HWK 5

1 Let $f\,:\,\mathbb{R}^d \to \mathbb{R}$ be a function. The eipgraph of f is the set

$$\mathbf{epi} f = \{(\mathbf{x}, y)\,:\,\mathbf{x} \in \mathbb{R}^d, y \geq f(\mathbf{x})\}.$$

Show that f is convex if and only if $\mathbf{epi} f$ is a convex set.

2 A convex combination of $\mathbf{z}_1, \ldots, \mathbf{z}_m \in \mathbb{R}^d$ is a linear combination of the form

$$\mathbf{w} = \sum_{i=1}^m \alpha_i \mathbf{z}_i$$

where $\alpha_i \geq 0$ for all i and $\sum_{i=1}^m \alpha_i = 1$. Show that a set is convex if and only if it contains all convex combinations of its elements. [Hint: Use induction on m.]

3 The Lorentz cone is the set

$$C = \left\{(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}\,:\,\|\mathbf{x}\|_2 \leq t\right\}.$$

Show that C is convex.

4 Show that if S_1 and S_2 are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum

$$S = \{(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2)\,|\,\mathbf{x} \in \mathbb{R}^m, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n, (\mathbf{x}, \mathbf{y}_1) \in S_1, (\mathbf{x}, \mathbf{y}_2) \in S_2\}.$$

5 Let $f : \mathbb{R}^d \to \mathbb{R}$ be a strictly convex function. Assume further that there is a global minimizer $\mathbf{x}^*$. Show that it is unique.