Lecture 1 : Overview. Conditional Expectation I.

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Sections 0, 4.8, 9], [Dur10, Section 5.1].

1 Stochastic processes

The course MATH 275B is an introduction to stochastic processes.

DEF 1.1 *A* stochastic process (SP) *is a collection* $\{X_t\}_{t\in\mathcal{T}}$ *of* $(E,\mathcal{E})$*-valued random variables on a triple* $(\Omega,\mathcal{F},\mathbb{P})$*, where* $\mathcal{T}$ *is an arbitrary* index set. *For a fixed* $\omega\in\Omega$*,* $\{X_t(\omega)\,:\,t\in\mathcal{T}\}$ *is called a* sample path.

EX 1.2 *When* $\mathcal{T}=\mathbb{N}$ *or* $\mathcal{T}=\mathbb{Z}_+$ *we have a* discrete-time SP. *For instance,*

- $X_1, X_2, \ldots$ *iid RVs*
- $\{S_n\}_{n\geq 1}$ *where* $S_n=\sum_{i\leq n} X_i$ *with* X_i *as above*

EX 1.3 *When* $\mathcal{T}=\mathbb{R}_+$*, we have a* continuous-time SP. *For instance,*

- $N_t=\sup\{n\geq 1\,:\,S_n\leq t\}$ *where* S_n *is as above with nonnegative* X_i*s*

In general, $\mathcal{T}$ does not need to represent time.

EX 1.4 *When* $\mathcal{T}$ *is finite, we have a* random vector. *Although seemingly simple, this example encapsulates many non-trivial SPs. For instance,*

- *Let* $V=\{1,\ldots,n\}$ *and* $E=\{e=(u,v)\,:\,u\neq v\in V\}$ *. Consider iid RVs* $X(e)$*,* $e\in E$*, distributed according to Bernoulli(p) for* $0\leq p\leq 1$*. Then* $G_p=(V,E_p)$*, where* $E_p=\{e\in E\,:\,X(e)=1\}$*, is called an* Erdos-Renyi random graph.

2 A Preview of Things to Come

Two main themes:

1. Beyond independence

2. Sample path properties

Here are a few important examples of processes and questions we will answer about them.

2.1 Random walks

DEF 1.5 *A* random walk (RW) *on* $\mathbb{R}^d$ *is an SP of the form:*

$$S_n = \sum_{i \leq n} X_i,\ n \geq 1$$

where the X_i*s are iid in* $\mathbb{R}^d$*.*

EX 1.6 *When* $d = 1$*, recall from MATH 275A that*

- *SLLN:* $n^{-1} S_n \to \mathbb{E}[X_1]$ *a.s. when* $\mathbb{E}|X_1| < +\infty$

- *CLT:*
$$\frac{S_n - n\mathbb{E}[X_1]}{\sqrt{n \mathrm{Var}[X_1]}} \Rightarrow N(0, 1),$$
when $\mathbb{E}[X_1^2] < \infty$*.*

These are examples of limit theorems. Sample path properties, on the other hand, involve properties of the sequence $S_1(\omega), S_2(\omega), \ldots$*. For instance, let* $A \subset \mathbb{R}^d$

- $\mathbb{P}[S_n \in A$ *for some* $n \geq 1]$*?*

- $\mathbb{P}[S_n \in A \text{ i.o.}]$*?*

- $\mathbb{E}[T_A]$*? where* $T_A = \inf\{n \geq 1 \,:\, S_n \in A\}$

2.2 Branching processes

DEF 1.7 *A* branching process *is an SP of the form:*

- *Let* $X(i, n)$*,* $i \geq 1$*,* $n \geq 1$*, be an array of iid* $\mathbb{Z}_+$*-valued RVs with finite mean* $\mu = \mathbb{E}[X(1, 1)] < +\infty$ *and* $\mathbb{P}[X(1, 1) = 0] > 0$

- $Z_0 = 1$, *and inductively,*

$$Z_n = \sum_{1 \leq i \leq Z_{n-1}} X(i, n)$$

EX 1.8 *Typical questions about branching processes are:*

- *Extinction:* $\mathbb{P}[Z_n = 0 \text{ for some } n \geq 1]$*?*
- *Exponential growth:* $M_n = \mu^{-n} Z_n \to ?$
- *Limit of expectations: when* $\mu < 1$ *we have* $\mathbb{E}[M_n] = 1$ *for all* n *yet* $\mathbb{E}[M_\infty] = 0$

2.3 Markov chains

The two previous examples are special cases of a large class of SPs.

DEF 1.9 *A* discrete-time countable-space Markov chain(MC) *is an SP of the form:*

- E *countable state space*
- μ *initial distribution, that is,* $\mu_i \geq 0$, $i \in E$, *and* $\sum_{i \in E} \mu_i = 1$
- $\{p_{ij}\}_{i,j \in E}$ *transition matrix, that is,* $p_{ij} \geq 0$, $i, j \in E$, *and* $\sum_{j \in E} p_{ij} = 1$ *for all* $i \in E$
- *Let* $Y(i, n)$, $i \in E$, $n \geq 1$, *be an array of iid RVs distributed according to* $p_{i \cdot}$
- *Define the process recursively by* $Z_0 = 0$, *and,*

$$Z_n = Y(Z_{n-1}, n)$$

3 Review of undergraduate conditional probability

3.1 Conditional probability

For two events A, B, the conditional probability of A given B is defined as

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]},$$

where we assume $\mathbb{P}[B] > 0$.

3.2 Conditional expectation

Let X and Z be RVs taking values $x_1, \ldots, x_m$ and $z_1, \ldots, z_n$ resp. The conditional expectation of X given $Z = z_j$ is given as

$$y_j \equiv \mathbb{E}[X \mid Z = z_j] = \sum_i x_i \mathbb{P}[X = x_i \mid Z = z_j].$$

We assume $\mathbb{P}[Z = z_j] > 0$.

As motivation for the general definition, we make the following observations:

- We can think of the conditional expectation as a RV $Y \equiv \mathbb{E}[X \mid Z]$ defined as follows:
$$Y(\omega) = y_j, \text{ on } G_j \equiv \{\omega : Z(\omega) = z_j\}.$$
- Then Y is $\mathcal{G}$-measurable where $\mathcal{G} = \sigma(Z)$.
- On sets in $\mathcal{G}$, the expectation of Y agrees with the expectation of X, that is,
$$\begin{aligned} \mathbb{E}[Y; G_j] &= y_j \mathbb{P}[G_j] \\ &= \sum_i x_i \mathbb{P}[X = x_i \mid Z = z_j] \mathbb{P}[Z = z_j] \\ &= \sum_i x_i \mathbb{P}[X = x_i, Z = z_j] \\ &= \mathbb{E}[X; G_j]. \end{aligned}$$
This is also true for all $G \in \mathcal{G}$ by summation.

4 Conditional expectation: definition, existence, uniqueness

4.1 Definition

DEF&THM 1.10 *Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub σ-field. Then there exists a (a.s.) unique $Y \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$ s.t.*

$$\mathbb{E}[Y; G] = \mathbb{E}[X; G], \ \forall G \in \mathcal{G}.$$

Such Y is called a version of $\mathbb{E}[X \mid \mathcal{G}]$.

Further reading

Kolmogorov's extension theorem [Dur10, Section A.3]. Radon-Nikodym theorem [Dur10, Section A.4].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.