Lecture 21 : Markov property

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.2], [Lig10, Section 1.7], [MP10, Section 2.1].

1 Filtrations

Recall:

DEF 21.1 (Filtration) *A* filtration *is a family* $\{\mathcal{F}(t)\,:\,t \geq 0\}$ *of sub-*σ*-fields such that* $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ *for all* $s \leq t$*.*

We will consider two natural filtrations for BM.

DEF 21.2 *Let* $\{B(t)\}$ *be a BM. Then we denote*

$$\mathcal{F}^0(t) = \sigma(B(s)\,:\,0 \leq s \leq t).$$

Moreover, we let

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s).$$

Clearly $\mathcal{F}^0(t) \subseteq \mathcal{F}^+(t)$*. The latter has the advantage of being* right-continuous*, that is,*

$$\bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon) = \mathcal{F}^+(t).$$

DEF 21.3 (Germ field) *The* germ σ-field *is* $\mathcal{F}^+(0)$*.*

EX 21.4 *Let* $B(t)$ *be a standard BM and define*

$$T = \inf\{t > 0\,:\,B(t) > 0\}.$$

Then $\{T = 0\} \in \mathcal{F}^+(0)$ *since*

$$\{T = 0\} = \bigcap_{n\geq 1} \{\exists 0 < \varepsilon < n^{-1},\ B(\varepsilon) > 0\}.$$

2 Markov property

The basic Markov property for BM is the following.

THM 21.5 (Markov property I) *Suppose that $\{B(t)\}$ is a BM started at x. Let $s \geq 0$. Then the process $\{B(s+t) - B(s)\}_{t\geq 0}$ is a BM started at 0 and is independent of the process $\{B(t)\,:\,0 \leq s \leq t\}$, that is, the σ-fields*

$$\sigma(B(s+t) - B(s)\,:\,t \geq 0),$$

and

$$\sigma(B(t)\,:\,0 \leq t \leq s),$$

are independent.

Proof: We have already proved that $\{B(s+t) - B(s)\}_{t\geq 0}$ is a BM started at 0.

Further, recall:

LEM 21.6 (Independence and π-systems) *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras and that $\mathcal{I}$ and $\mathcal{J}$ are π-systems (i.e., families of subsets stable under finite intersections) such that*

$$\sigma(\mathcal{I}) = \mathcal{G}, \quad \sigma(\mathcal{J}) = \mathcal{H}.$$

Then $\mathcal{G}$ and $\mathcal{H}$ are independent if and only if $\mathcal{I}$ and $\mathcal{J}$ are, i.e.,

$$\mathbb{P}[I \cap J] = \mathbb{P}[I]\mathbb{P}[J], \quad \forall I \in \mathcal{I}, J \in \mathcal{J}.$$

Note that sets of the form

$$\{\omega\,:\,B(t_j) \in A_j,\ 0 \leq t_j \leq t,\ j = 1, \ldots, n\},$$

for $A_j \in \mathcal{B}$ are a π-system generating $\mathcal{F}^0(t)$. Similarly for $\sigma(B(s+t) - B(s)\,:\,t \geq 0)$. Therefore the independence statement immediately follows from the independence of increments. ■

In fact, we can prove a stronger statement:

THM 21.7 (Markov property II) *Suppose that $\{B(t)\}$ is a BM started at x. Let $s \geq 0$. Then the process $\{B(s+t) - B(s)\}_{t\geq 0}$ is a BM started at 0 and is independent of $\mathcal{F}^+(s)$.*

Proof: By continuity,

$$B(t+s) - B(s) = \lim_n B(s_n + t) - B(s_n),$$

for a strictly decreasing sequence $\{s_n\}_n$ converging to s. But note that for any $0 \leq t_1 < \cdots < t_j$

$$(B(t_1 + s_n) - B(s_n), \ldots, B(t_j + s_n) - B(s_n)),$$

is independent of $\mathcal{F}^+(s) \subseteq \mathcal{F}^0(s_n)$ and so is the limit. ■

3 Applications

As a first application, we get the following.

THM 21.8 (Blumenthal's 0**-**1 **law)** *For any x, the germ σ-field $\mathcal{F}^+(0)$ of a BM started at x is trivial.*

Proof: Let

$$A \in \mathcal{F}^+(0) \subseteq \sigma(B(t) \,:\, t \geq 0) = \sigma(B(t) - x \,:\, t \geq 0).$$

By the previous theorem, the two σ-fields above are independent and therefore A is independent of itself, that is,

$$\mathbb{P}[A] = \mathbb{P}[A \cap A] = \mathbb{P}[A]^2,$$

or $\mathbb{P}[A] \in \{0, 1\}$. ■

We come back to our example.

EX 21.9 *Let $B(t)$ be a standard BM and define*

$$T = \inf\{t > 0 \,:\, B(t) > 0\}.$$

Then $\{T = 0\} \in \mathcal{F}^+(0)$ since

$$\{T = 0\} = \bigcap_{n \geq 1} \{\exists 0 < \varepsilon < n^{-1},\ B(\varepsilon) > 0\}.$$

Hence,

$$\mathbb{P}[T = 0] \in \{0, 1\}.$$

We show that it is 1 by showing that it is positive. Note that

$$\mathbb{P}[T \leq t] \geq \mathbb{P}[B(t) > 0] = \frac{1}{2},$$

for $t > 0$, by symmetry of the Gaussian. It also follows by continuity that

$$\inf\{t > 0 \,:\, B(t) = 0\} = 0,$$

almost surely.

An immediate application of Blumenthal's 0-1 law (by time inversion) is:

THM 21.10 (0**-**1 **law for tail events)** *Let $B(t)$ be a BM. Then the tail of B, that is,*

$$\mathcal{T} = \bigcap_{t \geq 0} \mathcal{G}(t) = \bigcap_{t \geq 0} \sigma(B(s) \,:\, s \geq t),$$

is trivial.

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.