Notes 16 : Martingales in $\mathcal{L}^p$

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References: [Wil91, Section 12], [Dur10, Section 5.4].

1 Martingales in $\mathcal{L}^2$

1.1 Preliminaries

DEF 16.1 *For $1 \leq p < +\infty$, we say that $X \in \mathcal{L}^p$ if*

$$\|X\|_p = \mathbb{E}[|X^p|]^{1/p} < +\infty.$$

By Jensen's inequality, for $1 \leq p \leq r < +\infty$ we have $\|X\|_p \leq \|X\|_r$ if $X \in \mathcal{L}^r$.

Proof: For $n \geq 0$, let

$$X_n = (|X| \wedge n)^p.$$

Take $c(x) = x^{r/p}$ on $(0, +\infty)$ which is convex. Then

$$(\mathbb{E}[X_n])^{r/p} \leq \mathbb{E}[(X_n)^{r/p}] = \mathbb{E}[(|X| \wedge n)^r] \leq \mathbb{E}[|X|^r].$$

Take $n \to \infty$ and use (MON). ■

DEF 16.2 *We say that X_n converges to X_∞ in $\mathcal{L}^p$ if $\|X_n - X_\infty\|_p \to 0$. By the previous result, convergence on $\mathcal{L}^r$ implies convergence in $\mathcal{L}^p$ for $r \geq p \geq 1$. (Moreover, by Chebyshev's inequality, convergence in $\mathcal{L}^p$ implies convergence in probability.)*

LEM 16.3 *Assume $X_n, X_\infty \in \mathcal{L}^1$. Then*

$$\|X_n - X_\infty\|_1 \to 0,$$

implies

$$\mathbb{E}[X_n] \to \mathbb{E}[X_\infty].$$

Proof: Note that

$$|\mathbb{E}[X_n] - \mathbb{E}[X_\infty]| \leq \mathbb{E}|X_n - X_\infty| \to 0.$$

■

DEF 16.4 *We say that $\{X_n\}_n$ is* bounded in $\mathcal{L}^p$ *if*

$$\sup_n \|X_n\|_p < +\infty.$$

1.2 $\mathcal{L}^2$ convergence

THM 16.5 *Let $\{M_n\}$ be a MG with $M_n \in \mathcal{L}^2$. Then $\{M_n\}$ is bounded in $\mathcal{L}^2$ if and only if*

$$\sum_{k\geq 1} \mathbb{E}[(M_k - M_{k-1})^2] < +\infty.$$

When this is the case, M_n converges a.s. and in $\mathcal{L}^2$. (In particular, it converges in $\mathcal{L}^1$.)

Proof:

LEM 16.6 (Orthogonality of increments) *Let $s \leq t \leq u \leq v$. Then,*

$$\langle M_t - M_s, M_v - M_u \rangle = 0.$$

Proof: Use $M_u = \mathbb{E}[M_v \,|\, \mathcal{F}_u]$, $M_t - M_s \in \mathcal{F}_u$ and apply the L^2 characterization of conditional expectations. ■

That implies

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{1\leq i\leq n} \mathbb{E}[(M_i - M_{i-1})^2],$$

proving the first claim.

By monotonicity of norms, M is bounded in L^2 implies M bounded in L^1 which, in turn, implies M converges a.s. Then using (FATOU) in

$$\mathbb{E}[(M_{n+k} - M_n)^2] = \sum_{n+1\leq i\leq n+k} \mathbb{E}[(M_i - M_{i-1})^2],$$

gives

$$\mathbb{E}[(M_\infty - M_n)^2] \leq \sum_{n+1\leq i} \mathbb{E}[(M_i - M_{i-1})^2].$$

The RHS goes to 0 which proves the second claim.

■

2 $\mathcal{L}^p$ convergence theorem

Recall:

LEM 16.7 (Markov's inequality) *Let $Z \geq 0$ be a RV. Then for $c > 0$*

$$c\mathbb{P}[Z \geq c] \leq \mathbb{E}[Z; Z \geq c] \leq \mathbb{E}[Z].$$

MGs provide a useful generalization.

