Lecture 10 : Uniform integrability

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References: [Wil91, Chapter 13], [Dur10, Section 4.5].

1 Uniform Integrability

LEM 10.1 *Let $Y \in L^1$. $\forall \varepsilon > 0$, $\exists K > 0$ s.t.*

$$\mathbb{E}[|Y|; |Y| > K] < \varepsilon.$$

Proof: Immediate by (MON) to $\mathbb{E}[|Y|; |Y| \leq K]$. ■

DEF 10.2 (Uniform Integrability) *A collection $\mathcal{C}$ of RVs on $(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly integrable (UI) if: $\forall \varepsilon > 0$, $\exists K > +\infty$ s.t.*

$$\mathbb{E}[|X|; |X| > K] < \varepsilon, \qquad \forall X \in \mathcal{C}.$$

THM 10.3 (Necessary and Sufficient Condition for L^1 Convergence) *Let $\{X_n\} \in L^1$ and $X \in L^1$. Then $X_n \to X$ in L^1 if and only if:*

- *$X_n \to X$ in prob*
- *$\{X_n\}$ is UI.*

Before giving the proof, we look at a few examples.

EX 10.4 (L^1-bddness is not sufficient) *Let $\mathcal{C}$ is UI and $X \in \mathcal{C}$. Note that*

$$\mathbb{E}|X| \leq \mathbb{E}[|X|; |X| \geq K] + \mathbb{E}[|X|; |X| < K] \leq \varepsilon + K < +\infty,$$

so UI implies L^1-bddness. But the opposite is not true by our last example.

EX 10.5 (L^p-bdd RVs) *Let $\mathcal{C}$ be L^p-bdd and $X \in \mathcal{C}$. Then*

$$\mathbb{E}[|X|; |X| > K] \leq \mathbb{E}[K^{1-p}|X|^p; |X| > K]] \leq K^{1-p}A \to 0,$$

as $K \to +\infty$.

EX 10.6 (Dominated RVs) *Assume $\exists Y \in L^1$ s.t. $|X| \leq Y$ $\forall X \in \mathcal{C}$. Then*

$$\mathbb{E}[|X|; |X| > K] \leq \mathbb{E}[Y; |X| > K] \leq \mathbb{E}[Y; Y > K],$$

and apply lemma above.

2 Proof of main theorem

Proof: We start with the if part. Fix $\varepsilon > 0$. We want to show that for n large enough:

$$\mathbb{E}|X_n - X| \leq \varepsilon.$$

Let $\phi_K(x) = \text{sgn}(x)[|x| \wedge K]$. Then,

$$\begin{aligned}\mathbb{E}|X_n - X| &\leq \mathbb{E}|\phi_K(X_n) - X_n| + \mathbb{E}|\phi_K(X) - X| + \mathbb{E}|\phi_K(X_n) - \phi_K(X)| \\ &\leq \mathbb{E}[|X_n|; |X_n| > K] + \mathbb{E}[|X|; |X| > K] + \mathbb{E}|\phi_K(X_n) - \phi_K(X)|.\end{aligned}$$

1st term $\leq \varepsilon/3$ by UI and 2nd term $\leq \varepsilon/3$ by lemma above. Check, by case analysis, that

$$|\phi_K(x) - \phi_K(y)| \leq |x - y|,$$

so $\phi_K(X_n) \to_P \phi_K(X)$. By bounded convergence for convergence in probability, the claim is proved.

LEM 10.7 (Bounded convergence theorem (convergence in probability version)) *Let $X_n \leq K < +\infty$ $\forall n$ and $X_n \to_P X$. Then*

$$\mathbb{E}|X_n - X| \to 0.$$

Proof:(Sketch) By

$$\mathbb{P}[|X| \geq K + m^{-1}] \leq \mathbb{P}[|X_n - X| \geq m^{-1}],$$

it follows that $\mathbb{P}[|X| \leq K] = 1$. Fix $\varepsilon > 0$

$$\begin{aligned}\mathbb{E}|X_n - X| &= \mathbb{E}[|X_n - X|; |X_n - X| > \varepsilon/2] + \mathbb{E}[|X_n - X|; |X_n - X| \leq \varepsilon/2] \\ &\leq 2K\mathbb{P}[|X_n - X| > \varepsilon/2] + \varepsilon/2 < \varepsilon,\end{aligned}$$

for n large enough. ■

Proof of only if part. Suppose $X_n \to X$ in L^1. We know that L^1 implies convergence in probability. So the first claim follows.

For the second claim, if $n \geq N$ (large enough),

$$\mathbb{E}|X_n - X| \leq \varepsilon.$$

We can choose K large enough so that

$$\mathbb{E}[|X_n|; |X_n| > K] < \varepsilon,$$

$\forall n \leq N$. So only need to worry about $n > N$. To use L^1 convergence, natural to write

$$\mathbb{E}[|X_n|; |X_n| > K] \leq \mathbb{E}[|X_n - X|; |X_n| > K] + \mathbb{E}[|X|; |X_n| > K].$$

First term $\leq \varepsilon$. The issue with the second term is that we cannot apply the lemma because the event involves X_n rather than X. In fact, a stronger version exists:

LEM 10.8 (Absolute continuity) *Let* $X \in L^1$. $\forall \varepsilon > 0$, $\exists \delta > 0$, *s.t.* $\mathbb{P}[F] < \delta$ *implies*

$$\mathbb{E}[|X|; F] < \varepsilon.$$

Proof: Argue by contradiction. Suppose there is ε and F_n s.t. $\mathbb{P}[F_n] \leq 2^{-n}$ and

$$\mathbb{E}[|X|; F_n] \geq \varepsilon.$$

By BC,

$$\mathbb{P}[H] \equiv \mathbb{P}[F_n \text{ i.o.}] = 0.$$

By (DOM) ,

$$\mathbb{E}[|X|; H] \geq \varepsilon,$$

a contradiction. ∎

To conclude note that

$$\mathbb{P}[|X_n| > K] \leq \frac{\mathbb{E}|X_n|}{K} \leq \frac{\sup_{n\geq N} \mathbb{E}|X_n|}{K} \leq \frac{\sup_{n\geq N} \mathbb{E}|X| + \mathbb{E}|X_n - X|}{K} < \delta,$$

uniformly in n for K large enough. We are done.

∎

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.