UW–Madison Math/CS 714

Methods of Computational Mathematics I

Initial value problems II

Instructor: Yue Sun (yue.sun@wisc.edu)

October 2, 2025

Convergence

As for boundary value problems, we would like to ensure that our numerical method will converge to the exact solution.

For a fixed time $T > 0$, we would like our numerical solution U^N to approach $u(T)$. Define $N = T/k$. Then convergence means that

$$\lim_{k \to 0,\, Nk=T} U^N = u(T).$$

It is possible that a method may converge for one problem but not for another. For a mathematical definition, we would like to ensure that a method converges for all problems in a reasonably large class and for all reasonable starting values.

Convergence

Let $U^n(k)$ be the numerical solution with step size k. For reasonable starting values, we might require that $U^\nu(k)$ should approximate $u(\nu k)$ for $\nu = 0, 1, \ldots, r-1$.

Since $k \to 0$, a weaker condition that we could impose is

$$\lim_{k \to 0} U^\nu(k) = \eta \tag{1}$$

for $\nu = 0, 1, \ldots, r-1$, where $u(0) = \eta$ is our initial condition.

Convergence

Definition. An r-step method is convergent if applying the method to any ODE $u'(t) = f(u(t), t)$ where $f(u, t)$ is Lipschitz continuous in u, and with any set of starting values satisfying Eq. (1), we obtain convergence in the sense that

$$\lim_{k \to 0,\, Nk = T} U^N = u(T)$$

for any $T > 0$ where the ODE has a unique solution.

Convergence for a test problem

Consider the linear ODE

$$u'(t) = \lambda u(t) + g(t)$$

where $u(t_0) = \eta$. This has general solution

$$u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau) d\tau.$$

Euler's method for the linear problem

Applying Euler's method to the test problem gives

$$U^{n+1} = U^n + k(\lambda U^n + g(t_n)) = (1 + k\lambda)U^n + kg(t_n).$$

The local truncation error is

$$\tau^n = \left(\frac{u(t_{n+1}) - u(t_n)}{k}\right) - (\lambda u(t_n) + g(t_n)) = \frac{k}{2}u''(t_n) + O(k^2).$$

Since

$$u(t_{n+1}) = (1 + k\lambda)u(t_n) + kg(t_n) + k\tau^n,$$

then defining global error as $E^n = U^n - u(t_n)$ gives

$$E^{n+1} = (1 + k\lambda)E^n - k\tau^n.$$

Euler's method for the linear problem

Applying this recursively shows that

$$\begin{aligned} E^n &= (1 + k\lambda)E^{n-1} - k\tau^{n-1} \\ &= (1 + k\lambda)\left[(1 + k\lambda)E^{n-2} - k\tau^{n-2}\right] - k\tau^{n-1} \end{aligned}$$

and

$$E^n = (1 + k\lambda)E^0 - k\sum_{m=1}^{n}(1 + k\lambda)^{n-m}\tau^{m-1}.$$

Note that $|1 + k\lambda| \leq e^{k|\lambda|}$ and hence

$$(1 + k\lambda)^{n-m} \leq e^{(n-m)k|\lambda|} \leq e^{nk|\lambda|} \leq e^{|\lambda|T}$$

where $t_n = nk \leq T$.

Euler's method for the linear problem

Then

$$
\begin{aligned}
|E^n| &\le e^{|\lambda| T}\left(|E_0| + k\sum_{m=1}^{n} |\tau^{m-1}|\right) \\
&\le e^{|\lambda| T}\left(|E_0| + nk \max_{m\in\{1,\ldots,n\}} |\tau^{m-1}|\right)
\end{aligned}
$$

Let $N = T/k$ be the number of timesteps to reach T. Define

$$
\|\tau\|_\infty = \max_{0\le n\le N-1} |\tau^n|
$$

where we expect from our previous analysis that

$$
\|\tau\|_\infty \approx \frac{k}{2}\|u''\|_\infty = O(k)
$$

where $\|u''\|_\infty$ is the maximum of u'' over $[0, T]$.

Euler's method for the linear problem

Then for $t = nk \le T$,

$$|E^n| \le e^{|\lambda| T}(|E^0| + T\|\tau\|_\infty).$$

From the condition about starting values, $|E^0| \to 0$ as $k \to 0$. Indeed, if U^0 is chosen to be $u(t_0)$, then $E^0 = 0$. Hence

$$|E^n| \le e^{|\lambda| T} T\|\tau\|_\infty = O(k)$$

as $k \to 0$. Hence the method converges and is first order accurate.

Euler's method for general problems

Suppose the same analysis is applied to $u' = f(u, t)$ where $f(u, t)$ is Lipschitz continuous with constant L. Then[1]

$$|E^n| \leq e^{LT} T \|\tau\|_\infty = O(k).$$

The bound exponentially diverges, although in practice the numerical errors are much smaller than this.

For general one-step methods, if the local truncation error is $O(k^p)$ then the global error will also be $O(k^p)$. Hence the method converges as $k \to 0$, which is referred to as zero-stability.

[1] See the ODE workshop slides for more detail.

