Lecture 16 : Subadditive Ergodic Theorem

MATH275B - Winter 2012 | *Lecturer: Sebastien Roch*

References: [Dur10, Section 6.4].

1 Subadditivity

DEF 16.1 *A sequence $\{\gamma_n\}_{n\geq 0}$ is* subadditive *if for all m, n:*

$$\gamma_{m+n} \leq \gamma_n + \gamma_m.$$

THM 16.2 (Limit of Subadditive Sequences) *If γ is subadditive then*

$$\frac{\gamma_n}{n} \to \inf_m \frac{\gamma_m}{m}.$$

Proof: Clearly

$$\liminf_n \frac{\gamma_n}{n} \geq \inf_m \frac{\gamma_m}{m}.$$

So STS

$$\limsup_n \frac{\gamma_n}{n} \leq \inf_m \frac{\gamma_m}{m}.$$

Fix m and write $n = km + \ell$ with $0 \leq \ell < m$. Applying the subadditivity repeatedly, we have

$$\gamma_n \leq k\gamma_m + \gamma_\ell,$$

so that

$$\frac{\gamma_n}{n} \leq \left(\frac{km}{km+\ell}\right)\frac{\gamma_m}{m} + \frac{\gamma_\ell}{n},$$

and the result follows by taking $n \to +\infty$.

■

EX 16.3 (Longest common subsequence) *Let $\{X_n\}$ and $\{Y_n\}$ be stationary sequences and let $L_{m,n}$ be the longest common subsequence on indices $m < k \leq n$. Clearly*

$$L_{0,m} + L_{m,n} \leq L_{0,n},$$

and $\gamma_n = -\mathbb{E}[L_{0,n}]$ is subadditive.

2 Statement of Subadditive Ergodic Theorem

THM 16.4 (Subadditive Ergodic Theorem) *Suppose* $\{X_{m,n}\}_{0\leq m<n}$ *satisfy:*

1. $X_{0,m} + X_{m,n} \geq X_{0,n}$.
2. $\{X_{nk,(n+1)k}, n \geq 1\}$ *is a stationary sequence for each* k.
3. *The distribution of* $\{X_{m,m+k}, k \geq 1\}$ *does not depend on* m.
4. $\mathbb{E}X_{0,1}^{+} < \infty$ *and for each* n, $\mathbb{E}X_{0,n} \geq \gamma_0 n$ *where* $\gamma_0 > -\infty$.

Then

- $\lim \mathbb{E}X_{0,n}/n = \inf_m \mathbb{E}X_{0,m}/m \equiv \gamma$.
- $X = \lim X_{0,n}/n$ *exists a.s. and in* L^1 *so* $\mathbb{E}X = \gamma$.
- *If all stationary sequences in* 2. *are ergodic then* $X = \gamma$ *a.s.*

Proof: See [Dur10]. ■

3 Examples

EX 16.5 (Age-dependent continuous-time branching process) *Start with one individual. Each individual dies independently after time* $T \sim F$ *and at that point produces* $K \sim \{p_k\}_k$ *offsprings (both with finite means). Let* $X_{0,m}$ *be the time of birth of the first individual from generation* m *and* $X_{m,n}$*, the time to the birth of the first descendant of that individual in generation* n*. We check the conditions:*

1. *Clearly*
$$X_{0,m} + X_{m,n} \geq X_{0,n}.$$
2. $\{X_{nk,(n+1)k}\}_n$ *is IID because we are looking at the descendants of a single individual (the first born) over* k *generations which are not overlapping.*
3. *The distribution of* $\{X_{m,m+k}\}_k$ *is independent of* m *for the same reason.*
4. *By nonnegativity and the finite mean of* F*, condition* 4. *is satisfied.*

So we can apply the thm. By the IID remark above in 2. *we get that the limit is trivial. See [Dur10] for a characterization of the limit.*

EX 16.6 (First-passage percolation) *Consider $\mathbb{Z}^d$ as a graph with edges connecting $x, y \in \mathbb{Z}^d$ if $\|x - y\|_1 = 1$. Assign to each edge a nonnegative random variable $\tau(e)$ corresponding to the time it takes to traverse e (in either direction). Define $t(x, y)$ (the passage time) as the minimum time to go from x to y. Let $X_{m,n} = t(mu, nu)$ where $u = (1, 0, \cdots, 0)$. We check the conditions:*

1. *Clearly*
$$X_{0,m} + X_{m,n} \geq X_{0,n}$$
2. *$\{X_{nk,(n+1)k}\}_n$ is stationary by translational symmetry.*
3. *The distribution of $\{X_{m,m+k}\}_k$ is independent of m for the same reason.*
4. *By nonnegativity and the finite mean of τ, condition 4. is satisfied.*

So we can apply the theorem. Enumerating the edges in some order, one can prove (check!) that the limit is tail-measurable and, by the IID assumption, is trivial. See [Dur10] for a characterization of the limit.

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[SS05] Elias M. Stein and Rami Shakarchi. *Real analysis*. Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces.

[Var01] S. R. S. Varadhan. *Probability theory*, volume 7 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2001.