Notes 19 : Martingale CLT

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References: [Bil95, Chapter 35], [Roc, Chapter 3].

Since we have not encountered weak convergence in some time, we first recall a number of definitions and results that will be useful to keep in mind throughout this lecture.

DEF 19.1 (Distribution function) *Let X be a RV on a triple $(\Omega, \mathcal{F}, \mathbb{P})$. The* law *of X is*

$$\mathcal{L}_X = \mathbb{P} \circ X^{-1},$$

which is a probability measure on $(\mathbb{R}, \mathcal{B})$. By the Uniqueness of Extension lemma, $\mathcal{L}_X$ is determined by the distribution function (DF) *of X*

$$F_X(x) = \mathbb{P}[X \leq x], \quad x \in \mathbb{R}.$$

DEF 19.2 (Convergence in distribution) *A sequence of DFs $(F_n)_n$* converges in distribution *(or weakly) to a DF F if*

$$F_n(x) \to F(x),$$

for all points of continuity x of F. If F_n and F are the DFs of X_n and X respectively, we write $X_n \Rightarrow X$.

THM 19.3 (Convergence in distribution v. in probability) *Suppose that $(X_n)_n$ is a sequence of RVs with $X_n \sim F_n$. The following holds:*

1. *If $X_n \to_P X_\infty$, then $X_n \Rightarrow X_\infty$.*
2. *If $X_n \Rightarrow c$, where c is a constant, then $X_n \to_P c$.*

LEM 19.4 (Converging together lemma) *If $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$, then $Z_n \Rightarrow X$.*

LEM 19.5 (Multiplying by a converging sequence) *If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ with $Y_n \geq 0$ and $c > 0$, then $X_n Y_n \Rightarrow cX$.*

DEF 19.6 (Characteristic function) *The* characteristic function *(CF) of a RV X is*

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)],$$

where the second equality is a definition and the expectations exist because they are bounded.

EX 19.7 (Gaussian distribution) *Let $X \sim \mathrm{N}(0,1)$. Then*

$$\phi_X(t) = e^{-t^2/2}.$$

THM 19.8 *If X_1 and X_2 are independent then*

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t).$$

LEM 19.9 *If DFs F and G have the same CF, then they are equal.*

THM 19.10 (Lévy's Continuity Theorem) *Let μ_n, $1 \leq n \leq \infty$ be probability measures (PM) with CFs ϕ_n.*

1. *If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t.*
2. *If $\phi_n(t)$ converges pointwise to a limit $\phi(t)$ that is continuous at 0 then the associated sequence of PMs μ_n converges weakly to a PM μ with CF ϕ.*

THM 19.11 (CLT) *Let $(X_n)_n$ be IID with $\mathbb{E}[X_1] = \mu$ and $\mathrm{Var}[X_1] = \sigma^2 < +\infty$.Then if $S_n = \sum_{k=1}^n X_k$*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow \mathrm{N}(0,1).$$

1 Martingale CLT

We have seen that MGs are, in some ways, generalizations of unbiased RWs. Hence, one might expect a CLT to hold under appropriate conditions (namely, Lindeberg-type conditions). That turns out to be the case, but the situation is more complex than for sums of independent RVs, as the next example suggests. In brief, mixtures of unbiased RWs are also MGs, in which case the limit may be a mixture of Gaussians.

EX 19.12 (Mixtures of RWs) *Recall that, if the bounded process $\{C_n\}_{n\geq 1}$ is predictable (i.e. $C_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$) and $\{S_n\}_{n\geq 0}$ is a MG, then the martingale transform $\{(C \bullet S)_n\}_{n\geq 0}$ with*

