Lecture 23 : Martingale property

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.5], [Lig10, Section 1.9], [MP10, Section 2.4].

1 Martingales

We first generalize MGs to continuous time.

DEF 23.1 (Continuous-time martingale) *A real-valued SP* $\{X(t)\}_{t\geq 0}$ *is a* martingale *with respect to a filtration* $\{\mathcal{F}(t)\}$ *if it is* adapted*, that is,* $X(t) \in \mathcal{F}(t)$ *for all* $t \geq 0$*, if* $E|X(t)| < +\infty$ *for all* $t \geq 0$*, and if*

$$\mathbb{E}[X(t) \,|\, \mathcal{F}(s)] = X(s),$$

almost surely, for all $0 \leq s \leq t$*.*

EX 23.2 *Let* $\{B(t)\}$ *be a standard BM. Then*

$$\begin{aligned}\mathbb{E}[B(t) \,|\, \mathcal{F}^+(s)] &= \mathbb{E}[B(t) - B(s) \,|\, \mathcal{F}^+(s)] + B(s)\\ &= \mathbb{E}[B(t) - B(s)] + B(s)\\ &= B(s),\end{aligned}$$

by the Markov property. Hence BM is a MG.

2 Optional stopping theorem

THM 23.3 (Optional stopping theorem) *Suppose* $\{X(t)\}_{t\geq 0}$ *is a continuous MG, and* $0 \leq S \leq T$ *are stopping times. If the process* $\{X(T \wedge t)\}_{t\geq 0}$ *is dominated by an integrable RV* X*, then*

$$\mathbb{E}[X(T) \,|\, \mathcal{F}(S)] = X(S),$$

almost surely.

Proof: Fix N and consider the discrete-time MG

$$X_n = X(T \wedge n2^{-N})$$

and the stopping times

$$S'_N = \lfloor 2^N S \rfloor + 1$$

and

$$T'_N = \lfloor 2^N T \rfloor + 1$$

with respect to the filtration

$$\mathcal{G}_n = \mathcal{F}(n2^{-N}).$$

The discrete-time optional stopping theorem gives

$$\mathbb{E}[X_{T'_N} \,|\, \mathcal{G}_{S'_N}] = X_{S'_N},$$

which is equivalent to

$$\mathbb{E}[X(T \wedge 2^{-N}T'_N) \,|\, \mathcal{F}(2^{-N}S'_N)] = \mathbb{E}[X(T) \,|\, \mathcal{F}(2^{-N}S'_N)] = X(T \wedge 2^{-N}S'_N).$$

For $A \in \mathcal{F}(S) \subseteq \mathcal{F}(2^{-N}S'_N)$, by the definition of the conditional expectation and the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}[X(T); A] &= \lim_N \mathbb{E}[\mathbb{E}[X(T) \,|\, \mathcal{F}(2^{-N}S'_N)]; A] \\ &= \mathbb{E}[\lim_N X(T \wedge 2^{-N}S'_N); A] \\ &= \mathbb{E}[X(S); A],\end{aligned}$$

where we used continuity. ■

3 Applications

A typical application is Wald's lemma.

THM 23.4 (Wald's lemma for BM) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that either:*

1. $\mathbb{E}[T] < +\infty$*, or*
2. $\{B(t \wedge T)\}$ *is dominated by an integrable RV.*

Then $\mathbb{E}[B(T)] = 0$.

Proof: The result under the second condition follows immediately from the optional stopping theorem with $S = 0$. We show that the first condition implies the second one.

Assume $\mathbb{E}[T] < +\infty$. Define

$$M_k = \max_{0 \leq t \leq 1} |B(t+k) - B(k)|,$$

and

$$M = \sum_{k=1}^{\lceil T \rceil} M_k,$$

and note that $|B(t \wedge T)| \leq M$.

Then

$$\begin{aligned} E[M] &= \sum_k \mathbb{E}[\mathbb{1}\{T > k-1\} M_k] \\ &= \sum_k \mathbb{P}[T > k-1] \mathbb{E}[M_k] \\ &= \mathbb{E}[M_0] \mathbb{E}[T+1] < +\infty \end{aligned}$$

by our result on the maximum from the previous lecture. ■

We state without proof:

THM 23.5 (Wald's second lemma) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that $\mathbb{E}[T] < +\infty$. Then*

$$\mathbb{E}[B(T)^2] = E[T].$$

Proof: The proof is based on the fact that $B(t)^2 - t$ is a MG. Consider

$$T_n = \inf\{t \geq 0 \,:\, |B(t)| = n\},$$

and take an appropriate limit. See [MP10] for details. ■

An immediate application of Wald's lemma gives:

THM 23.6 *Let $\{B(t)\}$ be a standard BM. For $a < 0 < b$ let*

$$T = \inf\{t \geq 0 \,:\, B(t) \in \{a, b\}\}.$$

Then

$$\mathbb{P}[B(T) = a] = \frac{b}{|a| + b}.$$

Moreover,

$$\mathbb{E}[T] = |a| b.$$

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.