Notes 29 : Brownian motion: martingale property

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References:[Dur10, Section 8.5, 8.6, 8.8], [MP10, Section 2.4, 5.1, 5.3].

Recall:

DEF 29.1 (Brownian motion) *The continuous-time stochastic process* $\{X(t)\}_{t\geq 0}$ *is a* standard Brownian motion *if it has almost surely continuous paths and stationary independent increments such that* $X(s+t) - X(s)$ *is Gaussian with mean* 0 *and variance* t.

DEF 29.2 (Filtration) *A* filtration *is a family* $\{\mathcal{F}(t) \,:\, t \geq 0\}$ *of sub-*σ*-fields such that* $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ *for all* $s \leq t$.

DEF 29.3 *Let* $\{B(t)\}$ *be a BM. Then we denote*

$$\mathcal{F}^0(t) = \sigma(B(s) \,:\, 0 \leq s \leq t).$$

Recall this is the smallest σ*-field such that each* $B(s,\cdot)$, $0 \leq s \leq t$, *is measurable. Moreover, we let*

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s).$$

Clearly $\mathcal{F}^0(t) \subseteq \mathcal{F}^+(t)$. *The latter has the advantage of being* right-continuous, *that is,*

$$\bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon) = \mathcal{F}^+(t).$$

DEF 29.4 (Germ field) *The* germ σ-field *is* $\mathcal{F}^+(0)$.

DEF 29.5 (Stopping time) *A RV* T *with values in* $[0,+\infty]$ *is a stopping time with respect to the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ *if for all* $t \geq 0$,

$$\{T \leq t\} \in \mathcal{F}(t).$$

THM 29.6 *If the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ *is right-continuous in the previous definition, then an equivalent definition is obtained by using a strict inequality.*

DEF 29.7 *Let T be a stopping time with respect to $\{\mathcal{F}^+(t)\}_{t\geq 0}$. Then we let*

$$\mathcal{F}^+(T) = \{A \,:\, A \cap \{T \leq t\} \in \mathcal{F}^+(t),\ \forall t \geq 0\}.$$

THM 29.8 (Strong Markov property) *Let $\{B(t)\}_{t\geq 0}$ be a BM and T, an almost surely finite stopping time. Then the process*

$$\{B(T+t) - B(T) \,:\, t \geq 0\},$$

is a BM started at 0 independent of $\mathcal{F}^+(T)$.

1 Martingales

We first generalize MGs to continuous time.

DEF 29.9 (Continuous-time martingale) *A real-valued SP $\{X(t)\}_{t\geq 0}$ is a* martingale *with respect to a filtration $\{\mathcal{F}(t)\}$ if it is* adapted*, that is, $X(t) \in \mathcal{F}(t)$ for all $t \geq 0$, if $E|X(t)| < +\infty$ for all $t \geq 0$, and if*

$$\mathbb{E}[X(t)\,|\,\mathcal{F}(s)] = X(s),$$

almost surely, for all $0 \leq s \leq t$.

EX 29.10 *Let $\{B(t)\}$ be a standard BM. Then*

$$\begin{aligned}\mathbb{E}[B(t)\,|\,\mathcal{F}^+(s)] &= \mathbb{E}[B(t) - B(s)\,|\,\mathcal{F}^+(s)] + B(s)\\ &= \mathbb{E}[B(t) - B(s)] + B(s)\\ &= B(s),\end{aligned}$$

by the Markov property. Hence BM is a MG.

2 Optional stopping theorem

We obtain a version of the optional stopping theorem for BMs.

THM 29.11 (Optional stopping theorem) *Suppose $\{X(t)\}_{t\geq 0}$ is a continuous MG, and $0 \leq S \leq T$ are stopping times. If the process $\{X(T \wedge t)\}_{t\geq 0}$ is dominated by an integrable RV X, then*

$$\mathbb{E}[X(T)\,|\,\mathcal{F}(S)] = X(S),$$

almost surely.

Proof: We proceed by discretization. Fix N and consider the discrete-time MG

$$X_n = X(T \wedge n2^{-N})$$

and the stopping times

$$S'_N = \lfloor 2^N S \rfloor + 1$$

and

$$T'_N = \lfloor 2^N T \rfloor + 1$$

with respect to the filtration

$$\mathcal{G}_n = \mathcal{F}(n2^{-N}).$$

The discrete-time optional stopping theorem gives

$$\mathbb{E}[X_{T'_N} \,|\, \mathcal{G}_{S'_N}] = X_{S'_N},$$

which is equivalent to

$$\mathbb{E}[X(T \wedge 2^{-N}T'_N) \,|\, \mathcal{F}(2^{-N}S'_N)] = \mathbb{E}[X(T) \,|\, \mathcal{F}(2^{-N}S'_N)] = X(T \wedge 2^{-N}S'_N).$$

For $A \in \mathcal{F}(S) \subseteq \mathcal{F}(2^{-N}S'_N)$, by the definition of the conditional expectation and the dominated convergence theorem,

$$\begin{aligned}
\mathbb{E}[X(T); A] &= \lim_N \mathbb{E}[\mathbb{E}[X(T) \,|\, \mathcal{F}(2^{-N}S'_N)]; A] \\
&= \mathbb{E}[\lim_N X(T \wedge 2^{-N}S'_N); A] \\
&= \mathbb{E}[X(S); A],
\end{aligned}$$

where we used continuity. The first line above is true for each N and therefore for the limit. ■

3 Applications

A typical application is Wald's lemma.

THM 29.12 (Wald's lemma for BM) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that either:*

1. $\mathbb{E}[T] < +\infty$, *or*
2. $\{B(t \wedge T)\}$ *is dominated by an integrable RV.*

Then $\mathbb{E}[B(T)] = 0$.

