Notes 12 : Random Walks

Math 733-734: Theory of Probability — *Lecturer: Sebastien Roch*

References: [Dur10, Section 4.1, 4.2, 4.3].

1 Random walks

DEF 12.1 *A* stochastic process (SP) *is a collection* $\{X_t\}_{t\in\mathcal{T}}$ *of* $(E,\mathcal{E})$*-valued random variables on a triple* $(\Omega,\mathcal{F},\mathbb{P})$*, where* $\mathcal{T}$ *is an arbitrary* index set*. For a fixed* $\omega\in\Omega$*,* $\{X_t(\omega)\,:\,t\in\mathcal{T}\}$ *is called a* sample path.

EX 12.2 *When* $\mathcal{T}=\mathbb{N}$ *or* $\mathcal{T}=\mathbb{Z}_+$ *we have a* discrete-time SP. *For instance,*

- $X_1, X_2, \ldots$ *iid RVs*
- $\{S_n\}_{n\geq 1}$ *where* $S_n=\sum_{i\leq n} X_i$ *with* X_i *as above*

We let

$$\mathcal{F}_n=\sigma(X_1,\ldots,X_n)$$

(the information known up to time n*).*

DEF 12.3 *A* random walk (RW) *on* $\mathbb{R}^d$ *is an SP of the form:*

$$S_n=S_0+\sum_{i\leq n} X_i,\ n\geq 1$$

where the X_i*s are iid in* $\mathbb{R}^d$*, independent of* S_0*. The case* X_i *uniform in* $\{-1,+1\}$ *is called simple random walk (SRW).*

EX 12.4 *When* $d=1$*, recall that*

- *SLLN:* $n^{-1}S_n\to\mathbb{E}[X_1]$ *when* $\mathbb{E}|X_1|<+\infty$
- *CLT:*
$$\frac{S_n-n\mathbb{E}[X_1]}{\sqrt{n\mathrm{Var}[X_1]}}\Rightarrow N(0,1),$$
when $\mathbb{E}[X_1^2]<\infty$.

These are examples of limit theorems. Sample path properties, on the other hand, involve properties of the sequence $S_1(\omega), S_2(\omega), \ldots$. For instance, let $A \subset \mathbb{R}^d$

- $\mathbb{P}[S_n \in A$ *for some* $n \geq 1]$*?*
- $\mathbb{P}[S_n \in A \text{ i.o.}]$*?*
- $\mathbb{E}[T_A]$*, where* $T_A = \inf\{n \geq 1 \,:\, S_n \in A\}$*?*

1.1 Stopping times

The examples above can be expressed in terms of stopping times:

DEF 12.5 *A random variable* $T \,:\, \Omega \to \overline{\mathbb{Z}}_+ \equiv \{0, 1, \ldots, +\infty\}$ *is called a* stopping time *if*

$$\{T \leq n\} \in \mathcal{F}_n,\ \forall n \in \overline{\mathbb{Z}}_+,$$

or, equivalently,

$$\{T = n\} \in \mathcal{F}_n,\ \forall n \in \overline{\mathbb{Z}}_+.$$

(To see the equivalence, note

$$\{T = n\} = \{T \leq n\} \setminus \{T \leq n-1\},$$

and

$$\{T \leq n\} = \cup_{i \leq n}\{T = i\}.)$$

A stopping time is a time at which one decides to stop the process. Whether or not the process is stopped at time n depends only on the history up to time n.

EX 12.6 *Let* $\{S_n\}$ *be a RW and* $B \in \mathcal{B}$*. Then*

$$T = \inf\{n \geq 1 \,:\, S_n \in B\},$$

is a stopping time. This example is also called the hitting time *of* B*. (Replacing the* $\inf$ *with a* $\sup$ *(over a finite time interval say) would be a typical example of something that is not a stopping time.)*

1.2 Wald's First Identity

Throughout, for $X_1, X_2, \ldots \in \mathbb{R}$

$$S_n = \sum_{i=1}^{n} X_i.$$

THM 12.7 *Let $X_1, X_2, \ldots \in L^1$ be iid with $\mathbb{E}[X_1] = \mu$ and let $T \in L^1$ be a stopping time. Then*

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T].$$

Proof: Let

$$U_T = \sum_{i=1}^{T} |X_i|.$$

Observe

$$\begin{aligned}
\mathbb{E}[U_T] &= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{T=n\}} \sum_{m=1}^{n} |X_m|\right] \\
&= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \mathbb{E}[|X_m|\mathbb{1}_{\{T=n\}}] \\
&= \sum_{m=1}^{\infty} \mathbb{E}[|X_m|\mathbb{1}_{\{T\geq m\}}] \\
&= \sum_{m=1}^{\infty} \mathbb{E}[|X_m|\mathbb{1}_{\{T\leq m-1\}^c}] \\
&= \sum_{m=1}^{\infty} \mathbb{E}|X_m|\mathbb{P}[T \geq m] \\
&= \mathbb{E}|X_1|\mathbb{E}[T] < \infty.
\end{aligned}$$

where we used that $X_m \perp \{T \leq m-1\} \in \mathcal{F}_{m-1}$ on the second to last line. Note that we've proved the theorem for nonnegative X_is. This calculation also justifies using Fubini for general RVs. ■

