Lecture 17 : Brownian motion: Definition

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References: [Dur10, Section 3.9, 8.1], [Lig10, Section 1.2-1.4], [MP10, Section 1.1, Appendix 12].

1 Random vectors

We first develop general tools to study multivariate distributions.

DEF 17.1 (Characteristic function) *The CF of a random vector $X = (X_1, \ldots, X_d)$ is given by, for $t \in \mathbb{R}^d$,*

$$\phi_X(t) = \mathbb{E}\left[\exp\left(i(t_1 X_1 + \cdots + t_d X_d)\right)\right].$$

As in the one-dimensional case, we have an inversion formula:

THM 17.2 (Inversion formula) *Let μ be the probability measure corresponding to the random vector $(X_1, \ldots, X_d)$, that is, for all $B \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mu(B) = \mathbb{P}[(X_1, \ldots, X_d) \in B].$$

If $A = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $\mu(\partial A) = 0$ then

$$\mu(A) = \lim_{T \to +\infty} (2\pi)^{-d} \int_{[-T,T]^d} \prod_{j=1}^{d} \psi_j(t_j)\phi(t)\mathrm{d}t,$$

where

$$\psi_j(s) = \frac{\exp(-isa_j) - \exp(-isb_j)}{is}.$$

Proof: Follows from the one-dimensional inversion formula. See [Dur10, Theorem 3.9.3]. ■

An important application of the previous formula is:

THM 17.3 *The RVs $X_1, \ldots, X_d$ are independent if and only if*

$$\phi_X(t) = \prod_{j=1}^{d} \phi_{X_j}(t_j),$$

for all $t \in \mathbb{R}^d$ where $X = (X_1, \ldots, X_d)$.

Proof: The "only if" part is obvious. The inversion formula and Fubini's theorem gives the "if" part. ■

DEF 17.4 *A sequence of random vectors X_n converges weakly to X_∞, denoted $X_n \Rightarrow X_\infty$, if*

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)],$$

for all bounded continuous functions f. The portmanteau theorem gives equivalent characterizations.

In terms of CFs, we have:

THM 17.5 (Convergence theorem) *Let X_n, $1 \leq n \leq \infty$, be random vectors with CFs ϕ_n. A necessary and sufficient condition for $X_n \Rightarrow X_\infty$ is that*

$$\phi_n(t) \to \phi_\infty(t),$$

for all $t \in \mathbb{R}^d$.

Proof: Follows from the one-dimensional result. See [Dur10, Theorem 3.9.4]. ■

We require one last definition:

DEF 17.6 (Covariance) *Let $X = (X_1, \ldots, X_d)$ be a random vector with mean $\mu = \mathbb{E}[X]$. The* covariance of X *is the $d \times d$ matrix Γ with entries*

$$\Gamma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)].$$

2 Multivariate Gaussian distribution

Recall:

DEF 17.7 (Gaussian distribution) *A* standard Gaussian *is a RV Z with CF*

$$\phi_Z(t) = \exp\left(-t^2/2\right),$$

and density

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right).$$

In particular, Z has mean 0 and variance 1. More generally,

$$X = \sigma Z + \mu,$$

is a Gaussian RV with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

We will need a multivariate generalization of the standard Gaussian.

DEF 17.8 (Multivariate Gaussian) *A d-dimensional standard Gaussian is a random vector $X = (X_1, \ldots, X_d)$ where the X_is are independent standard Gaussians. In particular, X has mean 0 and covariance matrix I. More generally, a random vector $X = (X_1, \ldots, X_d)$ is Gaussian if there is a vector b, a $d \times r$ matrix A and an r-dimensional standard Gaussian Y such that*

$$X = AY + b.$$

Then X has mean $\mu = b$ and covariance matrix $\Gamma = AA^T$. The CF of X is given by

$$\phi_X(t) = \exp\left(i \sum_{j=1}^{d} t_j \mu_j - \frac{1}{2} \sum_{j,k=1}^{d} t_j t_k \Gamma_{jk} \right).$$

From the CF and the theorems above, we get the following:

COR 17.9 (Independence) *Let $X = (X_1, \ldots, X_d)$ be a multivariate Gaussian. Then the X_is are independent if and only if $\Gamma_{ij} = 0$ for all $i \neq j$, that is, if they are* uncorrelated.

COR 17.10 (Convergence) *Let X_n be a sequence of random vectors with means μ_n and covariances Γ_n such that $X_n \to X_\infty$ a.s., $\mu_u \to \mu_\infty$, and $\Gamma_n \to \Gamma_\infty$. Then X_∞ is a multivariate Gaussian with mean μ_∞ and covariance matrix Γ_∞.*

COR 17.11 (Linear combinations) *The random vector $(X_1, \ldots, X_d)$ is multivariate Gaussian if and only if all linear combinations of its components are Gaussian.*

Finally:

THM 17.12 (Multivariate CLT) *Let $X_1, X_2, \ldots$ be IID random vectors with means μ and finite covariance matrix Γ. Let $S_n = \sum_{j=1}^n X_j$, Then*

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow Z,$$

where Z is a multivariate Gaussian with mean 0 and covariance matrix Γ.

Proof: Follows easily from one-dimensional result. See [Dur10, Theorem 3.9.6]. ■

3 Gaussian processes

DEF 17.13 (Gaussian process) *A continuous-time stochastic process* $\{X(t)\}_{t\geq 0}$ *is a* Gaussian process *if for all* $n \geq 1$ *and* $0 \leq t_1 < \cdots < t_n < +\infty$ *the random vector*

$$(X(t_1), \ldots, X(t_n)),$$

is multivariate Gaussian. The mean and covariance functions of X *are* $\mathbb{E}[X(t)]$ *and* $\mathrm{Cov}[X(s), X(t)]$ *respectively.*

4 Definition of Brownian motion

DEF 17.14 (Brownian motion: Definition I) *The continuous-time stochastic process* $X = \{X(t)\}_{t\geq 0}$ *is a* standard Brownian motion *if* X *is a Gaussian process with almost surely continuous paths, that is,*

$$\mathbb{P}[X(t) \textit{ is continuous in } t] = 1,$$

such that $X(0) = 0$,

$$\mathbb{E}[X(t)] = 0,$$

and

$$\mathrm{Cov}[X(s), X(t)] = s \wedge t.$$

More generally, $B = \sigma X + x$ *is a* Brownian motion started at x.

Further reading

Multivariate CLT in [Dur10, Section 2.9].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.