Notes 28 : Brownian motion: Markov property

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References:[Dur10, Section 8.2], [MP10, Section 2.1, 2.2].
Recall:

DEF 28.1 (Brownian motion: Definition I) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X is a Gaussian process with almost surely continuous paths, that is,*

$$\mathbb{P}[X(t) \textit{ is continuous in } t] = 1,$$

such that $X(0) = 0$,

$$\mathbb{E}[X(t)] = 0,$$

and

$$\mathrm{Cov}[X(s), X(t)] = s \wedge t.$$

More generally, $B = \sigma X + x$ is a Brownian motion started at x.

DEF 28.2 (Brownian motion: Definition II) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X has almost surely continuous paths and stationary independent increments such that $X(s+t) - X(s)$ is Gaussian with mean 0 and variance t.*

THM 28.3 (Existence) *Standard Brownian motion $B = \{B(t)\}_{t\geq 0}$ exists.*

1 Filtrations

Recall:

DEF 28.4 (Filtration) *A* filtration *is a family $\{\mathcal{F}(t) : t \geq 0\}$ of sub-σ-fields such that $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $s \leq t$.*

We will consider two natural filtrations for BM.

DEF 28.5 *Let $\{B(t)\}$ be a BM. Then we denote*

$$\mathcal{F}^0(t) = \sigma(B(s) \,:\, 0 \leq s \leq t).$$

Recall this is the smallest σ-field such that each $B(s, \cdot)$, $0 \leq s \leq t$, is measurable. Moreover, we let

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s).$$

Clearly $\mathcal{F}^0(t) \subseteq \mathcal{F}^+(t)$. The latter has the advantage of being right-continuous*, that is,*

$$\bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon) = \mathcal{F}^+(t).$$

DEF 28.6 (Germ field) *The* germ σ-field *is $\mathcal{F}^+(0)$.*

EX 28.7 *Intuitively, $\mathcal{F}^+(t)$ allows an "infinitesimal glance into the future." We give an example. Let $B(t)$ be a standard BM and define*

$$T = \inf\{t > 0 \,:\, B(t) > 0\}.$$

Then $\{T = 0\} \in \mathcal{F}^+(0)$ since

$$\{T = 0\} = \bigcap_{n\geq 1} \{\exists 0 < \varepsilon < n^{-1},\ B(\varepsilon) > 0\}.$$

By monotonicity you can start at any n and therefore this is in $\mathcal{F}^0(s)$ for every $s > 0$.

2 Markov property

The basic Markov property for BM is the following.

THM 28.8 (Markov property I) *Suppose that $\{B(t)\}$ is a BM started at x. Let $s \geq 0$. Then the process $\{B(s+t) - B(s)\}_{t\geq 0}$ is a BM started at 0 and is independent of the process $\{B(t) \,:\, 0 \leq s \leq t\}$, that is, the σ-fields*

$$\sigma(B(s+t) - B(s) \,:\, t \geq 0),$$

and

$$\sigma(B(t) \,:\, 0 \leq t \leq s),$$

are independent.

Proof: We have already proved that $\{B(s+t)-B(s)\}_{t\geq 0}$ is a BM started at 0.

Further, recall:

LEM 28.9 (Independence and π-systems) *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras and that $\mathcal{I}$ and $\mathcal{J}$ are π-systems (i.e., families of subsets stable under finite intersections) such that*

$$\sigma(\mathcal{I}) = \mathcal{G}, \quad \sigma(\mathcal{J}) = \mathcal{H}.$$

Then $\mathcal{G}$ and $\mathcal{H}$ are independent if and only if $\mathcal{I}$ and $\mathcal{J}$ are, i.e.,

$$\mathbb{P}[I \cap J] = \mathbb{P}[I]\mathbb{P}[J], \quad \forall I \in \mathcal{I}, J \in \mathcal{J}.$$

Note that sets of the form

$$\{\omega \,:\, B(t_j) \in A_j,\ 0 \leq t_j \leq t,\ j = 1, \ldots, n\},$$

for $A_j \in \mathcal{B}$ are a π-system generating $\mathcal{F}^0(t)$. Similarly for $\sigma(B(s+t)-B(s)\,:\, t \geq 0)$. Therefore the independence statement immediately follows from the independence of increments. ∎

In fact, we can prove a stronger statement:

