{\Omega}_h$. Hence, $U(x) \leq V(x) \leq a + 2cR^2$, for all $x \in \overline{\Omega}_h$. Since c was arbitrarily chosen in $\mathbb{R}^+$, we get that $U(x) \lesssim a$, for all $x \in \overline{\Omega}_h$.

Corollary 2.10 (Approximate Minimum Principle) *Let $h \approx 0$. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is internal and,*

$$\Delta_h U(x) \lesssim 0, \quad \forall x \in \Omega_h,$$
$$U(x) \gtrsim 0, \qquad \forall x \in \partial\Omega_h,$$

then $U(x) \gtrsim 0\ \forall x \in \overline{\Omega}_h$.

Proof. Apply the approximate maximum principle to $-U$.

Corollary 2.11 *Let* $h \approx 0$. *If* $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ *is internal and,*

$$\begin{aligned} \Delta_h U(x) &\approx 0, \quad \forall x \in \Omega_h, \\ U(x) &\approx 0, \qquad \forall x \in \partial\Omega_h, \end{aligned}$$

then $U(x) \approx 0 \; \forall x \in \overline{\Omega}_h$.

Proof. By the approximate maximum principle, $U(x) \lesssim 0$ on $\overline{\Omega}_h$. Also, applying the same result to $-U$, we get $U(x) \lesssim 0$ on $\overline{\Omega}_h$. Therefore, $U(x) \approx 0$, for all $x \in \overline{\Omega}_h$.

Now, we look for a standard version of the approximate maximum principle. To be able to formulate it, we consider a family of gridfunctions $U_h : \overline{\Omega}_h \to \mathbb{R}$, indexed in $h \in \mathbb{R}^+$.

Theorem 2.12 (Approximate Maximum Principle — Standard Form) *Let* $\{U_h : h \in \mathbb{R}^+\}$ *be a family of grid functions* $U_h : \overline{\Omega}_h \to \mathbb{R}$. *Then, for all* $\epsilon \in \mathbb{R}^+$, *there exists* $\delta \in \mathbb{R}^+$ *such that for all* $0 < h < \delta$ *the following statement holds. If*

$$\min_{x \in \Omega_h} \Delta_h U_h(x) > -\delta,$$

then

$$\max_{x \in \overline{\Omega}_h} U_h(x) < \epsilon + \max_{x \in \partial\Omega_h} U_h(x).$$

Proof.

Fix $\epsilon > 0$, and consider the set:

$$\begin{aligned} D \;=\; & \Big\{ \delta \in {}^*\mathbb{R}^+ : \forall h \in (0, \delta) \left(\min_{x \in \Omega_h} \Delta_h U_h(x) > -\delta \right) \\ & \qquad \Rightarrow \max_{x \in \overline{\Omega}_h} U_h(x) < \epsilon + \max_{x \in \partial\Omega_h} U_h(x) \Big\} \end{aligned}$$

D is internal and, by the approximate maximum principle, it includes the set of all positive infinitesimal δ. Therefore, by overspill, it must contain some standard $\delta > 0$. For that standard δ, the statement of the theorem is satisfied.

This version is actually equivalent to a weak form of Theorem (2.9), where we start with a one parameter family $\mathcal{U} = \{U_h : h \in \mathbb{R}^+\}$, and work with the internal family ${}^*\mathcal{U}$, instead of an individual internal gridfunction.

The following well-known result can be obtained as a corollary of the approximate maximum principle.

Corollary 2.13 (Analytical Maximum Principle) *Let $\Omega \in \mathbb{R}^2$ be bounded and open, and consider $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$. If $a \in \mathbb{R}$, and*

$$\begin{aligned} &\Delta u(x) \geq 0, \quad \forall x \in \Omega, \\ &u(x) \leq a, \qquad \forall x \in \partial\Omega, \end{aligned}$$

then $u(x) \leq a \ \forall x \in \overline{\Omega}$.

Proof.

Let $h \approx 0$, and consider $\overline{\Omega}_h$, Ω_h, $\partial\Omega_h$ and Δ_h as introduced before. Since u is C^2 in Ω, $\Delta_h u(x) \approx \Delta u(x) \geq 0$ at every x such that $\mathrm{dist}(x, \partial\Omega) \not\approx 0$. Consider the set:

$$E = \Big\{\delta \in {}^*\mathbb{R}^+ : \forall x \in \Omega_h \ \forall y \in {}^*\partial\Omega \ \ |x - y| > \delta \Rightarrow \Delta_h u(x) > -\delta\Big\}.$$

E is internal and includes all positive $\delta \not\approx 0$. Hence, it must contain some positive $\delta \approx 0$. Consider $\overline{\Omega}_h^\delta = \{x \in \overline{\Omega}_h : \forall y \in {}^*\partial\Omega \ \ |x - y| \geq \delta\}$, and let $\Omega_h^\delta = \{x \in \overline{\Omega}_h^\delta :$

"x has four neighbours"}, $\partial\Omega_h^\delta = \overline{\Omega}_h^\delta - \Omega_h^\delta$. Then:

$$\Delta_h u(x) > -\delta \approx 0, \quad \forall x \in \Omega_h^\delta,$$
$$u(x) \approx u({}^\circ x) \leq a, \quad \forall x \in \partial\Omega_h^\delta,$$

(note that ${}^*\mathrm{dist}(\partial\Omega_h^\delta, {}^*\partial\Omega) \lesssim \delta \approx 0$, so ${}^\circ x \in \partial\Omega$, for all $x \in \partial\Omega_h^\delta$). By the approximate maximum principle, we conclude that $u(x) \gtrsim a$, for all $x \in \overline{\Omega}_h^\delta$. Since ${}^\circ(\overline{\Omega}_h^\delta) = \overline{\Omega}$, this implies our result.

Corollary 2.14 *Let $\Omega \in \mathbb{R}^2$ be bounded and open and $f \in C(\partial\Omega, \mathbb{R})$. Then,*

$$\Delta u(x) = 0, \qquad x \in \Omega,$$
$$u(x) = f(x), \quad x \in \partial\Omega,$$

has no more than one solution in $C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$.

Proof. Let $v \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ be another solution. Then $\Delta(v - u) = 0$ in Ω and $(u - v)(x) = f(x) - f(x) = 0$ on $\partial\Omega$. By the analytical maximum principle (applied to $v - u$ and to $u - v$) we conclude that $u = v$ in $\overline{\Omega}$.

2.3 Existence and Uniqueness for the Classical Dirichlet Problem

We recall our Dirichlet problem (2.1):

$$\Delta u(x) = 0, \qquad x \in \Omega,$$
$$u(x) = f(x), \quad x \in \partial\Omega.$$

The main idea is to build a solution $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ by taking $h > 0$, $h \approx 0$, and letting

$$u(\,^\circ x) = \text{ st } U(x) \quad \forall x \in \overline{\Omega}_h \qquad (\text{i.e. } u = {}^\circ U),$$

where $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is the unique solution of:

$$\begin{aligned} \Delta_h U(x) &= 0, \quad\ \ x \in \Omega_h, \\ U(x) &= f_h(x), \quad x \in \partial\Omega_h. \end{aligned}$$

We show that u is a classical (or strong) solution to (2.1) in two steps:

Step 1: Show that U is S-continuous. That is, for all $x_1, x_2 \in \overline{\Omega}_h$ such that $x_1 \approx x_2$, $U(x_1) \approx U(x_2)$. From this, it follows that u is well defined and continuous.

Step 2: Show that $u \in C^2(\overline{\Omega}, \mathbb{R})$. (In fact, $u \in C^\infty(\overline{\Omega}, \mathbb{R})$).

To handle step 1, we begin by showing that if S-continuity of U fails, then it must also fail infinitesimally close to $\partial\Omega_h$. This will follow from the transfer of the following standard result:

Lemma 2.15 *Let $U : \overline{\Omega}_h \mapsto \mathbb{R}$ be the solution of (2.3). Then, for any $x, y \in \Omega_h$, there exist $\tilde{x}, \tilde{y} \in \overline{\Omega}_h$, with at least $y \in \partial\Omega_h$, and such that:*

$$|U(\tilde{y}) - U(\tilde{x})| \geq |U(y) - U(x)|\,.$$

Proof. Without loss of generality, $U(x) \geq U(y)$. Let $d = y - x$, that is, $d = (d_1, d_2) = (y_1 - x_1, y_2 - x_2)$. Let:

$$\overline{\Omega}_h^d = \left\{ x \in \mathbb{R}_h^2 : x \in \overline{\Omega}_h \wedge x + d \in \overline{\Omega}_h \right\};$$

also, define $\partial\Omega_h^d$ and Ω_h^d in the same way as was done for Ω_h and $\partial\Omega_h$. It is easy to see that, if $x \in \partial\Omega_h^d$, then x or $x+d$ is in $\partial\Omega_h$. Now, define $W^d : \Omega_h^d \mapsto \mathbb{R}$ by:

$$W^d(z) = U(z+d) - U(z)$$

Since $\Delta_h W^d(z) = \Delta_h U(z+d) - \Delta_h U(z) = 0$ for all $z \in \Omega_h^d$, then the (discrete) maximum principle assures that there is an $\tilde{x} \in \partial\Omega_h^d$ such that:

$$W^d(\tilde{x}) = U(\tilde{x}+d) - U(\tilde{x}) \geq U(x+d) - U(x) = U(y) - U(x)$$

Setting $\tilde{y} = \tilde{x} + d$, and since we have assumed that $U(y) \geq U(x)$, we get:

$$|U(\tilde{y}) - U(\tilde{x})| \geq |U(y) - U(x)|\,.$$

Since $\tilde{x} \in \partial\Omega_h$ or $\tilde{y} \in \partial\Omega_h$, the lemma is proved.

By the transfer of Lemma (2.15), we have that if U is not S-continuous, then the S-continuity condition must fail at some pair of points $x_1, x_2 \in \overline{\Omega}_h$, with at least $x_2 \in \partial\Omega_h$. Hence, to show that U is S-continuous, it is enough to show that:

$$\forall y \in \partial\Omega_h \; \forall x \in \overline{\Omega}_h \; y \approx x \Rightarrow U(y) \approx U(x).$$

To accomplish this, we introduce a nonstandard concept of *barrier*:

Definition 2.16 *Let $y \in \partial\Omega$ be standard. An internal and S-continuous function $b_y : {}^*\overline{\Omega}_h \to {}^*\mathbb{R}$ is called a* barrier *at y, iff, for all positive $h \approx 0$:*

(b1) *Let $b_{y,h}$ be the restriction of b_y to $\overline{\Omega}_h$. Then:*

$$\Delta_h b_{y,h}(x) \lesssim 0 \quad \text{for all } \; x \in \Omega_h;$$

(b2) $b_y(y) \approx 0$;

(b3) $b_y(x) \gtrapprox 0$, *for all* $x \in {}^*\overline{\Omega}$, *with* $x \not\approx y$.

If there exists a barrier at $y \in \partial\Omega$, *then* y *is called strongly regular.*

It follows from (b2) that:

$$b_{y,h}(x) \approx 0 \ , \ \ \forall x \in \overline{\Omega}_h \ , \ x \approx y.$$

Also, from (b3), we have:

$$b_{y,h} \gtrapprox 0 \ , \ \ \forall x \in \overline{\Omega}_h \ , \ x \not\approx y.$$

Proposition 2.17 *Let* $y \in \partial\Omega$. *If* $u \in C^2(\overline{\Omega}, \mathbb{R})$ *is such that*

$$\Delta u(x) \leq 0 \qquad \forall x \in \overline{\Omega}, \tag{2.5}$$

$$u(y) = 0, \tag{2.6}$$

$$u(x) > 0 \qquad \forall x \in \overline{\Omega} \ , \ x \neq y, \tag{2.7}$$

then u *is a barrier for* y.

Proof.

The barrier condition (b1) follows easily from the continuous differentiability of u on the compact set $\overline{\Omega}$, and from equation (2.5). (b2) and (b3) are simple consequences of (2.6), (2.7) and the continuity of u.

Example 2.18 *Let* $y \in \partial\Omega$ *satisfy an exterior circle condition. That is, there exists a ball,* $B_r(\xi)$ [6], *such that* $\overline{B}_r(\xi) \cap \overline{\Omega} = \{y\}$. *Then* $u(x) = \log \frac{|x-\xi|}{r}$ *satisfies*

[6] *As usual, let* $B_r(\xi) = \{x \in \mathbb{R} : |x - \xi| < r\}$ *and* $\overline{B}_r(\xi) = \{x \in \mathbb{R} : |x - \xi| \leq r\}$

the hypothesis of proposition (2.17). In Lemma (3.21), we show that the exterior circle condition implies a stronger barrier condition than Definition (2.16), for a general uniformly elliptic operator.

Given a barrier b_y, we can consider the function ${}^\circ b_y \in C(\overline{\Omega}, \mathbb{R})$. This function has the following properties:

(${}^\circ$b2) ${}^\circ b_y(y) = 0$

(${}^\circ$b3) ${}^\circ b_y(x) > 0$, for all $x \in \overline{\Omega} - \{y\}$

For each $\delta \in \mathbb{R}^+$, it follows, from the compactness of $\overline{\Omega} - B_\delta(y)$ that if $\overline{\Omega} - B_\delta(y) \neq \emptyset$, then b_y has a minimum, m, on this set; also, from (${}^\circ$b3), $m > 0$.

Lemma 2.19 *Let $f : \partial\Omega \to \mathbb{R}$ be a standard continuous function. Let $h > 0$, $h \approx 0$, and $U : \overline{\Omega}_h \mapsto {}^*\mathbb{R}$ internal be the unique solution of:*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad & x \in \Omega_h, \\ U(x) &= f_h(x), \quad & x \in \partial\Omega_h. \end{aligned}$$

Then, for all standard $y \in \partial\Omega$ such that there exists a barrier, b_y, and for all $x \in \overline{\Omega}_h$ such that ${}^\circ x = y$, we have $\operatorname{st} U(x) = f(y)$.

Proof. Let b_y be a barrier at y, and $b_{y,h}$ its restriction to $\overline{\Omega}_h$. Fix $\epsilon \in \mathbb{R}^+$. From continuity of f, we can find $\delta > 0$ such that:

$$\forall x \in \partial\Omega \;\; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon. \tag{2.8}$$

Let $M = \max_{x \in \partial\Omega} f(x)$. From our observations about ${}^\circ b_y$, we can find some $K \in \mathbb{R}^+$ such that if $|x - y| \geq \delta$, then $K\,{}^\circ b_y > 2M$ (just take $K = 2M/m$).

Translating this to $b_{y,h}$, we have:

$$\forall x \in \overline{\Omega}\ |x-y| > \delta \Rightarrow Kb_{y,h}(x) \gtrsim 2M. \tag{2.9}$$

Note that both δ and K depend only on the choice of ϵ (for fixed Ω and y). Consider the internal functions, defined on $\overline{\Omega}_h$ by:

$$\omega_\epsilon^-(x) = U(x) - f(y) - \epsilon - Kb_{y,h}(x);$$
$$\omega_\epsilon^+(x) = U(x) - f(y) + \epsilon + Kb_{y,h}(x).$$

They satisfy the following.

(i) For all $x \in \Omega_h$:

$$\Delta_h\omega_\epsilon^-(x) = -K\Delta_h b_{y,h}(x) \gtrsim 0;$$
$$\Delta_h\omega_\epsilon^+(x) = K\Delta_h b_{y,h}(x) \lesssim 0.$$

(ii) For all $x \in \partial\Omega_h$

$$\omega_\epsilon^-(x) = f_h(x) - f(y) - \epsilon - Kb_{y,h}(x) \approx f(x) - f(y) - \epsilon - Kb_{y,h}(x);$$
$$\omega_\epsilon^+(x) = f_h(x) - f(y) + \epsilon + Kb_{y,h}(x) \approx f(x) - f(y) + \epsilon + Kb_{y,h}(x).$$

But (ii) can be simplified in the following way:

Case 1: if $|x-y| < \delta$, and using (2.8):

$$\omega_\epsilon^-(x) \lesssim f(x) - f(y) - \epsilon \lesssim 0;$$
$$\omega_\epsilon^+(x) \gtrsim f(x) - f(y) + \epsilon \gtrsim 0.$$

Case 2: if $|x-y| \geq \delta$, and using (2.9):

$$\omega_\epsilon^-(x) \lesssim f(x) - f(y) - 2M \lesssim 0;$$
$$\omega_\epsilon^+(x) \gtrsim f(x) - f(y) + 2M \gtrsim 0.$$

By the approximate maximum principle:

$$U(x) - f(y) - \epsilon - Kb_{y,h}(x) = \omega_\epsilon^-(x) \lesssim 0 \lesssim \omega_\epsilon^+(x) = U(x) - f(y) + \epsilon + Kb_{y,h}(x).$$

Subtracting $U(x) - f(y)$ from all sides of the above inequalities yields:

$$|U(x) - f(y)| \lesssim \epsilon + Kb_{y,h}(x).$$

For $^\circ x = y$, we have from the barrier condition (b2) that $b_{y,h}(x) \approx 0$. Hence:

$$|U(x) - f(y)| \lesssim \epsilon.$$

Since ϵ was an arbitrarily chosen positive real number, we conclude that $U(x) \approx f(y)$, as wanted.

We can now conclude step 1.

Theorem 2.20 *Let $h \approx 0$ be positive. Let Ω be bounded open, with all points of $\partial\Omega$ h-regular, and $f : \partial\Omega \mapsto \mathbb{R}$ be a standard continuous function. Let $U : \overline{\Omega}_h \mapsto {}^*\mathbb{R}$ internal be the solution of:*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad && x \in \Omega_h, \\ U(x) &= f_h(x), && x \in \partial\Omega_h. \end{aligned}$$

Then, for all $x, \tilde{x} \in \overline{\Omega}_h$ such that $^\circ x = {}^\circ\tilde{x}$, $\mathrm{st}\, U(x) = \mathrm{st}\, U(\tilde{x})$.

Proof. Follows immediately from Lemma (2.15) and Lemma (2.19).

Theorem (2.20) implies that the (standard) function $u : \overline{\Omega} \mapsto \mathbb{R}$ defined by $u({}^\circ x) = \mathrm{st}\, U(x)$ is well defined and continuous. Thus, step 1 is now completed. We proceed to step 2.

Lemma 2.21 *Let Ω be bounded open. Let $f : \partial\Omega \mapsto \mathbb{R}$ be a (standard) continuous function. Let $h > 0$, $h \approx 0$, and $U : \overline{\Omega}_h \mapsto {}^*\mathbb{R}$ internal be the solution of:*

$$\Delta_h U(x) = 0, \quad x \in \Omega_h,$$
$$U(x) = f_h(x), \quad x \in \partial\Omega_h.$$

If for all $x, \tilde{x} \in \Omega_h$ such that ${}^\circ x = {}^\circ\tilde{x} \in \Omega$, $\operatorname{st} U(x) = \operatorname{st} U(\tilde{x})$, then $u = {}^\circ U \in C^\infty(\Omega, \mathbb{R})$ and $\Delta u(x) = 0\ \forall x \in \Omega$.

Proof. Let $\xi \in \Omega$, and choose $\epsilon > 0$ such that $B_\epsilon(\xi) \subset \Omega$. Then, a solution of

$$\Delta\bar{u}(x) = 0, \quad x \in B_\epsilon(\xi),$$
$$\bar{u}(x) = u(x), \quad x \in \partial B_\epsilon(\xi),$$

exists (since u is continuous on $\partial B_\epsilon(\xi)$), is unique and C^∞ in $B_\epsilon(\xi)$. The solution is actually given by Poisson's integral formula.

Consider the internal function $V = U - \bar{u}$. Since $\overline{u}$ is smooth in $B_\epsilon(\xi)$, $\Delta_h\overline{u}(x) \approx \Delta\overline{u}(x) = 0$ at every $x \in (B_\epsilon(\xi))_h$ whose distance to $\partial B_\epsilon(\xi)$ is not infinitesimal. Consider the set:

$$E = \Big\{\delta \in {}^*\mathbb{R} : 0 < \delta < \epsilon \wedge \forall x \in (B_\delta(\xi))_h\ |\Delta_h\overline{u}(x)| < \epsilon - \delta\Big\}.$$

E is internal and contains all $\delta \in {}^*(0, \epsilon)$ such that $\delta \not\approx \epsilon$. By overspill, E contains some $\delta \approx \epsilon$. Then, for all $x \in (B_\delta(\xi))_h$:

$$\Big|\Delta_h V(x)\Big| = \Big|\Delta_h U(x) - \Delta_h\overline{u}(x)\Big| = \Big|\Delta_h\overline{u}(x)\Big| < \epsilon - \delta \approx 0$$

In turn, for $x \in \partial\,(B_\delta(\xi))_h$, and since ${}^\circ x \in \partial B_\epsilon(\xi)$:

$$V(x) = U(x) - \bar{u}(x) \approx u({}^\circ x) - \bar{u}({}^\circ x) = 0$$

By Corollary (2.11), $U(x) \approx \bar{u}(x)$ for all $x \in (B_\delta(\xi))_h$. Hence, and taking standard parts, $u \equiv \bar{u}$ on $B_\epsilon(\xi) = {}^\circ(B_\delta(\xi))_h$. In particular, u is C^∞ at ξ and satisfies $\Delta u(\xi) = 0$. Since ξ was arbitrarily chosen in Ω, we get the desired result.

Theorem 2.22 *Let $h \approx 0$ be positive, and Ω be bounded open, with all points of $\partial\Omega$ h-regular. Let $f : \partial\Omega \to \mathbb{R}$ be a (standard) continuous function, and $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ (internal) be the solution of:*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad & x \in \Omega_h, \\ U(x) &= f_h(x), \quad & x \in \partial\Omega_h. \end{aligned}$$

Let $u = {}^\circ U$. Then $u \in C(\overline{\Omega}, \mathbb{R}) \cap C^\infty(\Omega, \mathbb{R})$, and u is the unique solution of:

$$\begin{aligned} \Delta u(x) &= 0, \quad & x \in \Omega, \\ u(x) &= f(x), \quad & x \in \partial\Omega. \end{aligned}$$

Proof. By Theorem (2.20), $u \in C(\overline{\Omega}, \mathbb{R})$. For $x \in \partial\Omega_h$, $U(x) = f_h(x)$. Hence, for each standard $y \in \partial\Omega$,

$$u(y) = \operatorname{st} U(x) = \operatorname{st} f_h(x) = \operatorname{st} f(\tilde{x}) = f(x),$$

for some $x \in \partial\Omega_h$, $\tilde{x} \in {}^*(\partial\Omega)$, with ${}^\circ x = {}^\circ\tilde{x} = y$. By Lemma (2.21) $u \in C^\infty(\Omega, \mathbb{R})$ and $\Delta u(x) = 0$ for all $x \in \Omega$. Uniqueness of solution follows from the analytical maximum principle.

We now look at a convergence result. For that, we need the strong nonstandard barrier condition. Our notion of convergence is based on the following norm.

Definition 2.23 *Let Ω be an open and bounded subset of $\mathbb{R}^2$. For each $h \in \mathbb{R}^+$, the L_h^∞ norm of a function $U : \overline{\Omega}_h \to \mathbb{R}^n$ is given by:*

$$\| U \|_{L_h^\infty} = \max_{x \in \overline{\Omega}_h} |U(x)|.$$

With $h \approx 0$, this gives an internal *norm, acting on internal gridfunctions $U : \overline{\Omega}_h \to {}^*\mathbb{R}$. $\| \cdot \|_{L_h^\infty} : \mathbb{R}^{\overline{\Omega}_h} \to {}^*\mathbb{R}^+$.

In the following, whenever needed, assume that any standard $U : \overline{\Omega}_h \to \mathbb{R}$ is extended as a stepfunction to ${}^*\overline{\Omega}$ [7]. Also extend internal gridfunctions $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ to ${}^*\overline{\Omega}$ in a similar way.

Theorem 2.24 *Let Ω be bounded open, with all points of $\partial\Omega$ strongly regular. Let $f : \partial\Omega \to \mathbb{R}$ be a (standard) continuous function. For each $h \in \mathbb{R}^+$, let $U_h : \overline{\Omega}_h \to \mathbb{R}$ be the solution of*

$$\begin{aligned} \Delta_h U_h(x) &= 0, \quad && x \in \Omega_h, \\ U_h(x) &= f_h(x), && x \in \partial\Omega_h. \end{aligned}$$

Let $u \in C(\overline{\Omega}, \mathbb{R}) \cap C^\infty(\Omega, \mathbb{R})$, be the solution of

$$\begin{aligned} \Delta u(x) &= 0, \quad && x \in \Omega, \\ u(x) &= f(x), && x \in \partial\Omega. \end{aligned}$$

then $\lim_{h \downarrow 0} \|U_h - u\|_{L_h^\infty} = 0$.

Proof. Let $h \approx 0$ be positive. Consider some $x \in {}^*\overline{\Omega}_h$. Then, using Theorem (2.22) and the continuity of u:

$$^\circ |U_h(x) - u(x)| = | {}^\circ U_h(x) - {}^\circ(u(x))| = |u({}^\circ x) - u({}^\circ x)| = 0.$$

[7] The extension to $\overline{\Omega}$ is done by taking $U(x_1, x_2) = U(\bar{x}_1, \bar{x}_2)$, where $(\bar{x}_1, \bar{x}_2)$ is the closest gridpoint to the left and down of (x_1, x_2), i.e.: $\bar{x}_i = \max\{nh \in h\mathbb{Z} : nh \leq x_i\}$, $i = 1, 2$.

Hence, for all $\epsilon \in \mathbb{R}^+$:

$$\|U_h - u\|_{L_h^\infty} < \epsilon.$$

This implies that $\|U_h - u\|_{L_h^\infty} \approx 0$. Since h was an arbitrary positive infinitesimal, we conclude that:

$$\lim_{h\downarrow 0} \|U_h - u\|_{L_h^\infty} = 0.$$

These results can easily be extended to the Laplace equation in $\mathbb{R}^n$. A convergence result for the Poisson equation, but with smoothness assumptions on the boundary of Ω and on the boundary conditions, can be found in Wendland [28]. Krylov [17] contains a generalization of this result to uniformly elliptic operators and a class of schemes on uniform grids; however, smoothness assumptions on the boundary are also required. In Chapter 3, we will consider uniformly elliptic operators and their discretizations, but without restricting our attention to uniform grids, and with only barrier-type conditions on the boundary. Error estimates will also be derived.

2.4 Nonstandard and Standard Barrier Conditions

The results of the previous section were obtained under the nonstandard barrier condition given by Definition (2.16). This condition looks quite different from the standard one. Nevertheless, a weakened version of our barrier condition is equivalent to the standard notion. Before we proceed, let us review some standard results, and the standard barrier condition. Let $u \in C^2(\Omega, \mathbb{R})$ and $B_\rho(\xi) \subset \Omega$.

A function $u \in C(\overline{\Omega}, \mathbb{R})$ is called superharmonic (in Ω), iff, for any $B_\rho(\xi) \subset \Omega$:

$$u(\xi) \geq \frac{1}{2\pi\rho} \int_{\partial B_\rho(\xi)} u(x)\, ds_x \quad . \tag{2.10}$$

Equivalently, u is superharmonic iff for all $\xi \in \Omega$ and $\rho \in \mathbb{R}^+$ such that $B_\rho(\xi) \in \Omega$, if w satisfies:

$$\begin{aligned} &\Delta w(x) = 0 \quad &&\forall x \in B_\rho(\xi), \\ &w(x) = u(x) &&\forall x \in \partial B_\rho(\xi), \end{aligned}$$

then $w(x) \leq u(x)$, $\forall x \in B_\rho(\xi)$. It follows from 2.10 that if Ω is connected, u is superharmonic in Ω, and u attains an interior minimum, then u is constant (minimum principle).

Definition 2.25 (Standard Barrier Condition) *Let $\Omega \subset \mathbb{R}^2$ be bounded and open. A point $y \in \partial\Omega$ satisfies the standard barrier condition (or we call it standardly regular) iff there exists a superharmonic function $b_y \in C(\overline{\Omega}, \mathbb{R})$ such that:*

$$\begin{aligned} &b_y(y) = 0\ ; \\ &b_y(x) > 0, \quad \forall x \in \overline{\Omega} - \{y\}. \end{aligned}$$

Given $f \in C(\partial\Omega, \mathbb{R})$, a superharmonic function in Ω, w, such that $w(x) \geq f(x)$ for all $x \in \partial\Omega$ is called a superfunction for f in Ω. From the minimum principle, if w is a superfunction of f in Ω, then $w(x) < \min_{y \in \partial\Omega} f(y)$, for all $x \in \Omega$, unless w is constant on some connected component of Ω. If f is not constant when restricted to the boundary of any connected component of Ω, then there exists a superfunction for f in Ω that is not constant on any connected component of Ω.

By Perron's method, if

$$\underline{\mathrm{u}}(x) = \inf\Big\{w(x) : \text{“}w \text{ is a superfunction for } f \text{ in } \Omega\text{”}\Big\}$$

and if $y \in \partial\Omega$ is standardly regular, then $\underline{\mathrm{u}} \in C^\infty(\Omega, \mathbb{R})$ and:

$$\begin{aligned} &\Delta\underline{\mathrm{u}}(x) = 0 && \forall x \in \Omega, \\ &f(y) = \underline{\mathrm{u}}(y) = \lim_{\substack{x \to y \\ x \in \Omega}} \underline{\mathrm{u}}(x). \end{aligned}$$

Define the standard interior of an internal set $A \subset {}^*\mathbb{R}^2$ by:

$$\text{S–int}\, A = \Big\{x \in A : \exists \delta \in \mathbb{R}^+ B_\delta(x) \subset A\Big\}^8.$$

The interior approximations to S–int A, given for each $\delta \in {}^*\mathbb{R}^+$ by

$$A^{-\delta} = \Big\{x \in A : B_\delta(x) \subset A\Big\}$$

are actually internal.

Definition 2.26 (Weak Nonstandard Barrier Condition) *Let $\Omega \subset \mathbb{R}^2$ be bounded and open and h be a positive infinitesimal. A point $y \in \partial\Omega$ is called h-weakly regular (or is said to satisfy the h-weak nonstandard barrier condition) iff, there exists an internal S-continuous function $\beta_{y,h} : \overline{\Omega}_h \to {}^*\mathbb{R}$, and a positive $\delta \approx 0$, such that:*

$$\begin{aligned} &\Delta_h \beta_{y,h}(x) \lesssim 0 && \forall x \in (({}^*\Omega)^{-\delta})_h, \\ &\beta_{y,h}(\tilde{y}) \approx 0, && \forall \tilde{y} \in \overline{\Omega}_h \ \ \tilde{y} \approx y \\ &\beta_{y,h}(x) \gtrapprox 0, && \forall x \in \overline{\Omega}_h \ \ x \not\approx y. \end{aligned}$$

y is called weakly regular (or said to satisfy the weak nonstandard barrier condition) iff for all positive $h \approx 0$, y is h-weakly regular.

[8] Note that S–int A is not, in general, internal

Theorem 2.27 *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and $y \in \partial\Omega$. Then, the following are equivalent:*

(a) y is standardly regular.

(b) y is weakly regular.

(c) There exists a positive $h \approx 0$ such that y is h-weakly regular.

Proof.

(a) $\Rightarrow$ (b).