$$M_n = (C \bullet S)_n \equiv \sum_{i\leq n} C_i(S_i - S_{i-1}),$$

is also a MG. Suppose S_n is SRW on the integers started at 0, that is, $S_n = \sum_{i=1}^n X_i$ where the X_is are IID uniform in $\{-1, +1\}$. Hence $S_i - S_{i-1} = X_i$. Let also X_0 be independent uniform in $\{1, 2\}$ and set $C_i = X_0$ for all $i \geq 1$. Then, with respect to the filtration $\mathcal{F}_n = \sigma(X_0, \ldots, n)$ for $n \geq 0$, $\{S_n\}_{n\geq 0}$ is a MG (because $\mathbb{E}[X_i] = 0$) and $\{C_n\}_{n\geq 1}$ is bounded and predictable. Hence $\{M_n\}$, as defined above, is a MG. Now note that $\{M_n\}$ is a mixture of two RWs, in the following sense. On $\{X_0 = 1\}$, which occurs with probability $1/2$, we have $M_n = \sum_{i=1}^n X_i$ and THM 19.11 implies that $M_n/\sqrt{n} \Rightarrow \mathrm{N}(0, 1)$. On the other hand, on $\{X_0 = 2\}$, which also occurs with probability $1/2$, we have instead $M_n = \sum_{i=1}^n 2X_i$ and THM 19.11 implies this time that $M_n/\sqrt{4n} \Rightarrow \mathrm{N}(0, 1)$—or $M_n/\sqrt{n} \Rightarrow \mathrm{N}(0, 4)$.

To handle this issue, we introduce the conditional variance. Let $\{M_n\}$ be a MG in $\mathcal{L}^2$ with corresponding MG difference $Z_n = M_n - M_{n-1}$ and let

$$\sigma_n^2 = \mathbb{E}[Z_n^2 \,|\, \mathcal{F}_{n-1}].$$

(Note that this is a RV.) Consider the stopping time (ST)

$$\tau_n = \inf\left\{m \geq 0 \,:\, \sum_{i=1}^m \sigma_i^2 \geq n\right\}. \tag{1}$$

This is indeed an ST since $\{\tau_n \leq m\} = \{\sum_{i=1}^m \sigma_i^2 \geq n\} \in \mathcal{F}_{m-1} \subseteq \mathcal{F}_m$. Under the assumption that $\sum_n \sigma_n^2 = +\infty$ with probability 1, we then have $\tau_n \uparrow +\infty$ a.s. when $n \to +\infty$.

THM 19.13 (A CLT for MGs with Bounded Increments) *Let $\{M_n\}$ be a MG with corresponding MG difference $Z_n = M_n - M_{n-1}$. Assume there is a constant $K < +\infty$ such that $|Z_n| \leq K$ with probability 1 for all n. Let $\sigma_n^2 = \mathbb{E}[Z_n^2 \,|\, \mathcal{F}_{n-1}]$ and assume that $\sum_n \sigma_n^2 = +\infty$ with probability 1. Let τ_n be as defined in (1). Then*

$$\frac{M_{\tau_n}}{\sqrt{n}} \Rightarrow \mathrm{N}(0, 1),$$

as $n \to +\infty$.

Going back to the example above:

EX 19.14 (Mixtures of RWs continued) *The conditions of THM 19.13 are satisfied with $K = 4$ by noting that $\sigma_n^2 \geq 1$ a.s. for all n. On $\{X_0 = 1\}$ we have $\tau_n = n$ for all n, while on $\{X_0 = 2\}$ we have $\tau_n = n/4$ for all n.*

To prove THM 19.13, we establish a generalization of the Lindeberg-Feller CLT. The setting involves a sequence of MGs $\{M_{n,m}\}_{m\geq 0}$ for $n \geq 1$ with respect to filtrations $\{\mathcal{F}_{n,m}\}_{m\geq 0}$. (If the original MG is only defined for $0 \leq m \leq r_n$, then one can extend it by letting $M_{n,m} = M_{n,r_n}$ and $\mathcal{F}_{n,m} = \mathcal{F}_{n,r_n}$ for all $m > r_n$. More generally, this setting accommodates a MG stopped at a sequence of stopping times diverging to infinity. See proof of THM 19.13 below.)

THM 19.15 (Martingale CLT) *For each n, let $\{M_{n,m}\}_{m\geq 0}$ be a MG in $\mathcal{L}^2$ with respect to filtration $\{\mathcal{F}_{n,m}\}_{m\geq 0}$ with corresponding MG difference $Z_{n,m} = M_{n,m} - M_{n,m-1}$ and conditional variance $\sigma_{n,m}^2 = \mathbb{E}[Z_{n,m}^2 \,|\, \mathcal{F}_{n,m-1}]$. Assume that, for each n, $M_{n,m}$ and $\Gamma_{n,m} \equiv \sum_{r=1}^{m} \sigma_{n,r}^2$ converge a.s. to a finite limit when $m \to +\infty$. Suppose that*

1. $\Gamma_{n,\infty} \equiv \sum_{m=1}^{+\infty} \sigma_{n,m}^2 \to 1$ *in probability as* $n \to +\infty$
2. $\forall \varepsilon > 0$, $\lim_n \sum_{m=1}^{+\infty} \mathbb{E}[Z_{n,m}^2; |Z_{n,m}| > \varepsilon] = 0$

Then

$$M_{n,\infty} \equiv \sum_{m=1}^{+\infty} Z_{n,m} \Rightarrow \mathrm{N}(0,1),$$

as $n \to +\infty$.

