Lecture 20 : Path properties II

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.1], [Lig10, Section 1.6], [MP10, Section 1.3].

1 Previous class

THM 20.1 *If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally α-Hölder continuous.*

Recall:

THM 20.2 (Scaling invariance) *Let $a > 0$. If $B(t)$ is a standard Brownian motion, then so is $X(t) = a^{-1}B(a^2t)$.*

THM 20.3 (Time inversion) *If $B(t)$ is a standard Brownian motion, then so is*

$$X(t) = \begin{cases} 0, & t = 0, \\ tB(t^{-1}), & t > 0. \end{cases}$$

LEM 20.4 (LLN) *Almost surely, $t^{-1}B(t) \to 0$ as $t \to +\infty$.*

2 Non-differentiability

So $B(t)$ grows slower than t. But the following lemma shows that its limsup grows faster than $\sqrt{t}$.

LEM 20.5 *Almost surely*

$$\limsup_{n\to+\infty} \frac{B(n)}{\sqrt{n}} = +\infty.$$

Proof: By (FATOU),

$$\mathbb{P}[B(n) > c\sqrt{n} \text{ i.o.}] \geq \limsup_{n\to+\infty} \mathbb{P}[B(n) > c\sqrt{n}] = \limsup_{n\to+\infty} \mathbb{P}[B(1) > c] > 0,$$

by the scaling property. Thinking of $B(n)$ as the sum of $X_n = B(n) - B(n-1)$, the event on the LHS is exchangeable and the Hewitt-Savage 0-1 law implies that it has probability 1. ■

DEF 20.6 (Upper and lower derivatives) *For a function f, we define the* upper *and* lower right derivatives *as*

$$D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h},$$

and

$$D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

We begin with an easy first result.

THM 20.7 *Fix $t \geq 0$. Then almost surely Brownian motion is not differentiable at t. Moreover, $D^* B(t) = +\infty$ and $D_* B(t) = -\infty$.*

Proof: Consider the time inversion X. Then

$$D^* X(0) \geq \limsup_{n \to +\infty} \frac{X(n^{-1}) - X(0)}{n^{-1}} = \limsup_{n \to +\infty} B(n) = +\infty,$$

by the lemma above. This proves the result at 0. Then note that $X(s) = B(t+s) - B(s)$ is a standard Brownian motion and differentiability of X at 0 is equivalent to differentiability of B at t. ■

In fact, we can prove something much stronger.

THM 20.8 *Almost surely, BM is nowhere differentiable. Furthermore, almost surely, for all t*

$$D^* B(t) = +\infty,$$

or

$$D_* B(t) = -\infty,$$

or both.

Proof: Suppose there is t_0 such that the latter does not hold. By boundedness of BM over $[0, 1]$, we have

$$\sup_{h \in [0,1]} \frac{|B(t_0 + h) - B(t_0)|}{h} \leq M,$$

for some $M < +\infty$. Assume t_0 is in $[(k-1)2^{-n}, k2^{-n}]$ for some k, n. Then for all $1 \leq j \leq 2^n - k$, in particular, for $j = 1, 2, 3$,

$$\begin{aligned} &|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \\ &\qquad \leq |B((k+j)2^{-n}) - B(t_0)| + |B(t_0) - B((k+j-1)2^{-n})| \\ &\qquad \leq M(2j+1)2^{-n}, \end{aligned}$$

by our assumption. Define the events

$$\Omega_{n,k} = \{|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n},\ j = 1,2,3\}.$$

It suffices to show that $\cup_{k=1}^{2^n-3}\Omega_{n,k}$ cannot happen for infinitely many n. Indeed,

$$\mathbb{P}\left[\exists t_0 \in [0,1],\ \sup_{h\in[0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \leq M\right] \leq \mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \text{ for infinitely many } n\right]$$

