Lecture 15 : Proof of the Ergodic Theorem (cont'd)

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References: [Var01, Chapter 6], [Dur10, Section 6.2], [SS05, Section 6.5].

1 Proof of Ergodic Theorem

We assume we have $(\Omega, \mathcal{F}, \mathbb{P})$, $f \in \mathcal{F}$, T a measure-preserving transformation, and we let $X_n(\omega) = f(T^n\omega)$ for all $n \geq 0$. It will be convenient to think of T as an operator of functions

$$Uf(\omega) = f(T\omega),$$

in which case $U^m f(\omega) = f(T^m\omega)$ and we define

$$A_n f = n^{-1}(I + \cdots + U^{n-1})f.$$

Recall:

LEM 15.1 *If $g, g' \in L^2$ then*

$$\langle Ug', Ug\rangle = \langle g', g\rangle.$$

THM 15.2 *Let $f \in L^1$. Then there is $\hat{f} \in \mathcal{I}$ s.t.*

$$A_n f \to \hat{f} \equiv \mathbb{E}[f \,|\, \mathcal{I}], \text{ a.s and in } L^1.$$

Proof: We first show a.s. convergence to a limit. We proceed as in the L^2 case. Fix ε and let

$$f = F + H = f_0 + (I - U)g_1' + H,$$

where $\|H\|_1 < \varepsilon$ includes both the L^1 and closure error terms. We show that $A_n F$ converges a.s. Note that

$$A_n F(\omega) = f_0(\omega) + n^{-1}(I - U^n)g_1'(\omega) = f_0(\omega) + \frac{g_1'(\omega)}{n} - \frac{g_1'(T^n\omega)}{n}.$$

To deal with the last term, note that

$$\sum_n \frac{g_1'(T^n\omega)^2}{n^2}$$

converges because its norm is bounded by $\|g_1'\|_2^2 \sum_n 1/n^2 < \infty$. To conclude let

$$E_\alpha = \{\lim_N \sup_{m,n\geq N} |A_n f - A_m f| > \alpha\}.$$

Note that

$$\mathbb{P}[E_\alpha] \leq \mathbb{P}[\lim_N \sup_{m,n\geq N} |A_n H - A_m H| > \alpha] \leq \mathbb{P}[2\sup_N |A_N H| > \alpha].$$

To conclude the proof of a.s. convergence, we need the following inequality which is similar to Doob's inequality.

LEM 15.3 (Wiener's Maximal Inequality) *For $f \in L^1$ and $\ell > 0$,*

$$\mathbb{P}\left[\sup_{j\geq 0} |A_j f| \geq \ell\right] \leq \frac{1}{\ell}\mathbb{E}|f|.$$

Proof: The proof is based on the so-called maximal ergodic lemma.

LEM 15.4 (Maximal Ergodic Lemma) *Let*

$$f_n^* = \sup_{1\leq j\leq n} f + \cdots + U^{j-1} f.$$

Then for all $n \geq 0$

$$\mathbb{E}[f; \{f_n^* \geq 0\}] \geq 0.$$

Apply the maximal ergodic lemma to $|f| - \ell$ and take $n \to \infty$. ■

Applying the lemma we have

$$\mathbb{P}[E_\alpha] \leq \mathbb{P}[2\sup_N |A_N H| > \alpha] \leq \frac{2}{\alpha}\mathbb{E}|H| < \frac{2\varepsilon}{\alpha},$$

so that $\mathbb{P}[E_\alpha] = 0$ for all α.