LEM 16.8 (Doob's submartingale inequality) *Let $\{Z_n\}$ be a nonnegative subMG. Then for $c > 0$*

$$c\mathbb{P}[\sup_{1\leq k\leq n} Z_k \geq c] \leq \mathbb{E}[Z_n; \sup_{1\leq k\leq n} Z_k \geq c] \leq \mathbb{E}[Z_n].$$

Proof: Divide $F = \{\sup_{1\leq k\leq n} Z_k \geq c\}$ according to the first time Z crosses c:

$$F = F_0 \cup \cdots \cup F_n,$$

where

$$F_k = \{Z_0 < c\} \cap \cdots \cap \{Z_{k-1} < c\} \cap \{Z_k \geq c\}.$$

Since $F_k \in \mathcal{F}_k$ and $\mathbb{E}[Z_n \,|\, \mathcal{F}_k] \geq Z_k$,

$$c\mathbb{P}[F_k] \leq \mathbb{E}[Z_k; F_k] \leq \mathbb{E}[Z_n; F_k].$$

Sum over k. ■

EX 16.9 (Kolmogorov's inequality) *Let $X_1, \ldots$ be independent RVs with $\mathbb{E}[X_k] = 0$ and $\mathrm{Var}[X_k] < +\infty$. Define $S_n = \sum_{k\leq n} X_k$. Then for $c > 0$*

$$\mathbb{P}[\max_{k\leq n} |S_k| \geq c] \leq c^{-2}\mathrm{Var}[S_n].$$

THM 16.10 (Doob's $\mathcal{L}^p$ inequality) *Let $p > 1$ and $p^{-1} + q^{-1} = 1$. Let $\{Z_n\}$ be a nonnegative subMG bounded in $\mathcal{L}^p$. Define*

$$Z^* = \sup_{k\geq 0} Z_k.$$

Then

$$\|Z^*\|_p \leq q \sup_k \|Z_k\|_p = q \uparrow \lim_k \|Z_k\|_p.$$

and $Z^* \in L^p$.

Proof: The last equality follows from (JENSEN). Let $Z_n^* = \sup_{k\leq n} Z_k$. By (MON) it suffices to prove:

LEM 16.11
$$\mathbb{E}[(Z_n^*)^p] \leq q^p\mathbb{E}[Z_n^p].$$

Proof: Recall the formula: for $Y \geq 0$ and $p > 0$
$$\mathbb{E}[Y^p] = \int_0^\infty py^{p-1}\mathbb{P}[Y \geq y]dy.$$

Then for $K > 0$ (note that $\{Z_n^* \wedge K \geq c\}$ is either $\{Z_n^* \geq c\}$ or empty (depending on whether K is smaller or bigger than c) so Doob's inequality still applies)

$$\begin{aligned}
\mathbb{E}[(Z_n^* \wedge K)^p] &= \int_0^\infty pc^{p-1}\mathbb{P}[Z_n^* \wedge K \geq c]dc \\
&\leq \int_0^\infty pc^{p-2}\mathbb{E}[Z_n; Z_n^* \wedge K \geq c]dc \\
&= \mathbb{E}\left[Z_n\left(\frac{p}{p-1}\right)\int_0^\infty (p-1)c^{p-2}\mathbb{1}[Z_n^* \wedge K \geq c]dc\right] \\
&= \mathbb{E}[qZ_n(Z_n^* \wedge K)^{p-1}] \\
&\leq q\mathbb{E}[Z_n^p]^{1/p}\mathbb{E}[(Z_n^* \wedge K)^p]^{1/q},
\end{aligned}$$

where we used that $(p-1)q = p$. Rearranging and using (MON) gives the result.
■ ■

THM 16.12 ($\mathcal{L}^p$ convergence) *Let $\{M_n\}$ be a MG bounded in $\mathcal{L}^p$ for $p > 1$. Then $M_n \to M_\infty$ a.s. and in $\mathcal{L}^p$.*

Proof: Note that $|M_n|$ is a subMG bounded in $\mathcal{L}^p$. In particular, it is bounded in $\mathcal{L}^1$ and $M_n \to M_\infty$ a.s. by the martingale convergence theorem. From the previous theorem,
$$|M_n - M_\infty|^p \leq (2\sup_k |M_k|)^p \in \mathcal{L}^1,$$
and by (DOM)
$$\mathbb{E}|M_n - M_\infty|^p \to 0.$$
■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.