THM 28.10 (Markov property II) *Suppose that $\{B(t)\}$ is a BM started at x. Let $s \geq 0$. Then the process $\{B(s+t)-B(s)\}_{t\geq 0}$ is a BM started at 0 and is independent of $\mathcal{F}^+(s)$.*

Proof: By continuity,

$$B(t+s) - B(s) = \lim_n B(s_n + t) - B(s_n),$$

for a strictly decreasing sequence $\{s_n\}_n$ converging to s. But note that for any $0 \leq t_1 < \cdots < t_j$

$$(B(t_1 + s_n) - B(s_n), \ldots, B(t_j + s_n) - B(s_n)),$$

is independent of $\mathcal{F}^+(s) \subseteq \mathcal{F}^0(s_n)$ and so is the limit. ∎

3 Applications

As a first application, we get the following.

THM 28.11 (Blumenthal's 0-1 law) *For any x, the germ σ-field $\mathcal{F}^+(0)$ of a BM started at x is trivial.*

Proof: Let

$$A \in \mathcal{F}^+(0) \subseteq \sigma(B(t) \,:\, t \geq 0) = \sigma(B(t) - x \,:\, t \geq 0).$$

By the previous theorem, the two σ-fields above are independent and therefore A is independent of itself, that is,

$$\mathbb{P}[A] = \mathbb{P}[A \cap A] = \mathbb{P}[A]^2,$$

or $\mathbb{P}[A] \in \{0, 1\}$. ■

We come back to our example.

EX 28.12 *Let $B(t)$ be a standard BM and define*

$$T = \inf\{t > 0 \,:\, B(t) > 0\}.$$

Then $\{T = 0\} \in \mathcal{F}^+(0)$ since

$$\{T = 0\} = \bigcap_{n \geq 1} \{\exists 0 < \varepsilon < n^{-1},\ B(\varepsilon) > 0\}.$$

Hence,

$$\mathbb{P}[T = 0] \in \{0, 1\}.$$

We show that it is 1 by showing that it is positive. Note that

$$\mathbb{P}[T \leq t] \geq \mathbb{P}[B(t) > 0] = \frac{1}{2},$$

for $t > 0$, by symmetry of the Gaussian. It also follows by continuity that

$$\inf\{t > 0 \,:\, B(t) = 0\} = 0,$$

almost surely.

An immediate application of Blumenthal's 0-1 law (by time inversion) is:

THM 28.13 (0-1 law for tail events) *Let $B(t)$ be a BM. Then the tail of B, that is,*

$$\mathcal{T} = \bigcap_{t \geq 0} \mathcal{G}(t) = \bigcap_{t \geq 0} \sigma(B(s) \,:\, s \geq t),$$

is trivial.

4 Stopping times

We first generalize stopping times to continuous time.

DEF 28.14 (Stopping time) *A RV T with values in $[0, +\infty]$ is a stopping time with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if for all $t \geq 0$,*

$$\{T \leq t\} \in \mathcal{F}(t).$$

The following theorem explains why we mostly work with the "augmented" $\mathcal{F}^+(t)$.

THM 28.15 *If the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is right-continuous in the previous definition, then an equivalent definition is obtained by using a strict inequality.*

Proof: One direction is obvious by monotonicity. Suppose the assumption is satisfied with a strict inequality. Then

$$\{T \leq t\} = \bigcap_{k=1}^{+\infty} \{T < t + k^{-1}\} \in \bigcap_{n=1}^{+\infty} \mathcal{F}(t + n^{-1}) = \mathcal{F}(t),$$

by right-continuity. ■

EX 28.16 *Let G be an open set. Then*

$$T = \inf\{t \geq 0 \,:\, B(t) \in G\},$$

is a stopping time with respect to $\{\mathcal{F}^+(t)\}$. Indeed, note

$$\{T < t\} = \bigcup_{s<t,\ s\in\mathbb{Q}} \{B(s) \in G\} \in \mathcal{F}^+(t),$$

by continuity of paths and the fact that G is open.

To define the strong Markov property, we will need the following.

DEF 28.17 *Let T be a stopping time with respect to $\{\mathcal{F}^+(t)\}_{t\geq 0}$. Then we let*

$$\mathcal{F}^+(T) = \{A \,:\, A \cap \{T \leq t\} \in \mathcal{F}^+(t),\ \forall t \geq 0\}.$$

The following lemma will be useful in extending properties about discrete-time stopping times to continuous time.

