Lecture 19 : Brownian motion: Path properties I

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.1], [Lig10, Section 1.5, 1.6], [MP10, Section 1.1, 1.2].

1 Invariance

We begin with some useful invariance properties. The following are immediate.

THM 19.1 (Time translation) *Let $s \geq 0$. If $B(t)$ is a standard Brownian motion, then so is $X(t) = B(t+s) - B(s)$.*

THM 19.2 (Scaling invariance) *Let $a > 0$. If $B(t)$ is a standard Brownian motion, then so is $X(t) = a^{-1}B(a^2 t)$.*

Proof: *Sketch.* We compute the variance of the increments:

$$\begin{aligned}\mathrm{Var}[X(t) - X(s)] &= \mathrm{Var}[a^{-1}(B(a^2t) - B(a^2s))] \\ &= a^{-2}(a^2t - a^2s) \\ &= t - s.\end{aligned}$$

■

THM 19.3 (Time inversion) *If $B(t)$ is a standard Brownian motion, then so is*

$$X(t) = \begin{cases} 0, & t = 0, \\ tB(t^{-1}), & t > 0. \end{cases}$$

Proof: *Sketch.* We compute the covariance function for $s < t$:

$$\begin{aligned}\mathrm{Cov}[X(s), X(t)] &= \mathrm{Cov}[sB(s^{-1}), tB(t^{-1})] \\ &= st\left(s^{-1} \wedge t^{-1}\right) \\ &= s.\end{aligned}$$

It remains to check continuity at 0. Note that

$$\left\{\lim_{t\downarrow 0} B(t) = 0\right\} = \bigcap_{m\geq 1}\bigcup_{n\geq 1}\left\{|B(t)| \leq 1/m,\ \forall t \in \mathbb{Q}\cap(0,1/n)\right\},$$

and

$$\left\{\lim_{t\downarrow 0} X(t) = 0\right\} = \bigcap_{m\geq 1}\bigcup_{n\geq 1}\left\{|X(t)| \leq 1/m,\ \forall t \in \mathbb{Q}\cap(0,1/n)\right\}.$$

The RHSs have the same probability because the distributions on all finite-dimensional sets —and therefore on the rationals—are the same. The LHS of the first one has probability 1. ■

Typical applications of these are:

COR 19.4 *For $a < 0 < b$, let*

$$T(a,b) = \inf\left\{t \geq 0\ :\ B(t) \in \{a,b\}\right\}.$$

Then

$$\mathbb{E}[T(a,b)] = a^2\mathbb{E}[T(1,b/a)].$$

In particular, $\mathbb{E}[T(-b,b)]$ is a constant multiple of b^2.

Proof: Let $X(t) = a^{-1}B(a^2 t)$. Then,

$$\begin{aligned}\mathbb{E}[T(a,b)] &= a^2\mathbb{E}[\inf\{t \geq 0,:\ X(t) \in \{1, b/a\}\}] \\ &= a^2\mathbb{E}[T(1,b/a)].\end{aligned}$$

■

COR 19.5 *Almost surely,*

$$t^{-1}B(t) \to 0.$$

Proof: Let $X(t)$ be the time inversion of $B(t)$. Then

$$\lim_{t\to\infty}\frac{B(t)}{t} = \lim_{t\to\infty} X(1/t) = X(0) = 0.$$

■

2 Modulus of continuity

By construction, $B(t)$ is continuous a.s. In fact, we can prove more.

DEF 19.6 (Hölder continuity) *A function f is said* locally α-Hölder continuous at x *if there exists $\varepsilon > 0$ and $c > 0$ such that*

$$|f(x) - f(y)| \leq c|x - y|^\alpha,$$

for all y with $|y - x| < \varepsilon$. We refer to α as the Hölder exponent and to c as the Hölder constant.

