Notes 11 : Infinitely divisible and stable laws

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References: [Dur10, Section 3.7, 3.8], [Shi96, Section III.6].

1 Infinitely divisible distributions

Recall:

EX 11.1 (Normal distribution) *Let $Z_1, Z_2 \sim \mathrm{N}(0,1)$ independent then*

$$\phi_{Z_1+Z_2}(t) = \phi_{Z_1}(t)\phi_{Z_2}(t) = e^{-t^2},$$

and $Z_1 + Z_2 \sim \mathrm{N}(0,2)$.

EX 11.2 (Poisson distribution) *Similarly, if $Y_1, Y_2 \sim \mathrm{Poi}(\lambda)$ independent then*

$$\phi_{Y_1+Y_2}(t) = \phi_{Y_1}(t)\phi_{Y_2}(t) = \exp\left(2\lambda(e^{it} - 1)\right),$$

and $Y_1 + Y_2 \sim \mathrm{Poi}(2)$.

DEF 11.3 (Infinitely divisible distributions) *A DF F is* infinitely divisible *if there for all $n \geq 1$, there is a DF F_n such that*

$$Z =_d X_{n,1} + \cdots + X_{n,n},$$

where $Z \sim F$ and the $X_{n,k}$'s are IID with $X_{n,k} \sim F_n$ for all $1 \leq k \leq n$.

THM 11.4 *A DF F can be a limit in distribution of IID triangular arrays*

$$Z_n = X_{n_1} + \cdots + X_{n,n},$$

if and only if F is infinitely divisible.

Proof: One direction is obvious. If F is infinitely divisible, then we can make $Z =_d Z_n$ for all n with $Z \sim F$.

For the other direction, assume $Z_n \Rightarrow Z$. Write

$$Z_{2n} = (X_{2n,1} + \cdots + X_{2n,n}) + (X_{2n,n+1} + \cdots + X_{2n,2n}) = Y_n + Y_n'.$$

Clearly Y_n and Y_n' are independent and identically distributed. We claim further that they are tight. Indeed

$$\mathbb{P}[Y_n > y]^2 = \mathbb{P}[Y_n > y]\mathbb{P}[Y_n' > y] \leq P[Z_{2n} > 2y]$$

and Z_n converges to a probability distribution by assumption. Now take a subsequence of (Y_n) so that $Y_{n_k} \Rightarrow Y$ (and hence $Y_{n_k}' \Rightarrow Y'$) then, using CFs,

$$Y_{n_k} + Y_{n_k}' \Rightarrow Y + Y',$$

where Y, Y' are independent. Since $Z_n \Rightarrow Z$ it follows that

$$Z =_d Y + Y'.$$

■

EX 11.5 (Normal distribution) *To test whether a DF F is infinitely divisible, it suffices to check that*

$$(\phi_F(t))^{1/n} \text{ is a CF, } \forall n.$$

In the $\mathrm{N}(0,1)$ *case,*

$$(\phi_F(t))^{1/n} = e^{-t^2/(2n)},$$

which is the CF of a $\mathrm{N}(0, 1\sqrt{n})$.

EX 11.6 (Poisson distribution) *Similarly, in the* $F \sim \mathrm{Poi}(\lambda)$ *case,*

$$(\phi_F(t))^{1/n} = \exp(\lambda(e^{it} - 1)/n),$$

is the CF of a $\mathrm{Poi}(\lambda/n)$.

EX 11.7 (Gamma distribution) *Let F be a* $\mathrm{Gamma}(\alpha, \beta)$ *distribution, that is, with density*

$$f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}\mathbb{1}_{x\geq 0},$$

for $\alpha, \beta > 0$, and CF

$$\phi(t) = \frac{1}{(1 - i\beta t)^\alpha}.$$

Hence

$$(\phi(t))^{1/n} = \frac{1}{(1 - i\beta t)^{\alpha/n}},$$

which is the CF of a $\mathrm{Gamma}(\alpha/n, \beta)$.

It is possible to characterize all infinitely divisble distributions. We quote the following without proof. See Breiman's or Feller's books.

