High-Dimensional Probability and Statistics

MATH/STAT/ECE 888: Topics in Mathematical Data Science

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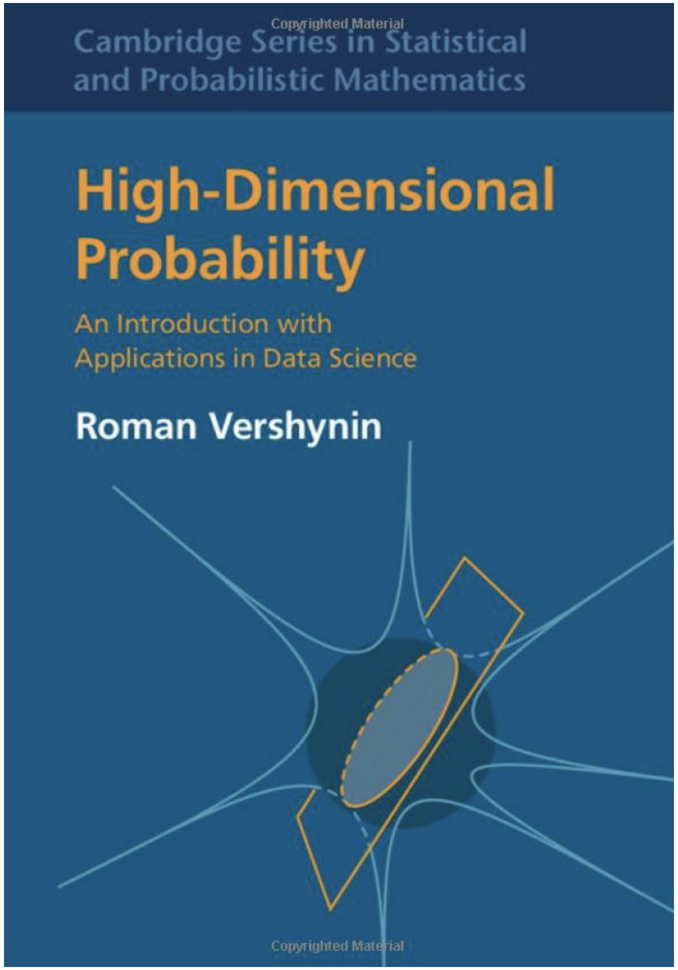
Lecture 12 (10/04/21)

Today's slides based on Vershynin



Isotropy

Definition

We might remember from a basic probability course how it is often convenient to assume that random variables have zero means and unit variances. This is also true in higher dimensions, where the notion of isotropy generalizes the assumption of unit variance.

Definition 3.2.1 (Isotropic random vectors). A random vector X in $\mathbb{R}^n$ is called *isotropic* if

$$\Sigma(X) = \mathbb{E}\, X X^{\mathsf{T}} = I_n$$

where I_n denotes the identity matrix in $\mathbb{R}^n$.

Recall that any random variable X with positive variance can be reduced by translation and dilation to the *standard score* – a random variable Z with zero mean and unit variance, namely

$$Z = \frac{X - \mu}{\sqrt{\operatorname{Var}(X)}}.$$

"Standardizing"

The following exercise gives a high-dimensional version of standard score.

Exercise 3.2.2 (Reduction to isotropy). ☕

(a) Let Z be a mean zero, isotropic random vector in $\mathbb{R}^n$. Let $\mu \in \mathbb{R}^n$ be a fixed vector and Σ be a fixed $n \times n$ positive-semidefinite matrix. Check that the random vector

$$X := \mu + \Sigma^{1/2} Z$$

has mean μ and covariance matrix $\mathrm{cov}(X) = \Sigma$.

(b) Let X be a random vector with mean μ and invertible covariance matrix $\Sigma = \mathrm{cov}(X)$. Check that the random vector

$$Z := \Sigma^{-1/2}(X - \mu)$$

is an isotropic, mean zero random vector.

This observation will allow us in many future results about random vectors to assume without loss of generality that they have zero means and are isotropic.

One-dimensional marginals

Lemma 3.2.3 (Characterization of isotropy). *A random vector X in $\mathbb{R}^n$ is isotropic if and only if*

$$\mathbb{E}\,\langle X, x\rangle^2 = \|x\|_2^2 \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof Recall that two symmetric $n \times n$ matrices A and B are equal if and only if $x^\mathsf{T} A x = x^\mathsf{T} B x$ for all $x \in \mathbb{R}^n$. (Check this!) Thus X is isotropic if and only if

$$x^\mathsf{T}\left(\mathbb{E}\,XX^\mathsf{T}\right)x = x^\mathsf{T} I_n x \quad \text{for all } x \in \mathbb{R}^n.$$

The left side of this identity equals $\mathbb{E}\,\langle X, x\rangle^2$ and the right side is $\|x\|_2^2$. This completes the proof. □

If x is a unit vector in Lemma 3.2.3, we can view $\langle X, x\rangle$ as a one-dimensional marginal of the distribution of X, obtained by projecting X onto the direction of x. Then X is isotropic if and only if *all one-dimensional marginals of X have unit variance*. Informally, this means that an isotropic distribution is extended evenly in all directions.