Course: Math 535 - Mathematical Methods in Data Science (MMiDS) - Spring 2024

HWK 9

1 Let $X \sim \mathrm{Mult}(n, \boldsymbol{\pi})$ with $n \geq 1$ and $\boldsymbol{\pi} \in \Delta_K$. Establish the following formulas:

a) $\mathbb{E}[X] = n\boldsymbol{\pi}$

b) $\mathrm{Var}[X] = n[\mathrm{Diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^T]$.

2 Show that $\mathrm{H}_{L_n}(\mathbf{w})$ is positive semidefinite, where L_n is the negative log-likelihood in the generalized linear model.

3 Let A, B, C be events such that $\mathbb{P}[B \cap C] > 0$.

a) First show that

$$\mathbb{P}[A|B \cap C] = \frac{\mathbb{P}[C|A \cap B]\,\mathbb{P}[A|B]}{\mathbb{P}[C|A \cap B]\,\mathbb{P}[A|B] + \mathbb{P}[C|A^c \cap B]\,\mathbb{P}[A^c|B]}.$$

b) Now suppose $B \perp\!\!\!\perp C|A$. Show that

$$\mathbb{P}[A|B \cap C] = \frac{\mathbb{P}[C|A]\,\mathbb{P}[A|B]}{\mathbb{P}[C|A]\,\mathbb{P}[A|B] + \mathbb{P}[C|A^c]\,\mathbb{P}[A^c|B]}.$$

4 Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ be discrete random vectors.

a) Show that $\mathbf{X} \perp\!\!\!\perp (\mathbf{Y}, \mathbf{Z})|\mathbf{W}$ implies that $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}|\mathbf{W}$ and $\mathbf{X} \perp\!\!\!\perp \mathbf{Z}|\mathbf{W}$.

b) Suppose that $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}|\mathbf{Z}$ and $\mathbf{X} \perp\!\!\!\perp \mathbf{Z}$. Show that $\mathbf{X} \perp\!\!\!\perp (\mathbf{Y}, \mathbf{Z})$.

5 Let $\mathbf{X} \in \mathbb{R}^d$ be a random vector with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ and let $B \in \mathbb{R}^{\ell \times d}$ be a deterministic matrix. Define the random vector $\mathbf{Y} = B\mathbf{X}$.

a) Compute $\mathbb{E}[\mathbf{Y}]$.

b) Compute $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$.

c) Compute $\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}]$.

6 Let the process $(\mathbf{X}_{0:T}, \mathbf{Y}_{1:T})$ have a joint density of the form

$$f_{\mathbf{X}_0}(\mathbf{x}_0) \prod_{t=1}^{T} f_{\mathbf{X}_t|\mathbf{X}_{t-1}}(\mathbf{x}_t|\mathbf{x}_{t-1}) f_{\mathbf{Y}_t|\mathbf{X}_t}(\mathbf{y}_t|\mathbf{x}_t).$$

Show that, for any $t = 1, \ldots, T$, $\mathbf{Y}_t$ is conditionally independent of $\mathbf{Y}_{1:t-1}$ given $\mathbf{X}_t$.

7 Consider the vector-valued function $\mathbf{f} = (f_1, \ldots, f_d) : \mathbb{R}^d \to \mathbb{R}^d$ defined as

$$f_i(\mathbf{x}) = x_i^3,$$

for all $\mathbf{x} \in \mathbb{R}^d$ and all $i = 1, \ldots, d$.

a) Compute the Jacobian $\mathbf{J_f}(\mathbf{x})$ for all $\mathbf{x}$.

b) When is $\mathbf{J_f}(\mathbf{x})$ invertible?

c) When is $\mathbf{J_f}(\mathbf{x})$ positive semidefinite?