ADDITIONAL EXERCISES FOR FINAL

AE 1 Consider the set of symmetric $n \times n$ matrices

$$\mathbf{S}^n = \left\{X \in \mathbb{R}^{n\times n} \,:\, X = X^T\right\}.$$

a) Show that $\mathbf{S}^n$ is a linear subspace of the vector space of $n \times n$ matrices in the sense that for all $X_1, X_2 \in \mathbf{S}^n$ and $\alpha \in \mathbb{R}$

$$X_1 + X_2 \in \mathbf{S}^n,$$

and

$$\alpha X_1 \in \mathbf{S}^n.$$

b) Show that there exists a basis of $\mathbf{S}^n$ of size $d = \binom{n}{2} + n$, that is, a collection of symmetric matrices $X_1, \ldots, X_d \in \mathbf{S}^n$ such that any matrix $Y \in \mathbf{S}^n$ can be written as a linear combination

$$Y = \sum_{i=1}^d \alpha_i X_i,$$

for some $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$, and such that furthermore the matrices $X_1, \ldots, X_d \in \mathbf{S}^n$ are linearly independent in the sense that

$$\sum_{i=1}^d \alpha_i X_i = \mathbf{0}_{n\times n} \quad \implies \quad \alpha_1 = \cdots = \alpha_d = 0.$$

AE 2 Let A, B and C be events. Use the product rule to show that

$$\mathbb{P}[A \cap B|C] = \mathbb{P}[A|B \cap C]\,\mathbb{P}[B|C].$$

In words, the conditional probabilities satisfy the product rule.

AE 3 Let A, B, C be events.

a) Assume that $\mathbb{P}[C] > 0$ and $A \perp\!\!\!\perp B|C$. Show that $A \perp\!\!\!\perp B^c|C$.

b) Assume that $\mathbb{P}[B \cap C], \mathbb{P}[A \cap C] > 0$. Show that $A \perp\!\!\!\perp B|C$ if and only if

$$\mathbb{P}[A|B \cap C] = \mathbb{P}[A|C] \quad \text{and} \quad \mathbb{P}[B|A \cap C] = \mathbb{P}[B|C].$$

AE 4 For $i = 1, \dots, K$, let p_i be a probability mass function over the finite set $\mathcal{S}_i \subseteq \mathbb{R}$ with mean μ_i and variance σ_i^2. Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_K) \in \Delta_K$. Suppose X is drawn from the mixture distribution

$$p_X(x) = \sum_{i=1}^{K} \pi_i p_i(x).$$

Establish the following formulas:

a) $\mu := \mathbb{E}[X] = \sum_{i=1}^{K} \pi_i \mu_i$

b) $\sigma^2 := \mathrm{Var}[X] = \sum_{i=1}^{K} \pi_i(\sigma_i^2 + \mu_i^2 - \mu^2)$.

AE 5 Suppose that X, Y have joint probability density function

$$f_{X,Y}(x, y) = c \exp\left[-\frac{x^2}{2} - \frac{(x-y)^2}{2}\right],$$

for $x, y \in \mathbb{R}^2$, for some constant $c > 0$.

a) Find the value of the constant c. [*Hint:* The order of integration matters. You can do this without doing complicated integrals.]

b) Find the marginal density functions of X and Y. [*Hint:* You can do this without doing complicated integrals.]

c) Determine whether X and Y are independent. Justify your answer.

AE 6 Prove the *Inverting a Block Matrix* lemma by directly computing BB^{-1} and $B^{-1}B$ using the formula for the product of block matrices.

AE 7 Let f be a real-valued function taking a matrix $A = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times n}$ as an input. Assume f is continuously differentiable in each entry of A. Consider the following matrix derivative

$$\frac{\partial f(A)}{\partial A} = \begin{pmatrix} \frac{\partial f(A)}{\partial a_{1,1}} & \cdots & \frac{\partial f(A)}{\partial a_{1,n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial a_{n,1}} & \cdots & \frac{\partial f(A)}{\partial a_{n,n}} \end{pmatrix}.$$

a) Show that, for any $B \in \mathbb{R}^{n \times n}$,

$$\frac{\partial \operatorname{tr}(B^T A)}{\partial A} = B.$$

b) Show that, for any $\mathbf{x} = (x_i)_i, \mathbf{y} = (y_i) \in \mathbb{R}^n$,

$$\frac{\partial \mathbf{x}^T A \mathbf{y}}{\partial A} = \mathbf{x}\mathbf{y}^T.$$

AE 8 Consider a square block matrix with the same partitioning of the rows and columns, that is,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A \in \mathbb{R}^{n \times n}$, $A_{ij} \in \mathbb{R}^{n_i \times n_i}$ for $i = 1, 2$ with the condition $n_1 + n_2 = n$. Show that the transpose can be written as

$$A^T = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}$$

by writing down the entries $(A^T)_{i,j}$ in terms of the blocks of A. Make sure to consider carefully all cases (e.g., $i \leq n_1$ and $j > n_1$, etc.).

AE 9 Let $A = (a_{i,j})_{i \in [n], j \in [m]} \in \mathbb{R}^{n \times m}$ and $B = (b_{i,j})_{i \in [p], j \in [q]} \in \mathbb{R}^{p \times q}$ be arbitrary matrices. Their Kronecker product, denoted $A \otimes B \in \mathbb{R}^{np \times mq}$, is the following matrix in block form

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,m}B \end{pmatrix}.$$

The Kronecker product satisfies the following properties (which follow from block formulas, but which you do not have to prove): 1) if A, B, C, D are matrices of such size that one can form the matrix products AC and BD, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$; 2) the transpose of $A \otimes B$ is $(A \otimes B)^T = A^T \otimes B^T$.

a) Show that if D_1 and D_2 are square diagonal matrices, then so is $D_1 \otimes D_2$.

b) Show that if Q_1 and Q_2 have orthonormal columns, so does $Q_1 \otimes Q_2$.

c) Let $A_1 = U_1 \Sigma_1 V_1^T$ and $A_2 = U_2 \Sigma_2 V_2^T$ be full SVDs of $A_1, A_2 \in \mathbb{R}^{n \times n}$ respectively. Compute a full SVD of $A_1 \otimes A_2$.

d) Let A_1 and A_2 be as in c). Show that the rank of $A_1 \otimes A_2$ is $\mathrm{rk}(A_1)\,\mathrm{rk}(A_2)$.

AE 10 Let $\sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^T$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be an SVD of $A \in \mathbb{R}^{n \times m}$.

a) How can one obtain the spectral decompositions of $A^T A$ and AA^T from $\sigma_j, \mathbf{u}_j, \mathbf{v}_j$, $j = 1, \ldots, r$?

b) How many zero eigenvalues do $A^T A$ and AA^T have?

c) What are the ranks of $A^T A$ and AA^T?