Lecture 19 : Brownian motion: Construction

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.1], [Lig10, Section 1.5], [MP10, Section 1.1].

1 Definition of Brownian motion

Recall:

DEF 19.1 (Brownian motion: Definition I) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X is a Gaussian process with almost surely continuous paths, that is,*

$$\mathbb{P}[X(t) \text{ is continuous in } t] = 1,$$

such that $X(0) = 0$,

$$\mathbb{E}[X(t)] = 0,$$

and

$$\mathrm{Cov}[X(s), X(t)] = s \wedge t.$$

More generally, $B = \sigma X + x$ is a Brownian motion started at x.

From the properties of the multivariate Gaussian, we get the following equivalent definition. We begin with a general definition.

DEF 19.2 (Stationary independent increments) *An SP $\{X(t)\}_{t\geq 0}$ has* stationary increments *if the distribution of $X(t) - X(s)$ depends only on $t - s$ for all $0 \leq s \leq t$. It has* independent increments *if the RVs $\{X(t_{j+1} - X(t_j)), 1 \leq j < n\}$ are independent whenever $0 \leq t_1 < t_2 < \cdots < t_n$ and $n \geq 1$.*

DEF 19.3 (Brownian motion: Definition II) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X has almost surely continuous paths and stationary independent increments such that $X(s+t) - X(s)$ is Gaussian with mean 0 and variance t.*

2 Construction of Brownian motion

Given that standard Brownian motion is defined in terms of finite-dimensional distributions, it is tempting to attempt to construct it by using Kolmogorov's Extension Theorem.

THM 19.4 (Kolmogorov's Extension Theorem: Uncountable Case) *Let*

$$\Omega_0 = \{\omega \,:\, [0,\infty) \to \mathbb{R}\},$$

and $\mathcal{F}_0$ be the σ-field generated by the finite-dimensional sets

$$\{\omega \,:\, \omega(t_i) \in A_i, 1 \leq i \leq n\},$$

for $A_i \in \mathcal{B}$. There is a unique probability measure ν on $(\Omega_0, \mathcal{F}_0)$ so that

$$\nu(\{\omega \,:\, \omega(0) = 0\}) = 1$$

and whenever $0 \leq t_1 < \cdots < t_n$ with $n \geq 1$ we have

$$\nu(\{\omega \,:\, \omega(t_i) \in A_i\}) = \mu_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n),$$

where the latter is the finite-dimensional distribution of standard Brownian motion.

See [Dur10]. The only problem with this approach is that the event

$$C = \{\omega \,:\, \omega(t) \text{ is continuous in } t\},$$

is not in $\mathcal{F}_0$. See Exercise 8.1.1 in [Dur10].

Instead, we proceed as follows. There are several constructions of Brownian motion. We present Lévy's contruction, as described in [MP10]. See [Dur10] and [Lig10] for further constructions.

THM 19.5 (Existence) *Standard Brownian motion $B = \{B(t)\}_{t\geq 0}$ exists.*

Proof: We first construct B on $[0,1]$. The idea is to construct the process on dyadic points and extend it linearly. Let

$$\mathcal{D}_n = \{k2^{-n} \,:\, 0 \leq k \leq 2^n\},$$

and

$$\mathcal{D} = \cup_{n=0}^{\infty} \mathcal{D}_n.$$

Note that $\mathcal{D}$ is countable and consider $\{Z_t\}_{t\in\mathcal{D}}$ a collection of independent standard Gaussians. We define $B(d)$ for $d \in \mathcal{D}_n$ by induction. First take $B(0) = 0$

and $B(1) = Z_1$. Note that $B(1) - B(0)$ is Gaussian with variance 1. Then for $d \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}$ we let

$$B(d) = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

By construction, $B(d)$ is independent of $\{Z_t \,:\, t \in \mathcal{D} \backslash \mathcal{D}_n\}$. Moreover, as a linear combination of zero-mean Gaussians, $B(d)$ is a zero-mean Gaussian.

We claim that the differences $B(d) - B(d - 2^{-n})$, for all $d \in \mathcal{D}_n \backslash \{0\}$, are independent Gaussians with variance 2^{-n} .

- We first argue about neighboring increments. Note that, for $d \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}$,

  $$B(d) - B(d - 2^{-n}) = \frac{B(d + 2^{-n}) - B(d - 2^{-n})}{2} + \frac{Z_d}{2 \cdot 2^{(n-1)/2}},$$

  and

  $$B(d + 2^{-n}) - B(d) = \frac{B(d + 2^{-n}) - B(d - 2^{-n})}{2} - \frac{Z_d}{2 \cdot 2^{(n-1)/2}},$$

  are Gaussians and they are independent by the following lemma. By induction the differences above are Gaussians with variance $2^{-(n-1)}$ and independent of Z_d.

  LEM 19.6 *If (X_1, X_2) is a standard Gaussian then so is $\frac{1}{\sqrt{2}}(X_1 + X_2, X_1 - X_2)$.*

- More generally, the two intervals are separated by $d \in \mathcal{D}_j$. Take a minimal such j. Then, by induction, the increments over the intervals $[d - 2^{-j}, d]$ and $[d, d + 2^{-j}]$ are independent. Moreover, the increments over the two intervals of length 2^{-n} of interest (included in the above intervals) are constructed from $B(d) - B(d - 2^{-j})$, respectively $B(d + 2^{-j}) - B(d)$, using a disjoint set of variables $\{Z_t \,:\, t \in \mathcal{D}_n\}$. That proves the claim by induction.

We now interpolate linearly between dyadic points. More precisely, let

$$F_0(t) = \begin{cases} Z_1, & t = 1, \\ 0, & t = 0, \\ \text{linearly}, & \text{in between.} \end{cases}$$

and for $n \geq 1$

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t, & t \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}, \\ 0, & t \in \mathcal{D}_{n-1}, \\ \text{linearly}, & \text{in between.} \end{cases}$$

We then have for $d \in \mathcal{D}_n$

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

We want to show that the resulting process is continuous on $[0, 1]$. We claim that the series

$$B(t) = \sum_{n=0}^{\infty} F_n(t),$$

is uniformly convergent. From a bound on Gaussian tails we saw last quarter,

$$\mathbb{P}[|Z_d| \geq c\sqrt{n}] \leq \exp\left(-c^2 n/2\right),$$

so that for c large enough

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{P}[\exists d \in \mathcal{D}_n, |Z_d| \geq c\sqrt{n}] &\leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(-c^2 n/2\right) \\ &< +\infty.\end{aligned}$$

By BC, there is N (random) such that $|Z_d| < c\sqrt{n}$ for all $d \in \mathcal{D}_n$ with $n > N$. In particular, for $n > N$ we have

$$\|F_n\|_\infty < c\sqrt{n} 2^{-(n+1)/2},$$

from which we get the claim.

To show that $B(t)$ has the correct finite-dimensional distributions, note that this is the case for $\mathcal{D}$ by the above argument. Since $\mathcal{D}$ is dense in $[0, 1]$ the result holds on $[0, 1]$ by taking limits and using the convergence theorem for Gaussians from the previous lecture.

Finally, we extend the process to $[0, +\infty)$ by gluing together independent copies of $B(t)$. ■

Further reading

Other constructions in [Dur10, Section8.1] and [Lig10, Section 1.5].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.