HWK 2

1. A Boolean n-vector is one for which all entries are either 0 or 1. Such vectors are used to encode whether each of n conditions holds, with $a_i = 1$ meaning that condition i holds.

a) Another common encoding of the same information uses the two values -1 and 1 for the entries. For example the Boolean vector $(0, 1, 1, 0)$ would be written using this alternative encoding as $(-1, 1, 1, -1)$. Suppose that $\mathbf{x}$ is a Boolean vector, and $\mathbf{y}$ is a vector encoding the same information using the values -1 and 1. Express $\mathbf{y}$ in terms of $\mathbf{x}$ using vector notation. Also, express $\mathbf{x}$ in terms of $\mathbf{y}$ using vector notation.

b) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are Boolean n-vectors. Give a simple word description of their squared Euclidean distance $\|\mathbf{x} - \mathbf{y}\|_2^2$ and justify it mathematically.

2. Consider the vectors

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; x_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; x_6 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the clusters

$$C_1 = \{1, 4\}, \; C_2 = \{2, 3, 5, 6\}.$$

Compute the optimal representatives of the clusters.

3. Use *Cauchy-Schwarz* to show that for any A, B it holds that

$$\|AB\|_F \leq \|A\|_F \|B\|_F.$$

4. Clustering a collection of vectors into $k = 2$ groups is called 2-way partitioning, since we are partitioning the vectors into 2 groups, with index sets G_1 and G_2. Suppose we run k-means with $k = 2$ on the n-vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$. Show that there is a nonzero vector $\mathbf{w}$ and a scalar v that satisfy

$$\mathbf{w}^T \mathbf{x}_i + v \geq 0, \forall i \in G_1, \quad \mathbf{w}^T \mathbf{x}_i + v \leq 0, \forall i \in G_2.$$

In other words, the affine function $f(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + v$ is greater than or equal to zero on the first group, and less than or equal to zero on the second group. This is called linear separation of the two groups. [*Hint:* Consider the function $\|\mathbf{x} - \mathbf{z}_1\|_2^2 - \|\mathbf{x} - \mathbf{z}_2\|_2^2$, where $\mathbf{z}_1$ and $\mathbf{z}_2$ are the group representatives.]

5. Is the following matrix of full column rank?

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Justify your answer.

6. Let $\beta_j \neq 0$ for all $j \in [m]$. Justify that $\mathrm{span}(\beta_1 \mathbf{w}_1, \ldots, \beta_m \mathbf{w}_m) = \mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_m)$. [Hint: An equality between sets is established by showing inclusion in both directions.]

7. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans U, then so does the list

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{n-1} - \mathbf{v}_n, \mathbf{v}_n\},$$

obtained by subtracting from each vector (except the last one) the following vector.

8. Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Show that, if for any $\mathbf{b} \in \mathbb{R}^n$ there exists a unique $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$, then A is nonsingular. [Hint: Consider $\mathbf{b} = \mathbf{0}$.]