Math 714: Assignment 0

This assignment will not be graded, and consists of several warm-up problems that can be used to test and refresh your mathematical and programming skills. You do not need to submit your answers.

1. Consider the recursive sequence $x_{k+1} = ax_k^2 + bx_k$ for $k \in \{0, 1, 2, 3, \ldots\}$.
   (a) If the sequence converges, to what value or values does it converge?
   (b) Suppose that x_0 is arbitrarily close to the value or values above. For what parameters (a, b) does the sequence x_k converge to that value?

2. The Chebyshev polynomials $T_k(x)$ can be defined using the recursive relation

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

   and $T_0(x) = 1$, $T_1(x) = x$. Evaluate and plot the Chebyshev polynomial of degree 5 at 101 evenly spaced points in the interval $x \in [-1, 1]$. Draw a 2D surface plot of the function $T_3(x)T_5(y)$ on a 101×101 grid on the domain $(x, y) \in [-1, 1]^2$.

3. The IEEE double-precision standard guarantees that for any mathematical operation $*$, the floating point operation $\circledast$ satisfies $x \circledast y = (1 + \delta)(x * y)$ where $|\delta| < \epsilon$ and ϵ is machine precision.
   (a) Calculate the minimum and maximum possible values of

$$S = \frac{3}{4+2} \tag{1}$$

   when evaluated using floating point arithmetic as $\tilde{S} = 3 \oslash (4 \oplus 2)$.
   (b) Show further that if $O(\epsilon^2)$ terms are neglected, then $|S - \tilde{S}| < \lambda\epsilon$, and determine the value of the constant λ.

4. For an invertible matrix A, define the condition number to be $\kappa(A) = \|A\| \, \|A^{-1}\|$ as discussed in the lectures. Assume that the matrix norm is defined using the Euclidean vector norm.
   (a) Find two 2×2 invertible matrices B and C such that $\kappa(B + C) < \kappa(B) + \kappa(C)$.
   (b) Find two 2×2 invertible matrices B and C such that $\kappa(B + C) > \kappa(B) + \kappa(C)$.
   (c) Suppose that A is a symmetric invertible matrix. Find $\kappa(A^2)$ in terms of $\kappa(A)$.[1]
   (d) Does the result from part (c) hold if A is not symmetric? Either prove the result, or find a counterexample.

[1] You may find it useful to recall that a symmetric matrix A can be written as $A = QDQ^{\mathsf{T}}$ where D is diagonal and Q is orthogonal.

(e) For invertible matrices B and C, prove that $\kappa(BC) \leq \kappa(B)\kappa(C)$. Find examples where $\kappa(BC) = \kappa(B)\kappa(C)$ and $\kappa(BC) < \kappa(B)\kappa(C)$.

5. The gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \tag{2}$$

and satisfies $(n-1)! = \Gamma(n)$ for integers n. Thus the function has the following values:

x	1	2	3	4	5	6
$\Gamma(x)$	1	1	2	6	24	120

In addition, $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$.

(a) For $k = 1, 2, 3, 4, 5$, calculate polynomials $p_k(x)$ of degree k that match $\Gamma(x)$ at the points $x = 1, 2, \ldots, k+1$. For each polynomial, evaluate the absolute error $|p_k(\frac{3}{2}) - \Gamma(\frac{3}{2})|$. Which of the polynomials p_k is most accurate?

(b) For $k = 1, 2, 3, 4, 5$, calculate polynomials $q_k(x)$ of degree k that match $\log(\Gamma(x))$ at the points $x = 1, 2, \ldots, k+1$. For each polynomial, evaluate the absolute error $|\exp(q_k(\frac{3}{2})) - \Gamma(\frac{3}{2})|$. Which of the polynomials q_k is most accurate?

(c) Examine the asymptotic behavior of $|p_k(\frac{3}{2}) - \Gamma(\frac{3}{2})|$ and $|q_k(\frac{3}{2}) - \Gamma(\frac{3}{2})|$ as $k \to \infty$.

6. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a fifth-differentiable function, and $h > 0$. By expanding Taylor series for $f(x-h)$ and $f(x+h)$, or otherwise, find coefficients $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$f''(x) = \frac{\alpha f(x-h) + \beta f(x) + \gamma f(x+h)}{h^2} + O(h). \tag{3}$$

(b) Suppose now that f' can also evaluated exactly. Find coefficients $a, b, c, r, s, t \in \mathbb{R}$ such that

$$f''(x) = \frac{af(x-h) + bf(x) + cf(x+h)}{h^2} + \frac{rf'(x-h) + sf'(x) + tf'(x+h)}{h} + O(h^4). \tag{4}$$

One approach is to expand Taylor series for f to terms in h^5, and Taylor series for f' to terms in h^4. Then the coefficients $\vec{b} = (a, b, c, r, s, t)$ can be found as the solutions of a linear system $A\vec{b} = \vec{q}$ for some $A \in \mathbb{R}^{6\times 6}$, and some $\vec{q} \in \mathbb{R}^6$.

(c) Consider the function

$$f(x) = e^{4 \sin x}. \tag{5}$$

Calculate f' and f'' analytically. Write a program to test your formulae for f'' in Eqs. (3) & (4) at $x = 1$, using grid spacings of $h = 2^{-k}$ for $k = 0, 1, 2, 3, \ldots, 23$.

For both formulae, make a log–log plot of the absolute error magnitude E as a function of h. In the regime where E is dominated by discretization error, fit the error to the form $E = Ch^p$ for some constants C and p. Discuss if the constants p for the two formulae are consistent with your answers to parts (a) and (b).