Let b_y be a standard barrier at y, and consider

$$\beta_y(x) = \sup\Big\{w(x) : \text{“}w \text{ is a subfunction for } b_y \text{ in } \Omega\text{”}\Big\}. \tag{2.11}$$

Each superfunction, w, in (2.11) satisfies $w(x) > \min_{z\in\partial\Omega} b_y(z) = 0$, for all $x \in \Omega$. Also, for all $x \in \partial\Omega$, $w(x) \geq b_y(x)$. So, for all $\delta \in \mathbb{R}^+$ sufficiently small so that $B_\delta(y) \cap \overline{\Omega} \neq \emptyset$, $\min_{x\in\overline{\Omega}-B_\delta(y)} w(x) > 0$. Hence:

$$\beta_y(x) > 0 \quad \forall x \in \overline{\Omega} - B_\delta(y).$$

Since this is true for all $\delta \in \mathbb{R}^+$, we conclude that:

$$\beta_y(x) > 0 \quad \forall x \in \overline{\Omega} - \{y\}. \tag{2.12}$$

By Perron's method, $\beta_y \in C^2(\Omega, \mathbb{R})$, and:

$$\Delta\beta_y(x) = 0 \quad \forall x \in \Omega, \tag{2.13}$$

$$\lim_{\substack{x\to y\\ x\in\Omega}} \beta_y(x) = f(y). \tag{2.14}$$

Consider any positive $h \approx 0$. Let $\beta_{y,h}$ be the restriction of β_y to $\overline{\Omega}_h$. From (2.12), it follows that, for every $x \in \overline{\Omega}_h$ such that $x \not\approx y$:

$$\beta_{y,h}(x) = \beta_y(x) \gtrsim\!\!\!\!/\; 0.$$

Furthermore, from (2.14), and if $\tilde{y} \in \overline{\Omega}_h$ is such that $\tilde{y} \approx y$

$$\beta_{y,h}(\tilde{y}) = \beta_y(\tilde{y}) \approx b_y(y) = 0.$$

It remains to show that $\Delta_h \beta_{y,h}(x) = 0$, $\forall x \in \left(({}^*\Omega)^{-\delta}\right)_h$, for some positive $\delta \approx 0$.

Consider the internal set:

$$D = \left\{\delta \in {}^*\mathbb{R}^+ : \forall x(({}^*\Omega)^{-\delta})_h : |\Delta_h \beta_{y,h}(x) - \Delta \beta_y(x)| < \delta\right\}. \tag{2.15}$$

Since β_y is C^2 in Ω, the above set contains all positive and noninfinitesimal δ. Hence, it must contain some infinitesimal. For that δ, and from (2.15):

$$\Delta_h \beta_{y,h}(x) \approx \Delta \beta_y(x) = 0, \qquad \forall x \in (({}^*\Omega)^{-\delta})_h.$$

(b) ⇒ (c).

This implication is obvious.

(c) ⇒ (a).

Consider the positive $h \approx 0$ such that *(c)* holds, and pick a weak nonstandard barrier, $\beta_{y,h} : \overline{\Omega}_h \to {}^*\mathbb{R}$ at y (relative to h). Now, let:

$$b_y({}^\circ x) = \operatorname{st} \beta_{y,h}(x) \quad \forall x \in \overline{\Omega}_h.$$

From the S-continuity of $\beta_{y,h}$, b_y is well-defined and continuous; it also satisfies:

$$b_y(y) = 0,$$
$$b_y(x) < 0, \quad \forall x \in \partial\Omega, x \neq y.$$

It remains to show that b_y is superharmonic. Choose any $\xi \in \Omega$ and $\rho \in \mathbb{R}^+$ such that $B_\rho(\xi) \in \Omega$, and let $B = B_\rho(\xi)$. Consider $u \in C^2(\overline{B}, \mathbb{R})$, such that $\Delta u(x) = 0$ in B and $u = b_y$ on ∂B. Then:

$$\begin{aligned} \Delta_h\Big(\beta_{y,h} - u\Big)(x) &\approx \Delta_h \beta_{y,h}(x) \lesssim 0 && \forall x \in B_h, \\ \Big(\beta_{y,h} - u\Big)(x) &\gtrsim \beta_{y,h}(x) - \beta_{y,h}(x) = 0 && \forall x \in \partial B_h. \end{aligned}$$

From the approximate maximum principle, $(\beta_{y,h} - u)(x) \gtrsim 0$, for all $x \in \overline{B}_h$. But then, for all $x \in \overline{B}_h$:

$$b_y({}^\circ x) = \operatorname{st} \beta_{y,h}(x) \geq \operatorname{st} u(x) = u({}^\circ x).$$

This shows that b_y is superharmonic in Ω.

What damage does the weakening of the nonstandard barrier condition do to the results of section 2.3? The results from step 1 (Lemma (2.15), Lemma (2.19) and Theorem (2.20)) of the existence proof require some minor changes.

If, for some positive $h \approx 0$, we consider δ as given by the barrier condition, we can use the maximum principle on $(({}^*\Omega)^{-\delta})_h$ and $\partial(({}^*\Omega)^{-\delta})_h$. Furthermore, ${}^\circ(({}^*\Omega)^{-\delta})_h = \overline{\Omega}$ and ${}^\circ\partial(({}^*\Omega)^{-\delta})_h = \partial\Omega$. So, in step 1, we only need to replace $\overline{\Omega}_h$ by $(({}^*\Omega)^{-\delta})_h \cup \partial(({}^*\Omega)^{-\delta})_h$, Ω_h by $(({}^*\Omega)^{-\delta})_h$ and $\partial\Omega_h$ by $\partial(({}^*\Omega)^{-\delta})_h$.

All we need now for the proofs in step 1 to be carried out as before is a construction of $f_h : \partial(({}^*\Omega)^{-\delta})_h \to {}^*\mathbb{R}$ such that:

$$f_h(x) \approx f({}^\circ x) \qquad \forall x \in \partial(({}^*\Omega)^{-\delta})_h. \tag{2.16}$$

For that, let:

$$f_h(x) = f(\tilde{x}),$$

where $\tilde{x} \in B_{2\delta}(x)$ is constructed as follows. Let $r = \text{dist}(x, {}^*\partial\Omega) \lesssim \delta$. Then, let $\tilde{x} = x + (r\cos\theta, r\sin\theta)$, where θ is the smallest hyperreal in ${}^*[0, 2\pi)$ such that $x + (r\cos\theta, r\sin\theta) \in {}^*\partial\Omega$. Now, (2.16) follows from the fact that ${}^\circ\tilde{x} = {}^\circ x \in \partial\Omega$ and $f \in C(\partial\Omega, \mathbb{R})$. Thus, we get:

Theorem 2.28 *Let Ω be bounded open, with all points of $\partial\Omega$ satisfying the standard barrier condition. Let $f : \partial\Omega \to \mathbb{R}$ be a (standard) continuous function, and consider a positive $h \approx 0$. Then there exists a positive $\delta \approx 0$ such that the following holds. Suppose $U : (\overline{({}^*\Omega)^{-\delta}})_h \to {}^*\mathbb{R}$ is the unique (internal) solution of:*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad x \in (({}^*\Omega)^{-\delta})_h, \\ U(x) &= f_h(x), \quad x \in \partial(({}^*\Omega)^{-\delta})_h. \end{aligned}$$

Then, $u \in C(\overline{\Omega}, \mathbb{R}) \cap C^\infty(\Omega, \mathbb{R})$ given by $u({}^\circ x) = \text{st}\, U(x)$, for all $x \in ({}^*\Omega)^{-\delta})_h$, is the unique solution of:

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega, \\ u(x) &= f(x), \quad x \in \partial\Omega. \end{aligned}$$

As for convergence, we can say the following:

Theorem 2.29 *Let Ω be bounded open, with all points of $\partial\Omega$ weakly regular. Let $f : \partial\Omega \to \mathbb{R}$ be a (standard) continuous function. If, for each $h \in \mathbb{R}^+$, $U : (\overline{\Omega^{-\delta}})_h \to \mathbb{R}$ is the solution of*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad x \in (\Omega^{-\delta})_h, \\ U(x) &= f_h(x), \quad x \in \partial(\Omega^{-\delta})_h, \end{aligned} \tag{2.17}$$

(for δ as given by Theorem (2.28)) and $u \in C(\overline{\Omega}, \mathbb{R}) \cap C^\infty(\Omega, \mathbb{R})$, is the solution of

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega, \\ u(x) &= f(x), \quad x \in \partial\Omega. \end{aligned}$$

then, for all compact $K \subset \Omega$, $\lim_{h \downarrow 0} \|U_h - u\|_{C(K,\mathbb{R})} = 0$, *that is,* U_h *converges uniformly to* u *on compact subsets of* Ω. *Furthermore for each* $x_0 \in \partial\Omega$, *there exists a function* $\delta : (0, h_0] \subset \mathbb{R}^+ \to \mathbb{R}$ *such that* $\lim_{h \downarrow 0} \delta(h) = 0$, *and*

$$u(x_0) = \lim_{\substack{h \downarrow 0 \\ x \to x_0 \\ x \in \Omega^{-\delta(h)}}} U_h(x)$$

Proof. Since $K \subset \Omega$ is compact, $\operatorname{dist}(K, \partial\Omega) > 0$. Therefore, $(\overline{({}^*\Omega)^{-\delta}})_h \supset {}^*K$. Hence, using the same argument as in the proof of Theorem (2.24), we get:

$$\lim_{h \downarrow 0} \|U_h - u\|_{L^\infty_h(K,\mathbb{R})} = 0.$$

As for the boundary convergence, for each positive $h \approx 0$, there exists a positive $\delta \in {}^*\mathbb{R}^+$ such that U_h satisfying the discrete problem (2.17) exists. By the internal definition principle and overspill, this statement should hold for $h \in {}^*(0, h_0]$, where $h_0 \not\approx 0$ is positive. Now, let $\delta(h)$ be the *infimum of the internal set D defined by equation (2.15) on page 35. Then $\lim_{h \downarrow 0} \delta(h) = 0$. Also, if $x_0 \in \partial\Omega$, $h_1 > 0$ is an infinitesimal, and $x_1 \in ({}^*\Omega)^{-\delta(h_1)}$, is such that ${}^\circ x_1 = x_0$, then $U_{h_1}(x_1) \approx u(x_0)$. Hence:

$$u(x_0) = \lim_{\substack{h \downarrow 0 \\ x \to x_0 \\ x \in \Omega^{-\delta(h)}}} U_h(x).$$

2.5 The Dirichlet Problem when the Barrier Condition Fails

The construction of a solution to the Dirichlet problem (2.1) employing finite differences, as carried out in the last section, can actually be used in a more general setting. We begin by showing that, even if there are nonregular boundary points in $\partial\Omega$, the solution U of (2.3) is always S-continuous in $\Omega_h \cap$ S–int ${}^*\Omega$.

Lemma 2.30 *Let $f : \partial\Omega \to \mathbb{R}$ be a standard continuous function. Let $h > 0$, $h \approx 0$, and $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ internal be the solution of:*

$$\begin{aligned} \Delta_h U(x) &= 0, \quad x \in \Omega_h, \\ U(x) &= f_h(x), \quad x \in \partial\Omega_h. \end{aligned} \tag{2.18}$$

Then, for all $x, \bar{x} \in \overline{\Omega}_h \cap$ S–int ${}^*\Omega$ such that ${}^\circ x = {}^\circ\bar{x}$, $\mathrm{st}\, U(x) = \mathrm{st}\, U(\bar{x})$.

Proof. Let:

$$M = \max_{x\in\partial\Omega_h} |f_h(x)| = \max_{x\in\overline{\Omega}_h} |U(x)|.[9]$$

Fix $x, \bar{x} \in \overline{\Omega} \cap$ S–int ${}^*\Omega$.

We begin by considering the special case $x = \bar{x} + (0, 2\delta)$. Without loss of generality, we may assume $\delta > 0$. Also, ${}^\circ x = {}^\circ\bar{x}$, implies that $\delta \approx 0$. A translation of the coordinate system does not change the form of equations (2.18) [10], so we may assume that $\bar{x} = (0, -\delta)$. With this assumption, $x = \bar{x} + (0, 2\delta) = (0, \delta)$. Note that ${}^\circ x = {}^\circ\bar{x} = (0, 0)$, so by our hypothesis, $(0, 0) \in$ S–int Ω. Therefore, there

[9] Equality of the two maxima follows from the discrete maximum principle.
[10] Δ_h is translation invariant.

exists a (standard) $d \in \mathbb{R}^+$ such that the "discrete square" of side d, ${}^*[-d,d]^2 \cap \Omega_h$, is contained in Ω_h. Consider the "discrete half-square":

$$Q_{d,h} = \Big\{(x_1,x_2) \in \mathbb{R}_h^2 : -d \le x_1 \le d \wedge 0 \le x_2 \le d\Big\}.$$

Define two gridfunctions:

$$\omega^{\pm}(x_1,x_2) = U(x_1,x_2) - U(x_1,-x_2) \pm \Psi(x_1,x_2),$$

where

$$\Psi(x_1,x_2) = \frac{2M}{d^2}\Big(x_1^2 + x_2(2d - x_2)\Big).$$

Note that, if $(x_1,x_2) \in Q_{d,h} \subset \Omega_h$, then $(x_1,-x_2) \in \Omega_h$. So $\omega^{\pm}$ and $\Delta_h\omega^{\pm}$ are well defined in $Q_{d,h}$.

A simple computation yields $\Delta\Psi = \frac{2M}{d^2}(2+0-2) = 0$. Hence, since $\Psi \in C^2(\Omega,\mathbb{R})$, $\Delta_h\Psi \approx \Delta\Psi = 0$, and so:

$$\Delta_h\omega^{\pm}(x_1,x_2) = \Delta_h U(x_1,x_2) - \Delta_h U(x_1,-x_2) \pm \Delta_h\Psi(x_1,x_2) \approx 0. \tag{2.19}$$

Now, let $(x_1,x_2) \in \partial Q_{d,h}$.

Case 1: $x_2 = 0$. Then

$$\Psi(x_1,x_2) = \frac{2Mx_1^2}{d^2},$$

and so:

$$\omega^{+}(x_1,x_2) = U(x_1,0) - U(x_1,0) + \Psi(x_1,x_2) = \frac{2Mx_1^2}{d^2} \ge 0,$$

$$\omega^{-}(x_1,x_2) = U(x_1,0) - U(x_1,0) - \Psi(x_1,x_2) = -\frac{2Mx_1^2}{d^2} \le 0.$$

Case 2: $x_1 = \pm d$. Then, since $0 \leq x_2 \leq d$:

$$\Psi(x_1, x_2) = \frac{2M}{d^2}\Big(d^2 + x_2(2d - x_2)\Big) \geq \frac{2M}{d^2} d^2 = 2M.$$

Therefore:

$$\omega^+(x_1, x_2) \geq -2M + \Psi(x_1, x_2) \geq -2M + 2M = 0, \tag{2.20}$$

$$\omega^-(x_1, x_2) \leq 2M - \Psi(x_1, x_2) \leq 2M - 2M = 0. \tag{2.21}$$

Case 3: $x_2 = d$. Then:

$$\Psi(x_1, x_2) = \frac{2M}{d^2}(x_1^2 + d^2) \geq 2M.$$

Therefore, equations (2.20) and (2.21) also hold.

So, we can conclude that, whenever $(x_1, x_2) \in \partial Q_{d,h}$,

$$\omega^+(x_1, x_2) \geq 0, \tag{2.22}$$

$$\omega^-(x_1, x_2) \leq 0. \tag{2.23}$$

From the approximate maximum principle, and using inequalities (2.19), (2.22) and (2.23), we get that, for all $(x_1, x_2) \in Q_{d,h}$:

$$\omega^+(x_1, x_2) \geq 0 \Rightarrow U(x_1, x_2) - U(x_1, -x_2) \geq -\Psi(x_1, x_2);$$

$$\omega^-(x_1, x_2) \leq 0 \Rightarrow U(x_1, x_2) - U(x_1, -x_2) \leq \Psi(x_1, x_2).$$

This means that:

$$|U(x_1, x_2) - U(x_1, -x_2)| \leq |\Psi(x_1, x_2)|.$$

In particular, with $x = (0, \delta)$ and $\bar{x} = (0 - \delta)$:

$$|U(x) - U(\bar{x})| \leq |\Psi(0, \delta)| \leq \frac{2M}{d^2}\delta(2d - \delta) \leq \frac{4M\delta}{d} \approx 0.$$

This concludes the proof of the special case $x = \bar{x} + (0, 2\delta)$.

The special case $x = \bar{x} + (2\delta, 0)$ follows from the above by switching the coordinate axis. The general case follows from the above two cases and the triangle inequality.

Lemmas (2.30) and (2.21) imply that the standard function

$$u({}^\circ x) = \begin{cases} \operatorname{st} U(x), & \text{if } {}^\circ x \in \Omega, \\ f({}^\circ x), & \text{if } {}^\circ x \in \partial\Omega, \end{cases}$$

is a well defined, C^∞ function on Ω which satisfies the Dirichlet problem (2.1). To show that it is the pointwise limit of the finite difference solutions (as $h \downarrow 0$), it is enough to establish that the solution obtained does not depend on the choice of $h \approx 0$, $h > 0$. For that, we use the following result, proved by Kellogg in [16] (Chapter 11, Section 20):

Theorem 2.31 (Kellogg) *Let $\Omega \subset \mathbb{R}^2$ be bounded open and $f : \partial\Omega \to \mathbb{R}$ continuous. Then $\partial\Omega$ contains standardly regular points. Also, there is a unique function $u \in C^\infty(\Omega, \mathbb{R})$ such that $\Delta u = 0$ in Ω and:*

$$\lim_{x \to x_0} u(x) = f(x_0)$$

for all standardly regular $x_0 \in \partial\Omega$.

Now, assume that h_1 and h_2 are positive infinitesimals, and U_{h_1} and U_{h_2} are the corresponding solutions of the discrete Dirichlet problem (2.3), extended as stepfunctions defined on all of ${}^*\overline{\Omega}$. Then, using Theorem (2.31), and by the same argument as in Theorem (2.24), U_h converges uniformly to u on compact subsets of Ω. Furthermore, for all standard regular $y \in \partial\Omega$,

$$f(y) = \lim_{\substack{h\downarrow 0\\ x\to y\\ x\in\Omega^{-\delta(h)}}} U_h(x),$$

where δ is some function such that $\lim_{h\downarrow 0}\delta(h) = 0$.

We summarize some of our results in the following standard statement:

Theorem 2.32 *Let Ω be bounded open. Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. For each $h \in \mathbb{R}^+$, let $U_h : \overline{\Omega} \to \mathbb{R}$ be the unique solution of the discrete Dirichlet problem (2.3) (extended to $\overline{\Omega}$ as a stepfunction). Then $u : \overline{\Omega} \to \mathbb{R}$ given by*

$$u(x) = \begin{cases} \lim\limits_{h\downarrow 0} U_h(x), & \text{if } x \in \Omega,\\ f(x), & \text{if } x \in \partial\Omega, \end{cases}$$

is well defined, $u \in C^\infty(\Omega, \mathbb{R})$ and is the unique function which solves the Dirichlet problem (2.1), and satisfies $\lim\limits_{x\to x_0} u(x) = f(x_0)$ for all standardly regular $x_0 \in \partial\Omega$.

Chapter 3

Discretizations of Elliptic Partial Differential Equations

In this chapter, we look at elliptic partial differential equations in the same spirit as our approach to the Laplace equation. Instead of specifying a discretization scheme, we work with a class of discretizations, defined in a convenient way. This will enable us to get some general results about convergence of discrete schemes. This chapter contains some examples which are only intended as illustrations of the definitions; the particular schemes here presented do not reflect the generality of our results.

Let $\mathcal{D} \subset \mathbb{R}^n$ be open. An operator, L, defined on $C^k(\mathcal{D}, \mathbb{R})$ by

$$Lu(x) = \sum_{m=0}^{k} \sum_{|\alpha|=m} a_\alpha(x) D^\alpha(x), \qquad \forall x \in \mathcal{D},\text{[1]}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

is called a linear differential operator of order k on $\mathcal{D}$. The functions a_α, defined on (at least) $\mathcal{D}$ are called the coefficient functions of L.

[1] We adhere to the convention that operators have precedence over function evaluation, so $Lu(x)$ means $(Lu)(x)$.

Let $\mathcal{D} \subset \mathbb{R}^n$ be open. $C^k(\overline{\mathcal{D}}, \mathbb{R})$ represents the set of functions on $C^k(\mathcal{D}, \mathbb{R})$ all of whose derivatives (of order 0 through k) admit continuous extensions to $\overline{\mathcal{D}}$. Throughout this chapter, a domain will be an open and bounded $\Omega \subset \mathbb{R}^n$.

Definition 3.1 *Let $\mathcal{D} \subset \mathbb{R}^n$ be open. Let L be a second order linear differential operator on $\mathcal{D}$, given by:*

$$Lu(x) = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) \quad \forall x \in \mathcal{D}, \tag{3.1}$$

where the coefficient functions, $a_{i,j}$, b_i and c are defined and bounded in $\mathcal{D}$ and the matrix $\mathbf{A}(x) = (a_{i,j}(x))_{i,j=1,\ldots,n}$ is symmetric, for all $x \in \mathcal{D}$. L is called uniformly elliptic iff, for all $x \in \mathcal{D}$, the eigenvalues $\lambda(x)$ of $\mathbf{A}(x)$ are all bigger than some $\mu > 0$ (with μ independent of x). This μ is called the ellipticity constant.

3.1 Consistent Schemes for Differential Operators

We now consider possible discrete versions of uniformly elliptic L, which we will denote by L_h. The simplest way to obtain these is by replacing the derivatives in L by finite differences. We get a family of linear difference operators indexed in $h > 0$, where h is a measure of the grid spacing.

Example 3.2 *Consider $\mathcal{D} = \mathbb{R}^2$ and $L = \Delta$ (the Laplacian operator). As seen in the previous chapter, we can define Δ_h by the equation:*

$$\begin{aligned} \Delta_h U(x_1, x_2) \quad = \quad & \frac{1}{h^2}\Big(U(x_1 + h, x_2) + U(x_1 - h, x_2) + U(x_1, x_2 + h) \\ & + U(x_1, x_2 - h) - 4U(x_1, x_2)\Big). \end{aligned}$$

Note that Δ_h can be applied to any *function $u : \mathbb{R}^2 \to \mathbb{R}$. In fact, $\{\Delta_h\}_{h\in\mathbb{R}^+}$ is a one-parameter family of operators, all defined in $\mathbb{R}^{\mathbb{R}^2}$. From its definition, it follows that these operators are linear, i.e.:*

$$\Delta_h(\alpha u + \beta v) = \alpha\Delta_h u + \beta\Delta_h v \quad \forall\alpha, \beta \in \mathbb{R}\ \forall u, v \in \mathbb{R}^{\mathbb{R}^2}.$$

Also, if $u \in C^2(\mathbb{R}^2, \mathbb{R})$, from Taylor's formula it follows that, for all x_0:

$$\lim_{h\downarrow 0, x\to x_0} \Delta_h u(x) = \Delta u(x_0).$$

We have worked with Δ_h defined in an appropriate set of functions, namely $\mathbb{R}^{\mathbb{R}_h^2\cap\Omega}$, where $\Omega \subset \mathbb{R}^2$ is open and bounded. We have seen that Δ_h defined in $\mathbb{R}^{\mathbb{R}_h^2\cap\Omega}$ satisfies a maximum principle.

Because of the three referred properties, $\{\Delta_h\}$ turned out to be a "good" choice. The second property is what we call consistency (with Δ, in this case). We will now formalize what it means to be a consistent scheme to a general linear partial differential operator. The discussion of maximum principles will be carried out in the next section.

We begin by looking at the special case of a linear differential operator (of any order) on $\mathcal{D} = \mathbb{R}^n$. This is easier to deal with, since it avoids the problem of discretizing the boundary of $\mathcal{D}$.

A *discretization* of $\mathbb{R}^n$ is a set, $\mathbb{R}_h^n$, such that for every bounded $B \subset \mathbb{R}^n$, $|\mathbb{R}_h^n \cap B| < \infty$. An example of this is a *uniform grid*:

$$\mathbb{R}_h^n = a + h\mathbb{Z}^n = \{a + h\alpha : \alpha \in \mathbb{Z}^n\},$$

where $a \in \mathbb{R}^n$. Other examples of possible $\mathbb{R}_h^n$ include:

1. Other grids (nonuniform):

$$\Big\{(nh, 2mh) : n, m \in \mathbb{Z}\Big\},$$

$$\Big\{(nh, mh^2) : n, m \in \mathbb{Z}\Big\},$$

$$\Big\{(nh, m^2h) : n, m \in \mathbb{Z}\Big\}.$$

2. A polar discretization, given by:

$$\Big\{(nh\cos(mh), nh\sin(mh)) : n, m \in \mathbb{Z}, 0 \leq m < 2\pi/h\Big\}.$$

Suppose $L_h : \mathbb{R}^{\mathbb{R}^n_h} \mapsto \mathbb{R}^{\mathbb{R}^n_h}$ is a linear operator, defined for each $h \in \mathbb{R}^+$. For the family $\{L_h\}_{h\in\mathbb{R}^+}$ to be a good approximation to L we need that, as h approaches 0, $\mathbb{R}^n_h$ approaches $\mathbb{R}^n$ and $L_h u$ approaches Lu. The appropriate definition is easy to formulate using nonstandard analysis. In this, and other similar definitions of this chapter, we consider ${}^*\mathcal{U}$, where $\mathcal{U} = \{L_h\}$. ${}^*\mathcal{U}$ is an internal family of linear maps, indexed in ${}^*\mathbb{R}^+$; each $L_h \in {}^*\mathcal{U}$ is a *linear map acting on a hyperfinite dimensional vector space of internal gridfunctions $U : \mathbb{R}^n_h \to \mathbb{R}$, with $\mathbb{R}^n_h$ a *discretization of ${}^*\mathbb{R}^n$. Roughly speaking, we will be stating that, whenever $h > 0$ and $h \approx 0$, the discretized operator and the domain of its gridfunctions resembles the corresponding analytical objects.

Definition 3.3 *Let L be a linear differential operator of order k, defined on $\mathbb{R}^n$, and let $\{\mathbb{R}^n_h\}$ be a family of discretizations of $\mathbb{R}^n$. The one parameter family $\{L_h\}_{h\in\mathbb{R}^+}$ of linear operators $L_h : \mathbb{R}^{\mathbb{R}^n_h} \mapsto \mathbb{R}^{\mathbb{R}^n_h}$ is said to be consistent with L iff for all positive $h \approx 0$:*

(a) ${}^\circ\mathbb{R}^n_h = \mathbb{R}^n$*;*

(b) $L_h u(x) \approx Lu(x), \quad \forall u \in C^k(\mathbb{R}^n, \mathbb{R}), \quad \forall x^{\text{finite}} \in \mathbb{R}^n_h.$

The condition above seems very strong but, in fact, the usual finite difference operators satisfy it. We now introduce the finite difference notation we will be using, and then show some general criteria for consistency of finite difference operators.

Definition 3.4 *Consider the discretization* $\mathbb{R}^n_h = a + h\mathbb{Z}^n$, *for some* $a \in \mathbb{R}^n$. *For each* $h \in \mathbb{R}^+$, *let* $U : \mathbb{R}^n_h \mapsto \mathbb{R}$.

(a) The shift operators, $\sigma^{\pm}_{i,h}$ *are given by:*

$$\begin{aligned} \sigma^{+}_{i,h} U(x) &= U(x_1, \ldots, x_i + h, \ldots, x_n), \\ \sigma^{-}_{i,h} U(x) &= U(x_1, \ldots, x_i - h, \ldots, x_n), \end{aligned}$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(b) The forward, backward and central difference operators are given by:

$$\begin{aligned} \delta^{+}_{i,h} &= \frac{1}{h}\left(\sigma^{+}_{i} - I\right); \\ \delta^{-}_{i,h} &= \frac{1}{h}\left(I - \sigma^{-}_{i}\right); \\ \delta_{i,h} &= \frac{1}{2h}\left(\sigma^{+}_{i} - \sigma^{-}_{i}\right). \end{aligned}$$

(c) The second central difference operator is given by:

$$\delta^2_{i,h} = \delta^{-}_{i}\delta^{+}_{i} = \frac{1}{h}\left(\sigma^{+}_{i} - 2I + \sigma^{-}_{i}\right).$$

When working with uniform grids, we will omit the subscript h when referring to individual shift and difference operators that occur anywhere in L_h. The following result establishes criteria of consistency of finite difference operators.

Proposition 3.5

(a) $\{\mathrm{id}_h\}$ *(the identity operator in* $\mathbb{R}^{\mathbb{R}_h^n}$*) is consistent with* id.

(b) $\{\delta_{i,h}^+\}$, $\{\delta_{i,h}^-\}$ *and* $\{\delta_{i,h}\}$ *are all consistent with* $\frac{\partial}{\partial x_i}$.

(c) $\{\delta_{i,h}^2\}$ *is consistent with* $\frac{\partial^2}{\partial x_i^2}$.

(d) Let L_1 *and* L_2 *be two linear differential operators of order* l_1 *and* l_2*, respectively. If the order of* $L_1 + L_2$*,* l*, equals* $\max\{l_1, l_2\}$*, and both* $\{L_{1,h}\}$ *and* $\{L_{2,h}\}$ *are consistent with, respectively,* L_1 *and* L_2*, then* $\{L_{1,h} + L_{2,h}\}$ *is consistent with* $L_1 + L_2$.

(e) Let $c \in C(\mathbb{R}^n, \mathbb{R})$*, and* $\{L_h\}$ *be consistent with* L*. Then* $\{cL_h\}$ *is consistent with* cL [2].

Proof.

(a) The proof is obvious.

(b) Consider a standard $u \in C^1(\mathbb{R}^n, \mathbb{R})$. Now take a positive $h \approx 0$ and $x^{\text{finite}} \in \mathbb{R}_h^n$. By the transfer of the intermediate value theorem we have that, for some $c \in {}^*\mathbb{R}$, with $x_i < c_i < x_i + h$:

$$\begin{aligned}
\delta_i^+ u(x) &= \frac{1}{h}\Big(u(x_1, \ldots, x_i + h, \ldots, x_n) - u(x_1, \ldots, x_n)\Big) \\
&= \frac{1}{h}\frac{\partial u}{\partial x_i}(x_1, \ldots, c_i, \ldots, x_n)h \\
&= \frac{\partial u}{\partial x_i}(x_1, \ldots, c_i, \ldots, x_n)
\end{aligned}$$

[2] *The sum of operators and product of operator by a function are defined in the usual way, i.e, pointwise.*

Since $h \approx 0$, we have that $c \approx x$. Hence, and since $\frac{\partial u}{\partial x_i}$ is continuous:

$$\delta_i^+ u(x) = \frac{\partial u}{\partial x_i}(x_1, \ldots, c_i, \ldots, x_n) \approx \frac{\partial u}{\partial x_i}(\,^\circ x) \approx \frac{\partial u}{\partial x_i}(x)$$

The proof is similar for δ_i^- and δ_i.

(c) Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$. Consider a positive $h \approx 0$ and $x^{\text{finite}} \in \mathbb{R}_h^n$. By the transfer of Taylor's theorem, we have that:

$$\begin{aligned}
\delta_i^2 u(x) &= \frac{1}{h^2}\Big(\sigma_i^+ u(x) + \sigma_i^- u(x) - 2u(x)\Big) \\
&= \frac{1}{h^2}\Bigg(u(x) + \frac{\partial u}{\partial x_i}(x)h + \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(c)h^2 \\
&\qquad + u(x) + \frac{\partial u}{\partial x_i}(x)(-h) + \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(d)(-h)^2 - 2u(x)\Bigg) \\
&= \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(c) + \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(d),
\end{aligned}$$

where $c = (x_1, \ldots, c_i, \ldots, x_n)$, $x_i < c_i < x_i + h$, and $d = (x_1, \ldots, d_i, \ldots, x_n)$, $x_i - h < d_i < x_i$. But $h \approx 0$, and so $^\circ c = \,^\circ x$ and $^\circ d = \,^\circ x$. Therefore, by continuity of $\frac{\partial^2 u}{\partial x_i^2}$:

$$\delta_i^2 u(x) \approx \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(\,^\circ x) + \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(\,^\circ x) \approx \frac{\partial^2 u}{\partial x_i^2}(x).$$

(d) Let $u \in C^l(\mathbb{R}^n, \mathbb{R}) \subset C^{l_1}(\mathbb{R}^n, \mathbb{R}) \cup C^{l_2}(\mathbb{R}^n, \mathbb{R})$. This inclusion follows from $l = \max\{l_1, l_2\}$). Take a positive $h \approx 0$. Then, for all finite $x \in \mathbb{R}_h^n$:

$$(L_{1,h} + L_{2,h})u(x) = L_{1,h}u(x) + L_{2,h}u(x) \approx L_1 u(x) + L_2 u(x).$$

(e) Let $u \in C^l(\mathbb{R}^n, \mathbb{R})$. Fix a positive $h \approx 0$. Then, taking a finite $x \in \mathbb{R}_h^n$, and using continuity of c and consistency of $\{L_h\}$ with L:

$$(cL_h)u(x) = c(x)L_h u(x) \approx c(x)Lu(x) = (cL)u(x).$$

Example 3.6 *Let:*

$$L = a_1\frac{\partial^2}{\partial x_1^2} + a_2\frac{\partial^2}{\partial x_2^2} + b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + c,$$

where $a_1, a_2, b_1, b_2, c \in C(\mathbb{R}^2, \mathbb{R})$. For all $h > 0$, let:

$$L_h = a_1\delta_1^2 + a_2\delta_2^2 + b_1\delta_1 + b_2\delta_2 + c.$$

Then $\{L_h\}$ is consistent with L. This follows from proposition (3.5) and the trivial fact that $\{\mathrm{id}_h\}$ (the identity operator in $\mathbb{R}^{\mathbb{R}^2_h}$) is consistent with id *(the identity on $\mathbb{R}^2$).*

The case of a linear differential operator of order k on an arbitrary open set $\mathcal{D} \in \mathbb{R}^n$ is more complicated, since it involves dealing with the boundary of $\mathcal{D}$. We start by discretizing $\overline{\mathcal{D}}$ by some $\overline{\mathcal{D}}_h \subset \overline{\mathcal{D}}$, such that $\overline{\mathcal{D}}_h \cap B$ is finite, for all bounded B. We use the overbar since $\overline{\mathcal{D}}_h$ is a discrete analogue of the topological closure of $\mathcal{D}$. We then consider a $\mathcal{D}_h \subset \overline{\mathcal{D}}_h$, appropriately chosen so that $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$ is well defined. The reason for this is better understood by an example.

