Lecture 28 : Random walks: recurrence

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 4.2].

1 Random walks and recurrence

DEF 28.1 *A* random walk (RW) *on $\mathbb{R}^d$ is an SP of the form:*

$$S_n = \sum_{i \leq n} X_i, \ n \geq 1$$

where the X_is are iid in $\mathbb{R}^d$.

EX 28.2 (SRW on $\mathbb{Z}^d$) *This is the special case:*

$$\mathbb{P}[X_i = e_j] = \mathbb{P}[X_i = -e_j] = \frac{1}{2d},$$

for all $j = 1, \ldots, d$ where e_j is the unit vector in the j-th direction.

DEF 28.3 *We say that $x \in \mathbb{R}^d$ is a recurrent value if, for all $\varepsilon > 0$, $\mathbb{P}[\|S_n - x\| < \varepsilon \text{ i.o.}] = 1$. Let V be the set of recurrent values. We say that S_n is transient if $V = \emptyset$, o.w. it is recurrent.*

2 SRW on $\mathbb{Z}$

Recall Stirling's formula:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

THM 28.4 (SRW on $\mathbb{Z}$) *SRW on $\mathbb{Z}$ is recurrent.*

Proof: First note the periodicity. So we look at S_{2n}. Then

$$\begin{aligned}
\mathbb{P}[S_{2n} = 0] &= \binom{2n}{n} 2^{-2n} \\
&\sim 2^{-2n} \frac{(2n)^{2n}}{(n^n)^2} \frac{\sqrt{2n}}{\sqrt{2\pi} n} \\
&\sim \frac{1}{\sqrt{\pi n}}.
\end{aligned}$$

So

$$\sum_m \mathbb{P}[S_m = 0] = \infty.$$

Denote

$$T_0^{(n)} = \inf\{m > T_0^{(n-1)} \,:\, S_m = 0\}.$$

By the strong Markov property $\mathbb{P}[T_0^{(n)} < \infty] = \mathbb{P}[T_0 < \infty]^n$. Note that

$$\begin{aligned}
\sum_m \mathbb{P}[S_m = 0] &= \mathbb{E}[\sum_m \mathbb{1}_{S_m=0}] \\
&= \mathbb{E}[\sum_n \mathbb{1}_{T_0^{(n)}<\infty}] \\
&= \sum_n \mathbb{P}[T_0^{(n)} < \infty] \\
&= \sum_n \mathbb{P}[T_0 < \infty]^n \\
&= \frac{1}{1 - \mathbb{P}[T_0 < \infty]}.
\end{aligned}$$

So $\mathbb{P}[T_0 < \infty] = 1$. ■

3 SRW on $\mathbb{Z}^2$

Now X_1 is in $\mathbb{Z}^2$ and $\mathbb{P}[X_1 = (1,0)] = \cdots = \mathbb{P}[X_1 = (0,-1)] = 1/4$.

THM 28.5 (SRW on $\mathbb{Z}^2$) *SRW on $\mathbb{Z}^2$ is recurrent.*

Proof: Let $R_n = (S_n^{(1)}, S_n^{(2)})$ where $S_n^{(i)}$ are independent SRW on $\mathbb{Z}$. Note that R_n is a SRW on $\mathbb{Z}^2$ rotated by 45 degrees. So the probability to be back at $(0,0)$ is the same as for two independent SRW on $\mathbb{Z}$ to be back at 0 simultaenously. Therefore,

$$\mathbb{P}[S_{2n} = (0,0)] = \mathbb{P}[S_{2n}^{(1)} = 0]^2 \sim \frac{1}{\pi n},$$

whose sum diverges. ■

4 SRW on $\mathbb{Z}^3$

Now X_1 is in $\mathbb{Z}^3$ and $\mathbb{P}[X_1 = (1,0,0)] = \cdots = \mathbb{P}[X_1 = (0,0,-1)] = 1/6$.

THM 28.6 (SRW on $\mathbb{Z}^3$) *SRW on $\mathbb{Z}^3$ is transient.*

Proof: Note, since the number of steps in opposite directions has to be equal,

$$\begin{aligned}\mathbb{P}[S_{2n}=0] &= 6^{-2n}\sum_{j,k}\frac{(2n)!}{(j!k!(n-k-j)!)^2}\\ &= 2^{-2n}\binom{2n}{n}\sum_{j,k}\left(3^{-n}\frac{n!}{j!k!(n-k-j)!}\right)^2\\ &\leq 2^{-2n}\binom{2n}{n}\max_{j,k}3^{-n}\frac{n!}{j!k!(n-k-j)!},\end{aligned}$$

where we used that $\sum_{j,k} a_{j,k}^2 \leq \max_{i,j} a_{j,k} \equiv a^*$ if $\sum_{j,k} a_{j,k} = 1$ and $a_{j,k} \geq 0$. Note that if $j < n/3$ and $k > n/3$ then

$$\frac{(j+1)!(k-1)!}{j!k!} = \frac{j+1}{k} \leq 1.$$

That implies that the term in the max is maximized when $j, k, (n-k-j)$ are roughly $n/3$. Using Stirling

$$\frac{n!}{j!k!(n-k-j)!} \sim \frac{n^n}{j^j k^k (n-k-j)^{n-k-j}}\sqrt{\frac{n}{jk(n-k-j)}}\frac{1}{2\pi} \sim C\frac{3^n}{n}.$$

Hence $\mathbb{P}[S_{2n}=0] \sim Cn^{-3/2}$ which is summable and $\mathbb{P}[T_0 < \infty] < 1$. ■

COR 28.7 *SRW on $\mathbb{Z}^d$ with $d > 3$ is transient.*

Proof: Let $R_n = (S_n^1, S_n^2, S_n^3)$. Let

$$U_m = \inf\{n > U_{m-1} \,:\, R_n \neq R_{U_{m-1}}\}.$$

Then R_{U_n} is a three-dimensional SRW. It visits $(0,0,0)$ only finitely many times whp. ■

5 RW in $\mathbb{R}^d$

Now X_1 is in $\mathbb{R}^d$. See [Dur10, Section 3.2] for a proof of:

- S_n is recurrent in $d = 1$ if $S_n/n \to 0$ in probability
- S_n is recurrent in $d = 2$ if $S_n/\sqrt{n} \Rightarrow$ Gaussian
- S_n is recurrent in $d \geq 3$ if it is truly three-dimensional (for all $\theta \neq 0$, $\mathbb{P}[X_1 \cdot \theta \neq 0] > 0$)

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.