THM 11.8 (Lévy-Khinchin Theorem) *A DF F is infinitely divisible if and only if $\phi_F(t)$ is of the form $e^{\psi(t)}$ with*

$$\psi(t) = it\beta - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \mu(\mathrm{d}x),$$

where $\beta \in \mathbb{R}$, $\sigma^2 \geq 0$, and μ is a measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\mu(\mathrm{d}x) < +\infty$. (The previous ratio is ≤ 1 so any finite measure will work.)

EX 11.9 (Normal distribution) *Taking $\beta = 0$, $\sigma^2 = 1$, and $\mu \equiv 0$ gives a* $\mathrm{N}(0,1)$.

EX 11.10 (Poisson distribution) *Taking $\sigma^2 = 0$, $\mu(\{1\}) = \lambda$, and $\beta = \int \frac{x}{1+x^2}\mu(\mathrm{d}x)$ gives a* $\mathrm{Poi}(\lambda)$.

EX 11.11 (Compound Poisson distribution) *Let W be a RV with CF $\phi(t)$. A compound Poisson RV with parameters λ and ϕ is*

$$Z = \sum_{i=1}^{N} W_i,$$

where the W_i's are IID with CF ϕ and $N \sim \mathrm{Poi}(\lambda)$. By direct calculation,

$$\phi_Z(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\phi(t)^n = \exp(\lambda(\phi(t) - 1)).$$

Z is infinitely divisible for the same reason that the Poisson distribution is. Taking $\sigma^2 = 0$, $\mu(\{1\}) = \lambda \times$ PM of Z, and $\beta = \int \frac{x}{1+x^2}\mu(\mathrm{d}x)$ gives ϕ_Z. (The previous integral exists for PMs because when x is large the ratio is essentially $1/x$.)

2 Stable laws

An important special case of infinitely divisible distributions arises when the triangular array takes the special form $X_k, \forall n, k,$, up to a normalization that depends on n. E.g., the simple form of the CLT.

DEF 11.12 *A DF F is* stable *if for all n*

$$Z =_d \frac{\sum_{i=1}^n W_i - b_n}{a_n},$$

where $Z, (W_i)_i \sim F$ and the W_i's are independent and $a_n > 0, b_n \in \mathbb{R}$ are constants.

Note that this applies to Gaussians but not to Poisson variables (which are integer-valued).

THM 11.13 *Let F be a DF and $Z \sim F$. A necessary and sufficient condition for the existence of constants $a_n > 0$, b_n such that*

$$Z_n = \frac{S_n - b_n}{a_n} \Rightarrow Z,$$

where $S_n = \sum_{i=1}^n X_i$ with X_i's IID is that F is stable.

Proof: We prove the non-trivial direction. Assume

$$Z_n = \frac{S_n - b_n}{a_n} \Rightarrow Z,$$

as above. Letting

$$S_n^{(1)} = \sum_{1 \leq i \leq n} X_i,$$

and

$$S_n^{(2)} = \sum_{n+1 \leq i \leq 2n} X_i.$$

Then

$$Z_{2n} = \frac{1}{a_n^{-1} a_{2n}} \left(\left\{ \frac{S_n^{(1)} - b_n}{a_n} \right\} + \left\{ \frac{S_n^{(2)} - b_n}{a_n} \right\} - \frac{2b_n - b_{2n}}{a_n} \right).$$

The LHS and both terms in bracket converge weakly. One has to show that the constants converge.

LEM 11.14 (Convergence of types) *If $W_n \Rightarrow W$ and there are constants $\alpha_n > 0$, β_n so that $W_n' = \alpha_n W_n + \beta_n \Rightarrow W'$ where W and W' are nondegenerate, then there are constants α and β so that $\alpha_n \to \alpha$ and $\beta_n \to \beta$.*

See [D] for proof. It requires an exercise done in homework. ■

Finally there is also a characterization for stable laws.

THM 11.15 (Lévy-Khinchin Representation) *A DF F is stable if and only if the CF $\phi_F(t)$ is of the form $e^{\psi(t)}$ with*

$$\psi(t) = it\beta - d|t|^\alpha \left(1 + i\kappa \operatorname{sgn}(t) G(t,\alpha)\right),$$

where $0 < \alpha \leq 2$, $\beta \in \mathbb{R}$, $d \geq 0$, $|\kappa| \leq 1$, and

$$G(t,\alpha) = \begin{cases} \tan \frac{1}{2}\pi\alpha & \textit{if } \alpha \neq 1, \\ \frac{2}{\pi} \log |t| & \textit{if } \alpha = 1. \end{cases}$$

The parameter α is called the index *of the stable law.*

EX 11.16 *Taking $\alpha = 2$ gives a Gaussian.*

Note that a DF is symmetric if and only if

$$\phi(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[e^{-itX}] = \overline{\phi(t)},$$

or, in other words, $\phi(t)$ is real.