Example 3.7 *Given some domain $\mathcal{D} \subset \mathbb{R}^n$, use the uniform grid to define:*

$$\overline{\mathcal{D}}_h = h\mathbb{Z}^n \cap \overline{\mathcal{D}}.$$

This is the discrete analogue of the topological closure of $\mathcal{D}$. The analogue of the open set, $\mathcal{D}$, is introduced as follows. All we need of these "open" sets is that $L_h u(x)$ can be evaluated, for any x on them. Note that, to compute $L_h u(x)$, we need the values of u at some neighbors of x on the lattice $h\mathbb{Z}^n$. The actual pattern of lattice neighbors of some $x \in h\mathbb{Z}^n$ depends on L_h. For example, take the discrete

Laplacian, Δ_h, computed with the gridpoints $h\mathbb{Z}^2$:

$$\Delta_h = \delta_1^2 + \delta_2^2 = \frac{1}{h^2}\left(\sigma_1^+ + \sigma_1^- + \sigma_2^+ + \sigma_2^- - 4I\right).$$

The pattern of neighbors for Δ_h is independent of x, and is depicted in Fig. (4).

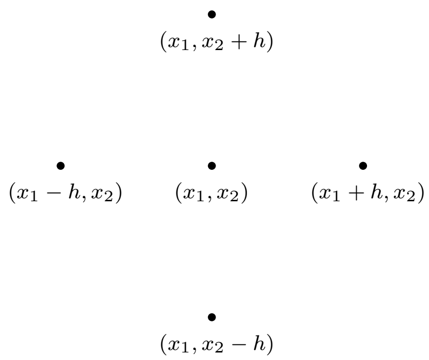

Figure 4: Pattern of neighbors for Δ_h.

For finite difference operators, which can always be defined in terms of (finitely many) shift operators, this pattern is completely determined by the shift operators occurring in L_h [3]*. So, for finite difference operators, we can get $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$ well defined by constructing $\mathcal{D}_h \subset \overline{\mathcal{D}}_h$ in any way that makes $\overline{\mathcal{D}}_h$ contain all the lattice neighbors, relative to L_h, of the points of $\mathcal{D}_h$. For example, for Δ_h, we may take:*

$$\begin{aligned}\mathcal{D}_h \quad = \quad & \Big\{(x_1, x_2) \in \overline{\mathcal{D}}_h : (x_1 + h, x_2) \in \overline{\mathcal{D}}_h \wedge (x_1 - h, x_2) \in \overline{\mathcal{D}}_h \\ & \quad \wedge (x_1, x_2 + h) \in \overline{\mathcal{D}}_h \wedge (x_1, x_2 - h) \in \overline{\mathcal{D}}_h \wedge \mathrm{dist}\,((x_1, x_2), \partial\mathcal{D}) \geq h\Big\}.\end{aligned}$$

[3] *This is a simple example. Other discretizations of L may lead to neighbor patterns that vary with x.*

We add the condition $\operatorname{dist}(x, \partial\mathcal{D}) \geq h$) *to ensure that small holes in* $\mathcal{D}$ *will carry over to* $\mathcal{D}_h$. *This way,* ${}^\circ(\overline{\mathcal{D}}_h - \mathcal{D}_h) \supset \partial\mathcal{D}$, *for* $h \approx 0$.

We have seen that a well-defined element of a scheme, $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$, must include in its definition the domain, $\overline{\mathcal{D}}_h$, of the gridfunctions it acts upon, and the domain, $\mathcal{D}_h$, of the gridfunctions in its range. So, for us, a scheme will be a one-parameter family, $\{L_h\}$, where L_h implicitly includes $\overline{\mathcal{D}}_h$ and $\mathcal{D}_h$. For any $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$, we let $\partial\mathcal{D}_h = \overline{\mathcal{D}}_h - \mathcal{D}_h$. As the notation suggests, this is the discrete analogue of $\partial\mathcal{D}$ that is uniquely determined by each L_h.

Definition 3.8 *Let* L *be a linear differential operator of order* k *on a domain* $\mathcal{D} \subset \mathbb{R}^n$. *A discrete scheme for* L *is a one parameter family* $\{L_h : h \in \mathbb{R}^+\}$, *where each* $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$ *is a well defined linear operator. The scheme is called consistent (with L) iff:*

(a) ${}^\circ\overline{\mathcal{D}}_h = \overline{\mathcal{D}}$;

(b) ${}^\circ\partial\mathcal{D}_h = \partial\mathcal{D}$;

(c) $L_h u(x) \approx Lu(x), \quad \forall u \in C^k(\mathcal{D}, \mathbb{R}), \forall x^{\text{finite}} \in \mathcal{D}_h$.

Proposition 3.9 *Let* L *be a linear differential operator of order* k *defined on an open* $\mathcal{D}$, *and let* $\{L_h : h \in \mathbb{R}^+\}$ *be a consistent discrete scheme for* L, *with* $L_h : \mathbb{R}^{\overline{\mathcal{D}}_h} \to \mathbb{R}^{\mathcal{D}_h}$. *Let* $\mathcal{E} \subset \mathcal{D}$ *be also open and* $\mathcal{E}_h \subset \overline{\mathcal{E}}_h \subset \overline{\mathcal{D}}$ *be such that* ${}^\circ\overline{\mathcal{E}}_h = \overline{\mathcal{E}}$ *and* ${}^\circ\partial\mathcal{E}_h = \partial\mathcal{E}$ *and the operators* $\tilde{L}_h : \mathbb{R}^{\overline{\mathcal{E}}_h} \to \mathbb{R}^{\mathcal{E}_h}$ *given by*

$$\tilde{L}_h U(x) = L_h U(x) \quad \forall U \in \mathbb{R}^{\mathcal{E}_h}\ \forall x \in \mathcal{E}_h$$

are well defined. Then, $\left\{\tilde{L}_h : h \in \mathbb{R}^+\right\}$ *is a consistent discrete scheme for* L *on* $\mathcal{E}$.

Proof. Follows directly from definition (3.8).

3.2 The Maximum Principle Condition and its Approximate Versions

Consistency is not, in general, enough to show convergence of discrete schemes. For elliptic operators, a convenient condition is a maximum principle.

Definition 3.10 *Let L be a linear differential operator of order k on some domain $\Omega \subset \mathbb{R}^n$. Let $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h\in\mathbb{R}^+}$ be a discrete scheme for L. $\{L_h\}$ is said to have a maximum principle (MP) iff, for all $\Gamma \subset \Omega_h$, for all $h \in \mathbb{R}^+$ sufficiently small, for all $U : \overline{\Omega}_h \to \mathbb{R}$, and all $a \in \mathbb{R}$, if*

(a) $L_h U(x) \geq 0 \quad \forall x \in \Gamma$,

(b) $U(x) \leq a \quad \forall x \in \overline{\Omega}_h - \Gamma$,

then $U(x) \leq a \ \ \forall x \in \overline{\Omega}_h$.

We now specialize our discussion to second order uniformly elliptic operators. Here is an important example of a scheme with a maximum principle.

Example 3.11 (A Discretization for Diagonal Uniformly Elliptic Operators) *Let L be given by:*

$$Lu(x) = \sum_{i=1}^{n} a_i(x)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) \quad \forall x \in \mathbb{R}^n, \tag{3.2}$$

where all coefficient functions are defined and continuous in $\mathbb{R}^n$. *Assume that* L *is uniformly elliptic with constant* $\mu \in \mathbb{R}^+$, *i.e., for all* $i = 1, \dots, n$ *and* $x \in \mathbb{R}^n$, $a_i(x) > \mu$. *Suppose also that* $c(x) \leq 0$, *for all* $x \in \mathbb{R}^n$.

Now, consider $\mathbb{R}^n_h = h\mathbb{Z}^n$, *and let:*

$$L_h u(x) = \sum_{i=1}^{n} a_i(x)\delta_i^2(x) + \sum_{i=1}^{n} b_i(x)\delta_i(x) + c(x)u(x) \quad \forall x \in \mathbb{R}^n_h.$$

From Proposition (3.5), $\{L_h\}$ *is consistent with* L. *Furthermore,* L_h *can be written as:*

$$\begin{aligned} L_h &= \sum_{i=1}^{n} \left(a_i\delta_i^2 + b_i\delta_i\right) + c \\ &= \sum_{i=1}^{n} \left(\frac{a_1}{h^2}\left(\sigma_i^+ + \sigma_i^- - 2I\right) + \frac{b_i}{2h}\left(\sigma_i^+ + \sigma_i^-\right)\right) + c \qquad (3.3) \\ &= \frac{1}{h^2}\sum_{i=1}^{n} \left(\left(a_i + \frac{h}{2}b_i\right)\sigma_i^+ + \left(a_i - \frac{h}{2}b_i\right)\sigma_i^- - 2a_i I\right) + c. \end{aligned}$$

Say Ω *is a domain. Let* $\overline{\Omega}_h = h\mathbb{Z}^n \cap \overline{\Omega}$. *If* $\{e_i : i = 1, \dots, n\}$ *is the canonical base of* $\mathbb{R}^n$, *let:*

$$\begin{aligned} \Omega_h = \Big\{ & x \in \overline{\Omega}_h : x + he_1 \in \overline{\Omega}_h \wedge x - he_1 \in \overline{\Omega}_h \wedge \dots \wedge \\ & x + he_n \in \overline{\Omega}_h \wedge x - he_n \in \overline{\Omega}_h \wedge \operatorname{dist}(x, \partial\Omega) \geq h \Big\}. \end{aligned}$$

Then, by Proposition (3.9), $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}$ *is a consistent discrete scheme for* L *on* $\overline{\Omega}$.

Now we show that $\{L_h\}$ *has MP. Fix* $\Gamma \subset \Omega_h$. *Let*

$$K = \max\Big\{|b_i(x)| : i = 1, \dots, n\,;\, x \in \overline{\Omega}\Big\},$$

and choose $\eta \in \mathbb{R}^+$ *sufficiently small so that* $\mu - \frac{\eta}{2}K \geq 0$. *This ensures that, for all* $i = 1, \dots, n$:

$$a_i(x) \pm \frac{h}{2}b_i(x) \geq 0 \qquad \forall h \in (0, \eta) \ \forall x \in \overline{\Omega}. \qquad (3.4)$$

Consequently, all coefficients of the shift operators in equation (3.3) are strictly positive functions, for $h \in (0, \eta)$. Let $U : \overline{\Omega}_h \to \mathbb{R}$ be such that $L_h U(x) \geq 0$, for all $x \in \Gamma$. If the maximum of U in $\overline{\Omega}_h$ does not occur in Γ, we are done. If not, let M be the maximum of U in $\overline{\Omega}_h$, and $y \in \Gamma$ satisfy $u(y) = M$. Then:

$$\begin{aligned} L_h U(y) &= \frac{1}{h^2}\sum_{i=1}^{n}\Big(\Big(a_i(y) + \tfrac{h}{2}b_i(y)\Big)U(y+he_i) + \Big(a_i(y) - \tfrac{h}{2}b_i(y)\Big)U(y-he_i) \\ &\qquad -2a_i(y)U(y)\Big) + c(y)U(y) \geq 0. \end{aligned}$$

Hence, and since $U(y) = M$:

$$\begin{aligned} \Big(a_1(y) + \tfrac{h}{2}b_1(y)\Big)U(y+he_1) &\geq M\sum_{i=1}^{n}2a_i(y) - \Big(a_1(y) - \tfrac{h}{2}b_1(y)\Big)U(y-h_1) \\ -\sum_{i=2}^{n}\Big(\Big(a_i(y) + \tfrac{h}{2}b_i(y)\Big)U(y+he_i) &+ \Big(a_i(y) - \tfrac{h}{2}b_i(y)\Big)U(y-he_i)\Big) - h^2c(y)M \end{aligned}$$

Using inequalities (3.4), and the fact that $U(y \pm he_i) \leq M$:

$$\begin{aligned} \Big(a_1(y) + \tfrac{h}{2}b_1(y)\Big)U(y+he_1) &\geq M\sum_{i=1}^{n}2a_i(y) - \Big(a_1(y) - \tfrac{h}{2}b_1(y)\Big)M \\ -\sum_{i=2}^{n}\Big(\Big(a_i(y) + \tfrac{h}{2}b_i(y)\Big)M &+ \Big(a_i(y) - \tfrac{h}{2}b_i(y)\Big)M\Big) - h^2c(y)M \end{aligned}$$

Canceling some terms yields:

$$\begin{aligned} \Big(a_1(y) + \tfrac{h}{2}b_1(y)\Big)U(y+he_1) &\geq M\sum_{i=1}^{n}2a_i(y) - \Big(a_1(y) - \tfrac{h}{2}b_1(y)\Big)M \\ &\quad -\sum_{i=2}^{n}2Ma_i(y) - h^2c(y)M \\ &= M\Big(2a_1(y) - a_1(y) + \tfrac{h}{2}b_1(y) - h^2c(y)\Big) \\ &= M\Big(a_1(y) + \tfrac{h}{2}b_1(y) - h^2c(y)\Big) \end{aligned}$$

Therefore, again from (3.4), and since $c(y) \leq 0$:

$$U(y+he_1) \geq M\left(1 - \frac{h^2c(y)}{a_1(y) + \frac{h}{2}b_1(y)}\right) \geq M$$

Since M is the maximum of U on $\overline{\Omega}_h$, we conclude that $U(y \pm he_1) = M$. Repeating this argument k times, we get that $U(y \pm khe_1) = M$. But, since Ω is bounded, $\Gamma \subset \Omega_h \subset \overline{\Omega}_h \subset \overline{\Omega}$ is finite, so for some k, $y \pm khe_1 \in \partial\overline{\Omega}_h \subseteq \overline{\Omega}_h - \Gamma$. This shows the maximum will also occur at $\overline{\Omega}_h - \Gamma$. Hence, $\{L_h\}$ has MP.

Theorem 3.12 (Approximate Maximum Principle) *Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, with ellipticity constant $\mu \in \mathbb{R}^+$. Suppose $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h \in \mathbb{R}^+}$ is a consistent discrete scheme for L that has MP. Now, let $h \in {}^*\mathbb{R}^+$, $h \approx 0$, and $\Gamma \subset \Omega_h$ internal. Let $a \in \mathbb{R}$ and $p \in \mathbb{R}^+$. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is an internal function such that*

$$\forall x \in \Gamma \; \exists \epsilon \approx 0 \;\; L_h U(x) \geq \epsilon h^p, \tag{3.5}$$

$$\forall x \in \overline{\Omega}_h - \Gamma \; \exists \delta \approx 0 \;\; U(x) \leq a + \delta h^p, \tag{3.6}$$

then for all $x \in \overline{\Omega}_h$, there exists $\eta \approx 0$, such that $U(x) \leq a + \eta h^p$.

Proof. First we construct a standard function, $w \in C^2(\mathbb{R}^n, \mathbb{R})$, as follows. Since Ω is bounded, we can choose an $\hat{x} \in \mathbb{R}^n$ such that for some $r \in \mathbb{R}^+$ we have, for all $x \in \overline{\Omega}_h$:

$$|x_i - \hat{x}_i| > r \qquad (i = 1, \ldots, n).$$

Also, there is an $R \in \mathbb{R}$ such that $\overline{\Omega} \subset B_R(\hat{x})$. Then, for all $x \in \overline{\Omega}_h$:

$$r < |x_i - \hat{x}_i| < R \qquad (i = 1, \ldots, n). \tag{3.7}$$

Define w by

$$w(x) \;=\; \sum_{i=1}^{n} (x_i - \hat{x}_i)^{2m+2},$$

where $m \in \mathbb{N}$ is a parameter to be fixed later. We have:

$$\begin{aligned}
\frac{\partial w}{\partial x_i} &= (2m+2)\,(x_i - \hat{x}_i)^{2m+1}\,, \\
\frac{\partial^2 w}{\partial x_i^2} &= \delta_{i,j}\;(2m+2)(2m+1)\,(x_i - \hat{x}_i)^{2m}\,.
\end{aligned}$$

Therefore:

$$\begin{aligned}
Lw(x) &= (2m+2)(2m+1)\sum_{i=1}^{n} a_{i,i}(x)\,(x_i - \hat{x}_i)^{2m} \\
&\quad + (2m+2)\sum_{i=1}^{n} b_i(x)\,(x_i - \hat{x}_i)^{2m+1} + \; c(x)\,(x_i - \hat{x}_i)^{2m+2}\,.
\end{aligned}$$

Using consistency of $\{L_h\}$ with L (and since $w \in C^2(\mathbb{R}^n, \mathbb{R})$) we have, for all $x \in \Omega_h$:

$$\begin{aligned}
L_h w(x) &\approx (2m+2)(2m+1)\sum_{i=1}^{n} a_{i,i}(x)\,(x_i - \hat{x}_i)^{2m} \\
&\quad + (2m+2)\sum_{i=1}^{n} b_i(x)\,(x_i - \hat{x}_i)^{2m+1} + \; c(x)\,(x_i - \hat{x}_i)^{2m+2} \\
&= \sum_{i=1}^{n}\Big((2m+2)(2m+1)a_{i,i}(x) + (2m+2)b_i(x)\,(x_i - \hat{x}_i) \\
&\qquad + c(x)\,(x_i - \hat{x}_i)^2\Big)\,(x_i - \hat{x}_i)^{2m}\,.
\end{aligned}$$

Consider the standard positive reals $M = \max\{|b_i(x)| : x \in \overline{\Omega}, i = 1, \ldots, n\}$ and $K = \max\{-c(x) : x \in \overline{\Omega}\}$. Using inequalities (3.7), and by transfer:

$$\begin{aligned}
L_h w(x) &\gtrsim \sum_{i=1}^{n}\big((2m+2)(2m+1)a_{i,i}(x) - (2m+2)MR - KR^2\big)\,r^{2m} \\
&= \left((2m+2)(2m+1)\sum_{i=1}^{n} a_{i,i}(x) - n(2m+2)MR - nKR^2\right) r^{2m} \\
&= \Big((2m+2)(2m+1)\,\mathrm{tr}\,\mathbf{A}(x) - n(2m+2)MR - nKR^2\Big) r^{2m}.
\end{aligned}$$

By the condition of uniform ellipticity, the eigenvalues of $\mathbf{A}(x)$ are strictly bigger than $\mu \in \mathbb{R}^+$, so:

$$\begin{aligned} L_h w(x) &> \Big((2m+2)(2m+1)n\mu - n(2m+2)MR - nKR^2\Big)r^{2m} \\ &= n\,r^{2m}\Big((2m+2)(2m+1)\mu - (2m+2)MR - KR^2\Big). \end{aligned}$$

Recall that $m \in \mathbb{N}$ is, up to this point, a free parameter. We now fix it so that:

$$(2m+2)(2m+1)\mu - (2m+2)MR - KR^2 > 0.$$

We can always find such standard m, since μ,M,K and R are all standard and μ is positive. Note that m depends only on Ω and the coefficient functions of L. With this choice of m, we get:

$$L_h w(x) \gtrapprox 0 \qquad \forall x \in \overline{\Omega}_h. \tag{3.8}$$

On the other hand, since $\overline{\Omega}_h - \Gamma \subset \overline{\Omega} \subset B_R(\hat{x})$, we have that for each $x \in \overline{\Omega}_h - \Gamma$:

$$w(x) < R^{2m+2} \tag{3.9}$$

We conclude that, for a fixed L, we can find $w \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying (3.8), and (3.9) with $R \in \mathbb{R}^+$.

Let $c \in \mathbb{R}^+$, and consider $V_c : \overline{\Omega}_h \to {}^*\mathbb{R}$ given by:

$$V_c(x) = U(x) + ch^p w(x), \qquad \forall x \in \overline{\Omega}_h.$$

Then, from our hypothesis about U:

(a) For all $x \in \Gamma$, and using (3.5) and (3.8):

$$L_h V_c(x) = L_h U(x) + ch^p L_h w(x) \geq h^p(\epsilon + cL_h w(x)) > 0.$$

(b) For all $x \in \overline{\Omega}_h - \Gamma$, and using (3.6) and (3.9):

$$V_c(x) = U(x) + ch^p w(x) < a + \left(\delta h^p + ch^p R^{2m+2}\right) = a + \left(\delta + cR^{2m+2}\right) h^p.$$

By the transfer of the maximum principle, $V_c(x) \leq a + (\delta + cR^{2m+2})\, h^p$ for all $x \in \overline{\Omega}_h$. Therefore, and since $w(x) \geq 0$ in $\overline{\Omega}_h$:

$$U(x) = V_c(x) - ch^p w(x) \leq V(x) \leq a + \left(\delta + cR^{2m+2}\right) h^p$$

But $c \in \mathbb{R}^+$ was arbitrary. Hence, the (internal) set of all $b \in {}^*\mathbb{R}^+$ such that, for all $x \in \overline{\Omega}_h$, $V(x) \leq a + bh^p$ contains all noninfinitesimal positive reals. Therefore, it must contain some $\eta \approx 0$. We conclude that:

$$U(x) \leq a + \eta h^p \qquad \forall x \in \Omega_h.$$

Corollary 3.13 (Approximate Minimum Principle) *Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, with ellipticity constant $\mu \in \mathbb{R}^+$. Suppose $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h \in \mathbb{R}^+}$ is a consistent discrete scheme for L that has MP. Let $a \in \mathbb{R}$ and $p \in \mathbb{R}^+$. Now, let $h \in {}^*\mathbb{R}^+$, $h \approx 0$, and $\Gamma \subset \Omega_h$ be internal. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is an internal function such that:*

$$\forall x \in \Gamma\ \exists \epsilon \approx 0\ \ L_h U(x) \leq \epsilon h^p;$$

$$\forall x \in \overline{\Omega}_h - \Gamma\ \exists \delta \approx 0\ \ U(x) \geq a + \delta h^p.$$

Then for all $x \in \overline{\Omega}_h$, there exists $\eta \approx 0$, such that $U(x) \geq a + \eta h^p$.

Proof. Apply the approximate maximum principle to $-U$.

Corollary 3.14 *Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, with ellipticity constant $\mu \in \mathbb{R}^+$. Suppose $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h \in \mathbb{R}^+}$ is a consistent discrete scheme for L that has MP. Let $a \in \mathbb{R}$ and $p \in \mathbb{N}$. Now, let $h \in {}^*\mathbb{R}^+$, $h \approx 0$, and let $\Gamma \subset \Omega_h$ be internal. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is an internal function such that:*

$$\forall x \in \Gamma \; \exists \epsilon \approx 0 \;\; L_h U(x) = \epsilon h^p;$$

$$\forall x \in \overline{\Omega}_h - \Gamma \; \exists \delta \approx 0 \;\; U(x) = a + \delta h^k.$$

Then for all $x \in \overline{\Omega}_h$, there exists $\eta \approx 0$, such that $U(x) = a + \eta h^p$.

Proof. Follows from the approximate maximum principle and from corollary (3.13).

By making $k = 0$ in the above results, we get:

Corollary 3.15 *Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, with ellipticity constant $\mu \in \mathbb{R}^+$. Suppose $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h \in \mathbb{R}^+}$ is a consistent discrete scheme for L that has MP. Let $a \in \mathbb{R}$. Now, let $h \in {}^*\mathbb{R}^+$, $h \approx 0$, and let $\Gamma \subset \Omega_h$ be internal. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is an internal function such that:*

$$\begin{array}{lll} L_h U(x) \gtrsim 0 & [\text{resp } \lesssim, \approx] & \forall x \in \Gamma; \\ U(x) \lesssim a & [\text{resp } \gtrsim, \approx] & \forall x \in \overline{\Omega}_h - \Gamma. \end{array}$$

Then $U(x) \lesssim a$ [resp $\gtrsim, \approx$], $\forall x \in \overline{\Omega}_h$.

3.3 The Discrete Dirichlet Problem

Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$. Let $f \in C(\Omega, \mathbb{R})$ and $g \in C(\partial\Omega, \mathbb{R})$. Then,

$$\begin{cases} Lu(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases} \tag{3.10}$$

is called an analytical Dirichlet problem on Ω. We will be using the triple $\langle L, f, g\rangle$ to refer to (3.10).

Here is the discrete counterpart of (3.10):

Definition 3.16 *Let $\langle L, f, g\rangle$ be a Dirichlet problem on a domain $\Omega \subset \mathbb{R}^n$. Then, the one parameter family $\{\langle L_h, g_h, f_h\rangle : h \in \mathbb{R}^+\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$ is called a discrete Dirichlet scheme (for $\langle L, f, g\rangle$ on Ω) iff $\{L_h\}$ is a discrete scheme for L, $f_h \in \mathbb{R}^{\overline{\Omega}_h}$, and $g_h \in \mathbb{R}^{\partial\Omega_h}$. The scheme is consistent if:*

(a) $\{L_h\}$ is consistent with L;

(b) $f_h(x) \approx f(x), \quad \forall h \approx 0, \quad \forall x \in \overline{\Omega}_h$;

(c) $g_h(x) \approx g({}^\circ x), \quad \forall h \approx 0, \quad \forall x \in \partial\Omega_h$.

The scheme has MP if $\{L_h\}$ has MP.

Given a consistent discrete scheme, $\{\langle L_h, g_h, f_h\rangle\}$, we can form a one parameter family of discrete problems:

$$\begin{cases} L_hU(x) = f_h(x), & \text{if } x \in \Omega_h, \\ U(x) = g_h(x), & \text{if } x \in \partial\Omega_h. \end{cases} \tag{3.11}$$

Note that $\overline{\Omega}_h$ is finite (for h standard), so (3.11) is a system of algebraic equations, with $|\overline{\Omega}_h|$ unknowns. The linearity of L_h implies that these are systems of linear equations. For specific L_h (and Ω_h,$\overline{\Omega}_h$), f_h and g_h can then be constructed. This is usually not hard, as the following example shows.

Example 3.17 *Consider the diagonal uniformly elliptic operator, given by (3.2) and L_h given by (3.3). Assume all coefficient functions of L and f are defined and continuous on $\overline{\Omega}$ and g is continuous on $\partial\Omega$. All points $x \in \partial\Omega_h$ satisfy:*

$$B_h(x) \cap \partial\Omega \neq \emptyset$$

So we may take

$$g_h(x) = g(b) \qquad \text{for some} \quad b \in B_h(x) \cap \partial\Omega.$$

A construction for giving the point b can be done in a similar way as was done, in the previous chapter, for the Laplacian. Then, for any $h \approx 0$, $g_h(x) = g(b) \approx g({}^\circ x)$. As for f_h, just take $f_h(x) = f(x)$, $\forall x \in \Omega_h$.

As the following proposition shows, the discrete Dirichlet problem is much easier to solve than the analytical one.

Proposition 3.18 *Let $\{\langle L_h, g_h, f_h\rangle : h \in \mathbb{R}^+\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a consistent Dirichlet scheme which has MP. Then the problem:*

$$\begin{cases} L_h U(x) = f_h(x), & \textit{if } x \in \Omega_h, \\ U(x) = g_h(x), & \textit{if } x \in \partial\Omega_h, \end{cases} \tag{3.12}$$

has a unique solution $U \in \mathbb{R}^{\overline{\Omega}_h}$.

Proof. The set of functions $U : \overline{\Omega}_h \to \mathbb{R}$, with the usual sum of functions and product by a scalar, is a finite dimensional vector space of dimension $|\overline{\Omega}_h|$, which is isomorphic to the space $\mathbb{R}^{|\overline{\Omega}_h|}$. So we can interpret U as a vector in $\mathbb{R}^{\overline{\Omega}_h}$. Since L_h is linear, equations (3.12) set up a system of linear equations

$$AU = b, \tag{3.13}$$

where A is an $|\overline{\Omega}_h| \times |\overline{\Omega}_h|$ matrix and $b \in \mathbb{R}^{|\overline{\Omega}_h|}$. Consider the linear map, M, given by:

$$\mathbb{R}^{|\overline{\Omega}_h|} \ni v \mapsto Av \in \mathbb{R}^{|\overline{\Omega}_h|}.$$

We show that M is one to one. Suppose $Av = 0$. From (3.12), this means that $f_h(x) = 0$, for all $x \in \Omega_h$ and $g_h(x) = 0$, for all $x \in \partial\Omega_h$ By MP, $v = 0$. Since our linear space is finite dimensional, M is also onto. So we can conclude that (3.12) has a unique solution.

3.4 Convergence of the Solutions of the Discrete Dirichlet Problem

Assume there exists a solution $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ of the analytical Dirichlet problem (3.10). Our objective is to show that the solutions of a family of discrete Dirichlet problems satisfying Definition (3.16) converge, as $h \downarrow 0$, to u. Our notion of convergence is based on the L_h^∞-norm, which is defined as follows.

Definition 3.19 *Let Ω be a domain, and $\overline{\Omega}_h \in \overline{\Omega}$ be a discretization for Ω, for some $h \in \mathbb{R}^+$. The L_h^∞ norm of a function $U : \overline{\Omega}_h \to \mathbb{R}^n$ is given by:*

$$\| U \|_{L_h^\infty} = \max_{x \in \overline{\Omega}_h} |U(x)|.$$

With $h \approx 0$, this gives an internal *norm, acting on internal gridfunctions $U : \overline{\Omega}_h \to {}^*\mathbb{R}$. If U_h is the unique solution of the discrete problem (3.12) corresponding to $h > 0$, and u is the solution of the analytical problem (3.11), we seek to show

$$\lim_{h \downarrow 0} \| U_h - u \|_{L_h^\infty} = 0 \text{ [4]},$$

by showing the equivalent statement

$$\| U_h - u \|_{L_h^\infty} \approx 0 \qquad \forall h > 0, h \approx 0.$$

So we fix $h > 0$, $h \approx 0$ and consider the unique internal function solving the discrete Dirichlet problem, (3.12), U_h. First we show that U_h is S-continuous at points infinitesimally close to $\partial\Omega_h$. Secondly, we use the approximate maximum principle to compare U_h with u. [5].

As in Chapter 2, we introduce a discrete concept of barrier. Since we are now studying an inhomogeneous equation ($Lu = f \not\equiv 0$) we need a condition (b1), which is stronger than the one used in Chapter 2.

Definition 3.20 *Let $\{L_h\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme consistent with the uniformly elliptic operator L on the domain Ω. Let $y \in \partial\Omega$. An internal S-continuous function $b_y : {}^*\overline{\Omega} \to {}^*\mathbb{R}$, is called a barrier at y iff, for all $h \approx 0$:*

(b1) $L_h b_y(x) \lesssim -1, \quad \forall x \in \Omega_h$;

(b2) $b_y(x) \approx 0, \quad \forall x \in \overline{\Omega}_h, x \approx y$;

(b3) $b_y(x) \gtrapprox 0, \quad \forall x \in \overline{\Omega}_h, x \not\approx y$.