THM 12.8 *Let $X_1, X_2, \ldots \in L^2$ be iid with $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}[X_1] = \sigma^2$ and let $T \in L^1$ be a stopping time. Then*

$$\mathbb{E}[S_T^2] = \sigma^2\mathbb{E}[T].$$

1.3 Application: Simple Random Walk

Let $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = 1/2$ and $T = \inf\{n \geq 1 \,:\, S_n \notin (a,b)\}$, where $a < 0 < b$. Let $S_0 = 0$. We first argue that $\mathbb{E}T < \infty$ a.s. Since $(b-a)$ steps to the right necessarily take us out of (a,b),

$$\mathbb{P}[T > n(b-a)] \leq (1 - 2^{-(b-a)})^n,$$

by independence of the $(b-a)$-long stretches, so that

$$\mathbb{E}[T] = \sum_{k\geq 0} \mathbb{P}[T > k] \leq \sum_{n} (b-a)(1-2^{-(b-a)})^n < +\infty,$$

by monotonicity. In particular $T < +\infty$ a.s.

By Wald's First Identity,

$$a\mathbb{P}[S_T = a] + b\mathbb{P}[S_T = b] = 0,$$

that is

$$\mathbb{P}[S_T = a] = \frac{b}{b-a} \qquad \mathbb{P}[S_T = b] = \frac{-a}{b-a}.$$

In other words, letting $T_a = \inf\{n \geq 1 \,:\, S_n = a\}$

$$\mathbb{P}[T_a < T_b] = \frac{b}{b-a}.$$

By monotonicity, letting $b \to \infty$

$$\mathbb{P}[T_a < \infty] \geq \mathbb{P}[T_a < T_b] \to 1.$$

Note that this is true for every T_x. In particular, we come back to where we started almost surely. This property is called *recurrence*. We will study recurrence more closely below.

Wald's Second Identity tells us that

$$\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T],$$

where $\sigma^2 = 1$ and

$$\mathbb{E}[S_T^2] = \frac{b}{b-a}a^2 + \frac{-a}{b-a}b^2 = -ab,$$

so that $\mathbb{E}T = -ab$.

2 Recurrence of SRW

We study the recurrence of SRW on $\mathbb{Z}^d$. Recal Stirling's formula:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}.$$

2.1 Strong Markov property

We will need an important property of stopping times.

DEF 12.9 *For a stopping time T, the σ-field $\mathcal{F}_T$ (the information known up to time T) is*

$$\mathcal{F}_T = \{A \,:\, A \cap \{T = n\} \in \mathcal{F}_n,\ \forall n\}.$$

THM 12.10 (Strong Markov property) *Let $X_1, X_2, \ldots$ be IID, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and T be a stopping with $\mathbb{P}[T < \infty] > 0$. On $\{T < \infty\}$, $\{X_{T+n}\}_{n\geq 1}$ is independent of $\mathcal{F}_T$ and has the same distribution as the original sequence.*

Proof: By the Uniqueness lemma, it suffices to prove

$$\mathbb{P}[A, T < \infty, X_{T+j} \in B_j,\ 1 \leq j \leq k] = \mathbb{P}[A, T < \infty] \prod_{j=1}^{k} \mathbb{P}[X_j \in B_j].$$

for all $A \in \mathcal{F}_T$, $B_1, \ldots, B_k \in \mathcal{B}$. Then sum up over the value of N and use the definition of $\mathcal{F}_T$. Indeed

$$\begin{aligned}\mathbb{P}[A, T = n, X_{T+j} \in B_j,\ 1 \leq j \leq k] &= \mathbb{P}[A, T = n, X_{n+j} \in B_j,\ 1 \leq j \leq k] \\ &= \mathbb{P}[A, T = n] \prod_{j=1}^{k} \mathbb{P}[X_j \in B_j].\end{aligned}$$

■

2.2 SRW on $\mathbb{Z}$

Let $S_0 = 0$ and $T_0 = \inf\{n > 0 \,:\, S_m = 0\}$. We give a second proof of:

THM 12.11 (SRW on $\mathbb{Z}$) *SRW on $\mathbb{Z}$ is recurrent.*

Proof: First note the periodicity. So we look at S_{2n}. Then

$$\begin{aligned}\mathbb{P}[S_{2n} = 0] &= \binom{2n}{n} 2^{-2n} \\ &\sim 2^{-2n} \frac{(2n)^{2n}}{(n^n)^2} \frac{\sqrt{2n}}{\sqrt{2\pi} n} \\ &\sim \frac{1}{\sqrt{\pi n}}.\end{aligned}$$

