WISCONSIN
UNIVERSITY OF WISCONSIN-MADISON

UW–Madison Math/CS 714
Methods of Computational Mathematics I

Iterative methods II

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September 25, 2025

Efficiently solving elliptic PDEs

Consider m=2, A is SPD
$\nabla\phi(u_{k-1}) = Au_{k-1} - f \equiv -r_{k-1}$

Direct solve: find u^* that solves the linear system $Au = f$

Minimization problem: find u^* that minimizes the quadratic function $\phi(u) = \frac{1}{2}u^T Au - u^T f$

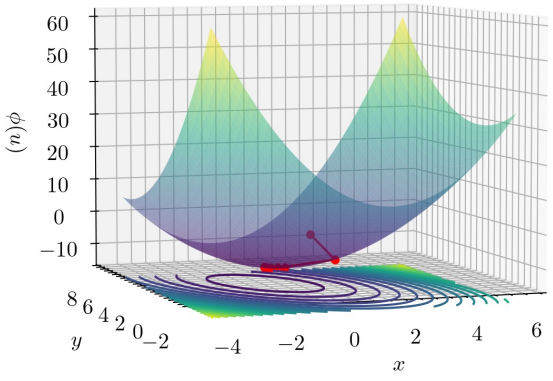
Steepest descent

Descent direction $-\nabla\phi(u_{k-1})$

Update rule $u_k = u_{k-1} - \alpha_{k-1}\nabla\phi(u_{k-1})$

Choose scalar α via a minimization problem (line search)
(Drop $k-1$ here for simplicity)

$$\min_{\alpha\in\mathbb{R},\alpha\geq 0} \phi(u_{k-1} - \alpha\nabla\phi(u_{k-1})) \Rightarrow \frac{d\phi(u+\alpha r)}{d\alpha} = 0 \Rightarrow \alpha = \frac{r^T r}{r^T A r}$$



At iteration $k-1$

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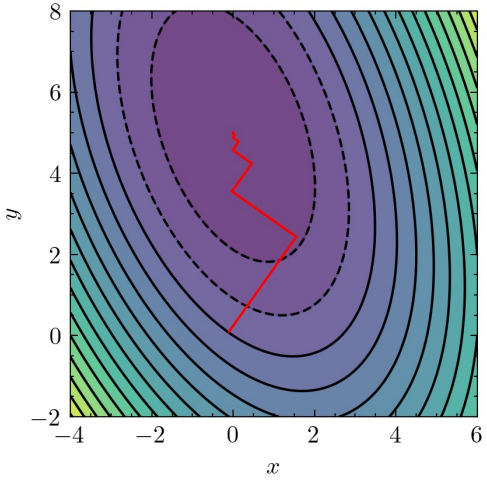
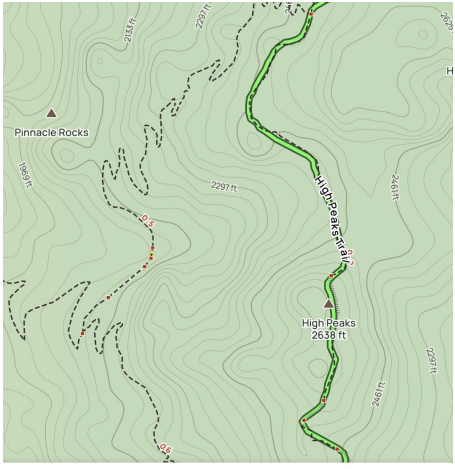


At iteration $k-1$

Steepest descent in a hike
at the Pinnacles National Park



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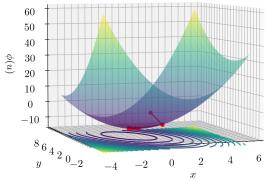
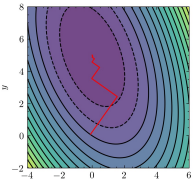
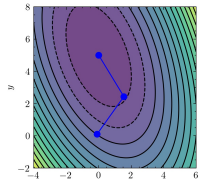
	Steepest descent	Conjugate gradient
Descent direction	$-\nabla\phi(u_{k-1})$	p_{k-1}
Update rule	$u_k = u_{k-1} - \alpha_{k-1}\nabla\phi(u_{k-1})$	$u_k = u_{k-1} + \alpha_{k-1}p_{k-1}$
Choose scalar α via a minimization problem (line search) (Drop $k-1$ here for simplicity)	$\min_{\alpha\in\mathbb{R},\alpha\geq 0} \phi(u_{k-1} - \alpha\nabla\phi(u_{k-1})) \Rightarrow \frac{d\phi(u+\alpha r)}{d\alpha} = 0 \Rightarrow \alpha = \frac{r^T r}{r^T A r}$	$\min_{\alpha\in\mathbb{R},\alpha\geq 0} \phi(u_{k-1} + \alpha p_{k-1}) \Rightarrow \frac{d\phi(u+\alpha p)}{d\alpha} = 0 \Rightarrow \alpha = \frac{p^T r}{p^T A p}$







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