Before giving the proof of the MG-CLT, we use it to establish THM 19.13.
Proof:(of THM 19.13) We construct a sequence of MGs as follows. Let $\mathcal{F}_{n,m} = \mathcal{F}_m$ and

$$M_{n,m} = \frac{M_{m\wedge\tau_n}}{\sqrt{n}}$$

so that

$$Z_{n,m} = \frac{Z_m}{\sqrt{n}} \mathbb{1}\{\tau_n \geq m\}.$$

This is indeed a MG for each n since τ_n is a ST (and multiplying by a constant does not affect the MG property). Moreover

$$\begin{aligned}\sigma_{n,m}^2 &= \mathbb{E}\left[\left(\frac{Z_m}{\sqrt{n}}\mathbb{1}\{\tau_n \geq m\}\right)^2 \middle| \mathcal{F}_{n,m-1}\right] \\ &= \frac{1}{n}\mathbb{1}\{\tau_n \geq m\}\,\mathbb{E}[Z_m^2 \,|\, \mathcal{F}_{m-1}] \\ &= \frac{\sigma_m^2}{n}\,\mathbb{1}\{\tau_n \geq m\},\end{aligned}$$

where we used that $\{\tau_n \geq m\} \in \mathcal{F}_{m-1}$. Summing over m and using the bound on Z_m, we get

$$1 \leq \Gamma_{n,\infty} = \sum_{m=1}^{\tau_n} \frac{\sigma_m^2}{n} \leq 1 + \frac{K^2}{n},$$

by the definition of τ_n. So as $n \to +\infty$, we have $\Gamma_{n,\infty} \to 1$. Also, for any $\varepsilon > 0$,

$$|Z_{n,m}| = \left|\frac{Z_m}{\sqrt{n}}\mathbb{1}\{\tau_n \geq m\}\right| \leq \frac{K}{\sqrt{n}} < \varepsilon, \qquad \forall m$$

whenever n is large enough so that

$$\lim_n \sum_{m=1}^{+\infty} \mathbb{E}[|Z_{n,m}|^2; |Z_{n,m}| > \varepsilon] = 0.$$

Therefore, the result follows from the MG-CLT after noting that

$$M_{n,\infty} = \frac{M_{\tau_n}}{\sqrt{n}},$$

where we used the fact that $\tau_n < +\infty$ a.s. by the assumption $\sum_n \sigma_n^2 = +\infty$. ■

We move on to the proof of the MG-CLT.

Proof:(of THM 19.15) We first assume that there is a constant c such that $\Gamma_{n,\infty} \leq c$, for all n. We seek to prove that the following is going to 0 as $n \to +\infty$:

$$\begin{aligned}\left|\mathbb{E}[e^{itM_{n,\infty}}] - e^{-t^2/2}\right| &\leq \Big|\mathbb{E}[e^{itM_{n,\infty}} - e^{itM_{n,\infty}}e^{-t^2/2}e^{t^2\Gamma_{n,\infty}/2}] \\ &\qquad + \mathbb{E}[e^{itM_{n,\infty}}e^{-t^2/2}e^{t^2\Gamma_{n,\infty}/2} - e^{-t^2/2}]\Big| \\ &\leq \mathbb{E}[|1 - e^{-t^2/2}e^{t^2\Gamma_{n,\infty}/2}|] + \left|\mathbb{E}[e^{itM_{n,\infty}}e^{t^2\Gamma_{n,\infty}/2} - 1]\right|.\end{aligned}$$

The first term above $\to 0$ by (BDD) and the assumption that $\Gamma_{n,\infty} \leq c$.