Zero-stability for multistep methods

For multistep methods, the previous results do not apply directly. Consider the following two-step example:

$$U^{n+2} - 3U^{n+1} + 2U^n = -kf(U^n).$$

The truncation error is

$$\begin{aligned}\tau^n &= \frac{u(t_{n+2}) - 3u(t_{n+1}) + 2u(t_n) - ku'(t_n)}{k} \\ &= \frac{5k}{2}u''(t_n) + O(k^2).\end{aligned}$$

so the method is consistent.

Stability of a multistep scheme

Consider applying this method to $u' = 0$ with $u(0) = 0$. The solution is given by

$$U^{n+2} - 3U^{n+1} + 2U^n = 0.$$

Two starting values are needed. If $U^0 = U^1 = 0$, then $U^n = 0$ for all n, and the numerical results match the exact solution.

But suppose that $U^1 = \epsilon \neq 0$ due to some numerical error. Then we can show that

$$U^n = \epsilon(2^n - 1)$$

which blows up exponentially. Similar results would be seen when applying this to any ODE, showing that the method is not stable.

Linear difference equations

The previous numerical example involves a linear difference equation which can be written generally as

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = 0.$$

Consider looking for a solution of the form $U^n = \zeta^n$ (where the n on the RHS is a power). Then

$$\sum_{j=0}^{r} \alpha_j \zeta^{n+j} = 0$$

and hence

$$\sum_{j=0}^{r} \alpha_j \zeta^{j} = \rho(\zeta) = 0,$$

where $\rho(\zeta)$ is the first characteristic polynomial introduced previously.

Linear difference equations

In general ρ can be factorized as

$$\rho(\zeta) = \alpha_r(\zeta - \zeta_1)(\zeta - \zeta_2)\dots(\zeta - \zeta_r)$$

where the roots ζ_j may be complex. Since the difference equation is linear, any linear combination of solutions will satisfy it.

Assuming the roots are all distinct, the general solution is

$$U^n = c_1\zeta_1^n + c_2\zeta_2^n + \dots + c_r\zeta_r^n.$$

Given initial data $U^0, U^1, \dots, U^{r-1}$, the constants c_j can be determined.

Linear difference equations

For the example difference equation $U^{n+2} - 3U^{n+1} + 2U^n = 0$, the characteristic polynomial factorizes as

$$\rho(\zeta) = 2 - 3\zeta + \zeta^2 = (\zeta - 1)(\zeta - 2),$$

so the roots are $\zeta_1 = 1$ and $\zeta_2 = 2$. The general solution is therefore

$$U^n = c_1 + c_2 2^n,$$

which matches is consistent with the solution presented previously.

In general, the presence of a root where $|\zeta_j| > 1$ will lead to an exponentially divergent solution.

Repeated roots

More generally, it is possible that some roots may be repeated. Suppose that the distinct roots are $\zeta_1, \ldots, \zeta_l$ where $l \leq r$, and let m_j be the multiplicity of ζ_j. Hence

$$\sum_{j=1}^{l} m_j = r$$

The general solution of the difference equation is

$$U^n = \sum_{j=1}^{l} p_j(n) \zeta_j^n$$

where p_j is a polynomial of degree $m_j - 1$.

Repeated roots example

Consider the linear multistep method

$$U^{n+2} - 2U^{n+1} + U^n = \frac{k(f(U^{n+2}) + f(U^n))}{2}.$$

The characteristic polynomial is

$$\rho(\zeta) = \zeta^2 - 2\zeta + 1 = (\zeta - 1)^2$$

and so $\zeta_1 = 1$ is a root with multiplicity 2. Therefore the general solution is

$$U^n = p_1(n)\zeta_1^n$$

where $p_1(n)$ is a polynomial of degree 1, so

$$U^n = c_1 + c_2 n.$$

In general, a repeated root with $|\zeta_j| = 1$ is enough to create an algebraic divergence.

The root condition

This leads to the follow theorem for determining zero-stability.

Theorem. An r-step linear multistep method is said to be zero-stable if the roots ζ_j of the characteristic polynomial $\rho(\zeta)$ satisfy:

- $|\zeta_j| \leq 1$ for all j,
- If ζ_j is a repeated root, then $|\zeta_j| < 1$.

Examples

The program *z_stabilty.py* integrates three different multistep schemes:

- Stable:

$$24U^{n+3} - 24U^{n+2} = k(9f(U^{n+3}) + 19f(U^{n+2}) - 5f(U^{n+1}) + f(U^n))$$

- Exponentially unstable:

$$\begin{aligned}11U^{n+3} + 27U^{n+2} - 27U^{n+1} - 11U^n \\ = 3k(f(U^{n+3}) + 9f(U^{n+2}) + 9f(U^{n+1}) + f(U^n))\end{aligned}$$

- Algebraically unstable:

$$U^{n+3} + U^{n+2} - U^{n+1} - U^n = 2k(f(U^{n+2}) + f(U^{n+1}))$$

Convergence

A theorem due to Dahlquist shows that for linear multistep methods

$$\text{(consistency)} \quad + \quad \text{(zero-stability)} \quad \Longleftrightarrow \quad \text{(convergence)}$$

This is an asymptotic statement about the case when $k \to 0$. In practice, obtaining a convergence may also depend on the size of the timestep k.