Lecture 13 : UI MGs: Optional Sampling Thm

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References: [Wil91, Appendix to Chapter 14], [Dur10, Section 4.7].

1 Review: Stopping times

Recall:

DEF 13.1 *A random variable* $T : \Omega \to \overline{\mathbb{Z}}_+ \equiv \{0, 1, \ldots, +\infty\}$ *is called a* stopping time *if*

$$\{T = n\} \in \mathcal{F}_n, \; \forall n \in \overline{\mathbb{Z}}_+.$$

EX 13.2 *Let* $\{A_n\}$ *be an adapted process and* $B \in \mathcal{B}$*. Then*

$$T = \inf\{n \geq 0 \,:\, A_n \in B\},$$

is a stopping time.

THM 13.3 (Optional Stopping Thm) *Let* $\{M_n\}$ *be a MG and* T *be a stopping time. Then* M_T *is integrable and*

$$\mathbb{E}[M_T] = \mathbb{E}[X_0].$$

if one of the following holds:

1. T *is bounded.*
2. M *is bounded and* T *is a.s. finite.*
3. $\mathbb{E}[T] < +\infty$ *and* M *has bounded increments.*
4. M *is UI.*

2 The σ-field $\mathcal{F}_T$

DEF 13.4 ($\mathcal{F}_T$) *Let T be a stopping time. Denote by $\mathcal{F}_T$ the set of all events F such that $\forall n \in \overline{\mathbb{Z}}_+$*

$$F \cap \{T = n\} \in \mathcal{F}_n.$$

The following two lemmas clarify the definition:

LEM 13.5 *$\mathcal{F}_T = \mathcal{F}_n$ if $T \equiv n$, $\mathcal{F}_T = \mathcal{F}_\infty$ if $T \equiv \infty$ and $\mathcal{F}_T \subseteq \mathcal{F}_\infty$ for any T.*

Proof: In the first case, note $F \cap \{T = k\}$ is empty if $k \neq n$ and is F if $k = n$. So if $F \in \mathcal{F}_T$ then $F = F \cap \{T = n\} \in \mathcal{F}_n$ and if $F \in F_n$ then $F = F \cap \{T = n\} \in F_n$. Moreover $\emptyset \in \mathcal{F}_n$ so we have proved both inclusions. This works also for $n = \infty$. For the third claim note

$$F = \cup_{k \in \overline{\mathbb{Z}}_+} F \cap \{T = n\} \in \mathcal{F}_\infty.$$

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LEM 13.6 *If X is adapted and T is a stopping time then $X_T \in \mathcal{F}_T$ (where we assume that $X_\infty \in \mathcal{F}_\infty$, e.g., $X_\infty = \liminf X_n$).*

Proof: For $B \in \mathcal{B}$

$$\{X_T \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$

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LEM 13.7 *If S, T are stopping times then $\mathcal{F}_{S \wedge T} \subseteq \mathcal{F}_T$.*

Proof: Let $F \in \mathcal{F}_{S \wedge T}$. Note that

$$F \cap \{T = n\} = \cup_{k \leq n} [(F \cap \{S \wedge T = k\}) \cap \{T = n\}] \in \mathcal{F}_n.$$

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3 Optional Sampling Theorem (OST)

THM 13.8 (Optional Sampling Theorem) *If M is a UI MG and S, T are stopping times with $S \leq T$ a.s. then $\mathbb{E}|M_T| < +\infty$ and*

$$\mathbb{E}[M_T \,|\, \mathcal{F}_S] = M_S.$$

Proof: Since M is UI, $\exists M_\infty \in \mathcal{L}^1$ s.t. $M_n \to M_\infty$ a.s. and in $\mathcal{L}^1$. We prove a more general claim:

LEM 13.9

$$\mathbb{E}[M_\infty \,|\, \mathcal{F}_T] = M_T.$$

Indeed, we then get the theorem by (TOWER) and (JENSEN).
Proof:(Lemma) Wlog we assume $M_\infty \geq 0$ so that $M_n = \mathbb{E}[M_\infty \,|\, \mathcal{F}_n] \geq 0 \ \forall n$. Let $F \in \mathcal{F}_T$. Then (trivially)

$$\mathbb{E}[M_\infty; F \cap \{T = \infty\}] = \mathbb{E}[M_T; F \cap \{T = \infty\}]$$

so STS

$$\mathbb{E}[M_\infty; F \cap \{T < +\infty\}] = \mathbb{E}[M_T; F \cap \{T < +\infty\}].$$

In fact, by (MON), STS

$$\mathbb{E}[M_\infty; F \cap \{T \leq k\}] = \mathbb{E}[M_T; F \cap \{T \leq k\}] = \mathbb{E}[M_{T\wedge k}; F \cap \{T \leq k\}],$$