THM 11.17 (Lévy-Khinchin representation: Symmetric case) *A DF F is symmetric and stable if and only if its CF is of the form $e^{-d|t|^\alpha}$ where $0 < \alpha \leq 2$ and $d \geq 0$. The parameter α is called the* index *of the stable law.*

$\alpha = 2$ is the Gaussian case. $\alpha = 1$ is the Cauchy case.

2.1 Domain of attraction

DEF 11.18 (Domain of attraction) *A DF F is in the* domain of attraction *of a DF G if there are constants $a_n > 0$ and b_n such that*

$$\frac{S_n - b_n}{a_n} \Rightarrow Z,$$

where $Z \sim G$ and

$$S_n = \sum_{k \leq n} X_k,$$

with $X_k \sim F$ and independent.

DEF 11.19 (Slowly varying function) *A function L is* slowly varying *if for all $t > 0$*

$$\lim_{x \to +\infty} \frac{L(tx)}{L(x)} = 1.$$

EX 11.20 *If* $L(t) = \log t$ *then*

$$\frac{L(tx)}{L(x)} = \frac{\log t + \log x}{\log x} \to 1.$$

On the other hand, if $L(t) = t^\varepsilon$ *with* $\varepsilon > 0$ *then*

$$\frac{L(tx)}{L(x)} = \frac{t^\varepsilon x^\varepsilon}{x^\varepsilon} \to t^\varepsilon.$$

THM 11.21 *Suppose* $(X_n)_n$ *are IID with a distribution that satisfies*

1. $\lim_{x\to\infty} \mathbb{P}[X_1 > x]/\mathbb{P}[|X_1| > x] = \theta \in [0,1]$
2. $\mathbb{P}[|X_1| > x] = x^{-\alpha}L(x)$,

where $0 < \alpha < 2$ *and* L *is slowly varying. Let* $S_n = \sum_{k\le n} X_k$,

$$a_n = \inf\{x \,:\, \mathbb{P}[|X_1| > x] \le n^{-1}\}$$

(Note that a_n *is roughly of order* $n^{1/\alpha}$*.) and*

$$b_n = n\mathbb{E}[X_1 \mathbb{1}_{|X_1|\le a_n}].$$

Then

$$\frac{S_n - b_n}{a_n} \Rightarrow Z,$$

where Z *has an* α*-stable distribution.*

This condition is also necessary. Note also that this extends in some sense the CLT to some infinite variance cases when $\alpha = 2$. See the infinite variance example in Section 3.4 on [D].

EX 11.22 (Cauchy distribution) *Let* $(X_n)_n$ *be IID with a density symmetric about* 0 *and continuous and positive at* 0. *Then*

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \Rightarrow \text{a Cauchy distribution}.$$

Clearly $\theta = 1/2$. *Moreover*

$$\mathbb{P}[1/X_1 > x] = \mathbb{P}[0 < X_1 < x^{-1}] = \int_0^{x^{-1}} f(y)\mathrm{d}y \sim f(0)/x,$$

and similarly for $\mathbb{P}[1/X_1 < -x]$. *So* $\alpha = 1$. *Clearly* $b_n = 0$ *by symmetry. Finally* $a_n \sim 2f(0)n$.

EX 11.23 (Centering constants) *Suppose* $(X_n)_n$ *are IID with for* $|x| > 1$

$$\mathbb{P}[X_1 > x] = \theta x^{-\alpha}, \quad \mathbb{P}[X_1 < -x] = (1-\theta)x^{-\alpha},$$

where $0 < \alpha < 2$*. In this case* $a_n = n^{1/\alpha}$ *and*

$$b_n = n \int_1^{n^{1/\alpha}} (2\theta - 1)\alpha x^{-\alpha} \mathrm{d}x \sim \begin{cases} cn & \alpha > 1 \\ cn \log n & \alpha = 1 \\ cn^{1/\alpha} & \alpha < 1. \end{cases}$$

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[Shi96] A. N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.