Course: Math 535 - Mathematical Methods in Data Science (MMiDS) - Spring 2024

PRACTICE MIDTERM 2

1 Answer the following short questions. You do not need to justify your answers.

a) State formally the linear regression problem on input data $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$.

b) Under what conditions on the data is there a unique solution to the linear regression problem?

2 Let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a list of m vectors in $\mathbb{R}^n$. Define

$$\mathcal{W} = \left\{\frac{\mathbf{v}_1 - \mathbf{v}_2}{\sqrt{2}}, \frac{\mathbf{v}_2 - \mathbf{v}_3}{\sqrt{2}}, \ldots, \frac{\mathbf{v}_{m-1} - \mathbf{v}_m}{\sqrt{2}}, \mathbf{v}_m\right\}.$$

Assume that $\mathcal{V}$ is an orthonormal list. Prove that the vectors in $\mathcal{W}$ have unit norm, but that $\mathcal{W}$ is **not** an orthonormal list. Make sure to justify your answer.

3 Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, let $\mathbf{a}_1, \mathbf{a}_2$ be vectors in $\mathbb{R}^d$, and let $b_1, b_2 \in \mathbb{R}$. Consider the following real-valued function of $\mathbf{x} \in \mathbb{R}^d$:

$$g(\mathbf{x}) = \frac{1}{2} f(\mathbf{a}_1^T \mathbf{x} + b_1) + \frac{1}{2} f(\mathbf{a}_2^T \mathbf{x} + b_2).$$

a) Compute the gradient of g **in vector form** in terms of the derivative f' of f. (By vector form, we mean that it is not enough to write down each element of $\nabla g(\mathbf{x})$ separately.)

b) Compute the Hessian of g **in matrix form** in terms of the first derivative f' and second derivative f'' of f. (By matrix form, we mean that it is not enough to write down each element of $H_g(\mathbf{x})$ or each of its columns separately.)

4 Let

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.$$

a) Write A in outer-product form $\mathbf{u}\mathbf{v}^T$.

b) Use a) to give a compact SVD of A. Make sure to justify your answer.

c) Give a full SVD of A. Make sure to justify your answer. [*Hint:* Complete the bases.]

5 Let $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ be an orthonormal list in $\mathbb{R}^n$ with $n > 3$. Let $Z \in \mathbb{R}^{n \times 3}$ be the matrix with columns $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

a) Prove that

$$H = I_{n \times n} - 2ZZ^T,$$

is an orthogonal matrix. Make sure to justify your answer.

b) Given $\varepsilon \neq 0$, consider the vector

$$\mathbf{u} = \frac{1}{\sqrt{1+\varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \mathbf{z}_1 \end{pmatrix} \in \mathbb{R}^{n+1}.$$

Prove that, if $O \in \mathbb{R}^{(n+1)\times(n+1)}$ is an orthogonal matrix, then $O\mathbf{u}$ is a unit vector. Make sure to justify your answer. [*Hint:* First show that $\mathbf{u}$ is a unit vector.]