To bound the second term we use a telescoping argument. Write

$$\begin{aligned} e^{itM_{n,\infty}} e^{t^2\Gamma_{n,\infty}/2} - 1 &= \sum_{m\geq 1} \left\{ e^{itM_{n,m}} e^{t^2\Gamma_{n,m}/2} - e^{itM_{n,m-1}} e^{t^2\Gamma_{n,m-1}/2} \right\} \\ &= \sum_{m\geq 1} e^{itM_{n,m-1}} e^{t^2\Gamma_{n,m}/2} \left\{ e^{itZ_{n,m}} - e^{-t^2\sigma^2_{n,m}/2} \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\left| \mathbb{E}[e^{itM_{n,\infty}} e^{t^2\Gamma_{n,\infty}/2} - 1] \right| \\ &\quad \leq \sum_{m\geq 1} \left| \mathbb{E}\left[e^{itM_{n,m-1}} e^{t^2\Gamma_{n,m}/2} \left\{ e^{itZ_{n,m}} - e^{-t^2\sigma^2_{n,m}/2} \right\} \right] \right| \\ &\quad \leq \sum_{m\geq 1} \left| \mathbb{E}\left[e^{itM_{n,m-1}} e^{t^2\Gamma_{n,m}/2} \mathbb{E}\left[e^{itZ_{n,m}} - e^{-t^2\sigma^2_{n,m}/2} \,\middle|\, \mathcal{F}_{n,m-1} \right] \right] \right| \\ &\quad \leq e^{ct^2} \sum_{m\geq 1} \mathbb{E}\left[\left| \mathbb{E}\left[e^{itZ_{n,m}} - e^{-t^2\sigma^2_{n,m}/2} \,\middle|\, \mathcal{F}_{n,m-1} \right] \right| \right] \end{aligned}$$

where we used that $M_{n,m-1}, \Gamma_{n,m} \in \mathcal{F}_{n,m-1}$ and the assumption that $\Gamma_{n,\infty} \leq c$.

We will use the following lemmas proved in a previous lecture.

LEM 19.16 *It holds that*

$$\left| \mathbb{E}\left[e^{itX} \right] - \left(1 + it\,\mathbb{E}[X] - \frac{t^2}{2}\,\mathbb{E}[X^2] \right) \right| \leq \mathbb{E}\left[\min\{|tX|^3, |tX|^2\} \right].$$

LEM 19.17 *If z is a complex number then*

$$|e^z - (1+z)| \leq |z|^2 e^{|z|}.$$

By the MG property, $\mathbb{E}[Z_{n,m} \,|\, \mathcal{F}_{n,m-1}]$. Also by definition $\mathbb{E}[Z^2_{n,m} \,|\, \mathcal{F}_{n,m-1}] = \sigma^2_{n,m}$. By LEM 19.16,

$$\begin{aligned} &\left| \mathbb{E}\left[e^{itZ_{n,m}} \,|\, \mathcal{F}_{n,m-1} \right] - \left(1 - \frac{t^2}{2}\sigma^2_{n,m} \right) \right| \\ &\quad \leq \mathbb{E}\left[|tZ_{n,m}|^3 \wedge |tZ_{n,m}|^2 \,|\, \mathcal{F}_{n,m-1} \right] \\ &\quad \leq \mathbb{E}[|tZ_{n,m}|^3; |Z_{n,m}| \leq \varepsilon \,|\, \mathcal{F}_{n,m-1}] + \mathbb{E}[|tZ_{n,m}|^2; |Z_{n,m}| > \varepsilon \,|\, \mathcal{F}_{n,m-1}] \\ &\quad \leq \varepsilon |t|^3 \mathbb{E}[Z^2_{n,m}; |Z_{n,m}| \leq \varepsilon \,|\, \mathcal{F}_{n,m-1}] + t^2 \mathbb{E}[Z^2_{n,m}; |Z_{n,m}| > \varepsilon \,|\, \mathcal{F}_{n,m-1}] \\ &\quad \leq \varepsilon |t|^3 \sigma^2_{n,m} + t^2 \mathbb{E}[Z^2_{n,m}; |Z_{n,m}| > \varepsilon \,|\, \mathcal{F}_{n,m-1}]. \end{aligned}$$