[4]The norm $\| U_h - u \|_{L_h^\infty}$, means $\| U_h - u|_{\overline{\Omega}_h} \|_{L_h^\infty}$.

[5]Whenever h is being fixed in a whole proof, we will drop the subscript h on U_h

A point $y \in \partial\Omega$ is called strongly regular if for every scheme $\{L_h\}$ consistent with L, there exists a barrier b_y.

Lemma 3.21 *Let L be a uniformly elliptic operator on a domain $\Omega \in \mathbb{R}^n$, with $c \leq 0$. If $y \in \partial\Omega$ satisfies an exterior sphere condition, then y is strongly regular.*

Proof. Let $\{L_h\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme consistent with L. Let $y \in \partial\Omega$ satisfy a exterior sphere condition, i.e., for some $\hat{x} \notin \overline{\Omega}$:

$$B_R(\hat{x}) \cap \overline{\Omega} = \{y\}, \qquad \text{with } \; R = |y - \hat{x}|.$$

Now, define $w : \overline{\Omega} \to \mathbb{R}$ by

$$w(x) = R^{-m} - r^{-m},$$

where

$$r = |x - \hat{x}| = \left(\sum_{i=1}^{n} (x_i - \hat{x}_i)^2 \right)^{1/2},$$

and m is a positive integer to be fixed later. Note that the exterior sphere condition implies that $R^{-m} > r^{-m}$, as long as $x \neq y$. Hence:

$$\begin{aligned} w(x) > 0 \qquad & \text{if} \quad x \in \overline{\Omega} - \{y\}, \\ w(x) = 0 \qquad & \text{if} \quad x = y. \end{aligned}$$

Now, let b_y be the restriction of w to $\overline{\Omega}_h$. Since w is a continuous standard function, conditions (b2) and (b3) follow easily. To verify condition (b1), we begin by calculating the partial derivatives of w. Then we use consistency to get an estimate for $L_h b_y(x)$.

The first partial derivatives are:

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \left(\sum_{i=1}^{n} (x_i - \hat{x}_i)^2 \right)^{-1/2} 2(x_i - \hat{x}_i) = (x_i - \hat{x}_i) r^{-1};$$

$$\frac{\partial w}{\partial x_i} = mr^{m-1}(x_i - \hat{x}_i)r^{-1} = m(x_i - \hat{x}_i)r^{-(m+2)}.$$

As for the second partial derivatives, we have:

$$\begin{aligned}
\frac{\partial^2 w}{\partial x_i \partial x_j} &= mr^{-(m+2)}\delta_{i,j} - m(x_i - \hat{x}_i)(m+2)r^{-(m+3)}(x_j - \hat{x}_j)r^{-1} \\
&= mr^{-(m+2)}\delta_{i,j} - m(m+2)(x_i - \hat{x}_i)(x_j - \hat{x}_j)r^{-(m+4)}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
Lw(x) &= mr^{-(m+2)}\sum_{i=1}^{n} a_{i,i}(x) - m(m+2)r^{-(m+4)}\sum_{i,j=1}^{n}(x_i - \hat{x}_i)(x_j - \hat{x}_j)a_{i,j}(x) \\
&\quad + mr^{-(m+2)}\sum_{i=1}^{n}(x_i - \hat{x}_i)b_i(x) + c(x)(R^{-m} - r^{-m}).
\end{aligned}$$

Note that $w \in C^2(\overline{\Omega}, \mathbb{R})$. Hence, using consistency of $\{L_h\}$ with L, we have, for all $h \approx 0$ and $x \in \Omega_h$:

$$\begin{aligned}
L_h w(x) &\approx mr^{-(m+2)}\sum_{i=1}^{n} a_{i,i}(x) - m(m+2)r^{-(m+4)}\sum_{i,j=1}^{n}(x_i - \hat{x}_i)(x_j - \hat{x}_j)a_{i,j}(x) \\
&\quad + mr^{-(m+2)}\sum_{i=1}^{n}(x_i - \hat{x}_i)b_i(x) + c(x)(R^{-m} - r^{-m}).
\end{aligned}$$

Since $c(x)(R^{-m} - r^{-m}) \leq 0$, we have:

$$\begin{aligned}
L_h w(x) &\lesssim mr^{-(m+2)}\Bigg(-(m+2)r^{-2}\sum_{i,j=1}^{n}(x_i - \hat{x}_i)(x_j - \hat{x}_j)a_{i,j}(x) \\
&\qquad + \sum_{i=1}^{n} a_{i,i}(x) + \sum_{i=1}^{n}(x_i - \hat{x}_i)b_i(x)\Bigg).
\end{aligned}$$

Since L is uniformly elliptic,

$$\sum_{i,j=1}^{n}(x_i - \hat{x}_i)(x_j - \hat{x}_j)a_{i,j}(x) \leq \mu \sum_{i=1} n(x_i - \hat{x}_i)^2 = \mu r^2,$$

where $\mu \in \mathbb{R}^+$ is the ellipticity constant of L. Hence:

$$L_h w(x) \lesssim mr^{-(m+2)}\left(-(m+2)r^{-2}\mu r^2 + \sum_{i=1}^{n}\Big(a_{i,i}(x) + (x_i - \hat{x}_i)b_i(x)\Big)\right).$$

Consider the standard nonnegative real:

$$M = \max \left\{ \left| \sum_{i=1}^{n} \Big(a_{i,i}(x) + (x_i - \hat{x}_i) b_i(x) \Big) \right| : x \in \overline{\Omega} \right\}.$$

This M exists finite since the coefficient functions of L are bounded. Hence:

$$L_h w(x) \lesssim m r^{-(m+2)} \Big(-(m+2)\mu + M \Big).$$

Let $r_0 \in \mathbb{R}^+$ be such that $B_{r_0}(\hat{x}) \supset \overline{\Omega}_h$. We fix $m \in \mathbb{N}$ so that:

$$m r_0^{-(m+2)} \Big(M - (m+2)\mu \Big) \leq -1.$$

This m depends only on Ω and the coefficient functions on L. For this choice of m, b_y satisfies:

$$L_h b_y(x) \lesssim -1 \qquad \forall x \in \Omega_h.$$

This shows (b1).

Given $y \in \partial\Omega$, and a barrier b_y at y, we can consider the function ${}^{\circ}b_y$. It satisfies:

(${}^{\circ}$b2) ${}^{\circ}b_y(y) = 0$;

(${}^{\circ}$b3) ${}^{\circ}b_y(x) > 0$, $\forall x \in \overline{\Omega} - \{y\}$.

Let $\delta \in \mathbb{R}^+$. It follows from compactness of $\overline{\Omega} - B_\delta(y)$ that ${}^{\circ}b_y$ has a minimum, which by (b3) must be positive.

Lemma 3.22 *Let L be a uniformly elliptic operator on a domain $\Omega \in \mathbb{R}^n$, with $c \leq 0$. Let $f : \Omega \to \mathbb{R}$ be bounded, and $g \in C(\partial\Omega, \mathbb{R})$. Let $\{L_h, f_h, g_h\}$, with*

$L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme with MP, consistent with the Dirichlet problem:

$$\begin{cases} Lu(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega. \end{cases}$$

Let $h \approx 0$, and let $U : \overline{\Omega}_h \to \mathbb{R}$ be the unique solution of the discrete Dirichlet problem:

$$\begin{cases} L_hU(x) = f_h(x), & \text{if } x \in \Omega_h, \\ U(x) = g_h(x), & \text{if } x \in \partial\Omega_h. \end{cases}$$

Then, for all $y \in \partial\Omega$ strongly regular, if $\Omega_h \ni x \approx y$, then $U(x) \approx f(y)$.

Proof. Let b_y be a barrier at y. Fix $\epsilon \in \mathbb{R}^+$. From continuity of g, we can find a $\delta \in \mathbb{R}^+$ such that, for all $x, y \in \partial\Omega$:

$$|x - y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon. \tag{3.14}$$

Define the standard real:

$$M = \max \{|g(x)| : x \in \partial\Omega\}. \tag{3.15}$$

From our observations about ${}^\circ b_y$, we can find $K \in \mathbb{R}^+$ such that, if $x, y \in \overline{\Omega}$, $|x - y| \geq \delta$, then $K\,{}^\circ b_y(x) > 2M$ (just take $K = 2M/A$, where $A = \min\{{}^\circ b_y(x) : x \in \partial\Omega - B_\delta(y)\} > 0$). Translating this to b_y, we have, for all $x, y \in \Omega_h \subset {}^*\Omega$:

$$|x - y| \geq \delta \Rightarrow Kb_y(x) \gtrsim 2M \tag{3.16}$$

Note that both δ and K depend only on ϵ.

Let

$$C = \max \left\{ \sup_{x\in\Omega} |f(x)|, K \right\}, \tag{3.17}$$

and consider the internal functions $\omega_\epsilon^\pm : \overline{\Omega}_h \to {}^*\mathbb{R}$ given by:

$$\omega_\epsilon^-(x) = U(x) - \epsilon - Cb_y(x),$$

$$\omega_\epsilon^+(x) = U(x) + \epsilon + Cb_y(x).$$

If $\hat{\epsilon}$ denotes the constant function $\hat{\epsilon}(x) = \epsilon$, then by consistency:

$$L_h\hat{\epsilon}(x) \approx L\hat{\epsilon}(x) = c(x)\epsilon \leq 0. \tag{3.18}$$

So, we have:

(a) For all $x \in \Omega_h$, and using consistency, property (b1) of the barrier b_y, and equations (3.18) and (3.17):

$$L_h\omega_\epsilon^-(x) \approx f_h(x) - c(x)\epsilon - CL_hb_y(x) \gtrsim f(x) + C \geq 0;$$

$$L_h\omega_\epsilon^+(x) \approx f_h(x) + c(x)\epsilon + CL_hb_y(x) \lesssim f(x) - C \leq 0.$$

(b) For all $x \in \partial\Omega_h$, and using consistency and equation (3.17):

$$\begin{aligned} \omega_\epsilon^-(x) &= g_h(x) - \epsilon - Cb_y(x) \\ &\lesssim g({}^\circ x) - \epsilon - Kb_y(x); \end{aligned}$$

$$\begin{aligned} \omega_\epsilon^+(x) &= g_h(x) + \epsilon + Cb_y(x) \\ &\gtrsim g({}^\circ x) + \epsilon + Kb_y(x). \end{aligned}$$

If $|{}^\circ x - y| < \delta$, then from property (b3) of the barrier b_y, and (3.14):

$$\omega_\epsilon^-(x) \lesssim g({}^\circ x) - \epsilon \lesssim g(y);$$

$$\omega_\epsilon^+(x) \gtrsim g({}^\circ x) + \epsilon \gtrsim g(y);$$

If $|{}^\circ x - y| \geq \delta$, then from (3.16) and (3.15):

$$\omega_\epsilon^-(x) \lesssim g({}^\circ x) - Kb_y(x) \lesssim g({}^\circ x) - 2M \lesssim g(y);$$

$$\omega_\epsilon^+(x) \gtrsim g({}^\circ x) + Kb_y(x) \gtrsim g({}^\circ x) + 2M \gtrsim g(y).$$

From (a) and (b) above, the functions $\omega_\epsilon^\pm$ are in the conditions of Corollary (3.15). Thus:

$$U(x) - \epsilon - Cb_y(x) = \omega_\epsilon^-(x) \lesssim f(y) \lesssim \omega_\epsilon^+ = U(x) + \epsilon + Cb_y(x)$$

Subtracting $U(x)$ on both sides of these two inequalities yields:

$$|U(x) - f(y)| \lesssim \epsilon + Cb_y(x)$$

For $x \approx y$, property (b2) of the barrier b_y asserts that $b_y(x) \approx 0$. Hence, $|U(x) - f(y)| \leq \epsilon$. Since ϵ was arbitrarily chosen in $\mathbb{R}^+$, we conclude that $U(x) \approx f(y)$.

Theorem 3.23 *Let L be a uniformly elliptic operator with $c \leq 0$, on a domain $\Omega \in \mathbb{R}^n$, all of whose boundary points are strongly regular. Let $f : \Omega \to \mathbb{R}$ be bounded, and $g \in C(\partial\Omega, \mathbb{R})$. Assume that the analytical Dirichlet problem,*

$$\begin{cases} Lu(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases}$$

has a solution, $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$. Let $\{L_h, f_h, g_h\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme with MP, consistent with the above analytical Dirichlet problem. For each $h \in \mathbb{R}^+$, let $U_h : \overline{\Omega}_h \to \mathbb{R}$ be the unique solution of the discrete Dirichlet problem:

$$\begin{cases} L_h U(x) = f_h(x), & \text{if } x \in \Omega_h, \\ U(x) = g_h(x), & \text{if } x \in \partial\Omega_h. \end{cases}$$

Then, for all positive $h \approx 0$, $\| U_h - u \|_{L_h^\infty} \approx 0$. Equivalently, $\lim\limits_{h \downarrow 0} \| U_h - u \|_{L_h^\infty} = 0$.

Proof. Fix $h \approx 0$. For each $\epsilon \in {}^*\mathbb{R}^+$, let:

$$\Gamma^\epsilon = \Big\{x \in \Omega_h : \text{dist}\,(x, \partial\Omega_h) > \epsilon\Big\}.$$

Consider the internal set:

$$D = \Big\{\epsilon \in \mathbb{R}^+ : \forall x \in \Gamma^\epsilon \; |L_h u(x) - Lu(x)| < \epsilon\Big\}.$$

Note that, for any $\epsilon > 0$ and noninfinitesimal, $u \in C^2({}^\circ\Gamma^\epsilon, \mathbb{R})$. Hence, by consistency of L_h with L, the set D contains all noninfinitesimal $\epsilon > 0$. Since D is internal, it follows that D contains some $\epsilon \approx 0$. Therefore, for this ϵ, and for all $x \in \Gamma^\epsilon$:

$$|L_h u(x) - Lu(x)| < \epsilon \Rightarrow L_h u(x) \approx Lu(x).$$

Hence:

$$L_h(U_h - u)(x) = L_h U_h(x) - L_h u(x) \approx f_h(x) - Lu(x) = f_h(x) - f(x) \approx 0.$$

On the other hand, for all $x \in \overline{\Omega}_h - \Gamma^\epsilon$, $\text{dist}\,(x, \partial\Omega) \lesssim \epsilon \approx 0$. Hence, using Lemma (3.22) and the continuity of u at $\partial\Omega$:

$$U_h(x) - u(x) \approx g({}^\circ x) - u(x) \approx g({}^\circ x) - g({}^\circ x) = 0.$$

By Corollary (3.15):

$$U_h(x) - u(x) \approx 0, \qquad \forall x \in \overline{\Omega}_h. \tag{3.19}$$

Since $\overline{\Omega}_h$ is hyperfinite, the ${}^*\max$ of $|U_h(x) - u(x)|$ on $\overline{\Omega}_h$ exists, and by equation (3.19), it must be infinitesimal. Hence:

$$\| \, U_h - u \, \|_{L_h^\infty} \approx 0.$$

Under suitable conditions on its data, it can be shown that the problem (3.10) has a unique solution. We quote the following result from the Schauder theory of linear elliptic equations (of second order).

Theorem 3.24 *Let L be uniformly elliptic in a domain Ω, with $c \leq 0$ and let f and the coefficients of L be bounded functions in $C^{2,\alpha}(\Omega)$, and $g \in C(\partial\Omega, \mathbb{R})$. Suppose that Ω satisfies an exterior sphere condition at every $y \in \partial\Omega$. Then, the Dirichlet problem,*

$$\begin{cases} Lu(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases}$$

has a unique solution $u \in C(\overline{\Omega}, \mathbb{R}) \cap C^{2,\alpha}(\Omega, \mathbb{R})$ [6].

The following is a corollary of Theorems (3.23) and (3.24).

Corollary 3.25 *Let L be uniformly elliptic in a domain Ω, with $c \leq 0$ and let f and the coefficients of L be bounded functions in $C^{2,\alpha}(\Omega)$, and $g \in C(\partial\Omega, \mathbb{R})$. Suppose that Ω satisfies an exterior sphere condition at every $y \in \partial\Omega$. Let $\{L_h, f_h, g_h\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme with MP, consistent with the Dirichlet problem $\langle L, f, g\rangle$ on Ω. For each $h \in \mathbb{R}^+$, let $U_h : \overline{\Omega}_h \to \mathbb{R}$ be the unique solution of the discrete Dirichlet problem:*

$$\begin{cases} L_h U(x) = f_h(x), & \text{if } x \in \Omega_h, \\ U(x) = g_h(x), & \text{if } x \in \partial\Omega_h. \end{cases}$$

Then, for all infinitesimal h, $\| U_h - u \|_{L^\infty_h} \approx 0$, where u is the unique solution of the analytical Dirichlet problem. Equivalently, $\lim_{h \downarrow 0} \| U_h - u \|_{L^\infty_h} = 0$.

[6] *For a proof of this, see for example [9].*

3.5 Accuracy of the Solutions of the Discrete Dirichlet Problem

The results of the last section are conceptually useful to justify the use of a (analytical) elliptic Dirichlet problem to model the limit, as $h \downarrow 0$, of a large class of discrete problems. But, for numerical applications, the point of view is entirely the opposite: one starts with the analytical problem, and then studies suitable discrete problems, for the purpose of calculation. It is no surprise that for numerical applications, convergence proofs are not very useful without error estimates.

The hypothesis on the Dirichlet problem from last section were too general to be able to get useful error estimates. The fact that we consider analytical problems where the solution may be no better than continuous at the boundary of Ω imply that $L_h U_h$ may be a very bad approximation of Lu near the boundary. So, L_h^∞ estimates may be very poor. The main result of this section handles this problem by putting some extra constraints on the analytical problem.

Before we proceed, lets introduce the "small oh" notation. Let $A \in {}^*\mathbb{R}$ and $\varphi : A \to {}^*\mathbb{R}$ be both internal. Let $p \in \mathbb{R}^+ \cup \{0\}$. Then

$$\varphi(h) = o(h^p)$$

means that, for all $h \approx 0$, there exists $\epsilon \approx 0$ such that $\varphi(h) = \epsilon h^p$. Additionally

$$\varphi(h) = \psi(h) + o(h^p)$$

means that $\varphi(h) - \psi(h) = o(h^p)$. It is clear that, if $c, d \in {}^*\mathbb{R}$ are finite, $\varphi(h) = o(h^p)$ and $\psi(h) = o(h^q)$, then:

$$c\varphi(h) + d\psi(h) = o(h^{\min\{p,q\}}).$$

With this notation, Corollary (3.15) can be written as

Corollary 3.26 *Let L be a uniformly elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, with ellipticity constant $\mu \in \mathbb{R}^+$. Suppose $\{L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}\}_{h\in\mathbb{R}^+}$ is a consistent discrete scheme for L that has MP. Now, let $h \in {}^*\mathbb{R}^+$, $h \approx 0$. If $U : \overline{\Omega}_h \to {}^*\mathbb{R}$ is an internal function such that, for some $a \in \mathbb{R}$ and some $k \in \mathbb{N}$:*

$$\forall x \in \Omega_h \; L_h U(x) = o(h^k);$$

$$\forall x \in \partial\Omega_h \; U(x) = a + o(h^k).$$

Then for all $x \in \overline{\Omega}_h$, $U(x) = a + o(h^k)$.

First we define accuracy of a discrete scheme for a Dirichlet problem.

Definition 3.27 *Let L be a linear differential operator of order k on a domain $\Omega \subset \mathbb{R}^n$. Let $\langle L, f, g\rangle$ be a Dirichlet problem on a domain $\Omega \subset \mathbb{R}^n$. Then, the discrete Dirichlet scheme $\{\langle L_h, g_h, f_h\rangle : h \in \mathbb{R}^+\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$ is called accurate of order $p \in \mathbb{R}^+$ (with respect to $\langle L, f, g\rangle$) on Ω iff it is consistent with $\langle L, f, g\rangle$ on Ω, and*

(a) $\forall x \in \partial\Omega_h \; \exists y \in {}^*\partial\Omega \quad \dfrac{|x-y|}{h^p} \approx 0;$

(b) $L_h u(x) = Lu(x) + o(h^p), \quad \forall h \approx 0, \; \forall u \in C^k(\overline{\mathcal{D}}, \mathbb{R}), \; \forall x^{\text{finite}} \in \mathcal{D}_h \;;$

(c) $f_h(x) = f(x) + o(h^p), \qquad \forall h \approx 0, \; \forall x \in \overline{\Omega}_h;$

(d) $g_h(x)) = g(x) + o(h^p), \qquad \forall h \approx 0, \; \forall x \in \partial\Omega_h;$

Theorem 3.28 *Let L be a uniformly elliptic operator on a domain $\Omega \in \mathbb{R}^n$ all whose boundary points are strongly regular, and with $c \leq 0$. Let $f : \Omega \to \mathbb{R}$ be bounded, and $g \in C(\partial\Omega, \mathbb{R})$. Assume the analytical Dirichlet problem,*

$$\begin{cases} Lu(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases}$$

has a unique solution, $u \in C^2(\overline{\Omega}, \mathbb{R})$. Let $\{L_h, f_h, g_h\}$, with $L_h : \mathbb{R}^{\overline{\Omega}_h} \to \mathbb{R}^{\Omega_h}$, be a discrete scheme with MP, consistent with the above analytical Dirichlet problem and accurate of order $p \in \mathbb{R}^+$. For each $h \in \mathbb{R}^+$, let $U_h : \overline{\Omega}_h \to \mathbb{R}$ be the unique solution of the discrete Dirichlet problem:

$$\begin{cases} L_h U(x) = f_h(x), & \text{if } x \in \Omega_h, \\ U(x) = g_h(x), & \text{if } x \in \partial\Omega_h. \end{cases}$$

Then, $\| U_h - u \|_{L_h^\infty} = o(h^p)$.

Proof. Let $h \approx 0$. If $x \in \Omega_h$, then since $u \in C^2(\overline{\Omega}, \mathbb{R})$, and using consistency:

$$\begin{aligned} L_h(U_h - u)(x) &= L_h U_h(x) - L_h u(x) \\ &= f_h(x) - Lu(x) + o(h^p) \\ &= f_h(x) - f(x) + o(h^p) \\ &= o(h^p). \end{aligned}$$

On the other hand, if $x \in \partial\Omega_h$, pick $y \in {}^*\partial\Omega$ such that $\frac{|x-y|}{h^p} \approx 0$. Then, since by our hypothesis, u must be Lipshitz continuous:

$$\begin{aligned} (U_h - u)(x) &= g_h(x) - u(x) \\ &= g_h(x) - g(y) + u(y) - u(x) \\ &= g_h(x) - g(y) + o(h^p) = o(h^p). \end{aligned}$$

By corollary (3.26):

$$U_h(x) - u(x) = o(h^p) \qquad \forall x \in \overline{\Omega}_h$$

Hence, $\| U_h - u \|_{L_h^\infty} = o(h^p)$.

Usually, the easier way to satisfy condition (a) of Definition (3.27) is to have a scheme where $\partial\Omega_h \subset \partial\Omega$. These schemes are not restricted to special domains (e.g., rectangular or spherical regions). By defining L_h by a different formula for points near the boundary, general domains can be discretized in this fashion. For an example of this type of schemes, see Wendland [28]

Chapter 4

Brouwer Degree

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$, and $b \in \mathbb{R}^n - \partial\Omega$. A function, $d(\varphi, \Omega, b)$, called the (Brouwer) degree of φ with respect to Ω and b, can be defined in such a way that it gives an *algebraic count* of the number of solutions of the equation

$$\varphi(x) = b,$$

on Ω. This will not be the usual count: some solutions add $+1$ to d, while others add -1.

As it turns out, the algebraic count has much nicer properties then the usual count. For example, the degree stays constant under some very large deformations of φ, making it possible to use it for existence proofs where the original problem is deformed into a much nicer one.

The construction of the Brouwer degree provided here has some similarities with the one in Rabinowitz [23]. The main differences in our construction are the derivation of a lemma showing that the gradient of the integral representation of the degree vanishes, and the extension of the degree from C^2 maps to continuous maps by the use of ${}^*C^2$ liftings. The later leads to a new formula for the Brouwer degree in the general case.

Throughout this chapter, Ω will denote a bounded and open subset of $\mathbb{R}^n$.

4.1 The Definition of Brouwer Degree

We begin by defining the degree in the following special setting, called the *nice case*. Suppose

$$\varphi \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, and

$$b \in \mathbb{R}^n - \Big(\varphi(\partial\Omega) \cup \varphi(S)\Big). \tag{4.2}$$

S, named the *singular set*, is given by:

$$S = \Big\{x \in \Omega : J_\varphi(x) = 0\Big\}.$$

Here, $J_\varphi(x) = \det \varphi'(x)$, where $\varphi'(x)$ denotes the Jacobian matrix of φ at $x \in \Omega$.

Under the conditions of the nice case

$$\forall x \in \varphi^{-1}(b) \quad J_\varphi(x) \neq 0,$$

so by the inverse function theorem, φ is a diffeomorphism from each neighborhood of $x \in \varphi^{-1}(b)$ to a neighborhood of $\varphi(x) = b$. It follows that:

(a) all solutions of $\varphi(x) = b$ in $\overline{\Omega}$ actually lie in Ω;

(b) all solutions of $\varphi(x) = b$ are isolated;

(c) if $\overline{\Omega}$ is compact, then $\varphi^{-1}(b)$ is finite.

The proof of (c) goes as follows. Since $\varphi^{-1}(b) \subset \overline{\Omega}$ is closed, it is compact. Assuming $\varphi^{-1}(b)$ is infinite, we can find an open neighborhood of each $x \in \varphi^{-1}(b)$, U_x, such that $\varphi|_{U_x}$ is $1-1$. From $\{U_x : x \in \varphi^{-1}(b)\}$, extract a finite subcovering of

$\varphi^{-1}(b)$, $\{U_{x_1}, \ldots, U_{x_n}\}$. Then, one of the U_{x_i} must have more than one $x \in \varphi^{-1}(b)$; but this contradicts the fact that φ is $1-1$ on the U_{x_i}.

Let sgn $: \mathbb{R} - \{0\} \to \mathbb{R}$ be given by

$$\operatorname{sgn} t = \frac{t}{|t|} = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Definition 4.1 *Under the conditions of the nice case, the Brouwer degree of φ with respect to Ω and b is given by:*

$$d(\varphi, \Omega, b) = \sum_{\xi \in \varphi^{-1}(b)} \operatorname{sgn} J_\varphi(\xi).$$

Example 4.2 *Let $R > 1$ and $\varphi : [-R, R] \to \mathbb{R}$ given by $\varphi(x) = x(x-1)(x+1)$. Computing the degree of $\varphi(x) = x(x-1)(x+1)$.*

Then:

$$d\Big(\varphi, (-R, R), b\Big) = \begin{cases} 0 & \text{if } b \notin \varphi([-R, R]), \\ 1 & \text{if } b \in \varphi((-R, R)) \text{ and } b \notin \varphi(S). \end{cases}$$

Note that for all $b \notin \varphi(S)$, $b \in \varphi([-R, R])$ stays constant for b on connected components of $\mathbb{R} - \partial\Omega$.

Recall that we want to define the degree:

(a) relative to all $b \notin \varphi(\partial\Omega)$ (this means being also able to define the degree relative to values of b in the image of the singular set);

(b) relative to all $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$.

An approximation argument is used to extend Definition 4.1. The first step is to derive an integral representation of the degree. Given $b \in \mathbb{R}^n$, $\epsilon > 0$, let

$j_{b,\epsilon} \in C^1(\mathbb{R}^n, \mathbb{R})$ be such that:

$$\overline{\text{supp } j_{b,\epsilon}} \subset B_\epsilon(b); \tag{4.3}$$

$$\int_{\mathbb{R}^n} j_{b,\epsilon}(x)\, dx = 1. \tag{4.4}$$

For all $\varphi \in C^1(\overline{\Omega}, \mathbb{R})$, and all $b \in \mathbb{R}^n$, we can define:

$$I_\epsilon(\varphi, \Omega, b) = \int_{\overline{\Omega}} j_{b,\epsilon}(\varphi(x)) J_\varphi(x)\, dx.$$

This integral is a weighted average, with weight function $j_{b,\epsilon}(\varphi(\cdot))$, of the values of $J_\varphi(x)$ on some small neighborhoods of the points ξ such that $\varphi(\xi) = b$.

Proposition 4.3 (E. Heinz) *For all* $b \in \mathbb{R}^n - \Big(\varphi(\partial\Omega) \cup \varphi(S)\Big)$, *there exists* $\overline{\epsilon} > 0$ *such that for all* $\epsilon \in (0, \overline{\epsilon})$

$$I_\epsilon(\varphi, \Omega, b) = d(\varphi, \Omega, b).$$

Proof.

Case 1: $b \notin \varphi(\overline{\Omega})$.

Take $\overline{\epsilon} = \text{dist}\,(b, \varphi(S))$. Now let $\epsilon \in (0, \overline{\epsilon})$ and assume there exists $x \in \overline{\Omega}$ such that $j_{b,\epsilon}(\varphi(x)) \neq 0$. Then $\varphi(x) \in B_\epsilon(b)$, so $\epsilon > \text{dist}\,(b, \varphi(S)) = \overline{\epsilon}$, which leads to a contradiction. Hence $\text{supp } j_{b,\epsilon}(\varphi(\cdot)) = \emptyset$ and consequently:

$$I_\epsilon(\varphi, \Omega, b) = 0 = \int_{\overline{\Omega}} j_{b,\epsilon}(\varphi(x)) J_\varphi(x)\, dx.$$

Case 2: $b \in \varphi(\Omega) - \varphi(S)$.

Say $\varphi^{-1}(b) = \{\xi_1, \ldots, \xi_k\}$. From our hypothesis about b, $J_\varphi(\xi_i) \neq 0$ for all $i \in \{1, \ldots, k\}$. Using the inverse function theorem, there exist (standard) open neighborhoods $U_i \ni \xi_i$ of each ξ_i and V_i of b such that $\varphi|_{U_i} : U_i \to V_i$ is $1-1$ and onto (for all $i \in \{1,\dots,k\}$).

Choose now any $\epsilon > 0$, $\epsilon \approx 0$. Since the V_i's are (standard) open neighborhoods of b, $B_\epsilon(b) \subset V_i$, for all $i \in \{1,\dots,k\}$. Let

$$N_{i,\epsilon} = \varphi^{-1}(B_\epsilon(b)),$$

and

$$\varphi_i = \varphi|_{N_{i,\epsilon}} : N_{i,\epsilon} \to B_\epsilon(b).$$

By construction, φ_i is onto. Also, φ_i is $1-1$ since it is a restriction of $\varphi|_{U_i}$. Furthermore:

(i) Each $N_{i,\epsilon}$ has infinitesimal diameter, for otherwise, the equation $\varphi(x) = b$ would have infinitely many solutions. Hence, each $N_{i,\epsilon}$ is an infinitesimal neighborhood of ξ_i, and so $N_{i,\epsilon} \cap N_{j,\epsilon} = \emptyset$, for all $i \neq j$.

