Notes 10 : Method of moments

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References: [Dur10, Section 3.3].

1 The moment problem

Suppose that

$$\int x^k \mathrm{d}F_n(x) \to \mu_k < \infty, \quad \forall k.$$

Then the following lemma guarantees the sequence of DFs is tight.

LEM 10.1 *If there is $\phi \geq 0$ so that $\phi(x) \to +\infty$ as $|x| \to \infty$ and*

$$C = \sup_n \int \phi(x)\mathrm{d}F_n(x) < \infty,$$

then F_n is tight.

Proof: Note that

$$C \geq (1 - F_n(M) + F_n(-M)) \inf_{|x|\geq M} \phi(x).$$

∎

So every subsequential limit is a DF. Moreover, by the next lemma, the moments of the limit are given by the μ_k's. (Take say $h(x) = x^k$ and $g(x) = x^{2k}$.)

LEM 10.2 *Suppose g, h are continuous with $g(x) > 0$ and*

$$\frac{|h(x)|}{g(x)} \to 0, \quad \text{as } |x| \to \infty.$$

If $F_n \Rightarrow F$ and

$$\int g(x)\mathrm{d}F_n(x) \leq C < \infty,$$

then

$$\int h(x)\mathrm{d}F_n(x) \to \int h(x)\mathrm{d}F(x).$$

Proof: We use the method of the single probability space. Take $Y_n \to Y$ a.s. with $Y_n \sim F_n$ and $Y \sim F$. The result then follows from Theorem 1.6.8 in [D] which is the same as the above for a.s. convergence. ■

So the sequence converges weakly if there is only one DF with these moments (because every subsequence has a further subsequence converging to this unique distribution). However, in general, there can be more than one DF with the same moments.

EX 10.3 (Lognormal density) *Let*

$$f_0(x) = \frac{1}{\sqrt{2\pi}x} \exp\left(-(\log x)^2/2\right), \quad x > 0,$$

and for $-1 \leq a \leq 1$ *let*

$$f_a(x) = f_0(x)[1 + a\sin(2\pi \log x)].$$

CLAIM 10.4

$$\int_0^\infty x^r f_0(x)\sin(2\pi\log x)\mathrm{d}x = 0, \quad r = 0, 1, 2, \ldots$$

Proof: *Indeed, let* $x = e^{s+r}$ *so that* $s = \log x - r$ *and* $\mathrm{d}s = \mathrm{d}x/x$ *and*

$$\begin{aligned}
&\int_0^\infty x^r f_0(x)\sin(2\pi\log x)\mathrm{d}x \\
&\quad = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \exp(rs + r^2)\exp(-(s+r)^2/2)\sin(2\pi(s+r))\mathrm{d}s \\
&\quad = (2\pi)^{-1/2}\exp(r^2/2)\int_{-\infty}^{+\infty} \exp(-s^2/2)\sin(2\pi s)\mathrm{d}s,
\end{aligned}$$

using that r *is integer.* ■

2 A sufficient condition

The following condition is sufficient.

THM 10.5 *If*

$$\limsup_k \frac{\mu_{2k}^{1/2k}}{2k} = r < \infty, \tag{1}$$

then there is at most one DF F *with*

$$\mu_k = \int x^k \mathrm{d}F(x),$$

for all positive integers k.

Proof: The idea of the proof is to show that two DFs with the same moments satisfying (1) must have the same CF.

Let F be any DF with moments μ_k.

CLAIM 10.6 *For any θ*

$$\phi(\theta + t) = \phi(\theta) + \sum_{m=1}^{\infty} \frac{t^m}{m!}\phi^{(m)}(\theta), \quad \forall |t| < \frac{1}{er}.$$

The result then follows by a continuation argument. Indeed, assume there is another DF G with the same moments and CF $\psi(t)$. Since $\phi(0) = \psi(0) = 1$, using induction we get that $\phi(t) = \psi(t)$ for $|t| \leq k/3r$ for all k. So they must be equal, and hence $F = G$. We prove the claim.

Proof: Recall:

LEM 10.7 *We have*

$$\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right).$$

Then

$$\left| e^{i\theta X} \left(e^{itX} - \sum_{m=0}^{n-1} \frac{(itX)^m}{m!} \right) \right| \leq \frac{|tX|^n}{n!}.$$

LEM 10.8 *If μ has $\int |x|^n \mu(\mathrm{d}x) < \infty$ then its CF $\phi(t)$ has continuous derivative of order n given by*

$$\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(\mathrm{d}x).$$

Let

$$\nu_k = \int |x|^k \mathrm{d}F(x).$$

Taking expectations above,

$$\left| \phi(\theta + t) - \phi(\theta) - t\phi'(\theta) - \cdots - \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(\theta) \right| \leq \frac{|t|^n}{n!}\nu_n.$$

Cauchy-Schwarz implies that $\nu_{2k+1}^2 \leq \mu_{2k}\mu_{2k+2}$ so that

$$\limsup_k \frac{\nu_k^{1/k}}{k} \leq r.$$

(Take a further subsequence with fixed parity. If it is even, we are done. Otherwise, use that the bounding subsequence has a limit and readjust the denominator and the exponent. See Billingsley.) Hence,

$$\nu_k \leq (r+\varepsilon)^k k^k \leq (r+\varepsilon)^k e^k k!.$$

■ ■

EX 10.9 (Lognormal density continued) *The moments of the lognormal are*

$$\mathbb{E}[X^n] = \mathbb{E}[e^{nZ}] = e^{n^2/2},$$

by taking $t = in$ in the CF of the normal.

DEF 10.10 (Carleman's condition) *A more general sufficient condition is the so-called* Carleman's condition

$$\sum_{k=1}^{\infty} \frac{1}{\mu_{2k}^{1/2k}} = \infty.$$

3 The method of moments

Finally, we get the following:

THM 10.11 *Suppose $\int x^k \mathrm{d}F_n(x)$ has a limit μ_k for each k with*

$$\limsup_k \frac{\mu_{2k}^{1/2k}}{2k} = r < \infty, \tag{2}$$

then F_n converges weakly to the unique distribution with these moments.

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.