By LEM 19.17,

$$\left|e^{-t^2\sigma_{n,m}^2/2} - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right| \leq (t^2\sigma_{n,m}^2/2)^2 e^{t^2\sigma_{n,m}^2/2} \leq t^4\sigma_{n,m}^2 e^{ct^2}\left(\sup_{m\geq 1}\sigma_{n,m}^2\right).$$

Plugging the previous two displays into the equation above, we get

$$\begin{aligned}
&\left|\mathbb{E}[e^{itM_{n,\infty}}e^{t^2\Gamma_{n,\infty}/2} - 1]\right| \\
&\quad \leq e^{ct^2}\sum_{m\geq 1}\mathbb{E}\left[\left|\mathbb{E}\left[e^{itZ_{n,m}} - e^{-t^2\sigma_{n,m}^2/2}\,\middle|\,\mathcal{F}_{n,m-1}\right]\right|\right] \\
&\quad \leq e^{ct^2}\left\{\varepsilon|t|^3 c + t^2\sum_{m=1}^{+\infty}\mathbb{E}[Z_{n,m}^2;|Z_{n,m}|>\varepsilon] + t^4 c e^{ct^2}\mathbb{E}\left[\sup_{m\geq 1}\sigma_{n,m}^2\right]\right\}.
\end{aligned}$$

(Note that the partial sum in the telescoping argument above, $e^{itM_{n,m}}e^{t^2\Gamma_{n,m}/2} - 1$, is bounded and therefore we can take the expectation inside by (BDD).) To bound the supremum, we note that

$$\begin{aligned}
\sigma_{n,m}^2 &\leq \varepsilon^2 + \mathbb{E}[Z_{n,m}^2;|Z_{n,m}|>\varepsilon\,|\,\mathcal{F}_{n,m-1}] \\
&\leq \varepsilon^2 + \sum_{i=1}^{+\infty}\mathbb{E}[Z_{n,i}^2;|Z_{n,i}|>\varepsilon\,|\,\mathcal{F}_{n,i-1}]
\end{aligned}$$

so

$$\mathbb{E}\left[\sup_{m\geq 1}\sigma_{n,m}^2\right] \leq \varepsilon^2 + \sum_{i=1}^{+\infty}\mathbb{E}[Z_{n,i}^2;|Z_{n,i}|>\varepsilon].$$

Taking $n \to +\infty$ and $\varepsilon > 0$ arbitrarily and using the second condition of the statement, we get

$$\left|\mathbb{E}[e^{itM_{n,\infty}}e^{t^2\Gamma_{n,\infty}/2} - 1]\right| \to 0,$$

for all t. That concludes the proof in the case $\Gamma_{n,\infty} \leq c$.

To remove the last assumption, multiply the difference $Z_{n,m}$ by the indicator that $\{\Gamma_{n,m} \leq c\} \in \mathcal{F}_{n,m-1}$ for some $c > 1$ and apply the argument above. Then note that $\mathbb{P}[\Gamma_{n,\infty} \leq c] \to 1$ as $n \to +\infty$ by the first condition of the statement and use the converging together lemma. See the details in [B]. ■

2 An application: autoregressive processes

Let $\{W_n\}_{n\geq 1}$ be a sequence of IID RVs with $|W_n| \leq c$ for all n for some constant $c < +\infty$. Assume that $\mathbb{E}[W_1] = 0$ and $\mathrm{Var}[W_1] = \nu^2$. For $\rho \in (0,1)$, consider the following autoregressive process (of order 1)

$$X_n = \rho X_{n-1} + W_n, \qquad \forall n \geq 1$$

and $X_0 = 0$. Assume that ρ is unknown and that we only observe the sequence $\{X_n\}$. To estimate ρ, we observe that

$$\mathbb{E}[X_n X_{n-1}] = \mathbb{E}[\rho X_{n-1}^2 + W_n X_{n-1}] = \rho \mathbb{E}[X_{n-1}^2],$$

where we used the independence of W_n and $X_{n-1} \in \sigma(W_1, \ldots, W_{n-1})$. Hence, a natural estimator for ρ from $\{X_n\}$ is

$$\hat{\rho}_n \equiv \frac{\frac{1}{n}\sum_{k=1}^n X_k X_{k-1}}{\frac{1}{n}\sum_{k=1}^n X_{k-1}^2}.$$

We use the MG-CLT to establish the asymptotic normality of $\hat{\rho}_n$.

