Lecture 13 : Stationary Stochastic Processes

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References: [Var01, Chapter 6], [Dur10, Section 6.1], [Bil95, Chapter 24].

1 Stationary stochastic processes

DEF 13.1 (Stationary stochastic process) *A real-valued process $\{X_n\}_{n\geq 0}$ is stationary if for every k, m*

$$(X_m, \ldots, X_{m+k}) \sim (X_0, \ldots, X_k).$$

EX 13.2 *IID sequences are stationary.*

1.1 Stationary Markov chains

1.1.1 Markov chains

DEF 13.3 (Discrete-time finite-space MC) *Let A be a finite space, μ a distribution on A and $\{p(i,j)\}_{i,j\in A}$ a transition matrix on E. Let $(X_n)_{n\geq 0}$ be a process with distribution*

$$\mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] = \mu(x_0)p(x_0, x_1)\cdots p(x_{n-1}, n_n),$$

for all $n \geq 0$ and $x_0, \ldots, x_n \in A$.

EX 13.4 (RW on a graph) *Let $G = (V, E)$ be a finite, undirected graph. Define*

$$p(i,j) = \frac{\mathbb{1}\{(i,j) \in E\}}{|\{N(i)\}|},$$

where

$$N(i) = \{j \,:\, (i,j) \in E\}.$$

This defines a RW on a graph as the finite MC with the above transition matrix (for each μ, an arbitrary distribution on V). More generally, any finite MC can be seen as a RW on a weighted directed graph.

EX 13.5 (Asymmetric SRW on an interval) *Let $(S_n)_{n\geq 0}$ be an asymmetric SRW with parameter $1/2 < p < 1$. Let $a < 0 < b$, $N = T_a \wedge T_b$. Then $(X_n)_{n\geq 0} = (S_{N\wedge n})_{n\geq 0}$ is a Markov chain.*

1.1.2 Stationarity

DEF 13.6 (Stationary Distribution) *A probability measure π on A is a stationary distribution if*

$$\sum_i \pi(i)p(i,j) = \pi(j),$$

for all $i, j \in A$. In other words, if $X_0 \sim \pi$ then $X_1 \sim \pi$ and in fact $X_n \sim \pi$ for all $n \geq 0$.

EX 13.7 (RW on a graph) *In the RW on a graph example above, define*

$$\pi(i) = \frac{|N(i)|}{2|E|}.$$

Then

$$\sum_{i \in V} \pi(i)p(i,j) = \sum_{i:(i,j) \in E} \frac{|N(i)|}{2|E|}\frac{1}{|N(i)|} = \frac{1}{2|E|}|N(j)| = \pi(j),$$

so that π is a stationary distribution.

EX 13.8 (ASRW on interval) *In the ASRW on $[a, b]$, $\pi = \delta_a$ and $\pi = \delta_b$ as well as all mixtures are stationary.*

EX 13.9 (Stationary Markov chain) *Let X be a MC on A (countable) with transition matrix $\{p_{ij}\}_{i,j \in A}$ and stationary distribution $\pi > 0$. Then X started at π is a stationary stochastic process. Indeed, by definition of π and induction*

$$X_0 \sim X_n,$$

for all $n \geq 0$. Then for all m, k by definition of MCs

$$(X_0, \ldots, X_k) \sim (X_m, \ldots, X_{m+k}).$$

1.2 Abstract setting

EX 13.10 (A canonical example) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A map $T : \Omega \to \Omega$ is said to be measure-preserving (for $\mathbb{P}$) if for all $A \in \mathcal{F}$,*

$$(\mathbb{P}[\omega : T\omega \in A] =)\mathbb{P}[T^{-1}A] = \mathbb{P}[A].$$

If $X \in \mathcal{F}$ then $X_n(\omega) = X(T^n\omega)$, $n \geq 0$, defines a stationary sequence. Indeed, for all $B \in \mathcal{B}(\mathbb{R}^{k+1})$

$$\begin{aligned}\mathbb{P}[(X_0, \ldots, X_k)(\omega) \in B] &= \mathbb{P}[(X_0, \ldots, X_k)(T^m\omega) \in B] \\ &= \mathbb{P}[(X_m, \ldots, X_{m+k})(\omega) \in B].\end{aligned}$$

Kolmogorov's extension theorem indicates that all real-valued stationary stochastic processes can be realized in the framework of the previous example.

THM 13.11 (Kolmogorov Extension Theorem) *Suppose we are given probability measure μ_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ s.t.*

$$\mu_{n+1}((a_0, b_0] \times \cdots \times (a_n, b_n] \times \mathbb{R}) = \mu_n((a_0, b_0] \times \cdots \times (a_n, b_n]),$$

for all n and $(n+1)$-dimensional rectangles. Then there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{Z}_+}, \mathcal{R}^{\mathbb{Z}_+})$ with marginals μ_n.

EX 13.12 (Revisiting stationary processes) *Let $\tilde{X}$ be a stationary process on $\mathbb{R}$. Then by the previous theorem, we can realize $\tilde{X}$ on $\mathbb{R}^{\mathbb{Z}_+}$ as*

$$X_n(\omega) = \omega_n.$$

The corresponding measure-preserving transformation is the shift

$$T\omega = (\omega_1, \ldots).$$

In particular, $X_n(\omega) = X_0(T^n\omega)$.

EX 13.13 *Returning the previous example:*

1. *The only invariant sets are $\emptyset, \Omega$ so that $\mathcal{I}$ is trivial and T is ergodic.*
2. *Both Ω_1 and Ω_2 are invariant so that if $\alpha, \beta \neq 0$ we have that T is not ergodic. Further, note that $\hat{f}$ is measurable with respect to $\mathcal{I} = \{\emptyset, \Omega_1, \Omega_2, \Omega\}$, that is, $\hat{f}$ is invariant.*

Next time, we will prove the ergodic theorem:

THM 13.14 *Let $f \in L^1$. Then there is $\hat{f} \in \mathcal{I}$ s.t.*

$$n^{-1} S_n \to \hat{f},$$

a.s and in L^1. In the ergodic case, $\hat{f} = \mathbb{E}[f]$.

EX 13.15 (IID RVs) *Let $X_n(\omega) = \omega_n$ are iid rvs. If A is invariant then $\{\omega : \omega \in A\} = \{\omega : T\omega \in A\} \in \sigma(X_1, \ldots)$ and by induction*

$$A \in \cap_{n \geq 0} \sigma(X_n, \ldots) = \mathcal{T},$$

where $\mathcal{T}$ is the tail σ-field. Thus $\mathcal{I} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is trivial by Kolmogorov's $0-1$ law, so is $\mathcal{I}$. Therefore T is ergodic and $\mathbb{E}[f \,|\, \mathcal{I}] = \mathbb{E}[f]$. Applying the ergodic thm to $f = X_0 \in L^1$ we get

$$n^{-1} \sum_{m=0}^{n-1} X_m(\omega) \to \mathbb{E}[X_0],$$

that is, we recover the SLLN.

References

[Bil95] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1995.

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Var01] S. R. S. Varadhan. *Probability theory*, volume 7 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2001.