(ii) From (i) and continuity of φ, $J_\varphi(x) \neq 0$ for all $x \in N_{i,\epsilon}$ and all $i \in \{1,\dots,k\}$

Since the set

$$\begin{aligned} E \;=\; \Big\{ & \epsilon \in {}^*\mathbb{R} \;:\; \epsilon > 0 \;\wedge\; \forall i,j \in \{1,\dots,k\}\; i \neq j \;\Rightarrow\; N_{i,\epsilon} \cap N_{j,\epsilon} = \emptyset \\ & \wedge\; \forall i \in \{1,\dots,k\}\, \forall x \in N_{i,\epsilon}\; J_\varphi(x) \neq 0 \Big\} \end{aligned}$$

is internal and contains all positive infinitesimals, it must contain a standard $\overline{\epsilon} > 0$. Note that this implies that any $\epsilon \in (0, \overline{\epsilon}]$ will also belong to E, since:

$$N_{i,\epsilon} = \varphi^{-1}(B_\epsilon(b)) \subset \varphi^{-1}(B_{\overline{\epsilon}}(b)) = N_{i,\overline{\epsilon}}.$$

Hence, for any standard $\epsilon \in (0, \overline{\epsilon}]$:

$$\begin{aligned} I_\epsilon &= \int_{\{x \in \overline{\Omega} : \varphi \in B_\epsilon(b)\}} j_{b,\epsilon}(\varphi(x))\, J_\varphi(x)\, dx \\ &= \int_{\cup_{i=1}^k N_{i,\epsilon}} j_{b,\epsilon}(\varphi(x))\, J_\varphi(x)\, dx \\ &= \sum_{i=1}^k \int_{N_{i,\epsilon}} j_{b,\epsilon}(\varphi(x))\, J_\varphi(x)\, dx. \end{aligned}$$

We now make the change of variables $y = \varphi(y)$, and get:

$$I_\epsilon = \sum_{i=1}^k \int_{B_\epsilon} \text{sgn } J_\varphi(\varphi^{-1}(y)) j_{b,\epsilon}(y)\, dy.$$

Since $\text{sgn } J_\varphi(x)$ stays constant in each $N_{i,\epsilon}$,

$$I_\epsilon = \sum_{i=1}^k \text{sgn } J_\varphi(\xi_i) \int_{B_\epsilon} j_{b,\epsilon}(y)\, dy = d(\varphi, \Omega, b).$$

We now consider a family $\{j_{0,\epsilon}\}$ satisfying (4.3) and (4.4). For simplicity of notation, we drop the 0 subscript and denote $j_{0,\epsilon}$ by j_ϵ. It turns out that for each $\beta \in \mathbb{R}^n$ and $\epsilon > 0$, the function $j_{\beta,\epsilon} \in C^1(\mathbb{R}^n, \mathbb{R})$ given by

$$j_{\beta,\epsilon}(y) = j_\epsilon(y - \beta) \quad \forall y \in \mathbb{R}^n,$$

satisfies conditions (4.3) and (4.4) with $\beta = b$. This observation will be useful to show the following result.

Lemma 4.4 *Let $\varphi \in C^2(\Omega, \mathbb{R}^n)$ and $\epsilon > 0$. Then, for all $b \in \mathbb{R}^n$ such that $\overline{B_\epsilon(b)} \subset \varphi(\Omega)$, $I_\epsilon(\varphi, \Omega, \cdot)$ is differentiable at b, and its gradient is identically 0.*

Proof.

Fix $\varphi, \Omega, \epsilon$ as above. For every b such that $B_\epsilon(b) \subset \Omega$:

$$I_\epsilon(\varphi, \Omega, b) = \int_{\overline{\Omega}} j_\epsilon(\varphi(x) - b) J_\varphi(x)\, dx.$$

Let $\delta^{\pm}_{j,h}$ denote the usual forward and backward finite difference operators, i.e.:

$$(\delta^{+}_{j,h} f)(b) = \frac{1}{h}\Big(f(b + he_j) - f(b)\Big);$$

$$(\delta^{-}_{j,h} f)(b) = \frac{1}{h}\Big(f(b) - f(b - he_j)\Big).$$

Take $h > 0$, $h \approx 0$. We have:

$$\begin{aligned}(\delta_{j,h} I_\epsilon)(\varphi, \Omega, b) &= \int_{\overline{\Omega}} \frac{1}{h}\Big(j_\epsilon(\varphi(x) - b - he_j) - j_\epsilon(\varphi(x) - b)\Big) J_\varphi(x)\, dx \\ &= \int_{\overline{\Omega}} -\left(\delta^{-}_{j,h} j_\epsilon\right)(\varphi(x) - b) J_\varphi(x)\, dx.\end{aligned}$$

Consider the function (for fixed φ, Ω, and ϵ):

$$K_j(b) = \int_{\overline{\Omega}} -\frac{\partial j_\epsilon}{\partial y_j}(\varphi(x) - b) J_\varphi(x)\, dx.$$

Note that $\overline{\text{supp}\ j_\epsilon} \subset B_\epsilon(0)$, so supp $\left(\delta^{-}_{j,h}\ j_\epsilon(\varphi(\cdot) - b)\right) \subset \varphi^{-1} B_\epsilon(b)$. Consequently, supp $\frac{\partial j_\epsilon}{\partial y_j}(\varphi(\cdot) - b) \subset \varphi^{-1}(B_\epsilon(b))$. Then, since $j_\epsilon \in C^1(\mathbb{R}^n, \mathbb{R})$, for every standard $r > 0$:

$$\begin{aligned}\left|\delta^{+}_{j,h} I_\epsilon(\varphi, \Omega, b) - K_j(b)\right| &\leq \int_{\overline{\Omega}} \left| -(\delta^{-}_{j,h} j_\epsilon)(\varphi(x) - b) + \frac{\partial j_\epsilon}{\partial y_j}(\varphi(x) - b)\right| |J_\varphi(x)|\, dx \\ &\leq r \int_{\varphi^{-1}(B_\epsilon(b))} |J_\varphi(x)|\, dx \ \leq\ rM.\end{aligned}$$

(where M is standard and does not depend on h). Hence $|\delta_{j,h} I_\epsilon(\varphi, \Omega, b) - K_j(b)| \approx 0$ for every $h \approx 0$. Therefore, $\frac{\partial}{\partial y_j} I_\epsilon(\varphi, \Omega, y)\big|_{y=b}$ exists and equals $K_j(b)$. To finish the proof, it is enough to show that $K_j(b) = 0$, $j \in \{1, 2, \ldots, n\}$.

We first work the case $n = 1$. This is necessary, since our general proof only works for $n \geq 2$. We have:

$$\begin{aligned} \frac{d}{dy} I_\epsilon(\varphi, \Omega, y)\bigg|_{y=b} &= -\int_{\Omega} \frac{dj_\epsilon}{dy}(\varphi(x) - b)\varphi'(x)\, dx \\ &= -\int_{\Omega} \frac{d}{dx}\Big(j_\epsilon(\varphi(x) - b)\Big)\, dx \end{aligned}$$

Since $B_\epsilon(b) \subset \varphi(\Omega)$ and $|\varphi(x) - b| = |\varphi(x) - \varphi(\xi_i)| < \epsilon$ for each $x \in N_i^\epsilon$ ($i = 1, \ldots, k$), $\operatorname{supp} j_\epsilon(\varphi(\cdot) - b) \subset\subset \Omega$. Therefore if we extend $j_\epsilon(\varphi(\cdot) - b)$ trivially to all $x \in \mathbb{R}$, this extension remains C^1. Hence for a sufficiently large R:

$$\frac{d}{dy} I_\epsilon(\varphi, \Omega, y)\bigg|_{y=b} = -\int_{-R}^{R} \frac{d}{dx}\Big(j_\epsilon(\varphi(x) - b)\Big)\, dx = 0.$$

For $n \geq 2$, the argument is similar, although it becomes more technical. The idea now is to show that for some $u^j \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with compact support contained in Ω,

$$\begin{aligned} \frac{d}{dy_j} I_\epsilon(\varphi, \Omega, y)\bigg|_{y=b} &= K_j \;=\; \int_{\Omega} \operatorname{div} u^j(x)\, dx \\ &= \int_{B_R(0)} \operatorname{div} u^j(x)\, dx \;=\; \int_{\partial B_R(0)} u^j(x) \cdot \nu\, dS \;=\; 0 \end{aligned}$$

(where R is sufficiently large so that $B_R(0) \supset \varphi(\Omega)$). The construction of the u^j's can be done as follows. Let $A_{ji}(x)$ be the cofactor matrix of $\varphi'(x)$, i.e., $A_{ji}(x) = (-1)^{i+j} M_{ji}(x)$, where $M_{ji}(x)$ is the minor of $\varphi'(x)$ corresponding to the entry $\frac{\partial \varphi_j}{\partial x_i}(x)$. Then let $u^j(x) = (u_1^j, \ldots, u_n^j) : \Omega \to \mathbb{R}^n$ be given by:

$$u_i^j(x) = -j_\epsilon(\varphi(x) - b) A_{ji}(x).$$

Since $\varphi \in C^2(\Omega, \mathbb{R}^n)$ and $j_\epsilon \in C^1(\mathbb{R}^n, \mathbb{R})$, we conclude that $u^j \in C^1(\Omega, \mathbb{R}^n)$. In fact, since

$$\operatorname{supp} u^j \subset \operatorname{supp}\,(j_\epsilon(\varphi(\cdot) - b)) \subset \varphi^{-1}(B_\epsilon(b)) \subset\subset \Omega,$$

the trivial extension of u^j to $\mathbb{R}^n$ remains C^1. So we assume u^j extended in this fashion.

Furthermore:

$$\begin{aligned} \mathrm{div\ u^j(x)} &= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} j_\epsilon(\varphi(x)-b))A_{ji}(x) \\ &= \underbrace{-\sum_{i,k=1}^{n} \frac{\partial j_\epsilon}{\partial y_k}(\varphi(x)-b)\frac{\partial \varphi_k}{\partial x_i}A_{ij}(x)}_{s_1(x)} \underbrace{-\sum_{i=1}^{n} j_\epsilon(\varphi(x)-b))\frac{\partial}{\partial x_i}A_{ji}(x)}_{s_2(x)}. \end{aligned}$$

The first term in the previous expression, $s_1(x)$, can be written as:

$$s_1(x) = -\sum_{k=1}^{n} \frac{\partial j_\epsilon}{\partial y_k}(\varphi(x)-b)\sum_{i=1}^{n} \frac{\partial \varphi_k}{\partial x_i}A_{ji}(x).$$

For $k = j$, the second sum is just equal to $\det \varphi'(\mathrm{x}) = \mathrm{J}_\varphi(\mathrm{x})$. For $k \neq j$, it becomes the determinant of a matrix with two identical rows, so it equals 0. Hence:

$$s_1(x) = -\frac{\partial j_\epsilon}{\partial y_j}(\varphi(x)-b)J_\varphi(x).$$

This is precisely the integrand in $K_j(b)$. So to finish the proof, it remains to show that the term

$$s_2(x) = \sum_{i=1}^{n} j_\epsilon(\varphi(x)-b))\frac{\partial}{\partial x_i}A_{ji}(x) = j_\epsilon(\varphi(x)-b)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}A_{ji}(x)$$

vanishes identically. This is achieved using the next lemma.

Lemma 4.5 *Let* $\psi \in C^2(\mathcal{O}, \mathbb{R}^p)$ *with* $\mathcal{O} \subset \mathbb{R}^{p+1}$ *open. Say* $x = (x_1, \ldots, x_{p+1}) \in \mathbb{R}^{p+1}$. *Let* $D_i = \det\,(\psi_{\mathrm{x_1}}, \ldots, \hat{\psi}_{\mathrm{x_i}}, \ldots, \psi_{\mathrm{x_p}})$ [1] *Then:*

$$\sum_{i=1}^{p+1} (-1)^i \frac{\partial D_i}{\partial x_i} = 0.$$

[1] $(\psi_{x_1}, \ldots, \hat{\psi}_{x_i}, \ldots, \psi_{x_p})$ *is a list of the columns of the matrix, with* $\hat{\psi}_{x_i}$ *omitted*

Assuming for now the lemma, we see that the sum in $s_2(x)$ is of the form of the lemma, if we take $p+1=n$, and $D_i(x) = M_{ji}(x)$, $\psi = \varphi$. If this is the case,

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_{ji}(x) = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial}{\partial x_i} M_{ji}(x) = (-1)^j \sum_{i=1}^{p+1} (-1)^i \frac{\partial}{\partial x_i} D_i(x) = 0.$$

Hence, $s_2(x) \equiv 0$ as we wanted, and we conclude that

$$\operatorname{div} u^j(x) = s_1(x) = -\frac{\partial j_\epsilon}{\partial y_j}(\varphi(x) - b) J_\varphi(x)$$

Proof of Lemma (4.5).

Let $D_{ij} = D_{ji} = \det(\psi_{\mathrm{x_i x_j}}, \psi_{\mathrm{x_0}}, \ldots, \hat{\psi}_{\mathrm{x_i}}, \ldots, \hat{\psi}_{\mathrm{x_j}}, \ldots, \psi_{\mathrm{x_p}})$. Applying the product rule to calculate the derivative of the determinant yields:

$$\frac{\partial}{\partial x_i} D_i = \sum_{j<i} \det(\ldots, \psi_{\mathrm{x_j x_i}}, \ldots, \hat{\psi}_{\mathrm{x_i}}, \ldots) + \sum_{\mathrm{j>i}} \det(\ldots, \hat{\psi}_{\mathrm{x_i}}, \ldots, \psi_{\mathrm{x_j x_i}}, \ldots).$$

Permutating columns in the matrices, we obtain:

$$\frac{\partial}{\partial x_i} D_i = \sum_{j<i} (-1)^j \det D_{ij} + \sum_{j>i} (-1)^{j-1} \det D_{ij} = \sum_{j=1}^{p+1} (-1)^j \alpha_{ij} D_{ij},$$

where

$$\alpha_{ij} = \begin{cases} 1 & \text{if } j < i \ , \\ 0 & \text{if } j = i \ , \\ -1 & \text{if } j > i \ . \end{cases}$$

Then

$$\sum_{i=1}^{p+1} (-1)^i \frac{\partial D_i}{\partial x_i} = \sum_{i=1}^{p+1} (-1)^{i+j} \alpha_{ij} D_{ij} = 0,$$

because the $i = j$ terms are zero and, for $i > j$, the (i,j) terms cancel the (j,i) terms (since $\alpha_{ij} = -\alpha_{ji}$, $D_{ij} = D_{ji}$.

With the aid of Lemma (4.4), we can now show a very important property of the degree:

Proposition 4.6 *Let $\varphi \in C^2(\Omega, \mathbb{R}^n)$, and B be a connected component of $\mathbb{R}^n - \varphi(\partial\Omega)$. Then, for all $b \in B - \varphi(S)$, $d(\varphi, \Omega, b)$ is constant.*

Proof.

From Lemma (4.4), for all $\epsilon > 0$, $I_\epsilon(\varphi, \Omega, b)$ is constant on connected components of $\mathbb{R}^n - \varphi(\partial\Omega^\epsilon)$, where $\partial\Omega^\epsilon = \{x \in \mathbb{R}^n \ : \ \text{dist}(\text{x}, \partial\Omega) \leq \epsilon\}$. Taking $\epsilon \approx 0$, we conclude that $I_\epsilon(\varphi, \Omega, b)$ stays constant on all standard points, b, belonging to the same component of $\mathbb{R}^n - \varphi(\partial\Omega)$. Since, from proposition (4.3), $d(\varphi, \Omega, b) = I_\epsilon(\varphi, \Omega, b)$ for all standard $b \notin \varphi(\partial\Omega) \cup \varphi(S)$ and $\epsilon \approx 0$, the proof is finished.

The following result establishes the "smallness" of $\varphi(S)$.

Theorem 4.7 *(**Sard's Theorem**) If $\mathcal{O} \subset \mathbb{R}^n$ is open, $f \in C^1(\mathcal{O}, \mathbb{R}^n)$, and $S = \{x \in \mathcal{O} \ : \ J_f(x) = 0\}$. Then $f(S)$ has measure 0.*

For a proof, see [23].

We are now ready to extend the definition of degree to all $b \in \mathbb{R}^n - \varphi(\partial\Omega)$.

Step 1: Extension of $d(\varphi, \Omega, b)$ to all $b \in \mathbb{R}^n - \varphi(\partial\Omega)$

Given $b \in \mathbb{R}^n - \varphi(\partial\Omega)$, let B be the connected component of $\mathbb{R}^n - \varphi(\partial\Omega)$, which contains b. Since B is a connected component of an open set, it should also be open. By Sard's theorem, and since the measure of $\varphi(B)$ is greater than 0, there must exist $\beta \in B$ such that $\beta \notin \varphi(S)$. Define

$$d(\varphi, \Omega, b) = d(\varphi, \Omega, \beta).$$

By Proposition (4.6), this is well-defined.

To deal with the extension of the definition to all $\varphi \in C(\overline{\Omega}, \mathbb{R})$, we need the following result. This is a continuity property with respect to φ.

Proposition 4.8 *Let $\varphi \in C^2(\Omega, \mathbb{R}^n)$, $b \notin \varphi(\partial\Omega) \cup \varphi(S)$ (both standard). Then, for all $\psi \in {}^*C^2(\Omega, \mathbb{R}^n)$ such that $\|\varphi - \psi\|_{C^1(\Omega,\mathbb{R}^n)} \approx 0$, $d(\psi, \Omega, b) = d(\varphi, \Omega, b)$.*

Proof.

$$\begin{aligned} d(\psi, \Omega, b) - d(\varphi, \Omega, b) &= \int_{\overline{\Omega}} j_{b,\epsilon}(\psi(x)) J_\psi(x)\, dx - \int_{\overline{\Omega}} j_{b,\epsilon}(\varphi(x)) J_\varphi(x)\, dx \\ &= \int_{\overline{\Omega}} j_{b,\epsilon}(\psi(x))(J_\psi(x) - J_\varphi(x))\, dx \\ &\quad + \int_{\overline{\Omega}} \Big(j_{b,\epsilon}(\psi(x)) - j_{b,\epsilon}(\varphi(x)) \Big) J_\varphi(x)\, dx \end{aligned}$$

Since $\|\varphi - \psi\|_{C^1} \approx 0$, $\|J_\psi(\cdot) - J_\varphi(\cdot)\|_{C^0} \approx 0$, so the first integral is infinitesimal. Also, since $\|\varphi - \psi\|_{C^1} \approx 0$ and $j_{b,\epsilon}$ is continuous, we have that $j_{b,\epsilon}(\psi(x)) - j_{b,\epsilon}(\varphi(x)) \approx 0$. So the second integral is also infinitesimal. Hence, $d(\psi, \Omega, b) \approx d(\varphi, \Omega, b)$. Since both values are in ${}^*\mathbb{Z}$, we conclude that $d(\psi, \Omega, b) = d(\varphi, \Omega, b)$.

<u>Step 2:</u> Extension of $d(\varphi, \Omega, b)$ to all $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$

First, we find a lifting $\psi \in {}^*C^2(\Omega, \mathbb{R}^n)$ of φ (in the sense that $\forall x \in \Omega \;\; \psi(x) \approx \varphi(x)$) and define

$$d(\varphi, \Omega, b) = d(\psi, {}^*\Omega, b).$$

To show that this is well-defined we must show that the right-hand side is independent of lifting used and is finite.

Let $\hat{\psi}$ be another lifting and define a homotopy $H \in {}^*C^2([0,1] \times \overline{\Omega}, \mathbb{R}^n)$ by

$$H(t, x) = t\hat{\psi}(x) + (1 - t)\psi(x).$$

Since $H(t,x) \approx \psi(x) \approx \hat{\psi}(x) \approx \varphi(x)$ and $\mathrm{dist}(b, \varphi(\partial\Omega))$ is standard positive, we get that $b \neq H(t,x)$ for all $x \in \partial\Omega$. Hence $d(H(t,\cdot), {}^*\Omega, b)$ is well defined for all $t \in {}^*[0,1]$. Let $t_0 \in {}^*[0,1]$ be the *supremum of the set of $t \in {}^*[0,1]$ such that $d(H(0,\cdot), {}^*\Omega, b) = d(H(t,\cdot), {}^*\Omega, b)$. We assume $t_0 < 1$, and derive a contradiction. Let $t \in {}^*[0,1]$. Then:

$$\begin{aligned} H(t,x) - H(t_0,x) &= (t-t_0)\hat{\psi}(x) + (1-t-1+t_0)\psi(x) \\ &= (t-t_0)(\hat{\psi}(x) - \psi(x)). \end{aligned} \tag{4.5}$$

Choose a positive $\delta \approx 0$ so that $t_0 + \delta, t_0 - \delta \in {}^*(0,1]$ and $\delta \|\hat{\psi} - \psi\|_{C^1} \approx 0$. Then, say for $\bar{t} = t_0 + \frac{\delta}{2}$, and using (4.5):

$$\|H(\bar{t},.) - H(t_0,.)\|_{C^1} = |\bar{t} - t_0| \|\hat{\psi} - \psi\|_{C^1} = \frac{\delta}{2}\|\hat{\psi} - \psi\|_{C^1} \approx 0.$$

Hence from Proposition (4.8), $d(H(0,.), {}^*\Omega, b) = d(H(\bar{t},.), {}^*\Omega, b)$. Since $\bar{t} > t_0$ this gives us a contradiction.

It remains to show that $d(\varphi, \Omega, b)$ is finite. Given any arbitrarily small infinitely large $M \in {}^*\mathbb{R}$, we can find a $\psi_M \in {}^*C^2(\Omega, \mathbb{R}^n)$ lifting of φ such that $\|\psi_M\|_{C^1} \leq M$ (just take a convolution of φ, extended to $\mathbb{R}^n$, with a Gaussian with sufficiently large infinitesimal standard deviation). Then:

$$\begin{aligned} |d(\varphi, \Omega, b)| &= |d(\psi_M, \Omega, b)| \\ &\leq \int_\Omega |j_{b,\epsilon}(\psi_M(x)||J_{\psi_M}(x)|\, dx \\ &\leq M \int_\Omega |j_{b,\epsilon}(\psi_M(x)|\, dx \\ &\approx M \int_\Omega |j_{b,\epsilon}(\varphi(x))|\, dx \\ &= Mc. \end{aligned}$$

(where c is finite). Since this is true for any infinitely large M, the result follows.

Using the conclusions of step 1 and step 2, we can show the following:

Theorem 4.9 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$ and $b \in \mathbb{R}^n - \varphi(\partial\Omega)$. Let $\psi \in {}^*C^2(\Omega, \mathbb{R}^n)$ be a lifting of φ. Then, there exists $\beta \approx b$ such that $J_\varphi(x) \neq 0$, for all $x \in \psi^{-1}(\beta)$. Furthermore, $d(\psi, {}^*\Omega, \beta)$ is independent of the choice of ψ and β as above.*

Proof. Let $\epsilon \in \mathbb{R}^+$ be such that $B_\epsilon(b) \subset \mathbb{R}^n - \varphi(\partial\Omega)$; this means that $r \stackrel{\text{def}}{=} \text{dist}\,(B_\epsilon(b), \varphi(\partial\Omega)) \in \mathbb{R}^+$. Hence, since ψ is a lifting of φ:

$$\text{dist}\,(B_\epsilon(b), \varphi({}^*\partial\Omega)) > 0 \quad {}^2.$$

Therefore, $B_\epsilon(b) \subset {}^*\mathbb{R}^n - \psi({}^*\partial\Omega)$. Let C_b be the *connected component of ${}^*\mathbb{R}^n - \psi({}^*\partial\Omega)$ containing b. Since $B_\epsilon(b)$ is *connected and contains b, we conclude that $B_\epsilon(b) \subset C_b$. Hence, every $\beta \approx b$ is in C_b. Also, by the transfer of Sard's theorem, there exists some $\beta \approx b$ such that $J_\psi(x) \neq 0$ for all $x \in \psi^{-1}(\beta)$. By our results from step 1 and step 2, we can conclude that $d(\psi, {}^*\Omega, \beta)$ is independent of the choice of ψ and β satisfying the above conditions.

Definition 4.10 (Nonstandard Definition of Degree) *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$ and $b \in \mathbb{R}^n - \varphi(\partial\Omega)$. Then,*

$$d(\varphi, \Omega, b) = \sum_{x \in \psi^{-1}(\beta)} \text{sgn}\; J_\psi(x)$$

[2]We consider the *ball, ${}^*B_\epsilon(b) = \{x \in {}^*\mathbb{R}^n : |x - b| < \epsilon\}$ in ${}^*\mathbb{R}^n$. Since $B_\cdot(\cdot)$ is a function, we will use our convention and omit the star on B.

where $\psi \in {}^*C^2(\Omega, \mathbb{R}^n)$ *is a lifting of* φ *and* $\beta \in {}^*\Omega$ *with* ${}^0\beta = b$ *is such that* $J_\psi(x) \neq 0$ *for all* $x \in \psi^{-1}(\beta)$.

4.2 Basic Properties of Degree

Theorem 4.11 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded and open,* $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$, *and* $b \in \mathbb{R}^n - \varphi(\partial\Omega)$. *Then* $d(\varphi, \Omega, b)$ *is defined and possesses the following properties:*

1. **Normalization**:
$$d(\mathrm{id}, \Omega, b) = \begin{cases} 1 & \text{if } b \in \Omega\ , \\ 0 & \text{if } b \notin \Omega. \end{cases}$$

2. **Continuity with respect to** φ:
$$\exists \epsilon \in \mathbb{R}^+\ \forall \hat{\varphi} \in C(\overline{\Omega}, \mathbb{R}^n)\ \|\hat{\varphi} - \varphi\| < \epsilon\ \Rightarrow\ d(\hat{\varphi}, \Omega, b) = d(\varphi, \Omega, b).$$

3. **Homotopy Invariance**: *let* $H \in C(\overline{\Omega} \times [0,1], \mathbb{R}^n)$ *such that* $b \notin H(\partial\Omega \times [0,1])$. *Then,* $d(H(\cdot, t), \Omega, b)$ *is independent of* t.

4. **Continuity with respect to** b: *if* b *and* β *belong to the same connected component of* $\mathbb{R}^n - \varphi(\partial\Omega)$, *then:*
$$d(\varphi, \Omega, b) = d(\varphi, \Omega, \beta).$$

5. **Additivity**: *let* $\Omega = \Omega_1 \cup \Omega_2$, Ω_1, Ω_2 *open,* $\Omega_1 \cap \Omega_2 = \emptyset$. *If* $b \notin \varphi(\partial\Omega_1) \cup \varphi(\partial\Omega_2)$ *then:*
$$d(\varphi, \Omega, b) = d(\varphi, \Omega_1, b) + d(\varphi, \Omega_2, b).$$

6. **Excision**: *if* $K \subset \overline{\Omega}$ *is closed and* $b \notin \varphi(K)$, *then*

$$d(\varphi, \Omega, b) = d(\varphi, \Omega - K, b).$$

Proof.

1. Easy computation.

2. Let d_0 be the degree function defined for C^2 maps, and d its extension to the general case. Fix $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$, and consider the set:

$$E = \left\{ \epsilon \in {}^*\mathbb{R}^+ : \forall \psi \in {}^*C^2(\Omega, \mathbb{R}^n)\ \|\psi - \varphi\|_{C^0} < \epsilon \Rightarrow d_0(\psi, {}^*\Omega, b) = d(\varphi, {}^*\Omega, b) \right\}.$$

   This set is internal and contains all positive infinitesimals. Hence, it must contain some standard $\epsilon > 0$. Say $\|\hat{\varphi} - \varphi\|_{C^0} < \frac{\epsilon}{2}$. If $\hat{\psi}$ is a ${}^*C^2$ lifting of $\hat{\varphi}$, then:

$$\|\hat{\psi} - \varphi\|_{C^0} \leq \|\hat{\psi} - \hat{\varphi}\|_{C^0} + \|\hat{\varphi} - \varphi\|_{C^0} \lesssim \frac{\epsilon}{2} < \epsilon.$$

   Hence,

$$d(\hat{\varphi}, \Omega, b) = d_0(\hat{\psi}, {}^*\Omega, b) = d(\varphi, \Omega, b).$$

3. Use the continuity property 2. to show that the *supremum of the set of all $\overline{t} \in {}^*[0,1]$ such that for all $t \in {}^*[0, \overline{t}]$, $d(H(\cdot, 0), {}^*\Omega, b) = d(H(\cdot, t), {}^*\Omega, b)$ equals 1.

4. For $\varphi \in C^2(\Omega, \mathbb{R}^n)$, this has already been done. For the general case, let $\psi \in {}^*C^2(\Omega, \mathbb{R}^n)$ be a lifting of φ. Then:

$$d(\varphi, \Omega, b) = d(\psi, \Omega, b) = d(\psi, \Omega, \beta) = d(\varphi, \Omega, \beta).$$

5. Let ψ be a ${}^*C^2$ lifting of φ and $\beta \in {}^*\Omega$ such that ${}^\circ\beta = b$ and $J_\psi(x) \neq 0$ for all $x \in \psi^{-1}(\beta)$. Then:

$$\begin{aligned} d(\varphi, \Omega, b) &= d(\psi, \Omega, \beta) \\ &= \sum_{x \in \psi^{-1}(\beta)} \text{sgn } J_\psi(x) \\ &= \sum_{x \in \psi^{-1}(\beta) \cap \Omega_1} \text{sgn } J_\psi(x) + \sum_{x \in \psi^{-1}(\beta) \cap \Omega_2} \text{sgn } J_\psi(x) \\ &= d(\psi, \Omega_1, \beta) + d(\psi, \Omega_2, \beta) \\ &= d(\varphi, \Omega_1, b) + d(\varphi, \Omega_2, b). \end{aligned}$$

6. Let ψ be a ${}^*C^2$ lifting of φ and $\beta \in {}^*\Omega$ such that ${}^\circ\beta = b$ and $J_\psi(x) \neq 0$ for all $x \in \psi^{-1}(\beta)$. Note that $b \notin \varphi(K)$ and K compact implies that $\beta \notin \psi({}^*K)$. Hence:

$$\begin{aligned} d(\varphi, \Omega, b) &= d(\psi, \Omega, \beta) \\ &= \sum_{x \in \psi^{-1}(\beta)} \text{sgn } J_\psi(x) \\ &= \sum_{x \in \psi^{-1}(\beta) \cap {}^*(\Omega - K)} \text{sgn } J_\psi(x) \\ &= d(\psi, {}^*(\Omega - K), \beta) = d(\varphi, \Omega - K, b) \end{aligned}$$

It can be shown that there is a unique $\mathbb{Z}$ valued function satisfying the hypothesis and conclusions of Theorem (4.11). That function is called the Brouwer Degree. For a proof of this see [23]. We conclude that Definition (4.10) is an alternative way of defining the Brouwer Degree.

Proposition 4.12 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$ and $b \in \mathbb{R}^n - \varphi(\partial\Omega)$. Then, for all $c \in \mathbb{R}^n$:*

$$d(\varphi, \Omega, b) = d(\varphi - c, \Omega, b - c)$$

Proof.

Let $\Psi \in C^2(\overline{\Omega}, \mathbb{R}^n)$ be a lifting of φ and $\beta \in {}^*\Omega$ such that ${}^\circ\beta = b$ and $\beta \notin \varphi(S)$. Then, and since

$$x \in (\Psi - c)^{-1}(b - c) \Leftrightarrow \Psi(x) - c = b - c \Leftrightarrow \Psi(x) = b \Leftrightarrow x \in \Psi^{-1}(b),$$

and $J_\varphi = J_{\varphi - c}$, we get

$$\begin{aligned} d(\varphi - c, \Omega, b - c) &= \sum_{s \in (\varphi - c)^{-1}(b - c)} J_{\varphi - c}(x) \\ &= \sum_{s \in \varphi^{-1}(b)} J_\varphi(x) \\ &= d(\varphi, \Omega, b). \end{aligned}$$

4.3 Some Elementary Applications

In this section assume the hypothesis of Theorem (4.11).

Corollary 4.13 *If $b \notin \varphi(\overline{\Omega})$ then $d(\varphi, \Omega, b) = 0$.*

Proof.

Using excision, $d(\varphi, \Omega, b) = d(\varphi, \emptyset, b) = 0$.

Corollary 4.14 *If $d(\varphi, \Omega, b) \neq 0$ then $\exists \xi \in \Omega$ such that $\varphi(\xi) = b$.*

Corollary 4.15 *Let* $(\Omega_i)_{i\in I}$ *be a family of disjoint open subsets of* Ω *such that* $\varphi^{-1}(b) \subset \cup_{i\in I}\Omega_i$. *Then* $d(\varphi, \Omega_i, b) = 0$ *except for finitely many* i *and*

$$d(\varphi, \Omega, b) = \sum_{i\in I} d(\varphi, \Omega_i, b).$$

Proof.

