ADDITIONAL EXERCISES FOR MIDTERM

AE 1 If $\mathbf{a}$ is a vector, then $\mathbf{a}_{r:s}$ is the vector of size $s-r+1$, with entries $a_r, \ldots, a_s$, i.e., $\mathbf{a}_{r:s} = (a_r, \ldots, a_s)^T$. The vector $\mathbf{a}_{r:s}$ is called a slice. As a more concrete example, if $\mathbf{z}$ is the 4-vector $(1, -1, 2, 0)^T$, the slice $\mathbf{z}_{2:3} = (-1, 2)^T$. Suppose the T-vector $\mathbf{x}$ represents a time series or signal. The quantity

$$\mathcal{D}(\mathbf{x}) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{T-1} - x_T)^2,$$

the sum of the differences of adjacent values of the signal, is called the Dirichlet energy of the signal. The Dirichlet energy is a measure of the roughness or wiggliness of the time series.

a) Express $\mathcal{D}(\mathbf{x})$ in vector notation using slicing.

b) How small can $\mathcal{D}(\mathbf{x})$ be? What signals $\mathbf{x}$ have this minimum value of the Dirichlet energy?

c) Find a signal $\mathbf{x}$ with entries no more than one in absolute value that has the largest possible value of $\mathcal{D}(x)$. Give the value of the Dirichlet energy achieved.

AE 2 A vector of length n can represent the number of times each word in a dictionary of n words appears in a document. For example, $(25, 2, 0)^T$ means that the first dictionary word appears 25 times, the second one twice, and the third one not at all. Suppose the n-vector $\mathbf{w}$ is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.

a) What is $\mathbf{1}^T\mathbf{w}$? Here $\mathbf{1}$ is an all-one vector of the appropriate size.

b) What does $w_{282} = 0$ mean?

c) Let $\mathbf{h}$ be the n-vector that gives the histogram of the word counts, i.e., h_i is the fraction of the words in the document that are word i. Use vector notation to express $\mathbf{h}$ in terms of $\mathbf{w}$. (You can assume that the document contains at least one word.)

AE 3 Show that, if $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times p}$, then $(BC)^T = C^T B^T$. [Hint: Check that the entries match.]

AE 4 Prove that $\mathrm{null}(B)$ is a linear subspace.

AE 5 Suppose that U_1 and U_2 are linear subspaces of vector space V. Show that $U_1 \cap U_2$ is a linear subspace of V. Is $U_1 \cup U_2$ always a subspace of V?

AE 6 Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, then so is the list

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n, \mathbf{v}_n\},$$

obtained by subtracting from each vector (except the last one) the following vector.

AE 7 Suppose $A, B \in \mathbb{R}^{n \times n}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Find a (nontrivial) linear system of equations satisfied by any $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{a}\|^2 + \|B\mathbf{x} - \mathbf{b}\|^2$.

AE 8 Let $\mathbf{g} : D \to \mathbb{R}^d$ for some open set $D \subseteq \mathbb{R}$. Assume that $\mathbf{g}$ is continuously differentiable everywhere on D and that further

$$\|\mathbf{g}(t)\|^2 = 1, \qquad \forall t \in D.$$

Show that $\mathbf{g}'(t)^T \mathbf{g}(t) = 0$ for all $t \in D$. [*Hint:* Use composition.]

AE 9 Prove the *Quadratic Bound for Strongly Convex Functions*. [*Hint:* Adapt the proof of the *Quadratic Bound for Smooth Functions*.]

AE 10 Let $A \in \mathbb{R}^{d \times d}$ be a symmetric matrix. Show that $A \preceq M I_{d \times d}$ if and only if the eigenvalues of A are at most M. Similarly, $m I_{d \times d} \preceq A$ if and only if the eigenvalues of A are at least m. [Hint: Observe that the eigenvectors of A are also eigenvectors of the identity matrix $I_{d \times d}$.]