CLAIM 19.18 *We have*

$$\sqrt{\frac{\Psi_\infty n}{\nu^2}}(\hat{\rho}_n - \rho) \Rightarrow \mathrm{N}(0,1),$$

where $\Psi_\infty \equiv \frac{\nu^2}{1-\rho^2}$.

Proof: Our starting point is the following observation:

$$\begin{aligned}(\hat{\rho}_n - \rho)\sum_{k=1}^n X_{k-1}^2 &= \sum_{k=1}^n X_k X_{k-1} - \rho \sum_{k=1}^n X_{k-1}^2 \\ &= \sum_{k=1}^n (X_k - \rho X_{k-1}) X_{k-1} \\ &= \sum_{k=1}^n W_k X_{k-1}.\end{aligned}$$

The latter is not a sum of independent RVs, but it turns out to be a MG with respect to the filtration $\mathcal{F}_n = \sigma(W_1, \ldots, W_n)$, as we show next.

LEM 19.19 *The process*

$$M_n = \sum_{k=1}^n W_k X_{k-1}, \qquad n \geq 0$$

is a MG.

LEM 19.20 *We have*

$$\Psi_n \equiv \frac{1}{n}\sum_{k=1}^{n} X_{k-1}^2 \to_P \frac{\nu^2}{1-\rho^2} \equiv \Psi_\infty.$$

Before proving the two claims, it will be helpful to write by induction

$$X_n = \sum_{k=1}^{n} \rho^{n-k} W_k, \tag{2}$$

and define $H_n(x) = \sum_{k=1}^{n} x^{n-k}$. Then $\mathrm{Var}[X_n] = \mathbb{E}[X_n^2] = \nu^2 H_n(\rho^2)$, where we used that the W_ks are independent and have mean 0 and variance ν^2. Note also that by (2) and the bound on $|W_n|$, we have the following bound

$$|X_n| \leq \sum_{k=1}^{n} \rho^{n-k}|W_k| \leq cH_n(\rho). \tag{3}$$

Proof:(of LEM 19.19) The process $\{M_n\}$ is adapted by (2) and integrable by the bounds on $|W_n|$ and $|X_n|$. Observe further that, since $X_{k-1} \in \mathcal{F}_{k-1}$,

$$\mathbb{E}\left[\sum_{k=1}^{n} W_k X_{k-1} \,\middle|\, \mathcal{F}_{n-1}\right] = \sum_{k=1}^{n-1} W_k X_{k-1} + X_{n-1}\mathbb{E}\left[W_n \,|\, \mathcal{F}_{n-1}\right] = \sum_{k=1}^{n-1} W_k X_{k-1}.$$

That concludes the proof of the claim. ■

Proof:(of LEM 19.20) We use Chebyshev's inequality. We know that

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} X_{k-1}^2\right] = \frac{1}{n}\sum_{k=1}^{n} \nu^2 H_{k-1}(\rho^2) \to \frac{\nu^2}{1-\rho^2}, \tag{4}$$

as $H_n(\rho^2) \to \frac{1}{1-\rho^2}$. We need a bound on the variance. Trivially

$$\mathrm{Var}[X_k^2] \leq \mathbb{E}[X_k^4] \leq \{cH_k(\rho)\}^4.$$

Moreover, for $k < \ell$, we write $X_\ell \equiv \rho^{\ell-k}X_k + Y_{\ell,k}$ where $Y_{\ell,k} \in \sigma(W_{k+1}, \ldots, W_\ell)$