THM 19.7 (Holder continuity) *If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally α-Hölder continuous.*

Proof:

LEM 19.8 *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \leq t \leq 1 - h$,*

$$|B(t + h) - B(t)| \leq C\sqrt{h \log(1/h)}.$$

Proof: Recall our construction of Brownian motion on $[0, 1]$. Let

$$\mathcal{D}_n = \{k2^{-n} \,:\, 0 \leq k \leq 2^n\},$$

and

$$\mathcal{D} = \cup_{n=0}^\infty \mathcal{D}_n.$$

Note that $\mathcal{D}$ is countable and consider $\{Z_t\}_{t\in\mathcal{D}}$ a collection of independent standard Gaussians. Let

$$F_0(t) = \begin{cases} Z_1, & t = 1, \\ 0, & t = 0, \\ \text{linearly}, & \text{in between.} \end{cases}$$

and for $n \geq 1$

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t, & t \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}, \\ 0, & t \in \mathcal{D}_{n-1}, \\ \text{linearly}, & \text{in between.} \end{cases}$$

Finally

$$B(t) = \sum_{n=0}^\infty F_n(t).$$

Each F_n is piecewise linear and its derivative exists almost everywhere. By construction, we have

$$\|F_n'\|_\infty \leq \frac{\|F_n\|_\infty}{2^{-n}}.$$

Recall that there is N (random) such that $|Z_d| < c\sqrt{n}$ for all $d \in \mathcal{D}_n$ with $n > N$. In particular, for $n > N$ we have

$$\|F_n\|_\infty < c\sqrt{n}2^{-(n+1)/2}.$$

Using the mean-value theorem, assuming $l > N$,

$$\begin{aligned}
|B(t+h) - B(t)| &\leq \sum_{n=0}^{\infty} |F_n(t+h) - F_n(t)| \\
&\leq \sum_{n=0}^{l} h\|F_n'\|_\infty + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty, \\
&\leq h\sum_{n=0}^{N} \|F_n'\|_\infty + ch\sum_{n=N}^{l} \sqrt{n}2^{n/2} + 2c\sum_{n=l+1}^{\infty} \sqrt{n}2^{-n/2}.
\end{aligned}$$

Take h small enough that the first term is smaller than $\sqrt{h\log(1/h)}$ and l defined by $2^{-l} < h \leq 2^{-l+1}$ exceeds N. Then approximating the second and third terms by their largest element gives the result. ■

We go back to the proof of the theorem. For each k, we can find an $h(k)$ small enough so that the result applies to the standard BMs

$$\{B(k+t) - B(k)\,:\, t \in [0,1]\},$$

and

$$\{B(k+1-t) - B(k+1)\,:\, t \in [0,1]\}.$$

Since there are countably many intervals $[k, k+1)$, such $h(k)$'s exist almost surely on all intervals simultaneously. Then note that for any $\alpha < 1/2$, if $t \in [k, k+1)$ and $h < h(k)$ small enough,

$$|B(t+h) - B(t)| \leq C\sqrt{h\log(1/h)} \leq Ch^\alpha (= Ch^{1/2}(1/h)^{(1/2-\alpha)}).$$

This concludes the proof. ■

In fact:

THM 19.9 (Lévy's modulus of continuity) *Almost surely,*

$$\limsup_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}} = 1.$$

For the proof, see [MP10].

This result is tight. See [MP10, Remark 1.21].

3 Non-Monotonicity

THM 19.10 *Almost surely, for all $0 < a < b < +\infty$, standard BM is not monotone on the interval $[a, b]$.*

Proof: It suffices to look at intervals with rational endpoints because any general non-degenerate interval of monotonicity must contain one of those. Since there are countably many rational intervals, it suffices to prove that any particular one has probability 0 of being monotone. Let $[a, b]$ be such an interval. Note that for any finite sub-division

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

the probability that each increment satisfies

$$B(a_i) - B(a_{i-1}) \geq 0, \qquad \forall i = 1, \ldots, n,$$

or the same with negative, is at most

$$2\left(\frac{1}{2}\right)^n \to 0,$$

as $n \to \infty$ by symmetry of Gaussians. ∎

More generally, we can prove the following. For a proof see [Lig10].

THM 19.11 *Almost surely, BM satisfies:*

1. *The set of times at which local maxima occur is dense.*
2. *Every local maximum is strict.*
3. *The set of local maxima is countable.*

Proof: Part (3). We use part (2). If t is a strict local maximum, it must be in the set

$$\bigcup_{n=1}^{+\infty} \left\{t \,:\, B(t,\omega) > B(s,\omega),\ \forall s,\ |s - t| < n^{-1}\right\}.$$

But for each n, the set must be countable because two such t's must be separated by n^{-1}. So the union is countable. ∎

Further reading

Other constructions in [Dur10, Section8.1] and [Lig10, Section 1.5]. Proof of modulus of continuity [MP10, Theorem 1.14].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.