Lecture 24 : Skorokhod embedding

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.6, 8.8], [Lig10, Section 1.10], [MP10, Section 5.1, 5.3].

1 Previous class

Recall:

THM 24.1 (Wald's lemma for BM) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that $\mathbb{E}[T] < +\infty$. Then*

$$\mathbb{E}[B(T)] = 0.$$

THM 24.2 (Wald's second lemma) *Let $\{B(t)\}$ be a standard BM and T a stopping time with respect to $\{\mathcal{F}^+(t)\}$ such that $\mathbb{E}[T] < +\infty$. Then*

$$\mathbb{E}[B(T)^2] = E[T].$$

THM 24.3 *Let $\{B(t)\}$ be a standard BM. For $a < 0 < b$ let*

$$T = \inf\{t \geq 0 \,:\, B(t) \in \{a, b\}\}.$$

Then

$$\mathbb{P}[B(T) = a] = \frac{b}{|a| + b}.$$

Moreover,

$$\mathbb{E}[T] = |a|b.$$

2 Skorokhod embedding

THM 24.4 (Skorokhod embedding) *Suppose $\{B(t)\}_t$ is a standard BM and that X is a RV with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] < +\infty$. Then there exists a stopping time T with respect to $\{\mathcal{F}^+(t)\}_t$ such that $B(T)$ has the law of X and $\mathbb{E}[T] = \mathbb{E}[X^2]$.*

The proof uses a binary splitting MG:

DEF 24.5 *A* $\{X_n\}_n$ *is* binary splitting *if, whenever the event*

$$A(x_0, \ldots, x_n) = \{X_0 = x_0, \ldots, X_n = x_n\},$$

for some $x_0, \ldots, x_n$*, has positive probability, then the RV* X_{n+1} *conditioned on* $A(x_0, \ldots, x_n)$ *is supported on at most two values.*

LEM 24.6 *Let* X *be a RV with* $\mathbb{E}[X^2] < +\infty$*. Then there is a binary splitting MG* $\{X_n\}_n$ *such that* $X_n \to X$ *almost surely and in* $\mathcal{L}^2$*.*

Proof:(of Lemma) The MG is defined recursively. Let

$$\mathcal{G}_0 = \{\emptyset, \Omega\},$$

and

$$X_0 = \mathbb{E}[X].$$

For $n > 0$, we let

$$\xi_n = \begin{cases} 1, & \text{if } X \geq X_n \\ -1, & \text{if } X < X_n, \end{cases}$$

and

$$\mathcal{G}_n = \sigma(\xi_0, \ldots, \xi_{n-1}),$$

and

$$X_n = \mathbb{E}[X \,|\, \mathcal{G}_n].$$

Then $\{X_n\}_n$ is a binary splitting MG. It remains to prove the convergence claim.

By (cJENSEN)

$$\mathbb{E}[X_n^2] \leq \mathbb{E}[X^2],$$

so $\{X_n\}_n$ is bounded in $\mathcal{L}^2$ and we have by Lévy's upward theorem

$$X_n \to X_\infty = \mathbb{E}[X \,|\, \mathcal{G}_\infty],$$

almost surely and in $\mathcal{L}^2$, where

$$\mathcal{G}_\infty = \sigma\left(\bigcup_i \mathcal{G}_i\right).$$

We need to show that $X = X_\infty$.

CLAIM 24.7 *Almost surely,*

$$\lim_n \xi_n(X - X_{n+1}) = |X - X_\infty|.$$

We first finish the proof of the lemma. Note that

$$\mathbb{E}[\xi_n(X - X_{n+1})] = \mathbb{E}[\xi_n \mathbb{E}[X - X_{n+1} \,|\, \mathcal{G}_{n+1}]] = 0.$$

Since $\{\xi_n(X - X_{n+1})\}_n$ is bounded in $\mathcal{L}^2$, the expectations converge and

$$\mathbb{E}|X - X_\infty| = 0.$$

Finally we prove the claim. If $X = X_\infty$, both sides are 0. If $X < X_\infty$, then for n large enough, $X < X_n$ and $\xi_n = -1$ by construction and the result holds. Similarly for the other case. ■

Proof:(of Theorem) Take a binary splitting MG as in the previous lemma. Since X_n conditioned on $A(x_0, \ldots, x_{n-1})$ is supported on two values, we can use the stopping time from last time and we get a sequence of stopping times

$$T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots \uparrow T$$

for some T such that

$$B(T_n) \sim X_n,$$

and

$$\mathbb{E}[T_n] = \mathbb{E}[B(T_n)^2].$$

By (MON) and $\mathcal{L}^2$ convergence

$$\mathbb{E}[T] = \lim_n \mathbb{E}[T_n] = \lim_n \mathbb{E}[X_n^2] = \mathbb{E}[X].$$

By continuity of paths,

$$B(T_n) \to B(T), \qquad \text{a.s.}$$

and

$$B(T) \sim X.$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.