$\forall k$. To conclude we make two observations:

1. $\boxed{F \cap \{T \leq k\} \in \mathcal{F}_{T\wedge k}.}$ Indeed if $n \leq k$

$$F \cap \{T \leq k\} \cap \{T \wedge k = n\} = F \cap \{T = n\} \in \mathcal{F}_n,$$

   and if $n > k$

$$= \emptyset \in \mathcal{F}_n.$$

2. $\boxed{\mathbb{E}[M_\infty \,|\, \mathcal{F}_{T\wedge k}] = M_{T\wedge k}.}$ Since $\mathbb{E}[M_\infty \,|\, \mathcal{F}_k] = M_k$, STS $\mathbb{E}[M_k \,|\, \mathcal{F}_{T\wedge k}] = M_{T\wedge k}$. But note that if $G \in \mathcal{F}_{T\wedge k}$

$$\mathbb{E}[M_k; G] = \sum_{l\leq k} \mathbb{E}[M_k; G\cap\{T\wedge k = l\}] = \sum_{l\leq k} \mathbb{E}[M_l; G\cap\{T\wedge k = l\}] = \mathbb{E}[M_{T\wedge k}; G]$$

   since $G \cap \{T \wedge k = l\} \in \mathcal{F}_l$.

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4 Example: Biased RW

DEF 13.10 *The* asymmetric simple RW *with parameter* $1/2 < p < 1$ *is the process* $\{S_n\}_{n\geq 0}$ *with* $S_0 = 0$ *and* $S_n = \sum_{k\leq n} X_k$ *where the* X_k*s are iid in* $\{-1, +1\}$ *s.t.* $\mathbb{P}[X_1 = 1] = p$*. Let* $q = 1 - p$*. Let* $\phi(x) = (q/p)^x$ *and* $\psi_n(x) = x - (p - q)n$*.*

THM 13.11 *Let $\{S_n\}$ as above. Let $a < 0 < b$. Define $T_x = \inf\{n \geq 0 \,:\, S_n = x\}$. Then*

1. *We have*
$$\mathbb{P}[T_a < T_b] = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}.$$
In particular, $\mathbb{P}[T_a < +\infty] = 1/\phi(a)$ and $\mathbb{P}[T_b < \infty] = 1$.

2. *We have*
$$\mathbb{E}[T_b] = \frac{b}{2p - 1}.$$

Proof: There are two MGs here:

$$\mathbb{E}[\phi(S_n) \,|\, \mathcal{F}_{n-1}] = p(q/p)^{S_{n-1}+1} + q(q/p)^{S_{n-1}-1} = \phi(S_{n-1}),$$

and

$$\mathbb{E}[\psi_n(S_n) \,|\, \mathcal{F}_{n-1}] = p[S_{n-1}+1-(p-q)(n)]+q[S_{n-1}-1-(p-q)(n)] = \psi_{n-1}(S_{n-1}).$$

Let $N = T_a \wedge T_b$. Now note that $\phi(S_{N\wedge n})$ is a bounded MG and therefore applying the MG property at time n and taking limits as $n \to \infty$ (using (DOM))

$$\phi(0) = \mathbb{E}[\phi(S_N)] = \mathbb{P}[T_a < T_b]\phi(a) + \mathbb{P}[T_a > T_b]\phi(b),$$

where we need to prove that $N < +\infty$ a.s. Indeed, since $(b - a)$ $+1$-steps always take us out of (a, b),

$$\mathbb{P}[T_b > n(b - a)] \leq (1 - q^{b-a})^n,$$

so that
$$\mathbb{E}[T_b] = \sum_{k\geq 0} \mathbb{P}[T_b > k] \leq \sum_n (b - a)(1 - q^{b-a})^n < +\infty.$$

In particular $T_b < +\infty$ a.s. and $N < +\infty$ a.s. Rearranging the formula above gives the first result. (For the second part of the first result, take $b \to +\infty$ and use monotonicity.)

For the third one, note that $T_b \wedge n$ is bounded so that

$$0 = \mathbb{E}[S_{T_b\wedge n} - (p - q)(T_b \wedge n)].$$

By (MON), $\mathbb{E}[T_b \wedge n] \uparrow \mathbb{E}[T_b]$. Finally, using

$$\mathbb{P}[-\inf_n S_n \geq -a] = \mathbb{P}[T_a < +\infty],$$

and the fact that $-\inf_n S_n \geq 0$ shows that $\mathbb{E}[-\inf_n S_n] < +\infty$. Hence, we can use (DOM) with $|S_{T_b\wedge n}| \leq \max\{b, -\inf_n S_n\}$.

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References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.