So

$$\sum_m \mathbb{P}[S_m = 0] = \infty.$$

Denote

$$T_0^{(n)} = \inf\{m > T_0^{(n-1)} \,:\, S_m = 0\}.$$

By the strong Markov property $\mathbb{P}[T_0^{(n)} < \infty] = \mathbb{P}[T_0 < \infty]^n$. Note that

$$\begin{aligned}
\sum_m \mathbb{P}[S_m = 0] &= \mathbb{E}\left[\sum_m \mathbb{1}_{\{S_m=0\}}\right] \\
&= \mathbb{E}\left[\sum_n \mathbb{1}_{\{T_0^{(n)}<\infty\}}\right] \\
&= \sum_n \mathbb{P}[T_0^{(n)} < \infty] \\
&= \sum_n \mathbb{P}[T_0 < \infty]^n \\
&= \frac{1}{1 - \mathbb{P}[T_0 < \infty]}.
\end{aligned}$$

So $\mathbb{P}[T_0 < \infty] = 1$. ■

2.3 SRW on $\mathbb{Z}^2$

Now X_1 is in $\mathbb{Z}^2$ and $\mathbb{P}[X_1 = (1,0)] = \cdots = \mathbb{P}[X_1 = (0,-1)] = 1/4$.

THM 12.12 (SRW on $\mathbb{Z}^2$) *SRW on $\mathbb{Z}^2$ is recurrent.*

Proof: Let $R_n = (S_n^{(1)}, S_n^{(2)})$ where $S_n^{(i)}$ are independent SRW on $\mathbb{Z}$. By rotating the plane by 45 degrees, one sees that the probability to be back at $(0,0)$ in SRW on $\mathbb{Z}^2$ is the same as that for two independent SRW on $\mathbb{Z}$ to be back at 0 simultaneously. Therefore,

$$\mathbb{P}[S_{2n} = (0,0)] = \mathbb{P}[S_{2n}^{(1)} = 0]^2 \sim \frac{1}{\pi n},$$

whose sum diverges. ■

2.4 SRW on $\mathbb{Z}^3$

Now X_1 is in $\mathbb{Z}^3$ and $\mathbb{P}[X_1 = (1,0,0)] = \cdots = \mathbb{P}[X_1 = (0,0,-1)] = 1/6$.

THM 12.13 (SRW on $\mathbb{Z}^3$) *SRW on $\mathbb{Z}^3$ is transient (that is, not recurrent).*

Proof: Note, since the number of steps in opposite directions has to be equal,

$$\begin{aligned}
\mathbb{P}[S_{2n} = 0] &= 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-k-j)!)^2} \\
&= 2^{-2n} \binom{2n}{n} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-k-j)!} \right)^2 \\
&\leq 2^{-2n} \binom{2n}{n} \max_{j,k} 3^{-n} \frac{n!}{j!k!(n-k-j)!},
\end{aligned}$$

where we used that $\sum_{j,k} a_{j,k}^2 \leq \max_{j,k} a_{j,k} \equiv a^*$ if $\sum_{j,k} a_{j,k} = 1$ and $a_{j,k} \geq 0$. Note that if $j < n/3$ and $k > n/3$ then

$$\frac{(j+1)!(k-1)!}{j!k!} = \frac{j+1}{k} \leq 1.$$

That implies that the term in the max is maximized when $j, k, (n-k-j)$ are roughly $n/3$. Using Stirling

$$\frac{n!}{j!k!(n-k-j)!} \sim \frac{n^n}{j^j k^k (n-k-j)^{n-k-j}} \sqrt{\frac{n}{jk(n-k-j)}} \frac{1}{2\pi} \sim C n^{-1} 3^n,$$

if j, k are close to $n/3$. Hence $\mathbb{P}[S_{2n} = 0] \sim C n^{-3/2}$ which is summable and $\mathbb{P}[T_0 < \infty] < 1$. Note that it implies that S_n visits 0 only finitely many times with probability 1 as the expectation of the number of visits to 0 is $\sum_m \mathbb{P}[S_m = 0]$ (which is then finite). ■

COR 12.14 *SRW on $\mathbb{Z}^d$ with $d > 3$ is transient.*

Proof: Let $R_n = (S_n^1, S_n^2, S_n^3)$. Let

$$U_m = \inf\{n > U_{m-1} \, : \, R_n \neq R_{U_{m-1}}\}.$$

Then R_{U_n} is a three-dimensional SRW. It visits $(0,0,0)$ only finitely many times a.s. and the walk is transient. Indeed $\mathbb{P}[T_0 < +\infty] = 1$ would imply $\mathbb{P}[T_0^{(n)} < +\infty] = 1$ for all n, which in turn would imply that $\mathbb{P}[S_n = 0 \text{ i.o.}] = 1$. ■

3 Arcsine laws

Reference: Section 4.3 in [D].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

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