$\varphi^{-1}(b) \subset \Omega$, is compact, so $\varphi^{-1}(b) \subset \Omega_{i_1} \cup \ldots \cup \Omega_{i_k}$. Hence, for all but finitely many i, $d(\varphi, \Omega_i, b) = 0$. Using excision and additivity:

$$d(\varphi, \Omega, b) = d(\varphi, \Omega_{i_1} \cup \ldots \cup \Omega_{i_k}, b) = \sum_{j=1}^{k} d(\varphi, \Omega_{i_j}, b) = \sum_{i\in I} d(\varphi, \Omega_i, b)$$

Corollary 4.16 *If* $\psi \in C(\overline{\Omega}, \mathbb{R}^n)$ *and* $\psi = \varphi$ *on* $\partial\Omega$ *then*

$$d(\psi, \Omega, b) = d(\varphi, \Omega, b).$$

Proof.

Since $\psi = \varphi$ on $\partial\Omega$, $b \notin \varphi(\partial\Omega) = \psi(\partial\Omega)$. Hence, $d(\psi, \Omega, b)$ is well defined. Let

$$H(x, t) = t\psi(x) + (1 - t)\varphi(x)$$

For all $x \in \partial\Omega$, $H(x, t) = t\varphi(x) + (1 - t)\varphi(x) = \varphi(x) \neq b$. Hence $d(H(t, .), \Omega, b)$ is defined for all t and by the homotopy invariance property, it is constant. So:

$$d(\psi, \Omega, b) = d(H(1, .), \Omega, b) = d(H(0, .), \Omega, b) = d(\varphi, \Omega, b).$$

Corollary 4.17 *There is no* $f \in C(\overline{B_1(0)}, \partial B_1(0))$ *such that* $f|_{\partial B_1(0))} = \mathrm{id}$ *(that is, there does not exist a (continuous) retraction from* $B_1(0)$ *to* $\partial B_1(0)$*)*

Proof.

Suppose there exist such mapping. By the previous corollary:

$$d(f, B_1(0), 0) = d(\mathrm{id}, B_1(0), 0) = 1.$$

Hence, by Corollary (4.14), $\exists \xi \in B_1(0) \ : \ f(\xi) = 0$. But by hypothesis, the range of f is a subset of $\partial B_1(0)$, and $f|_{\partial B_1(0)} = \mathrm{id}$. Hence, such ξ cannot exist.

Corollary 4.18 *If $f \in C(\overline{B_1(0)}, \overline{B_1(0)})$ then f has a fixed point.*

Proof.

Suppose $f(x) \neq x$, $\forall x \in \overline{B_1(0)}$. We derive a contradiction. For $t \in [0, 1)$, $tf(x) \in B_1(0)$. Hence, for all $x \in \partial B_1(0)$, and $t \in [0, 1)$, $x - tf(x) \neq 0$. Also, for $x \in \partial B_1(0)$ and $t = 1$, $x - tf(x) = x - f(x) \neq 0$, by our hypothesis. By the homotopy invariance property:

$$d(x - tf(x), B_1(0), 0) \equiv \text{const}.$$

So

$$d(\mathrm{id}, B_1(0), 0) = 1 = d(x - f(x), B_1(0), 0).$$

Hence, by (4.14), there exists ξ such that $\xi - f(\xi) = 0$ and this contradicts our assumption.

Chapter 5

Degree Theory in Nonstandard Hulls of Hyperfinite Dimensional Banach Spaces

Hyperfinite dimensional Banach spaces occur in situations where we want to study the behavior "in the limit" of some class of discrete problems. For example, we may be interested in studying if some class of finite difference schemes converge to the solution of a differential equation, as the increments on the independent variables approach zero.

For its strong properties, a notion of degree for nonstandard hulls of hyperfinite dimensional Banach spaces may be very useful.

5.1 The Degree in Finite Dimensional Banach Spaces

We start by showing that the Brouwer degree can be defined in any finite dimensional normed space.

Let V be an n-dimensional normed space. We can identify V with $\mathbb{R}^n$ in the usual manner. That is, we fix a basis $\{v_i\}_{i=1,\ldots,n}$ for V and identify every $v = \alpha_1 v_1 + \ldots + \alpha_n v_n \in V$ with $(\alpha_1, \ldots, \alpha_n)$. Since all norms in $\mathbb{R}^n$ are equivalent, the representation map preserves point-set topological properties of sets, and continuity and differentiability of functions [1]. So we can define $d(\varphi, \Omega, b)$ for any open and bounded $\Omega \subset V$, $\varphi \in C(\Omega, V)$ and $b \in V - \varphi(\partial\Omega)$ by computing the degree relative to the corresponding $\mathbb{R}^n$ representations.

However, if we use another basis, $\{\tilde{v}_i\}_{i=1,\ldots,n}$, we get a different representation. If $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ are representations corresponding to different basis, then $\tilde{x} = Mx$, where M is non-singular, and, assuming our identification relative to $\{v_i\}_{i=1,\ldots,n}$.

	representation relative to $\{v_i\}_{i=1,\ldots,n}$	representation relative to $\{\tilde{v}_i\}_{i=1,\ldots,n}$
Sets	Ω	$\tilde{\Omega} = \{Mx \ : \ x \in \Omega\}$
Maps	$\varphi : \Omega \to \mathbb{R}^n$	$\tilde{\varphi} : \tilde{\Omega} \to \mathbb{R}^n$ $\tilde{\varphi}(\tilde{x}) = M\varphi(M^{-1}\tilde{x})$

Then, if $\tilde{d}$ is the degree relative to the basis $\{\tilde{v}_i\}_{i=1,\ldots,n}$ (computed using the "tilde" representations, i.e., $\tilde{d}(\varphi, \Omega, b) = d(\tilde{\varphi}, \tilde{\Omega}, \tilde{b})$), what is the relation between d and $\tilde{d}$?

Lemma 5.1 *Let V be an n-dimensional normed space, and $\Omega \subset V$ be open and bounded. Let $\varphi \in C(\overline{\Omega}, V)$, and suppose $b \in V - \varphi(\partial\Omega)$. Let $\{v_i\}_{i=1,\ldots,n}$ and $\{\tilde{v}_i\}_{i=1,\ldots,n}$ be two bases of V, and identify V with its $\mathbb{R}^n$ representation relative to*

[1] Interpret the derivative of $\varphi : \Omega \subset V \to V$ at x as a Frèchet derivative, i.e., a bounded linear map L_x such that $\forall \epsilon > 0\ \exists \delta > 0\ \forall y \in V\ \|y\|_V < \delta \Rightarrow \|f(x+y) - f(x) - L_x y\|_V < \epsilon \|y\|_V$.

$\{v_i\}_{i=1,\dots,n}$. If $\tilde{d}$ is the degree relative to $\{\tilde{v}_i\}_{i=1,\dots,n}$, then

$$\tilde{d}(\varphi,\Omega,b)=d(\varphi,\Omega,b).$$

Proof.

Let $\varphi\in C(\overline{\Omega},V)$, and suppose $b\notin\varphi(\partial\Omega)$ (so $d(\varphi,\Omega,b)$ is well defined). Note that $\tilde{\varphi}(\tilde{x})=M\varphi(M^{-1}\tilde{x})$, where M is a constant matrix. Then

$$\tilde{b}=\tilde{\varphi}(\tilde{x}) \iff Mb=M\varphi(M^{-1}\tilde{x}) \iff b=\varphi(x), \tag{5.1}$$

so $\tilde{b}\notin\tilde{\varphi}(\partial\tilde{\Omega})$. We conclude that $\tilde{d}(\varphi,\Omega,b)$ is well-defined. By the chain rule, $\varphi\in C^k(A,\mathbb{R}^n)$ iff $\tilde{\varphi}\in C^k(\tilde{\Omega},\mathbb{R}^n)$.

To show the equality, we first consider the nice case, i.e., $\varphi:\overline{\Omega}\to\mathbb{R}^n$ such that $\varphi\in C^2(\Omega,\mathbb{R}^n)$ and $b\notin\varphi(\partial\Omega)\cup\varphi(S)$.

By the chain rule

$$\tilde{\varphi}'(\tilde{x})=M\varphi'(M^{-1}\tilde{x})M^{-1}=M\varphi'(x)M^{-1}.$$

Hence:

$$J_{\tilde{\varphi}}(\tilde{x})=(\det M)J_{\varphi}(x)(\det M^{-1})=J_{\varphi}(x). \tag{5.2}$$

In particular, from (5.1) and (5.2)

$$\begin{aligned} b\in\varphi(S) &\iff \exists x\in\varphi^{-1}(b)\ J_{\varphi}(x)=0\\ &\iff \exists\tilde{x}\in\tilde{\varphi}^{-1}(\tilde{b})\ J_{\tilde{\varphi}}(\tilde{x})=J_{\varphi}(x)=0\\ &\iff \tilde{b}\in\tilde{\varphi}(\tilde{S}). \end{aligned}$$

Therefore, $\tilde{d}$ can be computed by:

$$\tilde{d}(\varphi,\Omega,b) \;=\; d(\tilde{\varphi},\tilde{\Omega},\tilde{b})$$

$$
\begin{aligned}
&= \sum_{\tilde{x}\in\tilde{\varphi}^{-1}(\tilde{b})} sgn\ J_{\tilde{\varphi}}(x) \\
&= \sum_{x\in\varphi^{-1}(b)} sgn\ J_{\varphi}(x) \\
&= d(\varphi, \Omega, b).
\end{aligned}
$$

The general case follows easily from lifting φ and taking $\beta \approx b$ such that $\beta \notin \varphi(S)$.

5.2 The Definition of Degree

We now turn our attention to hyperfinite dimensional Banach spaces and their non-standard hulls. Let $(\mathbb{F}, \|\cdot\|)$ be an internal N-dimensional *Banach space, where $N \in {}^*\mathbb{N}$ is infinite.

We briefly review the nonstandard hull construction. Recall that $^*V(\mathbb{R})$ is transitive. So, since $\mathbb{F} \in {}^*V(\mathbb{R})$, then every $\alpha \in \mathbb{F}$ is internal [2]. The galaxy of $\alpha \in \mathbb{F}$ is the subset of $\mathbb{F}$ given by:

$$\mathrm{Gal}_{\mathbb{F}}(\alpha) = \Big\{\beta \in \mathbb{F} : \|\beta - \alpha\| < \infty\Big\}.$$

Here, for each $t \in {}^*\mathbb{R}$, "$t < \infty$" is an abbreviation for $\exists n \in \mathbb{N} : t \leq n$, i.e., t is not an infinitely large positive hyperreal [3]. For each $\alpha \in \mathbb{F}$, let:

$$^\circ\alpha = \Big\{\beta \in \mathbb{F} : \|\beta - \alpha\| \approx 0\Big\}.$$

The nonstandard hull of $\mathbb{F}$ is the vector space

$$\mathrm{Hull}(\mathbb{F}) = \Big\{{}^\circ\alpha : \alpha \in \mathrm{Gal}_{\mathbb{F}}(0)\Big\},$$

[2] More generally, this is true for $\mathbb{F} \in {}^*V(X)$, where X is any base set. However, in our applications, we will only need $^*V(\mathbb{R})$.

[3] Similarly, one can introduce the abbreviation "$t > -\infty$" for $\exists n \in \mathbb{N} : t \geq -n$.

endowed with the norm:

$$\|{}^\circ\alpha\|_{\mathrm{Hull}(\mathbb{F})} = \text{ st } \|\alpha\|.$$

This pair is a Banach space. If $A \subset \mathbb{F}$, let

$${}^\circ A = \Big\{{}^\circ\alpha : \alpha \in A \cap \mathrm{Gal}_{\mathbb{F}}(0)\Big\}.$$

To improve the readability of this section, we make the following conventions. The first three roman capital letters will denote internal subsets of $\mathbb{F}$. It would be quite cumbersome to write $\overset{({}^*-)}{A}$ or $({}^*\partial)A$, so we will drop the stars on the symbols of *topological operators. This is done with the understanding that, whenever a topological operator acts on an internal set, it means its star. For example:

$$\overline{A} \quad \text{means} \quad \overset{({}^*-)}{A},$$

the *closure of A. From the context, it will always be possible to assert the meaning of the topological operators.

If $A \subset \mathbb{F}$ is internal and *open, then for every $\Phi \in {}^*C(\overline{A}, \mathbb{F})$, and $\beta \in \mathbb{F} - \Phi(\partial A)$, $d(\Phi, A, \beta)$ is defined, and gives us an element of ${}^*\mathbb{Z}$.

Here is our setup:

1°) $\mathcal{E} = \mathrm{Hull}(\mathbb{F})$, where $\mathbb{F}$ is an N-dimensional internal Banach space, with $N \in {}^*\mathbb{N}$ infinite.

2°) $\overline{\Omega} = {}^\circ A \subset \mathcal{E}$, where $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ is internal, *open and $\partial\Omega = {}^\circ\partial A$ [4].

[4] $\partial\Omega = {}^\circ\partial A$ is necessary because a *open set may be quite nasty. In particular, we may have ${}^\circ\partial A = {}^\circ A$. We need to ensure that 4° implies $\beta \notin \Phi(\partial A)$, and this will be false if ∂A does not closely match $\partial\Omega$.

3°) $\varphi : \overline{\Omega} \to \mathcal{E}$ is neocontinuous. This means that there exists an internal and S-continuous $\Phi : \overline{A} \to \mathbb{F}$ such that

$$\forall \alpha \in \overline{A}\ \varphi(^\circ\alpha) = {}^\circ(\Phi(\alpha)).$$

We call such Φ a lifting of φ.

4°) $b \in \mathcal{E} - \varphi(\partial\Omega)$. Let $\beta \in \mathbb{F}$ be such that $b = {}^\circ\beta$.

The notion of a neocontinuous map was introduced by Fajardo and Keisler [8] in the more general setting of nonstandard hulls of metric spaces. It is a stronger property than continuity, but it is weaker than compactness (of maps). Despite this, neocontinuous maps retain some of the nice properties of compact maps. Now let us take a closer look at the sets involved in 2°. Define:

$$SB(\mathbb{F}) = \Big\{ A \subset \mathbb{F}\ :\ \text{“}A \text{ is internal”} \ \text{ and }\ \partial(^\circ A) = {}^\circ\partial A \Big\}.$$

Then the sets $\overline{\Omega} \subset \mathcal{E}$ satisfying 2° are of the form $\overline{\Omega} = {}^\circ A$, where $A \in SB(\mathbb{F})$ is *open and $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$.

Remark 5.2 *Let $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ be internal. The internal set*

$$R = \Big\{ r \in {}^*\mathbb{R}^+ : \forall \alpha \in A\ \|\alpha\| < r \Big\}$$

contains all infinitely large positive hyperreals. By overspill, it must contain some $r \in \mathbb{R}^+$. Hence, there exists $r \in \mathbb{R}^+$ such that $\|\alpha\| < r$, for all $\alpha \in A$.

Lemma 5.3 (Characterization of $SB(\mathbb{F})$) *Let $A \in \mathbb{F}$ be internal. Then $A \subset SB(\mathbb{F})$ iff*

$$\forall \alpha \in A \cap \mathrm{Gal}_{\mathbb{F}}(0)\ \Big(^\circ\alpha \in \operatorname{int}{}^\circ A\ \Rightarrow\ \exists \delta \in \mathbb{R}^+\ B_\delta(\alpha) \subset A \Big). \tag{5.3}$$

Proof.

We first show the condition is necessary. Let $A \in SB(\mathbb{F})$. Pick $\alpha \in A \cap \mathrm{Gal}_{\mathbb{F}}(0)$, with ${}^\circ\alpha \in \operatorname{int} {}^\circ A$. There exists $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon({}^\circ\alpha) \subset {}^\circ A$. Therefore:

$$ {}^\circ B_{\epsilon/2}(\alpha) \subset \operatorname{int} {}^\circ A \subset {}^\circ A. $$

In particular, there are no boundary points of ${}^\circ A$ in ${}^\circ B_{\epsilon/2}(\alpha)$. Hence, since $\partial({}^\circ A) = {}^\circ\partial A$

$$ \begin{aligned} {}^\circ B_{\epsilon/2}(\alpha) \cap \partial({}^\circ A) = \emptyset \iff & \; {}^\circ B_{\epsilon/2}(\alpha) \cap {}^\circ\partial A = \emptyset \\ \implies & \; B_{\epsilon/4}(\alpha) \cap \partial A = \emptyset. \end{aligned} $$

Therefore, either $B_{\epsilon/4}(\alpha) \subset A$ or $B_{\epsilon/4}(\alpha) \subset A^{\mathrm{c}}$. But $\alpha \in A$, so we must have $B_{\epsilon/4}(\alpha) \subset A$.

Now, let us show that condition (5.3) is sufficient. Let $A \subset \mathbb{F}$ be internal, and assume $A \notin SB(\mathbb{F})$. Then $\partial({}^\circ A) \neq {}^\circ\partial A$. Consider $\alpha \in \mathbb{F}$ such that ${}^\circ\alpha \in \partial({}^\circ A)$. For all $\epsilon \in \mathbb{R}^+$, $B_\epsilon(\alpha) \cap A \neq \emptyset$ and $B_\epsilon(\alpha) \cap A^{\mathrm{c}} \neq \emptyset$. But then $B_\epsilon(\alpha) \cap \partial A \neq \emptyset$, which means that ${}^\circ\alpha \in {}^\circ\partial A$. Hence, $\partial({}^\circ A) \subset {}^\circ\partial A$, so $\partial({}^\circ A) \subsetneqq {}^\circ\partial A$, i.e., there exists $\alpha \in \partial A$ such that ${}^\circ\alpha \in \operatorname{int} {}^\circ A$. Since $\alpha \in \partial A$, $B_\delta(\alpha) \not\subset A$ for all $\delta > 0$. Choose $\tilde{\alpha} \in A$ such that $\tilde{\alpha} \approx \alpha$. Then ${}^\circ\tilde{\alpha} = {}^\circ\alpha \in \operatorname{int} {}^\circ A$ but $\forall \delta \in \mathbb{R}^+ \; B_\delta(\tilde{\alpha}) \not\subset A$. Hence, condition (5.3) fails.

Proposition 5.4

(1) $\emptyset, \mathbb{F} \in SB(\mathbb{F})$

(2) $\forall \alpha \in \mathrm{Gal}_{\mathbb{F}}(0)\; \forall \delta > 0 \;\; B_\delta(\alpha) \in SB(\mathbb{F})$

Proof.

(*1*) is obvious and (*2*) is an easy consequence of Lemma (5.3).

Lemma 5.5 *Let* $\mathbb{F}$ *be hyperfinite dimensional and* $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ *be* *open *and internal. Let* $\Phi : \overline{A} \to \mathbb{F}$ *be S-continuous, with* $\Phi(\overline{A}) \subset \mathrm{Gal}_{\mathbb{F}}(0)$*. Then, there exists an S-continuous and* *continuous *map* $\Psi : \overline{A} \to \mathbb{F}$ *such that* $^\circ\Phi = {}^\circ\Psi$*.*

Proof.

The set $\overline{A}$ is *compact. Let $\epsilon > 0$, $\epsilon \approx 0$, and consider the *open covering of $\overline{A}$, $\{B_\epsilon(\alpha) : \alpha \in \overline{A}\}$. By *compactness, there exist $\alpha_1, \ldots, \alpha_M \in K$ (with $M \in {}^*\mathbb{N}$) such that:

$$\overline{A} \subset \bigcup_{i=1}^{M} B_\epsilon(\alpha_i). \tag{5.4}$$

Define $w_i : \mathbb{F} \to \mathbb{F}$ by:

$$w_i(\alpha) = \max\{0, \epsilon - \|\alpha - \alpha_i\|\}, \qquad \text{for } i = 1, \ldots, M.$$

Note that each w_i is *continuous, has support equal to $B_\epsilon(\alpha_i)$, and for all $\alpha \in \mathbb{F}$, $0 \le w_i(\alpha) \le \epsilon$. Now, let:

$$\Psi(\alpha) = \frac{\sum_{i=1}^{M} w_i(\alpha)\Phi(\alpha_i)}{\sum_{i=1}^{M} w_i(\alpha)} = \sum_{i=1}^{M} \frac{w_i(\alpha)\Phi(\alpha_i)}{\sum_{j=1}^{M} w_j(\alpha)}$$

By inclusion (5.4), for all $\alpha \in \overline{A}$, $\sum_{i=1}^{M} w_i(\alpha) \neq 0$, so Ψ is well-defined in $\overline{A}$, and $\frac{w_i(\cdot)\Phi(\alpha_i)}{\sum_{j=1}^{M} w_j(\cdot)}$ is *continuous. So Ψ is a *finite sum of *continuous functions and so it is *continuous. We also have:

$$\begin{aligned}\Psi(\alpha)-\Phi(\alpha) &= \frac{\sum_{i=1}^{M} w_i(\alpha)\,\Phi(\alpha_i)}{\sum_{i=1}^{M} w_i(\alpha)} - \frac{\sum_{i=1}^{M} w_i(\alpha)}{\sum_{i=1}^{M} w_i(\alpha)}\Phi(\alpha) \\ &= \frac{\sum_{i=1}^{M} w_i(\alpha)\Big(\Phi(\alpha_i)-\Phi(\alpha)\Big)}{\sum_{i=1}^{M} w_i(\alpha)}.\end{aligned}$$

Therefore:

$$\|\Psi(\alpha)-\Phi(\alpha)\| \leq \frac{\sum_{i=1}^{M} w_i(\alpha)\,\|\Phi(\alpha_i)-\Phi(\alpha)\|}{\sum_{i=1}^{M} w_i(\alpha)}.$$

Let $r \in \mathbb{R}^+$. Whenever $w_i(\alpha) \neq 0$, i.e., $\alpha \in B_\epsilon(\alpha_i)$, we have, just by S-continuity of Φ, that $\|\Phi(\alpha_i)-\Phi(\alpha)\| < r$. Hence:

$$\|\Psi(\alpha)-\Phi(\alpha)\| \leq \frac{\sum_{i=1}^{M} w_i(\alpha)\|\Phi(\alpha_i)-\Phi(\alpha)\|}{\sum_{i=1}^{M} w_i(\alpha)} \leq \frac{\sum_{i=1}^{M} w_i(\alpha) r}{\sum_{i=1}^{M} w_i(\alpha)} = r.$$

Since this is true for arbitrarily small $r \in \mathbb{R}^+$, we conclude that for all $x \in \overline{A}$, $\|\Psi(\alpha)-\Phi(\alpha)\| \approx 0$.

The preceding Lemma insures us that, without loss of generality, we can take the lifting of φ satisfying 3°) to be a *continuous function. We are now ready to give the definition of degree.

Definition 5.6 *Let* $A \in SB(\mathbb{F})$ *be* *open, *with* $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$, *and* $\Omega = \mathrm{int}^\circ A$. *Let* $\varphi : \overline{\Omega} \to \mathcal{E}$ *be neocontinuous with lifting* $\Phi : \overline{A} \to \mathbb{F}$. *Furthermore, let* $b \in \mathcal{E} - \varphi(\partial\Omega)$

and $\beta \in \mathbb{F}$ *such that* ${}^\circ\beta = b$. *Then:*

$$d(\varphi, \Omega, b) = d(\Phi, A, \beta). \tag{5.5}$$

To show that the degree on the right-hand side of (5.5) exists, we must show that $\beta \notin \Phi(\partial A)$. The proof goes as follows. Since $\partial\Omega = {}^\circ\partial A$, and $\partial A \subset \mathrm{Gal}_{\mathbb{F}}(0)$:

$$\varphi(\partial\Omega) = \varphi({}^\circ\partial A) = \left\{ {}^\circ(\Phi(\alpha)) \ : \ \alpha \in \partial A \right\} = {}^\circ(\Phi(\partial A)).$$

Hence $b \notin \varphi(\partial\Omega)$ implies that:

$$\forall \alpha \in \partial A \ \ \|\Phi(\alpha) - \beta\| \not\approx 0. \tag{5.6}$$

In particular, $\beta \notin \Phi(\partial A)$. We observe that, by Lemma (5.1), the degree on the right-hand side of (5.5) is independent of the basis of $\mathbb{F}$ chosen to get the $\mathbb{R}^N$ representation of $\mathbb{F}$, and the actual computation of the degree does not involve the norm of $\mathbb{F}$. This norm, however plays an important role in establishing the class of subsets of $\mathcal{E}$ where the degree can be defined.

To show that $d(\varphi, \Omega, b)$ is well defined, we must now prove that the right-hand side of (5.5) is independent of the choice of A, Φ and β satisfying our assumptions.

We begin by showing that the right hand-side of (5.5) is independent of the choice of A and $\Phi : \overline{A} \to \mathbb{F}$. Consider $A_1, A_2 \in SB(\mathbb{F})$ such that $\overline{\Omega} = {}^\circ A_1 = {}^\circ A_2$ and $\Phi_i : \overline{A}_i \to \mathbb{F}$, for $i = 1, 2$, both S-continuous and such that, for all $\alpha \in \overline{A}_i$, $\varphi({}^\circ\alpha) = {}^\circ(\Phi_i(\alpha))$, for $i = 1, 2$.

Lemma 5.7 *Under the above conditions:*

(1) $\overline{\Omega} = {}^\circ(A_1 \cap A_2)$.

(2) $^\circ\left(\overline{A}_i - (A_1 \cap A_2)\right) = \partial\Omega$, *for* $i = 1, 2$.

Proof.

(1) To prove the non-trivial inclusion, we note that, since $\overline{\Omega}$ is closed, it is enough to show that:

$$\Omega = \operatorname{int}\overline{\Omega} = \overline{\Omega} - \partial\Omega \subset {}^\circ(A_1 \cap A_2).$$

So, let $a \in \Omega$ and $\alpha_1 \in A_1$ such that $^\circ\alpha_1 = a$. By Lemma (5.3), there exists $\delta \in \mathbb{R}^+$ such that $B_\delta(\alpha_1) \subset A_1$. Say $\alpha_2 \in A_2$ is such that $^\circ\alpha_2 = a$; then $\|\alpha_1 - \alpha_2\| \approx 0$, so $\alpha_2 \in A_1 \cap A_2$ and $a \in {}^\circ(A_1 \cap A_2)$.

(2) We prove the equality for $i = 1$. (For $i = 2$, just switch A_2 with A_1 in the $i = 1$ case). One inclusion is easily established:

$$\partial\Omega = {}^\circ\partial A_1 = {}^\circ(\overline{A}_1 - A_1) \subset {}^\circ\left(\overline{A}_1 - (A_1 \cap A_2)\right).$$

For the other inclusion, consider $\alpha \in \overline{A}_1 - (A_1 \cap A_2)$. If $\alpha \in \partial A_1$, we are done, so we may assume that $\alpha \in A_1$. Then:

$$\alpha \in A_1 - (A_1 \cap A_2) = A_1 - A_2. \tag{5.7}$$

We want to show that $^\circ\alpha \in \partial\Omega$. Assume the opposite. Since $\alpha \in A_1$, $^\circ\alpha \in \overline{\Omega}$, so by our assumption, $^\circ\alpha \in \operatorname{int}\Omega$. Let $\tilde{\alpha} \in A_2$ be such that $^\circ\tilde{\alpha} = {}^\circ\alpha$. By Lemma (5.3), there exists $\delta \in \mathbb{R}^+$ such that $B_\delta(\tilde{\alpha}) \subset A_2$. Since $\|\alpha - \tilde{\alpha}\| \approx 0$, we conclude that $\alpha \in A_2$. Hence $\alpha \notin A_1 - A_2$, Thus contradicting equation (5.7). So, we must have $^\circ\alpha \in \partial\Omega$.

Consequently, since $b \notin \varphi(\partial\Omega)$ and

$$\partial\Omega = {}^\circ\left(\overline{A}_i - (A_1 \cap A_2)\right), \quad i = 1, 2,$$

given any $\alpha \in \overline{A}_i - (A_1 \cap A_2)$, with $i = 1, 2$, we have:

$$b \neq \varphi({}^\circ\alpha) = {}^\circ(\Phi_i(\alpha)).$$

Thus, $|\Phi_i(\alpha) - \beta| \not\approx 0$. Therefore:

$$\beta \notin \Phi_i(\overline{A}_i - (A_1 \cap A_2)).$$

Hence, using the transfer of the excision property:

$$d(\Phi_i, A_i, \beta) = d(\Phi_i, A_1 \cap A_2, \beta).$$

To conclude this argument, we still have to show that:

$$d(\Phi_1, A_1 \cap A_2, \beta) = d(\Phi_2, A_1 \cap A_2, \beta).$$

This follows from the next lemma.

Lemma 5.8 *Let Φ, $\Psi : \overline{A} \to \mathbb{F}$ be two liftings of φ. Then:*

$$d(\Phi, A, \beta) = d(\Psi, A, \beta).$$

Proof.

The transfer of the continuity of the Brouwer degree with respect to Φ reads:

$$\exists \epsilon > 0 \;\; \forall \Psi \in {}^*C(\overline{A}, \mathbb{F}) \;\; \|\Phi - \Psi\|_0 < \epsilon \Rightarrow d(\Phi, A, \beta) = d(\Psi, A, \beta), \tag{5.8}$$

where $\|\Phi\|_0 = {}^*\sup_{x \in \overline{A}} \|\Phi(x)\|$. Note that the ϵ in (5.8) may be infinitesimal. Consider a *continuous homotopy $H : {}^*[0,1] \times \overline{A} \to \mathbb{F}$ given by:

$$H(t, x) = t\Psi(x) + (1 - t)\Phi(x).$$

Since $H(t,x) \approx \Phi(x) \approx \Psi(x)$, if there exists $t \in {}^*[0,1]$ and $x \in \partial A$ such that $H(t,x) = \beta$, then $\Phi(x) \approx \beta$. That contradicts the fact that $b \in \varphi(\partial\Omega)$. Hence, $d(H(t,\cdot),A,\beta)$ is defined for all $t \in {}^*[0,1]$.

Let $t_0 \in {}^*[0,1]$ be the *supremum of the set of all $t \in {}^*[0,1]$ such that $d(H(0,\cdot),A,\beta) = d(H(t,\cdot),A,\beta)$. We assume that $t_0 < 1$, and derive a contradiction. Let $t \in {}^*[0,1]$. Then:

$$\begin{aligned} H(t,x) - H(t_0,x) &= (t-t_0)\Psi(x) + (1-t-1+t_0)\Phi(x) \\ &= (t-t_0)\Big(\Psi(x) - \Phi(x)\Big). \end{aligned}$$

Choose a positive $\delta \approx 0$ so that $t_0+\delta, t_0-\delta \in {}^*[0,1]$ and $\delta\|\Psi-\Phi\|_0 \le \epsilon$. Then, letting $\bar{t} = t_0 + \frac{\delta}{2}$:

$$\begin{aligned} |H(\bar{t},\cdot) - H(t_0,\cdot)\|_0 &= |\bar{t} - t_0|\ \|\Psi - \Phi\|_0 \\ &= \frac{\delta}{2}\ \|\Psi-\Phi\|_0 \\ &\le \frac{\epsilon}{2} \ < \ \epsilon. \end{aligned}$$

Hence, by (5.8), we have that $d(H(0,\cdot),A,\beta) = d(H(\bar{t},\cdot),A,\beta)$. Since $\bar{t} > t_0$, this gives a contradiction.

This ends the proof that the right-hand side of (5.5) is independent of the choice of A and Φ. It remains to show that it is independent of the choice of β.

Let $\hat{\beta}$ be such that ${}^\circ\hat{\beta} = {}^\circ\beta = b$, which means that $\|\hat{\beta} - \beta\| \approx 0$. To show that $d(\Phi,A,\beta) = d(\Phi,A,\hat{\beta})$ it is necessary to show that β and $\hat{\beta}$ belong to the same *connected component of $\mathbb{F} - \Phi(\partial A)$. Recall equation (5.6):

$$\forall \alpha \in \partial A \ \ \|\Phi(\alpha) - \beta\| \not\approx 0. \tag{5.9}$$

Let $\epsilon = 2|\hat{\beta} - \beta| \approx 0$. By (5.9), $B_\epsilon(\beta) \cap \Phi(\partial A) = \emptyset$. Therefore, $B_\epsilon(\beta)$ is contained in a *connected component of $\mathbb{F} - \Phi(\partial A)$, Since $\hat{\beta} \in B_\epsilon(\beta)$ the result follows.