It is clear that the limit satisfies $\hat{f}(\omega) = \hat{f}(T\omega)$. In fact, by the density of L^2 in L^1, writing $f = g_r + h_r$ with $g_r \in L^2$ and $\|h_r\|_1 < 1/r$, we have $\hat{f} = \hat{g}_r + \hat{h}_r$ and for $G \in \mathcal{I}$

$$\mathbb{E}[\hat{f}; G] = \mathbb{E}[\hat{g}_r; G] + \mathbb{E}[\hat{h}_r; G] = \mathbb{E}[g_r; G] + \mathbb{E}[\hat{h}_r; G] \to \mathbb{E}[f; G],$$

where we used the L^2 Ergodic Theorem and

$$\mathbb{E}|\hat{h}_r| \leq \liminf_n \mathbb{E}|A_n h_r| \leq \liminf_n n^{-1} \sum_{m=0}^{n-1} \mathbb{E}|U^m h_r| = \mathbb{E}|h_r| = 1/r,$$

by (FATOU).

A truncation argument gives the L^1 convergence (see [Dur10]). Let

$$f'_M = f\mathbb{1}_{|f|\leq M},$$

and $f''_M = f - f'_M$. By the ergodic theorem and the bounded convergence theorem

$$\mathbb{E}\left|\frac{1}{n}\sum_{m=0}^{n-1} f'_M(T^m\omega) - \mathbb{E}[f'_M \,|\, \mathcal{I}]\right| \to 0.$$

By stationarity and (cJENSEN),

$$\mathbb{E}\left|\frac{1}{n}\sum_{m=0}^{n-1} f''_M(T^m\omega) - \mathbb{E}[f''_M \,|\, \mathcal{I}]\right| \leq 2\mathbb{E}|f''_M| \to 0,$$

as $M \to +\infty$ by (DOM). The result follows.

■

2 Applications

Going back to Markov chains:

DEF 15.5 *Let*

$$T_i = \inf\{n \geq 1 \,:\, X_n = i\},$$

and

$$f_{ij} = \mathbb{P}_i[T_j < +\infty].$$

A chain is irreducible *if $f_{ij} > 0$ for all $i, j \in A$. A state i is recurrent if $f_{ii} = 1$ and is positive recurrent if $\mathbb{E}_i[T_i] < +\infty$.*

LEM 15.6 *If X is irreducible and finite, then every state is positive recurrent.*

THM 15.7 *Let X be an irreducible and positive recurrent MC. Then there exists a unique stationary distribution π. In fact,*

$$\pi(i) = \frac{1}{\mathbb{E}_i[T_i]} > 0.$$

EX 15.8 (MCs) *Let X be a MC on S.*

- *In the ASRW on $[a, b]$ the invariant sets are $\{a\}$ and $\{b\}$ and therefore T is not ergodic if π has positive mass on both of them.*

- *On the other hand, assume X is irreducible and positive recurrent with stationary distribution $\pi > 0$. Let $A \in \mathcal{I}$ and note that $\mathbb{1}_A \circ T^n = \mathbb{1}_A$. Then by the Markov property,*

$$\mathbb{E}[\mathbb{1}_A \,|\, \mathcal{F}_n] = \mathbb{E}[\mathbb{1}_A \circ T^n \,|\, \mathcal{F}_n] = h(X_n),$$

where $h(x) = \mathbb{E}_x[\mathbb{1}_A]$. By Levy's $0-1$ law the LHS $\to \mathbb{1}_A$. By irreducibility and recurrence, any $y \in S$ is visited i.o. and we must have $\mathbb{E}_x[\mathbb{1}_A] \equiv h(x) \equiv 0$ or 1. Therefore $\mathbb{P}[A] \in \{0,1\}$ and $\mathcal{I}$ is trivial. Then applying the Ergodic Theorem to $f(\omega) = g(X_0(\omega))$ where

$$\sum_y |g(y)|\pi(y) < +\infty,$$

we then have

$$n^{-1}\sum_{m=0}^{n-1} g(X_m(\omega)) \to \sum_y \pi(y)g(y).$$

- *Note finally that the RW on a bipartite graph shows that, even in the irreducible recurrent case, $\mathcal{I}$ may be smaller than $\mathcal{T}$.*

Further reading

See a different proof in [Dur10, Section 6.2].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[SS05] Elias M. Stein and Rami Shakarchi. *Real analysis*. Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces.

[Var01] S. R. S. Varadhan. *Probability theory*, volume 7 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2001.