Proof: The result under the second condition follows immediately from the optional stopping theorem with $S = 0$. We show that the first condition implies the second one.

Assume $\mathbb{E}[T] < +\infty$. Define

$$M_k = \max_{0 \leq t \leq 1} |B(t+k) - B(k)|,$$

and

$$M = \sum_{k=1}^{\lceil T \rceil} M_k,$$

and note that $|B(t \wedge T)| \leq M$.

Then

$$\begin{aligned} E[M] &= \sum_k \mathbb{E}[\mathbb{1}\{T > k-1\} M_k] \\ &= \sum_k \mathbb{P}[T > k-1] \mathbb{E}[M_k] \\ &= \mathbb{E}[M_0] \mathbb{E}[T+1] < +\infty \end{aligned}$$

by our result on the maximum from the previous lecture. ■

We state without proof:

THM 29.13 (Wald's second lemma) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that $\mathbb{E}[T] < +\infty$. Then*

$$\mathbb{E}[B(T)^2] = E[T].$$

Proof: The proof is based on the fact that $B(t)^2 - t$ is a MG. Consider

$$T_n = \inf\{t \geq 0 \,:\, |B(t)| = n\},$$

and take an appropriate limit. See [MP10] for details. ■

An immediate application of Wald's lemma gives:

THM 29.14 *Let $\{B(t)\}$ be a standard BM. For $a < 0 < b$ let*

$$T = \inf\{t \geq 0 \,:\, B(t) \in \{a, b\}\}.$$

Then

$$\mathbb{P}[B(T) = a] = \frac{b}{|a| + b}.$$

Moreover,

$$\mathbb{E}[T] = |a| b.$$

Note that the stopped process in bounded.

4 Skorokhod embedding

BM can be used to embed any square-integrable RV.

THM 29.15 (Skorokhod embedding) *Suppose $\{B(t)\}_t$ is a standard BM and that X is a RV with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] < +\infty$. Then there exists a stopping time T with respect to $\{\mathcal{F}^+(t)\}_t$ such that $B(T)$ has the law of X and $\mathbb{E}[T] = \mathbb{E}[X^2]$.*

The proof uses a binary splitting MG:

DEF 29.16 *A $\{X_n\}_n$ is* binary splitting *if, whenever the event*

$$A(x_0, \ldots, x_n) = \{X_0 = x_0, \ldots, X_n = x_n\},$$

for some $x_0, \ldots, x_n$, has positive probability, then the RV X_{n+1} conditioned on $A(x_0, \ldots, x_n)$ is supported on at most two values.

LEM 29.17 *Let X be a RV with $\mathbb{E}[X^2] < +\infty$. Then there is a binary splitting MG $\{X_n\}_n$ such that $X_n \to X$ almost surely and in $\mathcal{L}^2$.*

Proof: The MG is defined recursively. Let

$$\mathcal{G}_0 = \{\emptyset, \Omega\},$$

and

$$X_0 = \mathbb{E}[X].$$

For $n > 0$, we let

$$\xi_n = \begin{cases} 1, & \text{if } X \geq X_n \\ -1, & \text{if } X < X_n, \end{cases}$$

and

$$\mathcal{G}_n = \sigma(\xi_0, \ldots, \xi_{n-1}),$$

and

$$X_n = \mathbb{E}[X \,|\, \mathcal{G}_n].$$

Then $\{X_n\}_n$ is a binary splitting MG. It remains to prove the convergence claim.

By (cJENSEN)

$$\mathbb{E}[X_n^2] \leq \mathbb{E}[X^2],$$

so $\{X_n\}_n$ is bounded in $\mathcal{L}^2$ and we have by Lévy's upward theorem

$$X_n \to X_\infty = \mathbb{E}[X \,|\, \mathcal{G}_\infty],$$

almost surely and in $\mathcal{L}^2$, where

$$\mathcal{G}_\infty = \sigma\left(\bigcup_i \mathcal{G}_i\right).$$

We need to show that $X = X_\infty$.

CLAIM 29.18 *Almost surely,*

$$\lim_n \xi_n(X - X_{n+1}) = |X - X_\infty|.$$

We first finish the proof of the lemma. Note that

$$\mathbb{E}[\xi_n(X - X_{n+1})] = \mathbb{E}[\xi_n \mathbb{E}[X - X_{n+1} \,|\, \mathcal{G}_{n+1}]] = 0.$$

Since $\{\xi_n(X - X_{n+1})\}_n$ is bounded in $\mathcal{L}^2$, the expectations converge and

$$\mathbb{E}|X - X_\infty| = 0.$$

Finally we prove the claim. If $X = X_\infty$, both sides are 0. If $X < X_\infty$, then for n large enough, $X < X_n$ and $\xi_n = -1$ by construction and the result holds. Similarly for the other case. ■

Proof:(of Theorem) Take a binary splitting MG as in the previous lemma. Since X_n conditioned on $A(x_0, \ldots, x_{n-1})$ is supported on two values, we can use the stopping time from last time (for BM started at x_{n-1}) and we get a sequence of stopping times

$$T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots \uparrow T$$

for some T such that

$$B(T_n) \sim X_n,$$

and

$$\mathbb{E}[T_n] = \mathbb{E}[B(T_n)^2].$$

By (MON) and $\mathcal{L}^2$ convergence

$$\mathbb{E}[T] = \lim_n \mathbb{E}[T_n] = \lim_n \mathbb{E}[X_n^2] = \mathbb{E}[X].$$

By continuity of paths,

$$B(T_n) \to B(T), \qquad \text{a.s.}$$

and

$$B(T) \sim X.$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.