5.3 Basic Properties

Let $\Omega = \operatorname{int}{}^\circ A$, with $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$. The set of neocontinuous maps $\varphi : \overline{\Omega} \to \mathcal{E}$ forms a linear space, $nC(\overline{\Omega})$. For each $\varphi \in nC(\overline{\Omega})$, let:

$$\|\varphi\|_{nC(\overline{\Omega})} = \mathrm{st}\left({}^*\sup_{\alpha \in \overline{A}} \|\Phi(\alpha)\|\right) = \mathrm{st}\,\|\Phi\|_0,$$

where $\Phi : \overline{A} \to \mathbb{F}$ is a lifting of φ. Note that, since φ is well-defined with range in $\mathcal{E}$, $\|\Phi(\alpha)\| < \infty$, $\forall \alpha \in \overline{A}$. Hence $\|\Phi\|_0 < \infty$, which ensures that $\|\cdot\|_{nC(\overline{\Omega})}$ is a well-defined norm on $nC(\overline{\Omega})$, with range in $\mathbb{R}^+ \cup \{0\}$. This norm will be needed to state one of the basic properties of our degree.

Theorem 5.9 *Let $A \in SB(\mathbb{F})$ be *open, with $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$, and $\Omega = \operatorname{int}{}^\circ A$. Let $\varphi : \overline{\Omega} \to \mathcal{E}$ be neocontinuous and let $b \in \mathcal{E} - \varphi(\partial\Omega)$. Then $d(\varphi, \Omega, b)$ is defined and possesses the following properties:*

(1) **Normalization***:*

$$d(\mathrm{id}, \Omega, b) = \begin{cases} 1 & \text{if } b \in \Omega, \\ 0 & \text{if } b \notin \Omega. \end{cases}$$

(2) **Continuity with respect to** φ*: There exists $\epsilon \in \mathbb{R}^+$ such that, for all neocontinuous $\psi : \overline{\Omega} \to \mathcal{E}$ satisfying $\|\psi - \varphi\|_{nC(\overline{\Omega}} < \epsilon$ and $b \notin \psi(\partial\Omega)$,*

$$d(\psi, \Omega, b) = d(\varphi, \Omega, b).$$

(3) **Continuity with respect to** *b: If b and $\overline{b}$ belong to the same connected component of $\mathcal{E} - \varphi(\partial\Omega)$, then*

$$d(\varphi, \Omega, b) = d(\varphi, \Omega, \overline{b}).$$

(4) **Homotopy invariance***: Let $h : \overline{\Omega} \times [0,1] \to \mathcal{E}$ be neocontinuous and such that $b \notin h(\partial\Omega \times [0,1])$. Then $d(h(\cdot,t), \Omega, b)$ is independent of t.*

(5) **Additivity-excision***: Let $\Omega_1 = \operatorname{int}{}^\circ A_1$, $\Omega_2 = \operatorname{int}{}^\circ A_2$ where $A_1, A_2 \in SB(\mathbb{F})$ are* *open *and $A_1, A_2 \subset A$, with $A_1 \cap A_2 = \emptyset$. If $b \notin \varphi(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$ then*

$$d(\varphi, \Omega, b) = d(\varphi, \Omega_1, b) + d(\varphi, \Omega_2, b).$$

Before we prove the theorem, we begin by stating and showing the following.

Lemma 5.10 *Let $B \subset A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ be internal sets, with $\Omega = \operatorname{int}^\circ A$. Let $\varphi : \overline{\Omega} \to \mathcal{E}$ be neocontinuous, with $\Phi : \overline{A} \to \mathbb{F}$ a lifting of φ. Let $b \in \mathcal{E}$ be such that $b \notin \varphi({}^\circ B)$. If $\beta \in \mathbb{F}$ is such that ${}^\circ\beta = b$, then there exists $\delta \in \mathbb{R}^+$ such that*

$$\forall \alpha \in B^\delta \quad \|\beta - \Phi(\alpha)\| \geq \delta$$

In particular, $\beta \notin \Phi(B^\delta)$ [5].

Proof.

Assume the opposite. Then the sets

$$B_n = \left\{ \alpha \in B^{1/n} : \|\beta - \Phi(\alpha)\| < \frac{1}{n} \right\}$$

[5] *If $A \subset \mathbb{F}$ is internal and $\delta \in {}^*\mathbb{R}^+$, let $A^\delta = \{\alpha \in \mathbb{F} : \exists \beta \in A \; \|\alpha - \beta\| \leq \delta\}$.*

are all nonempty. Since $B_1 \supset B_2 \supset B_3 \supset \dots$, by ω_1-saturation, there exists $\gamma \in \bigcap_{n\in\mathbb{N}} B_n$. Therefore:

$$\gamma \in \bigcap_{n\in\mathbb{N}} B^{1/n} \quad \Rightarrow \quad \mathrm{dist}(\gamma, B) \approx 0,$$

and

$$\left(\forall n \in \mathbb{N}\ \|\beta - \Phi(\alpha)\| < \frac{1}{n}\right) \quad \Rightarrow \quad \|\beta - \Phi(\alpha)\| \approx 0.$$

Then, ${}^\circ\gamma \in {}^\circ B$ and $\varphi({}^\circ\gamma) = {}^\circ(\Phi(\gamma)) = {}^\circ\beta = b$. But this contradicts the fact that $b \notin \varphi({}^\circ B)$.

Proof of Theorem (5.9).

(1) Using the normalization property of the *finite dimensional degree:

$$d(\mathrm{id}, \Omega, b) = d(\mathrm{id}|_{\mathbb{F}}, A, \beta) = \begin{cases} 1 & \text{if } \beta \in A\ , \\ 0 & \text{if } \beta \notin \overline{A}. \end{cases}$$

If $b \in \Omega = \mathrm{int}\,\overline{\Omega}$, then $\beta \in A - \partial A$ (since ${}^\circ\partial A = \partial\Omega$), so $d(\mathrm{id}, \Omega, b) = 1$. If $b \notin \overline{\Omega}$ then $\beta \notin \overline{A}$, so $d(\mathrm{id}, \Omega, b) = 0$.

In the rest of the proof, let $\Phi \in {}^*C(\overline{A}, \mathbb{F})$ be an internal S-continuous lifting of φ and $\beta \in \mathbb{F}$ be such that ${}^\circ\beta = b$.

(2) We begin by invoking Lemma (5.10), with $B = \partial A$. Let δ be as in the Lemma. Now, take any neocontinuous $\psi : \overline{\Omega} \to \mathcal{E}$ such that $\|\psi - \varphi\|_{nC(\overline{\Omega})} < \frac{\delta}{4}$. Take $\Psi \in {}^*C(\overline{A}, \mathbb{F})$ to be an S-continuous lifting of ϕ. Then $\|\Psi - \Phi\|_0 < \frac{\delta}{2}$ and from Lemma (5.10), for all $\alpha \in \partial A$:

$$\|\beta - \Psi(\alpha)\| \geq \|\beta - \Phi(\alpha)\| - \|\Phi(\alpha) - \Psi(\alpha)\| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

Therefore:

$$\frac{\delta}{4} < \|{}^\circ\beta - {}^\circ(\Psi(\alpha))\| = \|b - \psi({}^\circ\alpha)\|.$$

So, we conclude that $b \notin \psi(\partial\Omega)$; as a consequence, $d(\psi, \Omega, b)$ is defined. It remains to show it equals $d(\varphi, \Omega, b)$. Consider the homotopy $H : \overline{A} \times {}^*[0,1] \to \mathbb{F}$ given by:

$$H(\alpha, t) = t\Phi(\alpha) + (1-t)\Psi(\alpha).$$

H is clearly S-continuous in $\overline{A} \times {}^*[0,1]$. Also:

$$\begin{aligned} \|\Phi(\alpha) - H(\alpha, t)\| &= \|(1-t)\Phi(\alpha) + (1-t)\Psi(\alpha)\| \\ &= (1-t)\|\Phi(\alpha) - \Psi(\alpha)\| \\ &< (1-t)\frac{\delta}{2} \leq \frac{\delta}{2}. \end{aligned}$$

Hence, for $\alpha \in \partial A$, and using the Lemma (5.10) again:

$$\|\beta - H(\alpha, t)\| \geq \|\beta - \Phi(\alpha)\| + \|\Phi(\alpha) - H(\alpha, t)\| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

So $\beta \notin H(\partial A \times {}^*[0,1])$. By the homotopy invariance of the Brouwer degree:

$$\begin{aligned} d(\varphi, \Omega, b) &= d(\Phi, A, \beta) = d(H(\cdot, 0), A, \beta) \\ &= d(H(\cdot, 1), A, \beta) = d(\Psi, A, \beta) = d(\psi, \Omega, b). \end{aligned}$$

(3) From Proposition (4.12):

$$d(\Phi, A, \beta) = d(\Phi - \beta, A, 0).$$

Hence, and since $\Phi - \beta$ lifts $\varphi - b$:

$$d(\varphi, \Omega, b) = d(\Phi - \beta, A, 0) = d(\varphi - b, \Omega, 0).$$

Consider the function $d(\varphi - \cdot, \Omega, 0) : \mathcal{E} - \varphi(\partial\Omega) \to {}^*\mathbb{Z}$. From property (2) of this theorem (continuity with respect to φ), $d(\varphi - \cdot, \Omega, 0)$ is continuous. Since it is ${}^*\mathbb{Z}$-valued, it must be constant on connected components of $\mathcal{E} - \varphi(\partial\Omega)$.

(4) Let $H \in {}^*C(\overline{A} \times {}^*[0,1], \mathbb{F})$ be an S-continuous lifting of h. Since by hypothesis, $\forall t \in [0,1]$ $b \notin h(\partial\Omega, t)$, by Lemma (5.10), $\mathrm{dist}(H(\partial A, t), \beta) \not\approx 0$ (for all $t \in [0,1]$. Using the S-continuity of H, we conclude that $\beta \notin H(\partial A \times {}^*[0,1])$). Now, using the homotopy invariance of the Brouwer degree:

$$d(h(\cdot, t), \Omega, b) = d(H(\cdot, t), A, \beta) \equiv \text{constant}.$$

(5) Apply Lemma (5.10) (to φ, Ω, b, Φ, A, β), with $B = \overline{A} - (A_1 \cup A_2)$. Note that $b \notin \varphi({}^\circ B)$. Otherwise, for some $\alpha \in A - (A_1 \cup A_2)$, $\varphi({}^\circ\alpha) = b$. But $\alpha \in A - (A_1 \cup A_2)$ implies that ${}^\circ\alpha \in {}^\circ A$ and ${}^\circ\alpha \notin \mathrm{int}\,{}^\circ A_1$, ${}^\circ\alpha \notin \mathrm{int}\,{}^\circ A_2$. This means that ${}^\circ\alpha \in \overline{\Omega} - (\Omega_1 \cup \Omega_2)$, so $b \in \varphi(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$. This leads to a contradiction. So $b \notin \varphi({}^\circ B)$; hence, the conditions of the Lemma (5.10) are satisfied. Therefore, $\beta \notin \Phi(\overline{A} - (A_1 \cup A_2))$. Using now the additivity-excision property of the Brouwer degree:

$$d(\varphi, \Omega, b) = d(\Phi, A, \beta) = d(\Phi, A_1, \beta) + d(\Phi, A_2, \beta) = d(\varphi, \Omega_1, b) + d(\varphi, \Omega_2, b).$$

5.4 Some Elementary Applications

We begin by showing some results similar to corollaries (4.13)-(4.18).

Corollary 5.11 *If $b \notin \varphi(\overline{\Omega})$, then $d(\varphi, \Omega, b) = 0$*

Proof.

Use additivity-excision with $A_1 = A_2 = \emptyset$.

Corollary 5.12 *(***Basic Existence Criteria***):*

If $d(\varphi, \Omega, b) \neq 0$ then there exists $x \in \Omega$ such that $\varphi(x) = b$.

Proof.

Use Corollary (5.11).

Our degree theory actually shows the following stronger result

Corollary 5.13 *(***Extended Existence Criteria***):*

Let $\Omega = {}^{\circ}A$, with $A \in SB(\mathbb{F})$, $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ and A is *open. *Let $\varphi : \overline{\Omega} \to \mathcal{E}$ be neocontinuous and $b \in \mathcal{E} - \varphi(\partial\Omega)$. If $d(\varphi, \Omega, b) \neq 0$, then there exists $x \in \Omega$ such that $\varphi(x) = b$. Furthermore, if $\Phi : \overline{A} \to \mathbb{F}$ is* any **continuous lifting of φ, and $\beta \in \mathbb{F} - \Phi(\partial A)$ such that ${}^{\circ}\beta = b$, then there exists $\alpha \in A$ such that $\Phi(\alpha) = \beta$.*

Proof.

The first part of the statement is Corollary (5.12). As for the second, it follows from the fact that the degree is well-defined. Thus:

$$0 \neq d(\varphi, \Omega, b) = d(\Phi, \Omega, \beta),$$

for any lifting Φ of φ, and ${}^{\circ}\beta = b$. From Corollary (4.14) (basic existence property for the Brouwer degree), the result follows.

Corollary 5.14 *(***Fixed Point Theorem***): Let $A \in SB(\mathbb{F})$, $A \subset \mathrm{Gal}_{\mathbb{F}}(0)$ be* *open*. Let $\Omega = \mathrm{int}\,^\circ A$ and assume $\overline{\Omega}$ is a convex neighborhood of 0. Let $\varphi : \overline{\Omega} \to \overline{\Omega}$ be neocontinuous. Then φ has a fixed point in $\overline{\Omega}$. Furthermore if $\Phi : \overline{A} :\to \mathbb{F}$ is* any **continuous lifting of φ, then Φ has a fixed point in $\overline{A}$.*

Proof.

Without loss of generality, φ has no fixed points on $\partial\Omega$ (otherwise we are done). So $d(\mathrm{id} - \varphi, \Omega, 0)$ is defined. Consider a homotopy:

$$h(t, x) = x - t\varphi(x).$$

It is easily seen that h is neocontinuous on $\overline{\Omega} \times [0, 1]$. Also, for all $x \in \partial\Omega$, $h(x, 1) = x - \varphi(x) \neq 0$ (by assumption), and for $t \in [0, 1)$, from the convexity of Ω, $t\varphi(x) \in \Omega$, so $h(x, t) = x - t\varphi(x) \neq 0$. Hence, by homotopy invariance

$$d(\mathrm{id} - \varphi, \Omega, 0) = d(h(\cdot, 1), \Omega, 0) = d(h(\cdot, 0), \Omega, 0) = d(\mathrm{id}, \Omega, 0) = 1.$$

Now, apply Corollary (5.13) to get the result.

Corollary 5.15 *(***Perturbation lemma***): Let $R \in \mathbb{R}^+$, and $\varphi : B_R(0) \times [0, 1] \to \mathcal{E}$ be neocontinuous and such that $\varphi(x, 0) \equiv c \in \mathcal{E}$. Assume there exists $r \in (0, R)$ verifying:*

$$\forall \theta \in [0, 1]\,, \forall u \in B_R(0) \;\; u = \varphi(u, \theta) \;\Rightarrow\; \|u\|_{\mathcal{E}} \leq r. \tag{5.10}$$

Then, for all $\theta \in [0, 1]$, φ has a fixed point in $B_R(0) \subset \mathcal{E}$. Furthermore, if $\Phi : B_R(0) \subset \mathbb{F} \to \mathbb{F}$ is a *continuous lifting of φ, then Φ has a fixed point in $B_R(0) \subset \mathbb{F}$.

Proof.

Set up a homotopy $h(x,\theta) = x - \varphi(x,\theta)$. If $x \in \partial B_R(0)$, i.e., $\|x\|_{\mathcal{E}} = r < R$, then by (5.10) $h(x,\theta) \neq 0$ for all $\theta \in [0,1]$. Hence $0 \notin h(\partial B_R(0) \times [0,1])$. By homotopy invariance:

$$d(h(\cdot,\theta), B_R(0), 0) = \text{const} = d(h(\cdot,0), B_R(0), 0).$$

Since $h(\cdot,0)$ is the map $\mathrm{id}_{\mathcal{E}} + c$ it is easy to see that, the degree on the right hand side is 1. Hence:

$$d(h(\cdot,\theta), B_r(0), 0) = 1.$$

The result now follows from Corollary (5.13).

Chapter 6

The Boundary Value Problem for Newton's Law of Motion

In this chapter, we use the degree theory in nonstandard hulls of hyperfinite dimensional Banach spaces to show convergence of schemes for a nonlinear test problem. The key result is Corollary (5.15), which provides us convergence results from appropriate a priori bounds.

Our test problem consists of the equation

$$x'' = f(x', x, t), \tag{6.1}$$

where $x : I \to \mathbb{R}^n$, and $I = [a, b] \subset \mathbb{R}$, $a < b$. We assume $f : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is continuous and bounded. We impose boundary conditions at $t = a$ and $t = b$:

$$x(a) = x_0 \in \mathbb{R}^n; \qquad x(b) = x_1 \in \mathbb{R}^n. \tag{6.2}$$

For this problem, the a priori bounds will be relatively easy to establish [1].

Equation (6.1) can be interpreted as Newton's law of motion [2]. We are looking for solutions of (6.1) which pass through x_0 when $t = a$ and through x_1 when $t = b$.

[1] To generalize the results of this chapter to a nonlinear elliptic Dirichlet problem for the equation $Lu = f(\nabla u, u, t)$, we need a priori (discrete) bounds for a discretization of the linear elliptic equation.

[2] Variables have been scaled so that, in equation (6.1), all the mass of each particle equals 1.

As the following example shows, there is no guaranteed uniqueness of solution for (6.1)-(6.2), even when f is smooth.

Example 6.1 *Let $x : \mathbb{R} \to \mathbb{R}$ such that*

$$x'' = -\pi^2 x; \qquad x(0) = x(1) = 0.$$

Any function of the form $x(t) = A\sin(\pi t)$, with $A \in \mathbb{R}$ solves this problem.

Lack of uniqueness complicates the study of convergence of finite difference (or other discrete) schemes. Different schemes may converge to different solutions (if they converge at all), and we may not have a priori information to establish to which solution a particular scheme converges.

Therefore, the direct approach, i.e., to show the existence of an actual solution as an appropriate limit of discrete ones, may be the best way to handle this type of problem. In addition, it gives us more information about the "meaning" of the differential equation problem, in that it shows it as the limiting case of appropriate discrete schemes. In this chapter we will use the setup in chapter 5.

We hope that our methods can later be generalized to the PDE analogue of (6.1), i.e., $Lu = f(Du, u, \cdot)$, with L a (linear) uniformly elliptic operator.

6.1 A Discretization of the Boundary Value Problem

We look for a discretized version of equation (6.1). It does not need to be a particularly accurate one; at the end we will get a result valid for a large class of discretizations which is, in some sense, close to the one we are about to introduce.

First we work in the standard universe. Let a, $b \in \mathbb{R}$, $b > a$. To discretize $I = [a, b]$, we consider a positive $h \in \mathbb{R}$, of the form $h = \frac{b-a}{N}$ with $N \in \mathbb{N} - \{0\}$. Then, define:

$$\begin{aligned} [a, b]_h &= \Big\{a, a+h, \ldots, a+Nh = b\Big\} \\ &= (a + h\mathbb{N}) \cap [a, b]. \end{aligned}$$

For all other h, just let the h-discretization be the $1/N$-discretization, where N is the unique positive integer satisfying:

$$\frac{1}{N} \leq h < \frac{1}{N+1}.$$

Thus, and without loss of generality, we need only define the discretizations for the case $h = 1/N$.

Let:

$$\begin{aligned} a_h &= \min I_h &= a, \\ b_h &= \max I_h &= b. \end{aligned}$$

Note that, for α, $\beta \in I_h$, the discretized interval $[\alpha, \beta]_h$ is contained in I_h. For convenience, for each $\beta \geq \alpha$, $\alpha, \beta \in I_h$, define:

$$\begin{aligned} [\alpha, \beta)_h &= [\alpha, \beta]_h - \{\beta\}; \\ (\alpha, \beta]_h &= [\alpha, \beta]_h - \{\alpha\}; \\ (\alpha, \beta)_h &= [\alpha, \beta]_h - \{\alpha, \beta\}. \end{aligned}$$

It is obvious that some of these "intervals" may be the empty set.

For standard h, the set of all gridfunctions $X : I_h \to \mathbb{R}^n$ is just $(\mathbb{R}^n)^{I_h}$. With the pointwise sum and product, this is a finite dimensional linear space.

Now, we move to the nonstandard universe, and consider $h \approx 0$. From a previous observation, it is enough to consider h of the form $h = \dfrac{1}{N}$, with $N \in {}^*\mathbb{N}$. Then, $a_h = a < b = b_h$. We need an *internal* linear space of grid functions. It is well-known that given internal sets A, B, the set of all internal functions $g : A \to B$ is an internal set, which is denoted by A^B. In particular, we let $({}^*\mathbb{R}^n)^{I_h} \overset{\text{def}}{=} \mathbb{F}_h$ be the internal set of all internal $X : I_h \to {}^*\mathbb{R}^n$. By transfer, any internal norm for $\mathbb{F}_h$ makes it into a *Banach space.

Given $X \in \mathbb{F}_h$, we introduce the discrete counterparts of the derivatives to be the corresponding (right) difference quotients:

$$\delta_h^+ X(t_0) = \frac{1}{h}\Big(X(t_0 + h) - X(t_0)\Big).$$

The following shorter notations will be often used:

$$\begin{aligned} X_h^{(0)} &= X \\ X_h' &= \delta_h^+ X \\ X_h'' &= \delta_h^+ \delta_h^+ X = (\delta_h^+)^2 X \\ &\vdots \\ X_h^{(n)} &= (\delta_h^+)^n X \end{aligned}$$

To be able to compute $X_h^{(n)}(t)$, we need the values of $X(t), X(t+h), \ldots, X(t+nh)$, so the domain of $X_h^{(n)}(t)$ is $[a_h, b_h - nh]_h$.

The discretized version of problem (6.1)-(6.2) that we will be needing for our proofs is:

$$\begin{cases} X_h'' = f(X_h', X, t) \quad \text{for } t \in I_h - \{b_h, b_h - h\}, \\ X(a_h) = x_0, \\ X(b_h) = y_0, \end{cases} \tag{6.3}$$

In general, this is a nonlinear system of N equations.

A small collection of straightforward results concerning this little discrete "calculus" is now in order. They are well known from the literature on difference equations. Some of them will turn out to be quite useful in this chapter.

Proposition 6.2 *Let $h > 0$ and I_h be a discretized interval. Let $X, Y \in \mathbb{F}_h$, and $c, d \in {}^*\mathbb{R}$. Then:*

(1) $\Big(cX + dY\Big)'_h = cX'_h + dY'_h$;

(2) $\Big(X \cdot Y\Big)'_h(t) = X(t+h) \cdot Y'_h(t) + X'_h(t) \cdot Y(t), \quad$ *for all* $t \in [a_h, b_h)_h$.

The discrete counterpart of the integral is just the Riemann sum, given by

$$\sum_{t \in [\alpha,\beta)_h} X(t)h = h\Big(X(\alpha) + X(\alpha+h) + \ldots + X(\beta - h)\Big),$$

for $\beta > \alpha$. To simplify our calculations, let $\sum_{t\in\emptyset} X(t)h = 0$. As an relevant example, we compute the Riemann sum of the constant function, $F(t) = A \in {}^*\mathbb{R}^n$:

$$\sum_{t \in [\alpha,\beta)_h} Ah = h\frac{\beta - \alpha}{h}A = (\beta - \alpha)A. \tag{6.4}$$

The Riemann sum inherits all the properties of summation. We choose to prove the following well known proposition, just to illustrate the simplicity of our notation.

Proposition 6.3 *Let $h > 0$ and I_h a discretized interval. Let $\alpha, \beta \in I_h$, $\alpha \leq \beta$ and $X, Y \in \mathbb{F}_h$. then:*

(1) Fundamental Identity:

$$\sum_{t \in [\alpha,\beta)_h} X'_h(t)h = X(\beta) - X(\alpha).$$

(2) Summation by Parts Formula:

$$\sum_{t\in[\alpha,\beta)_h} X'_h(t)\cdot Y(t)h = X(t)\cdot Y(t)\Big|_{t=\alpha}^{\beta} - \sum_{t\in[\alpha,\beta)_h} X(t+h)\cdot Y'_h(t)h.$$

Proof.

To prove identity (1), we have:

$$\sum_{t\in[\alpha,\beta)_h} X'_h(t)h = \sum_{t\in[\alpha,\beta)_h} \Big(X(t+h)-X(t)\Big).$$

The last sum telescopes to $X(\beta)-X(\alpha)$.

To show the summation by parts formula, (2), start with Proposition (6.2), part (2):

$$(X\cdot Y)'_h(t) = X(t+h)\cdot Y'_h(t) + X'_h(t)\cdot Y(t).$$

Summing both sides, from α to β, yields:

$$\sum_{t\in[\alpha,\beta)_h} (X\cdot Y)'_h(t)h = \sum_{t\in[\alpha,\beta)_h} X(t+h)\cdot Y'_h(t)h + \sum_{t\in[\alpha,\beta)_h} X'_h(t)\cdot Y(t)h.$$

We now apply the fundamental identity, and get:

$$(X\cdot Y)(t)\Big|_{t=\alpha}^{\beta} = \sum_{t\in[\alpha,\beta)_h} X(t+h)\cdot Y'_h(t)h + \sum_{t\in[\alpha,\beta)_h} X'_h(t)\cdot Y(t)$$

which is equivalent to the desired formula.

To finish this section, we study possible internal *norms for $\mathbb{F}_h$. We know that all (internal) *norms for $\mathbb{F}_h$ are *equivalent, i.e.:

$$\exists c_1, c_2 \in {}^*\mathbb{R}^+ \;\; c_1\,|||X||| \le \|X\| \le c_2\,|||X|||$$

(for any *norms $\|\cdot\|$, $|||\cdot||| : \mathbb{F}_h \to {}^*\mathbb{R}$). Anyway, this fact does not seem to be very useful since if, say, c_1 is infinitesimal, it may happen that a galaxy of $(\mathbb{F}_h, |||\cdot|||)$

is contained in a monad of $(\mathbb{F}_h, \|\cdot\|)$. Hence, different norms in $\mathbb{F}_h$ may lead to completely different nonstandard hulls of $\mathbb{F}_h$. Here are some useful ones:

(1) **The L_h^∞-norm or max norm:**

$$\|X\|_{L_h^\infty} = \max_{t \in I_h} |X(t)| \quad ^3 .$$

(2) **The L_h^p-norm:**

$$\|X\|_{L_h^p} = \left(\sum_{t \in I_h} (X(t))^p h \right)^{1/p} ,$$

where $p \geq 1$ is standard [4].

The L_h^∞-norm is useful for looking at uniform convergence, while the L_h^p-norms give weaker notions of convergence.

To study convergence to a solution of (6.1)-(6.2) we need, as well, to get a handle on the size of the difference quotients. If we use the L_h^∞ norm to measure the size of X and its difference quotients up to order k, we get the discrete equivalent of the C^k norm:

$$\|X\|_k = \sum_{j=1}^{k} \|X_h^{(j)}\|_{L_h^\infty} = \|X_h\|_{L_h^\infty} + \|X_h'\|_{L_h^\infty} + \ldots + \|X_h^{(k)}\|_{L_h^\infty}. \tag{6.5}$$

Recall that the domain of $X_h^{(i)}$ is $[a_h, b_h - nh]_h$. Only for the purpose of computing the $\|X_h^{(i)}\|_{L_h^\infty}$, consider each $X_h^{(i)}$ trivially extended to I_h.

If we use an L_h^p norm in (6.5) (in place of the L_h^∞ norm), we get a discrete analogue of a Sobolev norm. In this chapter, we will be interested in finding strong solutions of (6.1), i.e., we want to find $x \in C^2([a,b], \mathbb{R}^n)$. For that, (6.5) is an appropriate choice.

[3] where $|\cdot|$ is the euclidean norm in ${}^*\mathbb{R}^n$.

[4] the case $p = 2$ is just the *euclidean norm in $\mathbb{F}_h$, renormalized (by multiplication with $\sqrt{h}$) in order to make the galaxy of 0 interesting.

6.2 S-differentiability of Functions in $\mathbb{F}_h$

This section contains some results needed to pass from the discrete solutions of (6.3) to the continuous solutions of (6.1)-(6.2). Some definitions of S-continuity in the literature require that the range of the function contains only finite points. In our case we do not require finiteness, so $X \in \mathbb{F}_h$ is S-continuous means only that, for all $t_1, t_2 \in I_h$:

$$t_1 \approx t_2 \Rightarrow {}^\circ(X(t_1)) = {}^\circ(X(t_2)).$$

In [26], Stroyan shows a converse of Taylor's formula, which can be used to show smoothness of the standard part of an internal map. Our approach relies on a generalization of a notion of S-differentiability. As with S-continuity, the following S-differentiability condition for $X \in \mathbb{F}_h$ does not assume finiteness of X'_h.

Definition 6.4 *Let $X \in \mathbb{F}_h$. X is S-differentiable (abbrev. S-C^1) iff for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that:*

$$\forall t_0, t \in I_h, t_0 \neq b_h \;\; |t - t_0| < \delta \;\Rightarrow\; \Big|X(t) - X(t_0) - X'_h(t_0)(t - t_0)\Big| < \epsilon |t - t_0|.$$

Proposition 6.5 *Let $X \in \mathbb{F}_h$. Then, the following are equivalent:*

(i) *X is S-C^1;*

(ii) *X'_h is S-continuous;*

(iii) *For all $t_0, t \in I_h$, with $t_0 \neq b_h$, if $t \approx t_0$ then:*

$$\exists \epsilon \approx 0 \;\; \Big|X(t) - X(t_0) - X'_h(t_0)(t - t_0)\Big| = \epsilon |t - t_0|.$$

Proof.

(i) ⇒ (iii).

Let $|t - t_0| \approx 0$. If $t = t_0$, the inequality is obvious. If $0 \neq |t - t_0| \approx 0$ then for any $\epsilon \in \mathbb{R}^+$, it follows from (i) that

$$\Big|X(t) - X(t_0) - X'_h(t_0)(t - t_0)\Big| < \epsilon|t - t_0|.$$

Then

$$\Big|X(t) - X(t_0) - X'_h(t_0)(t - t_0)\Big| \;/\; |t - t_0| \;\approx\; 0,$$

as wanted.

(iii) ⇒ (ii).

Let $t_1,\, t_2 \in I_h$, be such that $t_1 \approx t_2$. Then, for some $\epsilon_1,\, \epsilon_2 \approx 0$:

$$\begin{aligned}
&\Big|\Big(X'_h(t_2) - X'_h(t_1)\Big)(t_2 - t_1)\Big| \\
&\qquad = \Big|X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1) + X(t_1) - X(t_2) - X'_h(t_2)(t_1 - t_2)\Big| \\
&\qquad \leq \Big|X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1)\Big| + \Big|X(t_1) - X(t_2) - X'_h(t_2)(t_1 - t_2)\Big| \\
&\qquad = \epsilon_1|t_2 - t_1| + \epsilon_2|t_1 - t_2|.
\end{aligned}$$

Therefore

$$\Big|X'_h(t_2) - X'_h(t_1)\Big| \leq \epsilon_1 + \epsilon_2 \approx 0.$$

(ii) ⇒ (i).

Let $t_1,\, t_2 \in I_h$, with $|t_1 - t_2| \approx 0$. Using equation (6.4) and the fundamental identity, we have:

$$\Big|X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1)\Big| \;=\; \Big|\sum_{t \in [t_1, t_2)_h} X'_h(t)h - X'_h(t_1)(t_2 - t_1)\Big|$$

$$
\begin{aligned}
&= \Big| \sum_{t\in[t_1,t_2)_h} X_h'(t)h - \sum_{t\in[t_1,t_2)_h} X_h'(t_1)h \Big| \\
&\leq \Big| \sum_{t\in[t_1,t_2)_h} \Big(X_h'(t) - X_h'(t_1)\Big)h \Big| \\
&\leq \sum_{t\in[t_1,t_2)_h} \Big|X_h'(t) - X_h'(t_1)\Big| h.
\end{aligned}
$$

Say $\epsilon \in \mathbb{R}^+$. By the S-continuity of X_h',

$$
\Big|X_h'(t) - X_h'(t_1)\Big| < \epsilon,
$$

and so:

$$
\Big|X(t_2) - X(t_1) - X_h'(t_1)(t_2 - t_1)\Big| < \sum_{t\in[t_1,t_2)_h} \epsilon h = \epsilon(t_2 - t_1). \tag{6.6}
$$

So the set of all δ such that, whenever $|t_1 - t_2| < \delta$, inequality (6.6) holds, includes all positive infinitesimals. Since this set is internal, it must contain some $\delta \in \mathbb{R}^+$.