LEM 28.18 *The following hold:*

1. *If T_n is a sequence of stopping times with respect to $\{\mathcal{F}(t)\}$ such that $T_n \uparrow T$, then so is T.*

2. *Let T be a stopping time with respect to $\{\mathcal{F}(t)\}$. Then the following are also stopping times:*

$$T_n = (m+1)2^{-n} \text{ if } m2^{-n} \leq T < (m+1)2^{-n}.$$

Proof: Note that in the first case

$$\{T \leq t\} = \bigcap_{n=1}^{+\infty} \{T_n \leq t\},$$

and in the second case

$$\{T_n \leq t\} \subseteq \{T \leq t\}.$$

■

EX 28.19 *Let F be a closed set. Then*

$$T = \inf\{t \geq 0 \,:\, B(t) \in F\},$$

is a stopping time with respect to $\{\mathcal{F}^+(t)\}$. (Writing this as a countable event is not entirely obvious (because you could "touch" at an irrational time). But if you do, there will be rational times at which you are arbitrarily close by continuity. Consider open neighborhoods of F and take a limit.)

5 Strong Markov property

THM 28.20 (Strong Markov property) *Let $\{B(t)\}_{t\geq 0}$ be a BM and T, an almost surely finite stopping time. Then the process*

$$\{B(T+t) - B(T) \,:\, t \geq 0\},$$

is a BM started at 0 independent of $\mathcal{F}^+(T)$.

Proof: The idea of the proof is to discretize the stopping time, sum over all possibilities and use the Markov property. Let T_n be a discretization of T as above. Let

$$B_k(t) = B(t + k2^{-n}) - B(k2^{-n}),$$

and

$$B_*(t) = B(t + T_n) - B(T_n).$$

Suppose $E \in \mathcal{F}^+(T_n)$. Then for every "finite-dimensional" event A we have, by the Markov property and time translation invariance,

$$\begin{aligned}
\mathbb{P}[\{B_* \in A\} \cap E] &= \sum_{k=1}^{+\infty} \mathbb{P}[\{B_k \in A\} \cap E \cap \{T_n = k2^{-n}\}] \\
&= \sum_{k=1}^{+\infty} \mathbb{P}[B_k \in A]\mathbb{P}[E \cap \{T_n = k2^{-n}\}] \\
&= \mathbb{P}[B \in A] \sum_{k=1}^{+\infty} \mathbb{P}[E \cap \{T_n = k2^{-n}\}] \\
&= \mathbb{P}[B \in A]\mathbb{P}[E].
\end{aligned}$$

That is, B_* is independent of $\mathcal{F}^+(T_n)$. (Actually need to argue that the distribution of B and B_* are the same, but this is clear by taking $E = \Omega$.) Since $\mathcal{F}^+(T) \subseteq \mathcal{F}^+(T_n)$, B_* is also independent of $\mathcal{F}^+(T)$. Moreover, $T_n \downarrow T$ so that by continuity $\{B(t+T) - B(T)\}_{t \geq 0}$ is itself independent of $\mathcal{F}^+(T)$. The same argument shows that the increments have the correct distribution. (That is, it is true for all n by the above, and then true in the limit.) ■

6 Applications

We discuss one application.

THM 28.21 (Reflection principle) *Let $\{B(t)\}_{t\geq 0}$ be a standard BM and T, a stopping time. Then the process*

$$B^*(t) = B(t)\mathbb{1}\{t \leq T\} + (2B(T) - B(t))\mathbb{1}\{t > T\},$$

called BM reflected at T, *is also a standard BM.*

Proof: Follows immediately from the strong Markov property and symmetry. ■

A remarkable consequence is the following.

THM 28.22 *Let $\{B(t)\}$ be a standard BM and let*

$$M(t) = \max_{0 \leq s \leq t} B(s).$$

Then, if $a > 0$,

$$\mathbb{P}[M(t) \geq a] = 2\mathbb{P}[B(t) \geq a] = \mathbb{P}[|B(t)| \geq a].$$

Proof: Let

$$T = \inf\{t \geq 0 \,:\, B(t) = a\}.$$

Then we have the disjoint union

$$\begin{aligned}\{M(t) \geq a\} &= \{B(t) \geq a\} \cup \{B(t) < a, M(t) \geq a\} \\ &= \{B(t) \geq a\} \cup \{B^*(t) > a\}.\end{aligned}$$

(Just note the two implications—ignoring the event $\{B^*(t) = a\}$ which has 0 probability.) ■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.