But by the independence of increments

$$\begin{aligned}
\mathbb{P}[\Omega_{n,k}] &= \prod_{j=1}^{3} \mathbb{P}[|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n}] \\
&\leq \mathbb{P}\left[|B(2^{-n})| \leq \frac{7M}{2^n}\right]^3 \\
&= \mathbb{P}\left[\left|\frac{1}{\sqrt{2^{-n}}} B\left(\left[\sqrt{2^{-n}}\right]^2\right)\right| \leq \frac{7M}{\sqrt{2^{-n}}\cdot 2^n}\right]^3 \\
&= \mathbb{P}\left[|B(1)| \leq \frac{7M}{\sqrt{2^n}}\right]^3 \\
&\leq \left(\frac{7M}{\sqrt{2^n}}\right)^3,
\end{aligned}$$

because the density of a standard Gaussian is bounded by $1/2$. Hence

$$\mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right] \leq 2^n \left(\frac{7M}{\sqrt{2^n}}\right)^3 = (7M)^3 2^{-n/2},$$

which is summable. The result follows from BC. ■

3 Quadratic variation

Recall:

DEF 20.9 (Bounded variation) *A function $f\ :\ [0,t] \to \mathbb{R}$ is of* bounded variation *if there is $M < +\infty$ such that*

$$\sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| \leq M,$$

for all $k \geq 1$ *and all partitions* $0 = t_0 < t_1 < \cdots < t_k = t$*. Otherwise, we say that it is of* unbounded variation.

THM 20.10 (Quadratic variation) *Suppose the sequence of partitions*

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = t,$$

is nested*, that is, at each step one or more partition points are added, and the* mesh

$$\Delta(n) = \sup_{1 \leq j \leq k(n)} \{t_j^{(n)} - t_{j-1}^{(n)}\},$$

converges to 0*. Then, almost surely,*

$$\lim_{n \to +\infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t.$$

Proof: By considering subsequences, it suffices to consider the case where one point is added at each step. Let

$$X_{-n} = \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.$$

Let

$$\mathcal{G}_{-n} = \sigma(X_{-n}, X_{-n-1}, \ldots)$$

and

$$\mathcal{G}_{-\infty} = \bigcap_{k=1}^{\infty} \mathcal{G}_{-k}.$$

CLAIM 20.11 *We claim that* $\{X_{-n}\}$ *is a reversed MG.*

Proof: We want to show that

$$\mathbb{E}[X_{-n+1} \,|\, \mathcal{G}_{-n}] = X_{-n}.$$

In particular, this will imply by induction

$$X_{-n} = \mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-n}].$$

Assume that, at step n, the new point s is added between the old points $t_1 < t_2$. Write

$$X_{-n+1} = (B(t_2) - B(t_1))^2 + W,$$

and

$$X_{-n} = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2 + W,$$

where W is independent of the other terms. We claim that

$$\mathbb{E}[(B(t_2) - B(t_1))^2 \,|\, (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2] \\ = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2,$$

which follows from the following lemma.

LEM 20.12 *Let $X, Z \in \mathcal{L}^2$ be independent and assume Z is symmetric. Then*

$$\mathbb{E}[(X + Z)^2 \,|\, X^2 + Z^2] = X^2 + Z^2.$$

Proof: By symmetry of Z,

$$\begin{aligned} \mathbb{E}[(X + Z)^2 \,|\, X^2 + Z^2] &= \mathbb{E}[(X - Z)^2 \,|\, X^2 + (-Z)^2] \\ &= \mathbb{E}[(X - Z)^2 \,|\, X^2 + Z^2]. \end{aligned}$$

Taking the difference we get

$$\mathbb{E}[XZ \,|\, X^2 + Z^2] = 0.$$

■

The fact that X_{-n} is a reversed MG follows from the argument above. (Exercise.) ■

We return to the proof of the theorem. By Lévy's Downward Theorem,

$$X_{-n} \to \mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}],$$

almost surely. Note that $\mathbb{E}[X_{-1}] = \mathbb{E}[X_{-n}] = t$. Moreover, by (FATOU), the variance of the limit

$$\begin{aligned} \mathbb{E}[(\mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}] - t)^2] &\leq \liminf_n \mathbb{E}[(X_{-n} - t)^2] \\ &\leq \liminf_n \mathrm{Var}\left[\sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2\right] \\ &= \liminf_n 3 \sum_{j=1}^{k(n)} (t_j^{(n)} - t_{j-1}^{(n)})^2 \\ &\leq 3t \liminf_n \Delta(n) \\ &= 0. \end{aligned}$$

So finally

$$\mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}] = t.$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.