Lemma 6.6 *Let $X \in \mathbb{F}_h$ be such that $\|X_h'\|_{L_h^\infty}$ is finite. Then X is S-continuous.*

Proof.

Say $\|X_h'\|_{L_h^\infty} < M$, with M finite. Let $t_1, t_2 \in I$, $t_1 \approx t_2$. Using the fundamental identity and equation(6.4):

$$
\begin{aligned}
|X(t_2) - X(t_1)| &= \Big| \sum_{t\in[t_1,t_2)_h} X_h' h \Big| \\
&\leq \sum_{t\in[t_1,t_2)_h} |X_h'|\, h \\
&\leq \sum_{t\in[t_1,t_2)_h} Mh = M(t_1 - t_2).
\end{aligned}
$$

Theorem 6.7 *Let $X \in \mathbb{F}_h$ be S-C^1, with $\|X\|_1 < \infty$. Then, the function $x : I \to \mathbb{R}^n$ given by*

$$x(^\circ t) = {}^\circ(X(t)), \quad \text{for } t \in I_h, \tag{6.7}$$

is well-defined, belongs to $C^1(I, \mathbb{R}^n)$, and:

$$x'(^\circ t) = {}^\circ(X'_h(t)), \quad \text{for } t \in I_h. \tag{6.8}$$

Proof.

By Lemma (6.6), X is S-continuous, and each $X(t)$ is finite since $\|X\|_1 < \infty$; hence x is well defined by (6.7), and $x \in C(I, \mathbb{R}^n)$. To prove differentiability, first note that X'_h is S-continuous (this follows form Proposition (6.5)), and $X'_h(t)$ is finite since $\|X\|_1 < \infty$. Hence, the function on the right-hand side of equation (6.8),

$$y(^\circ t) \stackrel{\text{def}}{=} {}^\circ(X'_h(t)) \quad \text{for } t \in I_h,$$

is well-defined and continuous. Fix $\epsilon \in \mathbb{R}^+$ and pick $\delta \in \mathbb{R}^+$ such that, for all $t_2, t_1 \in I_h$, with $t_1 \neq b_h$:

$$|t_2 - t_1| < \delta \;\Rightarrow\; \Big|X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1)\Big| < \frac{\epsilon}{2}|t_2 - t_1|. \tag{6.9}$$

Then, whenever $|t_2 - t_1| < \delta$ and $^\circ t_2 \neq {}^\circ t_1$ (i.e. $0 \not\approx |t_2 - t_1| < \delta$), we have, using the S-continuity of the functions involved:

$$\begin{aligned}
&\Big|x(^\circ t_2) - x(^\circ t_1) - y(^\circ t_1)(^\circ t_2 - {}^\circ t_1)\Big| \\
&\qquad = \Big|{}^\circ(X(t_2)) - {}^\circ(X(t_1)) - {}^\circ\left((X'_h)(t_1)\right){}^\circ(t_2 - t_1)\Big| \\
&\qquad = \Big|{}^\circ\Big(X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1)\Big)\Big| \\
&\qquad = {}^\circ\Big|\Big(X(t_2) - X(t_1) - X'_h(t_1)(t_2 - t_1)\Big)\Big|.
\end{aligned}$$

Using (6.9), and since $\epsilon \in \mathbb{R}^+$ and $|t_2 - t_1| \not\approx 0$:

$$^\circ\left|\left(X(t_2) - X(t_1) - X_h'(t_1)(t_2 - t_1)\right)\right| \leq {}^\circ\left(\frac{\epsilon}{2}|t_2 - t_1|\right) = \frac{\epsilon}{2}|^\circ t_2 - {}^\circ t_1| < \epsilon|^\circ t_2 - {}^\circ t_1|$$

This shows the (standard) statement:

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall s, \hat{s} \in I\ \ 0 < |s - \hat{s}| < \delta \Rightarrow |x(s) - x(\hat{s}) - y(\hat{s})(s - \hat{s})| < \epsilon|s - \hat{s}|,$$

which implies the (uniform) differentiability of x in I, with $x' = y$.

Corollary 6.8 *Let $X \in \mathbb{F}_h$ and $k \in \mathbb{N}$. If $\|X\|_k < \infty$ and $X_h^{(k)}$ is S-continuous, then the function $x : I \to \mathbb{R}^n$ given by*

$$x(^\circ t) = {}^\circ(X(t)), \quad \text{for } t \in I_h,$$

is well-defined. Furthermore, $x \in C^k(I, \mathbb{R}^n)$, with

$$x^{(j)}(^\circ t) = {}^\circ(X_h^{(j)}(t)), \quad \text{for } t \in I_h,\ j \in \{1, \ldots, k\},$$

and $\|x\|_{C^k(\mathbb{R},\mathbb{R}^n)} = {}^\circ\left(\|X\|_h\right)$.

Proof.

The proof is by induction on k. For $k = 0$ the result follows from both the S-continuity of X, and $\|X\|_0 < \infty$. Now assume the result holds at some $k \in \mathbb{N}$. Since $\|X\|_{k+1} < \infty$, we have $\|X_h^{(k+1)}\|_{L_h^\infty} < \infty$, so by Lemma (6.6), $X_h^{(k)}$ is S-continuous. Hence, using the induction hypothesis,

$$x(^\circ t) = {}^\circ(X(t)), \quad \text{for } t \in I_h,$$

is well-defined and $x \in C^k(I, \mathbb{R}^n)$, with:

$$x^{(j)}(^\circ t) = {}^\circ\left(X_h^{(j)}(t)\right) \quad \text{for } t \in I_h,\ j \in \{1, \ldots, k\}.$$

Now, consider the function:

$$y(^\circ t) = {}^\circ\left(X_h^{(k)}(t)\right) = x^{(k)}(^\circ t) \quad \text{for } \ t \in I_h, \ j \in \{1, \dots, k\}.$$

We know that $\|X_h^{(k)}\|_1 = \|X_h^{(k)}\|_{L_h^\infty} + \|X_h^{(k+1)}\|_{L_h^\infty} \leq \|X\|_{k+1} < \infty$ and the function $X_h^{(k+1)}$ is S-continuous. Applying Theorem (6.7), we get that y is well-defined, $y \in C^1(I, \mathbb{R}^n)$, and

$$y'(^\circ t) = {}^\circ(X_h^{(k+1)}(t)) \quad \text{for } t \in I_h, \ j \in \{1, \dots, k\}.$$

But since $y = x^{(k)}$, we must have $x \in C^{k+1}(I, \mathbb{R}^n)$ and $x^{(k+1)} = y'$, i.e.

$$x^{(k+1)}(^\circ t) = {}^\circ\left(X_h^{(k+1)}(t)\right) \quad \text{for } \ t \in I_h.$$

As for the norm estimate, the case $k = 0$ is easy. For the induction step:

$$\begin{aligned}
\|x\|_{C^{k+1}} &= \|x\|_{C^k} + \left\|x^{(k+1)}\right\|_{C^0} \\
&= {}^\circ(\|X\|_k) + \left\|x^{(k+1)}\right\|_{C^0} \\
&= {}^\circ(\|X\|_k) + {}^\circ\left(\|X_h^{(k+1)}\|_{L_h^\infty}\right) \\
&= {}^\circ\left(\|X\|_{k+1}\right).
\end{aligned}$$

Corollary 6.9 *Suppose $x = {}^\circ X \in \mathrm{Hull}(\mathbb{F}_h, \|\cdot\|_k)$, where $X \in \mathbb{F}_h$ and $X_h^{(k)}$ is S-continuous. Then the function $\hat{x} : I \to \mathbb{R}^n$ given by*

$$\hat{x}(^\circ t) = {}^\circ(X(t)), \quad \text{for } t \in I_h,$$

is well-defined and depends only on x. Furthermore, $\hat{x} \in C^k(I, \mathbb{R}^n)$, with

$$\hat{x}^{(j)}(^\circ t) = {}^\circ\left(X_h^{(j)}(t)\right) \quad \text{for } t \in I_h, \ j \in \{1, \dots, k\},$$

and $\|\hat{x}\|_{C^k} = \|x\|_{\mathrm{Hull}(\mathbb{F}_h, \|\cdot\|_k)}$.

Proof.

By Corollary (6.8), the function $\hat{x}$ is well-defined, belongs to $C^k(I,\mathbb{R}^n)$, and $\|\hat{x}\|_{C^k} = {}^\circ(\|X\|_k)$. We just have to show that $\hat{x}$ is independent of the choice of X. So let Y be such that $Y_h^{(k)}$ is S-continuous and:

$$\|X-Y\|_k = \|X-Y\|_{L_h^\infty} + \|X_h'-Y_h'\|_{L_h^\infty} + \ldots + \|X_h^{(k)}-Y_h^{(k)}\|_{L_h^\infty} \approx 0.$$

This implies that

$$\left|X^{(j)}(t)_h - Y_h^{(j)}(t)\right| \approx 0 \qquad \forall\, t\in I_h \ \ \forall j\in\{0,\ldots,k\}.$$

So using X or Y to compute $\hat{x}$ gives exactly the same result. Also:

$$\|\hat{x}\|_{C^k} = {}^\circ\Big(\|X\|_k\Big) = \|x\|_{\mathrm{Hull}(\mathbb{F}_h,\|\cdot\|_k)}.$$

Remark 6.10 *At this point, we want to stress that the general discrete schemes for second order differential equations, as introduced in Chapter 3, can be used with the degree theory of Chapter 5. To clarify this point, let $\mathbb{R}_h^n$ be a discrete scheme for $\mathbb{R}^n$. Let h be a positive infinitesimal and assume that $\mathbb{R}_h^n$ is consistent with $\mathbb{R}^n$, i.e., ${}^\circ(\mathbb{R}_h^n)=\mathbb{R}^n$. Let $\mathcal{O}\subset\mathbb{R}^n$ be open. We can then form a *finite dimensional internal space $\mathbb{F}_h=\mathbb{R}^{\overline{\mathcal{O}}_h}$. Considering operators*

$$\delta_{i,h}:\mathbb{R}^{\overline{\mathcal{O}}_h}\to\mathbb{R}^{\mathcal{O}_h},$$

$$\delta_{ij,h}^2:\mathbb{R}^{\overline{\mathcal{O}}_h}\to\mathbb{R}^{\mathcal{O}_h},$$

such that $\delta_{i,h}$ is consistent with $\dfrac{\partial}{\partial x_i}$ and $\delta_{ij,h}^2$ is consistent with $\dfrac{\partial^2}{\partial x_i\partial x_j}$, we can introduce a norm for $\mathbb{F}_h$ as follows:

$$\|U\|_2 = \|U\|_{L_h^\infty} + \sum_{i=1}^n \|\delta_{i,h}U\|_{L_h^\infty} + \sum_{i,j=1}^n \|\delta_{ij,h}^2 U\|_{L_h^\infty}.$$

Take $\mathcal{E} = \mathrm{Hull}(\mathbb{F}_h, \|\cdot\|_2)$*. From consistency,* ${}^\circ\overline{\mathcal{O}}_h = \overline{\mathcal{O}}$ *and* ${}^\circ\partial\mathcal{O}_h = \partial\mathcal{O}$*. As with the special case treated in Chapter 6, it can be shown that* $\mathcal{E}$ *contains an isomorphic copy of* $C^2(\overline{\mathcal{O}})$ *(with* C^2 *norm). More specifically, for each* $U \in \mathrm{Gal}_{\mathbb{F}_h}(0)$ *such that each* $\delta_{ij,h}U$ *is S-continuous, then*

$$\begin{aligned} {}^\circ U &= u \in C^2(\overline{\mathcal{O}}), \\ {}^\circ(D_h U) &= Du, \\ {}^\circ(D_h^2 U) &= D^2 u. \end{aligned}$$

Here, $D_h = (\delta_{1,h}, \ldots, \delta_{1,h})$*, (the "discrete gradient"), and* $D_h^2 = \left(\delta_{ij,h}^2\right)_{i,j=1\ldots,n}$ *(the "discrete second derivative operator").*

6.3 Bounds for a Discrete Linear Problem

We look for an explicit formula for the solution of a simple linear version of problem (6.17):

$$\begin{cases} X_h''(t) = G(t), & \text{for } t \in [\alpha, \beta - 2h]_h, \\ X(\alpha) = A, \\ X(\beta) = B. \end{cases}$$

where $G \in \mathbb{F}_h$, and $A, B \in {}^*\mathbb{R}^n$.

Proposition 6.11 *Let* $\alpha, \beta \in I_h$*,* $\beta > \alpha$*, and* $G \in \mathbb{F}_h$*. Consider the problem:*

$$\begin{cases} X_h''(t) = G(t) & \text{for } t \in [\alpha, \beta - 2h]_h, \\ X(\alpha) = X(\beta) = 0 \end{cases} \tag{6.10}$$

Then, $X(\cdot) : [\alpha, \beta]_h \to {}^*\mathbb{R}^n$ *given by*

$$X(t) = C(t - \alpha) + \sum_{s \in (\alpha, t)_h} (t - s)G(s - h)h, \tag{6.11}$$

with

$$C = \frac{-1}{\beta - \alpha} \sum_{s \in (\alpha, t)_h} (\beta - s) G(s-h) h, \tag{6.12}$$

is the unique internal solution to (6.10).

Proof.

We start by checking that X satisfies the difference equation in (6.10):

$$\begin{aligned}
h^2 X_h''(t) &= X(t+2h) - 2X(t+h) + X(t) \\
&= C(t-\alpha) + 2hC + \sum_{s \in (\alpha, t+2h)_h} (t-s) G(s-h) h + \sum_{s \in (\alpha, t+2h)_h} 2hG(s-h)h \\
&\quad -2C(t-\alpha) - 2hC - 2 \sum_{s \in (\alpha, t+h)_h} (t-s) G(s-h) h - 2 \sum_{s \in (\alpha, t+h)_h} hG(s-h)h \\
&\quad +C(t-\alpha) + \sum_{s \in (\alpha, t)_h} (t-s) G(s-h) h \\
&= (t-(t+h))G(t)h + 2hG(t)h - (t-t)G(t-h)h \\
&= -h^2 G(t) + 2h^2 G(t) \ = \ h^2 G(t),
\end{aligned}$$

for $t = \alpha, \alpha+1, \ldots, \beta - 2h$.

This shows that X, as given by (6.11), satisfies the difference equation in (6.10) (with arbitrary C). Also, $X(\alpha) = 0$. As for the other boundary condition, and using (6.12):

$$X(\beta) = C(\beta - \alpha) + \sum_{s \in (\alpha, \beta)_h} (\beta - s) G(s-h) h = 0.$$

Uniqueness follows from the fact that the scheme $\left\{ \delta_h^- \delta_h^+ : h \in \mathbb{R}^+ \right\}$, with $\Omega_h = [\alpha, \beta]_h$ and $\partial \Omega_h = \{\alpha, \beta\}$, has a maximum principle.

Proposition 6.12 *Let $\alpha, \beta \in I_h$ with $\beta > \alpha$, $A, B \in {}^*\mathbb{R}^n$, and $G \in \mathbb{F}_h$ be internal. Consider the problem:*

$$\begin{cases} X_h''(t) = G(t), & \textit{for } t \in [\alpha, \beta - 2h]_h, \\ X(\alpha) = A, \\ X(\beta) = B. \end{cases} \tag{6.13}$$

Then X, defined in $[\alpha, \beta]_h$ by

$$X(t) = X_0(t) + \frac{t-\alpha}{\beta-\alpha}B + \frac{\beta - t}{\beta - \alpha}A, \tag{6.14}$$

and with $X_0(t)$ given by (6.11) and (6.12), is its unique internal solution.

Proof.

An easy computation shows that $X_1(t) = \frac{t-\alpha}{\beta-\alpha}B + \frac{\beta-t}{\beta-\alpha}A$ defines a solution to the homogeneous problem:

$$\begin{cases} X_h''(t) = 0 & \text{for } t \in [\alpha, \beta - 2h]_h, \\ X(\alpha) = A, \\ X(\beta) = B. \end{cases}$$

Hence, $X(t) = X_0(t) + X_1(t)$ solves problem (6.13). By maximum principle for $\left\{\delta_h^- \delta_h^+ : h \in \mathbb{R}^+\right\}$, the solution is unique.

Lemma 6.13 *Let $\alpha, \beta \in I_h$ with $\beta > \alpha$, $A, B \in {}^*\mathbb{R}^n$, and $G \in \mathbb{F}_h$ be such that $\|G\|_{L_h^\infty}$ is finite. Consider the problem:*

$$\begin{cases} X_h''(t) = G(t), & \textit{for } t \in [\alpha, \beta - 2h]_h, \\ X(\alpha) = A, \\ X(\beta) = B. \end{cases} \tag{6.15}$$

If $X : [\alpha, \beta]_h \to {}^*\mathbb{R}^n$ *is the unique internal solution of this problem, and* $\|X\|_2 < \infty$*, then:*

$$\|X\|_2 \lesssim C_1 + C_2 \|G\|_{L_h^\infty},$$

where $C_1 = \frac{|B-A|}{\beta-\alpha} + |A| + |B|$ *and* $C_2 = \frac{3}{2}(\beta - \alpha) + (\beta - \alpha)^2$. [5]

Proof.

By Lemma (6.6), X_h' and X are S-continuous. To prove the estimate in (i), we use the obtained solution formula for (6.15) :

$$\|X\|_{L_h^\infty} \leq \frac{t-\alpha}{\beta-\alpha}|B| + \frac{\beta-t}{\beta-\alpha}|A| + |C|(\beta-\alpha) + \|G\|_{L_h^\infty} \sum_{s\in(\alpha,t)_h} (t-s)h.$$

The above hyperfinite Riemann sum can be estimated by:

$$\sum_{s\in(\alpha,t)_h} (t-s)h \approx \int_\alpha^t (t-s)\, ds = \frac{(t-\alpha)^2}{2} \leq \frac{(\beta-\alpha)^2}{2}.$$

Therefore:

$$\|X\|_{L_h^\infty} \lesssim |A| + |B| + |C|(\beta-\alpha) + \frac{(\beta-\alpha)^2}{2}\|G\|_{L_h^\infty},$$

where C is given by (6.12). Estimating C now, yields:

$$\begin{aligned} |C| &\leq \frac{\|G\|_{L_h^\infty}}{(\beta-\alpha)} \sum_{s\in(\alpha,t)_h} (t-s)h \\ &\lesssim \frac{\|G\|_{L_h^\infty}}{(\beta-\alpha)} \frac{(\beta-\alpha)^2}{2} \\ &= \frac{1}{2}(\beta-\alpha)\|G\|_{L_h^\infty}. \end{aligned}$$

Hence:

$$\|X\|_{L_h^\infty} \lesssim |A| + |B| + (\beta-\alpha)^2\|G\|_{L_h^\infty}.$$

[5] *Note that both* C_1 *and* C_2 *depend only on the boundary data of the linear problem.*

To estimate $\|X_h'\|_{L_h^\infty}$ we, first compute X_h':

$$
\begin{aligned}
hX_h'(t) &= X(t+h) - X(t) \\
&= \frac{1}{\beta-\alpha}\Big((t+h-\alpha)B + (\beta-t-h)A - (t-\alpha)B - (\beta-t)A\Big) \\
&\quad + C(t+h-\alpha) - C(t-\alpha) \\
&\quad + \sum_{s\in(\alpha,t+h)_h} (t+h-s)G(s-h)h - \sum_{s\in(\alpha,t)_h} (t-s)G(s-h)h \\
&= h\bigg(\frac{B-A}{\beta-\alpha} + C + G(t-h)h + \sum_{s\in(\alpha,t)_h} G(s-h)h\bigg)
\end{aligned}
$$

Then, and using our previous estimate of C again:

$$
\begin{aligned}
\|X_h'\|_{L_h^\infty} &\leq \frac{|B-A|}{\beta-\alpha} + |C| + \|G\|_{L_h^\infty} \sum_{s\in(\alpha,t+h)_h} h \\
&\lesssim \frac{|B-A|}{\beta-\alpha} + \frac{1}{2}(\beta-\alpha)\|G\|_{L_h^\infty} + (\beta-\alpha)\|G\|_{L_h^\infty} \\
&= \frac{|B-A|}{\beta-\alpha} + \frac{3}{2}(\beta-\alpha)\|G\|_{L_h^\infty}.
\end{aligned}
$$

From the equation $X_h'' = G$, we have $\|X_h''\|_{L_h^\infty} = \|G\|_{L_h^\infty}$. So, and putting together the previous estimates, we conclude that:

$$
\begin{aligned}
\|X\|_2 &= \|X\|_{L_h^\infty} + \|X_h'\|_{L_h^\infty} + \|X_h''\|_{L_h^\infty} \\
&\lesssim \frac{|B-A|}{\beta-\alpha} + |A| + |B| + \frac{3}{2}(\beta-\alpha)\|G\|_{L_h^\infty} + (\beta-\alpha)^2\|G\|_{L_h^\infty} \\
&= C_1 + C_2\|G\|_{L_h^\infty}.
\end{aligned}
$$

Assuming, in addition, that G is S-continuous, then $X_h'' = G$ becomes S-continuous as well.

6.4 An Existence and Convergence Result

We recall the main problem of this chapter,

$$\begin{cases} x'' = f(x', x, t) \quad \text{for } t \in I, \\ x(a) = x_0, \\ x(b) = x_1, \end{cases} \tag{6.16}$$

and its discrete counterpart,

$$\begin{cases} X''_h(t) = f(X'_h(t), X(t), t) \quad \text{for } t \in [a_h, b_h - 2h]_h, \\ X(a_h) = x_0, \\ X(b_h) = y_0. \end{cases} \tag{6.17}$$

To solve the problem (6.17), we are going to look at fixed points of a map defined as follows. Given $Y : I_h \to {}^*\mathbb{R}^n$ and $\theta \in {}^*[0,1]$, consider the *linear* problem:

$$\begin{cases} X''_h(t) = \theta f(Y'_h(t), Y(t), t) \quad \text{for } t \in [a_h, b_h - 2h]_h, \\ X(a) = x_0, \\ X(b) = y_0. \end{cases} \tag{6.18}$$

From Proposition (6.12), there is a unique solution to (6.18); it is actually given using formulas (6.14), (6.10) and (6.11), if we take $G = G_\theta \overset{def}{=} \theta f(Y'_h, Y, \cdot)$. Hence, we can define $\Phi : \mathbb{F}_h \times {}^*[0,1] \to \mathbb{F}_h$ so that $\Phi(Y, \theta)$ is the unique solution of (6.18). Then, a fixed point of $\Phi(\cdot, 1)$ is a solution of problem (6.18).

Our next task is to get bounds on $\Phi(X, \theta)$, that will make it possible to apply Corollary (5.15).

Proposition 6.14 *Let $\Phi : \mathbb{F}_h \to \mathbb{F}_h$ be the map defined above. Then, there exists a finite $r > 0$ such that Φ satisfies:*

(i) $\|\Phi(X, \theta)\|_2 \le r$;

(ii) Φ *is S-continuous (with respect to the norm* $\|(X,\theta)\| = \|X\|_2 + |\theta|$*).*

Proof.

Since $f : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is bounded, $\|f\|_{L^\infty_h} < \infty$. Hence, by Lemma (6.13):

$$\|\Phi(X,\theta)\|_2 \lesssim C_1 + C_2\theta\|f\|_{L^\infty_h}.$$

where C_1 and C_2 are finite. So, just pick $r \gtrapprox C_1 + C_2\theta\|f\|_{L^\infty_h}$.

As for (ii), consider $X, Y \in \mathbb{F}_h$ such that $\|X - Y\|_2 \approx 0$, and $\theta_1, \theta_2 \in {}^*[0,1]$ such that $\theta_1 \approx \theta_2$. Then, $\Phi(X,\theta_1) - \Phi(Y,\theta_2)$ is the solution, Z, of:

$$\begin{cases} Z_h''(t) = \theta_1 f(X_h'(t), X(t), t) - \theta_2 f(Y_h'(t), Y(t), t) \stackrel{\text{def}}{=} G(t) \quad \text{for } t \in [a, b-2h]_h, \\ Z(a) = X(a) - Y(a) = 0, \\ Z(b) = X(b) - Y(b) = 0. \end{cases}$$

Since $f : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is continuous and bounded, its star is S-continuous. Hence, G is S-continuous. Since $\|X - Y\|_2 \approx 0$, and $\theta_1 \approx \theta_2$, for all $t \in [a, b-h]_h$:

$$\begin{aligned} G(t) &= \theta_1 f(X_h'(t), X(t), t) - \theta_2 f(Y_h'(t), Y(t), t) \\ &\approx \theta_2 f(X_h'(t), X(t), t) - \theta_2 f(X_h'(t), X(t), t) \\ &= 0 \end{aligned}$$

Consequently, $\|G\|_{L^\infty_h} \approx 0$. Using Lemma (6.13):

$$\|Z\|_2 \lesssim \left(\frac{3}{2}(b-a) + (b-a)^2\right)\|G\|_{L^\infty_h} \approx 0.$$

Hence:

$$\|\Phi(X,\theta_1) - \Phi(Y,\theta_2)\| = \|Z\| \approx 0.$$

We now consider the non-standard hull of $(\mathbb{F}_h, \|\cdot\|_2)$. Let $\mathcal{E} = \mathrm{Hull}(\mathbb{F}_h, \|\cdot\|_2)$, and take $R > r$ finite, where r is given by Proposition (6.14). Let $\Omega = \mathrm{int}\,{}^\circ B_R(0) \subset \mathcal{E}$. By the same proposition, the map $\varphi \stackrel{\text{def}}{=} {}^\circ\Phi : \overline{\Omega} \times [0,1] \to \mathcal{E}$ is well-defined. Also, φ and Ω satisfy the setup of our degree theory.

We recall that Φ was obtained from an approximation scheme that uses right difference quotients. We do not want our results to be related only to this particular scheme, which is great for computing estimates like the one in Lemma (6.13), but is not particularly accurate for numerical computation.

However, many other schemes just give rise to other liftings of φ. Here is an example of a more accurate one.

Example 6.15 (Using central difference quotients) *In the discrete problem (6.17), we replace the one-sided differences by central differences and get:*

$$\begin{cases} \frac{X(t-h)-2X(t)+X(t+h)}{h^2} = f\left(\frac{X(t+h)-X(t-h)}{2h}, X(t), t\right) & \text{for } t \in (a,b)_h, \\ X(a) = x_0, \\ X(b) = y_0. \end{cases} \tag{6.19}$$

Note that

$$X_h''(t-h) = \frac{X(t-h) - 2X(t) + X(t+h)}{h^2},$$

and

$$\frac{1}{2}\Big(X_h'(t-h) + X_h'(t)\Big) = \frac{X(t+h) - X(t-h)}{2h},$$

so, problem (6.19) is equivalent to:

$$\begin{cases} X_h''(t) = f\left(\frac{1}{2}\left(X_h'(t) + X_h'(t+h)\right), X(t+h), t+h\right) & \text{for } t \in [a, b-2h]_h, \\ X(a) = x_0, \\ X(b) = y_0. \end{cases} \tag{6.20}$$

As before, define $\Psi : B_R(0) \times {}^*[a,b] \to \mathbb{F}_h$ — *with the same* R *as for* Φ *and* $B_R(0) \subset \mathbb{F}_h$) — *so that* $X = \Psi(Y,\theta)$ *is the solution of:*

$$\begin{cases} X_h''(t) = \theta f\left(\frac{1}{2}\left(Y_h'(t) + Y_h'(t+h)\right), Y(t+h), t+h\right) & \text{for } t \in [a, b-2h]_h, \\ X(a) = x_0, \\ X(b) = y_0. \end{cases}$$

To determine $\|\Psi - \Phi\|_2$*, we use Lemma (6.13). Fix* θ *and* Y*, and let:*

$$Z = \Psi(Y,\theta) - \Phi(Y,\theta).$$

Then, Z *satisfies*

$$Z_h''(t) = \theta f\left(\tfrac{1}{2}\left(Y_h'(t) + Y_h'(t+h)\right), Y(t+h), t+h\right) - \theta f\left(Y_h'(t), Y(t), t\right) \approx 0,$$

for all $t \in [a, b-2h]_h$*, and*

$$Z(a) = Z(b) = 0.$$

By Lemma (6.13):

$$\|\Psi(Y,\theta) - \Phi(Y,\theta)\|_2 = \|Z\|_2 \approx 0.$$

Therefore, ${}^\circ\Psi = {}^\circ\Phi = \varphi$*, as desired.*

The last example shows that φ actually is approximated by many more than the scheme we did work out for the best part of this chapter. Our degree theory will enable us to show that any scheme on $\mathbb{F}_h$ which approximates φ has a fixed point which approximates a solution of (6.16).

Theorem 6.16 *Let* Ψ *be* **any** **continuous lifting of* φ *on* $\mathbb{F}_h$*. Then, for any finite* $R > r$*:*

(i) There exists a fixed point of $\varphi(\cdot, 1)$ *in* $B_r(0)$.

(ii) There exists a fixed point of $\Psi(\cdot, 1)$ *in* $B_R(0) \subset \mathbb{F}_h$.

(iii) Let X *be a fixed point of* $\Psi \in B_R(0)$. *Then, the function* $x : [a, b] \to \mathbb{R}^n$ *given by*

$$x({}^\circ t) = {}^\circ(X(t)) \quad \forall\, t \in [a, b]_h,$$

is well-defined, belongs to $C^2([a, b], \mathbb{R}^n)$, *and is a solution of:*

$$\begin{cases} x''(t) = f(x'(t), x(t), t) \quad \textit{for } t \in [a, b], \\ x(a) = x_0, \\ x(b) = y_0. \end{cases}$$

Proof.

We are in the conditions of Corollary (5.15). Hence, there exists $\chi \in B_R(0) \subset \mathcal{E}$ such that $\chi = \varphi(\chi, 1)$. This shows (i). As for (ii), and since Ψ is a lifting of φ, the same Corollary ensures that there exists $X \subset B_R(0) \subset \mathbb{F}_h$ such that $X = \Psi(X, 1)$.

Now, and using the lifting Φ of φ previously defined in this section:

$$\|X - \Phi(X, 1)\|_2 \leq \|X - \Psi(X, 1)\|_2 + \|\Psi(X, 1) - \Phi(X, 1)\|_0 \approx 0. \tag{6.21}$$

Hence

$$\begin{aligned} X(a) &\approx \Phi(X, 1)(a) = x_0 \\ X(b) &\approx \Phi(X, 1)(b) = y_0 \end{aligned}$$

and

$$X_h''(t) \approx \Phi(X, 1)_h''(t) = f(X_h'(t), X(t), t),$$

for all $t \in [a, b-2h]_h$. This means that X_h'' is S-continuous. Hence, by Corollary (6.8), $x : [a,b] \to \mathbb{R}^n$ given by

$$x({}^\circ t) = {}^\circ(X(t)) \quad \forall\, t \in [a,b]_h,$$

is well-defined, and is in $C^2([a,b], \mathbb{R}^n)$, with

$$\begin{aligned} x'({}^\circ t) &= {}^\circ(X_h'(t)), \\ x''({}^\circ t) &= {}^\circ(X_h''(t)), \end{aligned}$$

for all $t \in [a,b]_h$. Therefore:

$$\begin{aligned} x(a) &= {}^\circ(X(a)) &= x_0, \\ x(b) &= {}^\circ(X(b)) &= y_0. \end{aligned}$$

Also:

$$\begin{aligned} x''({}^\circ t) &= {}^\circ(X_h''(t)) = {}^\circ\Big(f(X_h'(t), X(t), t)\Big) \\ &= f\Big({}^\circ(X_h'(t)), {}^\circ(X(t)), {}^\circ t\Big) = f\Big(x'({}^\circ t), x({}^\circ t), {}^\circ t\Big), \end{aligned}$$

for all $t \in [a,b]_h$. This concludes the proof.

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