and $H_\ell(\rho^2) = \rho^{2(\ell-k)} H_k(\rho^2) + H_{\ell-k}(\rho^2)$. Hence,

$$\begin{aligned}
&\mathrm{Cov}\left[X_k^2, X_\ell^2\right] \\
&= \mathbb{E}\left[(X_k^2 - \nu^2 H_k(\rho^2))(X_\ell^2 - \nu^2 H_\ell(\rho^2))\right] \\
&= \mathbb{E}[(X_k^2 - \nu^2 H_k(\rho^2)) \\
&\qquad \times ((\rho^{\ell-k} X_k + Y_{\ell,k})^2 - \nu^2[\rho^{2(\ell-k)} H_k(\rho^2) + H_{\ell-k}(\rho^2)])] \\
&= \mathbb{E}[(X_k^2 - \nu^2 H_k(\rho^2)) \\
&\qquad \times (\rho^{2(\ell-k)} X_k^2 + 2\rho^{\ell-k} X_k Y_{\ell,k} + Y_{\ell,k}^2 - \nu^2[\rho^{2(\ell-k)} H_k(\rho^2) + H_{\ell-k}(\rho^2)])] \\
&= \rho^{2(\ell-k)} \mathbb{E}[(X_k^2 - \nu^2 H_k(\rho^2))^2] \\
&\leq \rho^{2(\ell-k)} \{c H_k(\rho)\}^4.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Var}\left[\frac{1}{n}\sum_{k=1}^n X_{k-1}^2\right] &= \frac{2}{n^2} \sum_{0\leq k\leq \ell\leq n-1} \rho^{2(\ell-k)} \{c H_k(\rho)\}^4 \\
&= \frac{2c^4}{n^2(1-\rho)^4} \sum_{0\leq k\leq \ell\leq n-1} \rho^{2(\ell-k)} \\
&\leq \frac{2c^4 n}{n^2(1-\rho)^4} \sum_{0\leq m\leq +\infty} \rho^{2m} \\
&\leq \frac{2c^4}{n(1-\rho)^4(1-\rho^2)} \\
&\equiv \frac{K}{n}.
\end{aligned}$$

By Chebyshev's inequality,

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{k=1}^n X_{k-1}^2 - \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^n X_{k-1}^2\right]\right| \geq \delta\right] \leq \frac{K/n}{\delta^2} \to 0,$$

as $n \to +\infty$. Together with (4), that proves the claim. ■

We apply the MG-CLT with

$$M_{n,m} = \frac{M_{m\wedge n}}{\sqrt{n\Psi_\infty \nu^2}},$$

and $\mathcal{F}_{n,m} = \mathcal{F}_m$. Then

$$Z_{n,m} = \frac{M_m - M_{m-1}}{\sqrt{n\Psi_\infty \nu^2}} \mathbb{1}\{m \leq n\} = \frac{W_m X_{m-1}}{\sqrt{n\Psi_\infty \nu^2}} \mathbb{1}\{m \leq n\},$$

and

$$\begin{aligned}\sigma_{n,m}^2 &= \mathbb{E}[Z_{n,m}^2 \,|\, \mathcal{F}_{n,m-1}] \\ &= \mathbb{1}\{m \leq n\} \frac{X_{m-1}^2}{n\Psi_\infty \nu^2}\, \mathbb{E}[W_m^2 \,|\, \mathcal{F}_{m-1}] \\ &= \mathbb{1}\{m \leq n\} \frac{X_{m-1}^2}{n\Psi_\infty \nu^2} \nu^2,\end{aligned}$$

and

$$\Gamma_{n,\infty} = \frac{1}{\Psi_\infty} \frac{1}{n} \sum_{m=1}^{n} X_{m-1}^2 = \frac{\Psi_n}{\Psi_\infty} \to_P 1,$$

as $n \to +\infty$ by LEM 19.20. To check the second condition in the MG-CLT, we note that

$$|Z_{n,m}| = \left| \frac{W_m X_{m-1}}{\sqrt{n\Psi_\infty \nu^2}} \mathbb{1}\{m \leq n\} \right| \leq \frac{c(cH_n(\rho))}{\sqrt{n\Psi_\infty \nu^2}} \leq \frac{c^2}{(1-\rho)\sqrt{\Psi_\infty \nu^2}} \times \frac{1}{\sqrt{n}} \leq \varepsilon,$$

for all n large enough. The MG-CLT then says that

$$\frac{M_n}{\sqrt{n\Psi_\infty \nu^2}} \Rightarrow \mathrm{N}(0,1)$$

or put differently, that

$$\sqrt{n}(\hat{\rho}_n - \rho) \times \frac{1}{\sqrt{\Psi_\infty \nu^2}} \times \frac{1}{n} \sum_{k=1}^{n} X_{k-1}^2 \Rightarrow \mathrm{N}(0,1).$$

By LEM 19.5 and LEM 19.20, we finally have

$$\sqrt{\frac{\Psi_\infty n}{\nu^2}} (\hat{\rho}_n - \rho) \Rightarrow \mathrm{N}(0,1).$$

That concludes the proof of CLAIM 19.18. ■

References

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